Coherent Motions in the Fermi-Pasta-Ulam Model

C. Eugene Wayne*
Department of Mathematics and Statistics
Boston University

June 21, 2008

Lecture 1: The Fermi-Pasta-Ulam model; introduction and KdV approximation

Outline:

1. Models of coupled oscillators
2. Two main classes
   (a) Models with on-site potentials; e.g. Frenkel-Kontorova or discrete NLS
   (b) Models with only nearest neighbor coupling; e.g. FPU type models
3. Focus on:
   (a) localized oscillations
   (b) traveling waves
4. Numerical experiments of Kruskal and Zabusky and the discovery of solitons.
5. The formal approximation of the FPU model by the KdV equation
6. A general method of justifying approximations by modulation equations
7. Details of the approximation proof in the case of the FPU model.

*The work of the author is supported in part by the NSF under grant number DMS-0405724

Lecture 2: Discrete Breathers

Outline:

1. Periodic solutions in nonlinear equations and discrete breaters
2. Periodic solutions in the linearized FPU model
3. Approaches for constructing discrete breathers in the FPU model:
   (a) Variational Methods
   (b) Methods based on the Implicit Function Theorem
      i. Continue solutions from the “anti-integrable” limit, rather than the linear problem
   (c) Center-manifold methods
      i. Toda’s “dual formulation” of the FPU equations.
      ii. A dynamical systems on the space of loops
      iii. Infinite dimensional map on the space of Fourier coefficients
      iv. The equation on the center-manifold.
      v. Possible multi-scale expansions for breathers.

Lecture 3: Traveling Waves in the FPU model

Outline:

1. The Toda Lattice
   (a) Explicit form of the one and two-soliton solutions.
2. Traveling waves in more general FPU models
   (a) Variational Approach
   (b) Center-manifold Approach
   (c) ”Renormalization” approach
3. The Friesecke-Pego renormalization approach
   (a) Relation of the traveling wave profile to the KdV equation
   (b) Stability of traveling waves
      i. What kind of stability can one expect?
   (c) Modulation equations
   (d) Linear stability implies nonlinear stability

Lecture 4: Counterpropagating two-solitons in the FPU model

Outline:

1. Linear Estimates:
   (a) Bäcklund Transformations for the Toda model.
   (b) A perturbation argument for general FPU solitons in the KdV limit.
2. The two-soliton problem
   (a) Localizing the perturbation
   (b) Controlling the interaction
   (c) Differences with stability problems
3. Statement of results
4. Sketch of proof

References

References

[1] D.K. Campbell, P. Rosenau, and G.M. Zaslavsky. Introduction: The Fermi-Pasta-Ulam problem – the first fifty years. *Chaos*, 15:1–4, 2005.

[2] J.-P. Eckmann and C. E. Wayne. Propagating fronts and the center manifold theorem. *Comm. Math. Phys.*, 136(2):285–307, 1991.

[3] S. Flach and C. R. Willis. Discrete breathers. *Phys. Rep.*, 295(5):181–264, 1998.

[4] G. Friesecke and R. L. Pego. Solitary waves on FPU lattices. I. Qualitative properties, renormalization and continuum limit. *Nonlinearity*, 12(6):1601–1627, 1999.

[5] G. Friesecke and R. L. Pego. Solitary waves on FPU lattices. II. Linear implies nonlinear stability. *Nonlinearity*, 15(4):1343–1359, 2002.

[6] G. Friesecke and R. L. Pego. Solitary waves on Fermi-Pasta-Ulam lattices. III. Howland-type Floquet theory. *Nonlinearity*, 17(1):207–227, 2004.

[7] G. Friesecke and R. L. Pego. Solitary waves on Fermi-Pasta-Ulam lattices. IV. Proof of stability at low energy. *Nonlinearity*, 17(1):229–251, 2004.

[8] Gero Friesecke and Jonathan A. D. Wattis. Existence theorem for solitary waves on lattices. *Comm. Math. Phys.*, 161(2):391–418, 1994.

[9] Aaron Hoffman and C. Eugene Wayne. Counterpropagating two-soliton solutions in the FPU lattice. Preprint; arXiv:0806.1637, 2008.

[10] Aaron Hoffman and C. Eugene Wayne. A simple proof of the stability of solitary waves in the Fermi-Pasta-Ulam model near the KdV limit. In preparation, 2008.

[11] Gérard Iooss. Travelling waves in the Fermi-Pasta-Ulam lattice. *Nonlinearity*, 13(3):849–866, 2000.

[12] Gérard Iooss and Klaus Kirchgässner. Travelling waves in a chain of coupled nonlinear oscillators. *Comm. Math. Phys.*, 211(2):439–464, 2000.

[13] G. James and Y. Sire. Center manifold theory in the context of infinite one-dimensional lattices. In G. Gallavotti, editor, *The Fermi-Pasta-Ulam Problem. A Status Report*, volume 728 of *Lecture Notes in Physics*, pages 207–238. Springer Verlag, 2008.

[14] Guillaume James. Existence of breathers on FPU lattices. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(6):581–586, 2001.

[15] R. S. MacKay and S. Aubry. Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators. *Nonlinearity*, 7(6):1623–1643, 1994.

[16] Yvan Martel and Frank Merle. Asymptotic stability of solitons for subcritical generalized KdV equations. *Arch. Ration. Mech. Anal.*, 157(3):219–254, 2001.

[17] Yvan Martel and Frank Merle. Asymptotic stability of solitons of the gKdV equations with general nonlinearity. *Math. Ann.*, 341(2):391–427, 2008.

[18] Tetsu Mizumachi. Asymptotic stability of lattice solitons in the energy space. Preprint; arXiv:0711.2134, 2007.

[19] Tetsu Mizumachi and Robert L. Pego. Asymptotic stability of toda lattice solitons. Preprint, 2007.

[20] Alexander Pankov. *Travelling waves and periodic oscillations in Fermi-Pasta-Ulam lattices*. Imperial College Press, London, 2005.

[21] Guido Schneider and C. Eugene Wayne. Counter-propagating waves on fluid surfaces and the continuum limit of the Fermi-Pasta-Ulam model. In *International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999)*, pages 390–404. World Sci. Publ., River Edge, NJ, 2000.

[22] Moikazu Toda. *The Theory of Nonlinear Lattices*, volume 20 of *Springer Series in Solid-State Sciences*. Springer Verlag, 2nd edition, 1989.

[23] Morikazu Toda. Vibration of a chain with nonlinear interaction. *J. Phys. Soc. Japan*, 22:431–436, 1967.

[24] Morikazu Toda. Studies of a non-linear lattice. *Phys. Rep.*, 18C(1):1–123, 1975.

[25] Morikazu Toda and Miki Wadati. A soliton and two solitons in an exponential lattice and related equations. *J. Phys. Soc. Japan*, 34(1):18–25, 1972.

[26] Morikazu Toda and Miki Wadati. A canonical transformation for the exponential lattice. *J. Phys. Soc. Japan*, 39(5):1204–1211, 1975.

[27] M. I. Weinstein. Excitation thresholds for nonlinear localized modes on lattices. *Nonlinearity*, 12(3):673–691, 1999.

[28] N. J. Zabusky and M. D. Kruskal. Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.*, 15(6):240–243, Aug 1965.