Corrigendum: Justification of the NLS approximation for a quasilinear water wave model

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The publication [1] contains error estimates for the NLS approximation for a quasilinear water wave model where the right hand side looses only half a derivative when written as a first order system. In order to control the solutions on the natural time scale of the NLS equation the quadratic terms of the water wave model are eliminated with the help of a normal form transform. The normal form transformation is of the form identity plus a “small” term which looses half a derivative and this complicates the computation of the inverse of the mapping. In [1], the inverse was constructed with the help of energy estimates. However, one of these estimates contained an error which we correct here.

The model problem considered in [1] is given in Fourier space by

$$\partial_t^2 \hat{u}(k,t) = -\omega^2(k)\hat{u}(k,t) - \omega^2(k)(\hat{u} * \hat{u})(k,t), \tag{1}$$

where $\omega(k)^2 = k \tanh k$ with $\omega(k) > 0$ for $k > 0$ and $\omega(k) < 0$ for $k < 0$. This equation was chosen so that (1) and the water wave problem have the same linear dispersion relation and such that the right hand side loses only half a derivative when written as a first order system - again, having in mind the similar properties of the Lagrangian formulation of the water wave problem.

In constructing the normal form transform in [1] we lost a factor i in the calculation which makes it necessary to modify the argument for the inversion of the normal form transform. The changes are as follows:

We write (1) as the first order system yields

$$\partial_t \hat{u}(k,t) = i\omega(k)\hat{v}(k,t), \qquad \partial_t \hat{v}(k,t) = i\omega(k)\hat{u}(k,t) + i\omega(k)(\hat{u} * \hat{u})(k,t), \tag{2}$$

If we now set

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

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this system is transformed into

$$
\begin{aligned}
\partial_t \widehat{U}_1(k,t) &= i\omega(k)\widehat{U}_1(k,t) + \int \sum_{m,n\in\{1,2\}} \widehat{\alpha}^1_{mn}(k)\widehat{U}_m(k-\ell,t)\widehat{U}_n(\ell,t)\,\mathrm{d}\ell, \\
\partial_t \widehat{U}_2(k,t) &= -i\omega(k)\widehat{U}_2(k,t) + \int \sum_{m,n\in\{1,2\}} \widehat{\alpha}^2_{mn}(k)\widehat{U}_m(k-\ell,t)\widehat{U}_n(\ell,t)\,\mathrm{d}\ell
\end{aligned}
\tag{3}
$$

where $\widehat{\alpha}^1_{11}(k) = \mathcal{O}(\sqrt{|k|})$ for $|k| \to \infty$, for instance $\widehat{\alpha}^j_{mn}(k) = i\omega(k)/\sqrt{2}$. (This system of equations is equivalent to (13) in [1]. But we have chosen a transformation matrix with real coefficients to make it easier to keep track of the factors of "i" in the subsequent computation. By this choice the NLS approximation $\varepsilon\Psi$ has only $\mathcal{O}(\varepsilon)$-terms in the first component which are of the form $\Psi_1 = \Psi_+ + \Psi_-$ where $\Psi_+ = \overline{\Psi}_-$ and where in Fourier space $\widehat{\Psi}_\pm$ is concentrated in an $\mathcal{O}(\varepsilon)$-neighborhood of the wave number $\pm k_0$. The error $\varepsilon^\beta\vartheta R_1 = \widehat{U}_1 - \varepsilon\Psi$ and $\varepsilon^\beta\vartheta R_2 = \widehat{U}_2 + \mathcal{O}(\varepsilon^2)$ satisfies

$$
\begin{aligned}
\partial_t \widehat{R}_1(k,t) &= i\omega(k)\widehat{R}_1(k,t) + 2\varepsilon \int \sum_{n\in\{1,2\}} \widehat{\vartheta}^{-1}(k)\widehat{\alpha}^1_{1n}(k)\widehat{\Psi}_1(k-\ell,t)\widehat{\vartheta}(\ell)\widehat{R}_n(\ell,t)\,\mathrm{d}\ell + h.o.t., \\
\partial_t \widehat{R}_2(k,t) &= -i\omega(k)\widehat{R}_2(k,t) + 2\varepsilon \int \sum_{n\in\{1,2\}} \widehat{\vartheta}^{-1}(k)\widehat{\alpha}^2_{1n}(k)\widehat{\Psi}_1(k-\ell,t)\widehat{\vartheta}(\ell)\widehat{R}_n(\ell,t)\,\mathrm{d}\ell + h.o.t.,
\end{aligned}
$$

with $\widehat{\vartheta}(k) = \min(\varepsilon + (1-\varepsilon)|k|/\delta, 1)$ (see (25) of [1]). In order to prove estimates on an $\mathcal{O}(\varepsilon^{-2})$ time scale one eliminates the terms of $\mathcal{O}(\varepsilon)$ with a near identity change of variables, namely

$$
\widetilde{R}_{j_1} = R_{j_1} + \varepsilon B^+_{j_1}(\Psi,R) + \varepsilon B^-_{j_1}(\Psi,R) + h.o.t.\ , \quad j_1 = 1,2\ , \tag{4}
$$

where

$$
\widehat{B}^\pm_{j_1}(\Psi,R) = \sum_{j_2=1}^{2} \int \widehat{b}^\pm_{j_1;j_2}(k)\widehat{\Psi}_\pm(k-\ell)\widehat{R}_{j_2}(\ell)d\ell, \tag{5}
$$

and where $\widehat{R}_{j_3}$ refers to the j_3-th component of the error $\widehat{R}$. As shown in [1], the kernel function $\hat{b}^\pm_{j_1;j_2}(k)$ has the form:

$$
\hat{b}^\pm_{j_1;j_2}(k) = \frac{2\widehat{P}^1(k)\widehat{\alpha}^{j_1}_{1,j_2}(k)}{(i\omega_{j_1}(k) - i\omega_1(\pm k_0) - i\omega_{j_2}(k\mp k_0))}\frac{\widehat{\vartheta}_0(k\mp k_0)}{\widehat{\vartheta}(k)} \tag{6}
$$

with $\omega_{1,2}(k) = \pm\omega(k)$, $\widehat{P}^1(k)\widehat{\vartheta}_0(k\mp k_0)\widehat{\vartheta}^{-1}(k)$ uniformly bounded for all $k\in\mathbb{R}$, and with $\widehat{\alpha}^{j_1}_{1,j_1}(k) = \mathcal{O}(\sqrt{|k|})$ for $|k|\to\infty$. (For the derivation of the formula see equation (52) and the discussion leading up to it in [1].) The growth of the kernel function with k in (7) means that the the normal form transformation is unbounded on Sobolev spaces, and the near identity change of variables cannot be inverted with the help of Neumann's series. In [1] it has been pointed out that an inversion is possible with the help of energy estimates if the $\widehat{b}^{\pm}_{j_1;j_2}(k)$ are purely imaginary. However, in [1], it was overlooked that $\widehat{\alpha}^{j_1}_{1,j_1}(k)$ is pure imaginary (see the remark following (3) above) and hence the kernel on the right hand side of (6) is real-valued which means that the argument in [1] must be modified.

In order to extract the relevant terms in (7) we remark as in [1] that $\hat{b}^{\pm}_{j_1;j_2}(k) = \mathcal{O}(1)$ for $|k| \to \infty$ if $j_1 \neq j_2$ since in this case the numerator and denominator both grow at a rate $\mathcal{O}(\sqrt{|k|})$. Thus, the associated part of the normal form transformation is bounded and we can concentrate on those terms with $j_1 = j_2$ which are the only ones leading to a loss of smoothness. We next use the fact that

$$i\omega_{j_1}(k) - i\omega_1(\pm k_0) - i\omega_{j_2}(k \mp k_0)) = -i\omega_1(\pm k_0) + \mathcal{O}(1/\sqrt{|k|})$$

for $|k| \to \infty$ and that $\widehat{P}^1(k)\frac{\widehat{\vartheta}_0(k \mp k_0)}{\widehat{\vartheta}(k)} - 1$ has compact support. This means if we reconsider the transformations (7), they can be written as

$$\widetilde{R}_{j_1} = R_{j_1} + \int \widehat{b}^{+}_{j_1}(k)\widehat{\Psi}_{+}(k-\ell)\widehat{R}_{j_1}(\ell)d\ell + \int \widehat{b}^{-}_{j_1}(k)\widehat{\Psi}_{-}(k-\ell)\widehat{R}_{j_1}(\ell)d\ell + \mathbf{B}(\Psi, R). \qquad (7)$$

where

$$\hat{b}^{\pm}_{j_1}(k) \;=\; \frac{2\widehat{\alpha}^{j_1}_{1,j_1}(k)}{-i\omega_1(\pm k_0)} = (-1)^{j_1}\frac{\sqrt{2}\widehat{\omega}(k)}{\mp\omega_1(k_0)} \in \mathbb{R} \qquad (8)$$

and $\mathbf{B}(\Psi, R)$ is a bounded transformation. The key observation is that $\hat{b}^{+}_{j_1}(k) = -\hat{b}^{-}_{j_1}(k)$. As a consequence we have

$$\begin{aligned} & \int \widehat{b}^{+}_{j_1}(k)\widehat{\Psi}_{+}(k-\ell)\widehat{R}_{j_1}(\ell)d\ell + \int \widehat{b}^{-}_{j_1}(k)\widehat{\Psi}_{-}(k-\ell)\widehat{R}_{j_1}(\ell)d\ell \\ = & \int \widehat{b}^{+}_{j_1}(k)(\widehat{\Psi}_{+}(k-\ell) - \widehat{\Psi}_{-}(k-\ell))\widehat{R}_{j_1}(\ell)d\ell \\ = & \; 2\int i\widehat{b}^{+}_{j_1}(k)\widehat{\Im(\Psi)}(k-\ell)\widehat{R}_{j_1}(\ell)d\ell \end{aligned}$$

Since $\Im(\Psi)$, the imaginary part of Ψ, is real-valued and $i\widehat{b}^{+}_{j_1}(k)$ purely imaginary we are in exactly the situation covered by the energy estimates in [1, Section 5] for the inversion of the normal form transform. Hence, this modification shows that the normal form transformation is invertible as claimed in [1]. Note that the statements of the theorems in [1] are unchanged by this correction.

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References

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