An Introduction to KAM Theory

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1 Introduction

Over the past thirty years, the Kolmogorov-Arnold-Moser (KAM) theory has played an important role in increasing our understanding of the behavior of non-integrable Hamiltonian systems. I hope to illustrate in these lectures that the central ideas of the theory are, in fact, quite simple. With this in mind, I will concentrate on two examples and will forego generality for concreteness and (I hope) clarity. The results and methods which I will present are well-known to experts in the field but I hope that by collecting and presenting them in as simple a context as possible I can make them somewhat more approachable to newcomers than they are often considered to be.

The outline of the lectures is as follows. After a short historical introduction, I will explain in detail one of the simplest situations where the KAM techniques are used – the case of diffeomorphisms of a circle. I will then go on to discuss the theory in its original context, that of nearly-integrable Hamiltonian systems.

The problem which the KAM theory was developed to solve first arose in celestial mechanics. More than 300 years ago, Newton wrote down the differential equations satisfied by a system of massive bodies interacting through gravitational forces. If there are only two bodies, these equations can be explicitly solved and one finds that the bodies revolve on Keplerian ellipses about their center of mass. If one considers a third body (the "three-body-problem"), no exact solution exists – even if, as in the solar system, two of the bodies are much lighter then the third. In this case, however, one observes that the mutual gravitational force between these two "planets" is much weaker than that between either planet and the sun. Under these circumstances one can try to solve the problem perturbatively, first ignoring the interactions between the planets. This gives an **integrable** system, or one which can be solved explicitly, with each planet revolving around the sun oblivious of the other's existence. One can then try to systematically include the interaction between the planets in a perturbative fashion. Physicists and astronomers used this method extensively throughout the nineteenth century, developing series expansions for the solutions of these equations in the small parameter represented by the ratio of the mass of the planet to the mass of the sun. However, the convergence of these series was never established – not even when the King of Sweden offered a very substantial prize to anyone who succeeded in doing so. The difficulty in establishing the convergence of these series comes from the fact that the terms in the series have **small denominators** which we shall consider in some detail later in these lectures. One can obtain some physical insight into the origin of these convergence problems in the following way. As one learns in an elementary course in differential equations, a harmonic oscillator has a certain natural frequency at which it oscillates. If one subjects such an oscillator to an external force of the same frequency as the natural frequency of the oscillator, one has **resonance** effects and the motion of the oscillator becomes unbounded. Indeed, if one has a typical nonlinear oscillator, then whenever the perturbing force has a frequency that is a rational multiple of the natural frequency of the oscillator, one will have resonances, because the nonlinearity will generate oscillations of all multiples of the basic driving frequency.

In a similar way, one planet exerts a periodic force on the motion of a second, and if the orbital periods of the two are commensurate, this can lead to resonance and instability. Even if the two periods are not exactly commensurate, but only approximately so the effects lead to convergence problems in the perturbation theory.

It was not until 1954 that A. N. Kolmogorov [8] in an address to the ICM in Amsterdam suggested a way in which these problems could be overcome. His suggestions contained two ideas which are central to all applications of the KAM techniques. These two basic ideas are:

- Linearize the problem about an approximate solution and solve the linearized problem – it is at this point that one must deal with the small denominators.

- Inductively improve the approximate solution by using the solution of the linearized problem as the basis of a Newton's method argument.

These ideas were then fleshed out, extended, and applied in numerous other contexts by V. Arnold and J. Moser, ([1], [9]) over the next ten years or so, leading to what we now know as the KAM theory.

As I said above, we will consider the details of this procedure in two cases. The first, the problem of showing that diffeomorphisms of a circle are conjugate to rotations, was chosen for its simplicity – the main ideas are visible with fewer technical difficulties than appear in other applications. We will then look at the KAM theory in its original setting of small perturbations of integrable Hamiltonian systems. I'll attempt to parallel the discussion of the case of circle diffeomorphisms as closely as possible in order to keep our focus on the main ideas of the theory and ignore as much as possible the additional technical complications which arise in this context.

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2 Circle Diffeomorphisms

Let us begin by discussing one of the simplest examples in which one encounters small denominators, and for which the KAM theory provides a solution. It may not be apparent for the moment what this problem has to do with the problems of celestial mechanics discussed in the introduction, but almost all of the difficulties encountered in that problem also appear in this context but in ways which are less obscured by technical difficulties – this is, if you like, our warm-up exercise.

We will consider orientation preserving diffeomorphisms of the circle, or equivalently, their lifts to the real line:

$$\phi : \mathbf{R}^1 \to \mathbf{R}^1$$

$$\phi(x) = x + \tilde{\eta}(x) \quad \text{with} \quad \tilde{\eta}(x+1) = \tilde{\eta}(x) \quad \text{and} \quad \tilde{\eta}'(x) > -1 \quad .$$

We wish to consider ϕ as a dynamical system, and study the behavior of its "orbits" – *i.e.* we want to understand the behavior of the sequences of points $\{\phi^{(n)}(x)(\text{mod}1)\}_{n=0}^{\infty}$, where $\phi^{(n)}$ means the n-fold composition of ϕ with itself. Typical questions of interest are whether or not these orbits are periodic, or dense in the circle.

The simplest such diffeomorphism is a rotation $R_\alpha(x) = x + \alpha$. Note that we understand "everything" about its dynamics. For instance, if α is rational, all the orbits of R_α are periodic, and none are dense. However, we would like to study more complicated dynamical systems than this. Thus we will suppose that

$$\phi(x) = x + \alpha + \eta(x) \quad , \tag{1}$$

where as before, $\eta(x+1) = \eta(x)$ and $\eta'(x) > -1$. As I said in the introduction, I will not attempt to consider the most general case, but rather will focus on simplicity of exposition. Thus I will consider only **analytic** diffeomorphisms. Define the strips $S_\sigma = \{z \in \mathbf{C} \mid |Imz| < \sigma\}$. Then I will assume that

$$\begin{aligned} \eta \in B_\sigma = & \{\eta \mid \eta(z) \quad \text{is} \quad \text{analytic} \quad \text{on} \quad S_\sigma, \\ & \eta(x+1) = \eta(x) \quad \text{and} \quad \sup_{|Imz|<\sigma} |\eta(z)| \equiv \|\eta\|_\sigma < \infty\} \quad . \end{aligned}$$

Note that one can assume that $\sigma < 1$, without loss of generality.

Our goal in this section will be to understand the dynamics of $\phi(x) = x + \alpha + \eta(x)$ when η has small norm. One way to do this is to show that the dynamics of ϕ are "like" the dynamics of a system we understand – for instance, suppose that we could find a change of variables which transformed ϕ into a pure rotation. Then since we understand the dynamics of the rotation, we would also understand those of ϕ. If we express this change of variables as $x = H(\xi)$, where $H(\xi+1) = 1 + H(\xi)$ preserves the periodicity of ϕ, then we want to find H such that

$$H^{-1} \circ \phi \circ H(\xi) = R_\rho(\xi) \quad ,$$

or equivalently

$$\phi \circ H(\xi) = H \circ R_\rho(\xi) \quad . \tag{2}$$

Such a change of variables is said to conjugate ϕ to the rotation R_ρ.

Remark 2.1 *The relationship between this problem and the celestial mechanics questions discussed in the introduction now becomes more clear. In that case we wanted to understand the extent to which the motion of the solar system when we included the effects of the gravitational interaction between the various planets was "like" that of the simple Kepler system.*

In order to answer this question we need to introduce an important characteristic of circle diffeomorphisms, the **rotation number**

Definition 2.1 *The rotation number of ϕ is*

$$\rho(\phi) = \lim_{n\to\infty} \frac{\phi^{(n)}(x) - x}{n} \quad .$$

Remark 2.2 *It is a standard result of dynamical systems theory that for any homeomorphism of the circle the limit on the right hand side of this equation exists and is independent of x. (See [6], p. 296.)*

Remark 2.3 *Note that from the definition of the rotation number, it follows immediately that for any homeomorphism H, the map $\tilde{\phi} = H^{-1} \circ \phi \circ H$ has the same rotation number as ϕ. (Since $\tilde{\phi}^{(n)} = H^{-1} \circ \phi^{(n)} \circ H$, and the initial and final factors of H and H^{-1} have no effect on the limit.)*

As a final remark about the the rotation number we note that if $\phi(x) = x + \alpha + \eta(x)$, then an easy induction argument shows that $\rho(\phi) = \alpha + \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \eta \circ \phi^{(j)}(x)$. In particular, if $\alpha = \rho$, we have $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \eta \circ \phi^{(j)}(x) = 0$, so we have proved:

Lemma 2.1 *If $\phi(x) = x + \rho + \eta(x)$ has rotation number ρ, then there exists some x_0 such that $\eta(x_0) = 0$.*

We must next ask about the properties we wish the change of variables H to have. If we only demand that H be a homeomorphism, then **Denjoy's Theorem** ([6] p. 301) says that if the rotation number of ϕ is irrational, we can always find an H which conjugates ϕ to a rotation. However, if we want more detailed information about the dynamics it makes sense to ask that H have additional smoothness. In fact, it is natural to ask that H be as smooth as the diffeomorphism itself – in this case, analytic. (There will, in general, be some loss of smoothness even in this case. We will find, for example, that while there exists an analytic conjugacy function, H, its domain of analyticity will be somewhat smaller than that of ϕ.) Surprisingly, the techniques which

Denjoy used fail completely in this case, and the answer was not known until the late fifties when Arnold applied KAM techniques to answer the question in the case when η is small. Even more surprisingly, in order to even state Arnold's theorem, we have to discuss a little number theory.

Any irrational number can be approximated arbitrarily well by rational numbers, and in fact, **Dirichlet's Theorem** even gives us an estimate of how good this approximation is. More precisely, it says that given any irrational number ρ, there exist infinitely many pairs of integers (m, n) such that $|\rho - (m/n)| < 1/n^2$. On the other hand, most irrational numbers can't be approximated much better than this.

Definition 2.2 *The real number ρ is of type (K, ν) if there exist positive numbers K and ν such that $|\rho - (m/n)| > K|n|^{-\nu}$, for all pairs of integers (m, n).*

Proposition 2.1 *For every $\nu > 2$, almost every irrational number ρ is of type (K, ν) for some $K > 0$.*

Proof: The proof is not difficult, but would take us a bit out of our way. The details can be found in [3], page 116, for example. Note also, that we can assume without loss of generality that $K \leq 1$, since if ρ is of type $(\tilde{K}, \nu)$ for some $\tilde{K} > 1$, it is also of type $(1, \nu)$.

Theorem 2.1 (Arnold's Theorem [1]) *Suppose that ρ is of type (K, ν). There exists $\epsilon(K, \nu, \sigma) > 0$ such that if $\phi(x) = x + \rho + \eta(x)$ has rotation number ρ, and $\|\eta\|_\sigma < \epsilon(K, \nu, \sigma)$, then there exists an analytic and invertible change of variables $H(x)$ which conjugates ϕ to R_ρ.*

As mentioned above, Arnold's proof of this theorem used the KAM theory. The proof can be broken into two main parts – an analysis of a linearized equation, and a Newton's method iteration step. These same two steps will reappear in the next section when we discuss nearly integrable Hamiltonian systems, and they are characteristic of almost all applications of the KAM theory.

Remark 2.4 *It may seem that by assuming that the diffeomorphism is of the form $\phi(x) = x + \rho + \eta(x)$, where ρ is the rotation number of ϕ, we are considering a less general situation than that described above in which we allowed ϕ to have the form $x + \alpha + \eta(x)$. As we shall see below, there is no real loss of generality in this restriction.*

Step 1: Analysis of the Linearized equation

Note that since $\|\eta\|_\sigma$ is small, the diffeomorphism ϕ is "close" to the pure rotation R_ρ. Thus, we might hope that if a change of variables H which satisfies (2) exists is would be close to the identity *i.e.* $H(x) = x + h(x)$, where h is "small". If we make this assumption and substitute this form of H in (2), we find that h should satisfy the equation

$$h(x+\rho) - h(x) = \eta(x + h(x)) \tag{3}$$

If we now expand both sides of this equation, retaining only terms of first order in the (presumably) small quantities h and η, we find:

$$h(x+\rho) - h(x) = \eta(x) \tag{4}$$

Since all the functions in this equation are periodic, and the equation is linear in the unknown function h, we can immediately write down a (formal) solution for the coefficients in the Fourier series of h. If $\hat{\eta}(n)$ is the n^{th} Fourier coefficient of η, then the n^{th} Fourier coefficient of h is

$$\hat{h}(n) = \frac{\hat{\eta}(n)}{e^{2\pi i n\rho} - 1} \ , \qquad n \neq 0 \ \ . \tag{5}$$

In just a moment, we will address whether or not the function

$$h(x) = \sum_{n\neq 0} \hat{h}(n) e^{2\pi i n x} = \sum_{n \neq 0} \frac{\hat{\eta}(n)}{e^{2\pi i n \rho} - 1} e^{2\pi i n x}$$

makes any sense, however, we first note that even if (5) defines a well-behaved function, it will not solve (4) but rather:

$$h(x+\rho) - h(x) = \eta(x) - \int_0^1 \eta(x)dx = \eta(x) - \hat{\eta}(0) \ \ . \tag{6}$$

This is because the zeroth Fourier coefficient of h drops out of (4). The fact that h does not solve (4) will complicate the estimates below. The problems with showing that (5) converges arise due to the presence of the factors of $e^{2\pi i n\rho} - 1$ in the denominator of the summands, and these are the (in)famous **small denominators** which plagued celestial mechanics in the last century and which the KAM theory finally overcame. We first note that if ρ is rational, there is little hope that the sum defining h will converge since the denominators in this sum will vanish for the infinitely many n for which $\rho n = m$ for some $m \in \mathbf{Z}$. Thus, we can only hope for success if $\rho \notin \mathbf{Q}$. If ρ is irrational, the denominator will still be large whenever $n\rho \approx m$. However, by assuming that ρ is of type (K,ν), we have some control over how close to zero the denominator can become. In fact, the following lemma immediately allows us to estimate $h(x)$.

Lemma 2.2 *If ρ is of type (K,ν), then*

$$|e^{2\pi i n\rho} - 1| = |e^{2\pi i m}(e^{2\pi i(\rho n - m)} - 1)| \geq 4K|n|^{-(\nu-1)} \ \text{ if } \ n \neq 0 \ \ .$$

Proof: Since ρ is of type (K,ν), we know that $|\rho n - m| \geq K|n|^{-(\nu-1)}$ and the lemma follows by writing $|e^{2\pi i(\rho n - m)} - 1| = 2|\sin(\pi(\rho n - m))|$, and then using the fact that $|\sin(\pi x)| \geq 2|x|$, for $|x| \leq 1/2$.

□

The other fact which we must use to estimate h is the fact that since $\eta \in B_\sigma$, Cauchy's theorem immediately gives an estimate on its Fourier coefficients of the form $|\hat{\eta}(n)| \leq \|\eta\|_\sigma e^{-2\pi\sigma|n|}$. Combining this remark with Lemma 2.2, we see that if $|Imz| \leq \sigma - \delta$, one has

$$\begin{aligned} |h(z)| &= |\sum_{n\neq 0} \frac{\hat{\eta}(n)e^{2\pi i n z}}{e^{2\pi i n \rho} - 1}| \leq \sum_{n\neq 0} \frac{|n|^{(\nu-1)}}{4K} \|\eta\|_\sigma e^{-2\pi\sigma|n|} e^{2\pi|n|(\sigma-\delta)} \\ &\leq \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu} \|\eta\|_\sigma \quad , \end{aligned}$$

where $\Gamma(\nu) = \int_0^\infty x^{\nu-1}e^{-x}dx$, and we have assumed that $2\pi(K+1)\delta \leq 4\pi\delta < 1$. Thus we have proven,

Proposition 2.2 *If ρ is of type (K,ν), and $\eta \in B_\sigma$, then $h(x)$, defined by (5) is an element of $B_{\sigma-\delta}$ for any $\delta > 0$, and if $4\pi\delta < 1$, we have the estimate:*

$$\|h\|_{\sigma-\delta} \leq \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu} \|\eta\|_\sigma \quad .$$

Remark 2.5 *Note that we do* **not** *get an estimate on h in B_σ – we lose some analyticity. This is why we can't use an ordinary Implicit Function Theorem to solve (2). Indeed, if we were to attempt to proceed with an ordinary iterative method to solve (2), we would find that we gradually lost* **all** *of the analyticity of our approximate solution. This is where the second "big idea" of the KAM theory enters the picture, namely:*

Step2: Newton's Method in Banach Space

Recall that Newton's method says that if you want to find the roots of some nonlinear equation, you should take an approximate solution and then use a linear approximation to the function whose roots you seek to improve your approximation to the solution. You then use this improved approximation as your new starting point and iterate this procedure. In the present circumstance, we regard ϕ as an approximation to R_ρ and then use the linear approximation, $H(x) = x + h(x)$, to the conjugating function to improve that approximation. Recall that if $h(x)$ had solved (2) exactly, then $H^{-1} \circ \phi \circ H = R_\rho$. Our hope is that if we use the $H(x)$ that comes from solving (6), $H^{-1} \circ \phi \circ H$ will be closer to R_ρ than ϕ was and then we can iterate this process.

The first thing we must do is check that H is invertible. Since $H(z) = z + h(z)$, H will be invertible with analytic inverse on any domain on which $\|h'\| < 1$. Cauchy's theorem and Proposition 2.2 imply that $\|h'\|_{\sigma-2\delta} \leq \frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma$, so we conclude

Lemma 2.3 *If $2\pi\Gamma(\nu)\|\eta\|_\sigma < K(2\pi\delta)^{(\nu+1)}$ and $4\pi\delta < 1$, then $H(z)$ has an analytic inverse on the image of $S_{\sigma-2\delta}$.*

Remark 2.6 *Note that if we combine the inequalities in Lemma 2.3 and Proposition 2.2, we find $\|h\|_{\sigma-\delta} < \delta$. Thus, if $z \in S_{\sigma-2\delta}$, $H(z) \in S_{\sigma-\delta}$. Furthermore, H maps the real axis into itself, and the images of the lines $Imz = \pm(\sigma - 2\delta)$ lie outside the strip $S_{\sigma-3\delta}$. With this information it is easy to show that $Range(H|_{S_{\sigma-2\delta}}) \supset S_{\sigma-3\delta}$, so that $H^{-1}(z)$ is defined for all $z \in S_{\sigma-3\delta}$.*

In addition to knowing that the inverse exists, we need an estimate on its properties which the following proposition provides.

Proposition 2.3 *If*

$$2\pi\Gamma(\nu)\|\eta\|_\sigma < K(2\pi\delta)^{(\nu+1)} \quad \text{and} \quad 4\pi\delta < 1 \quad ,$$

then $H^{-1}(z) = z - h(z) + g(z)$, where

$$\|g\|_{\sigma-4\delta} \leq \frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}}\|\eta\|_\sigma^2 \quad .$$

Proof: If we define $g(z)$ by $g(z) = H^{-1}(z) - z + h(z)$, then we see that

$$z = H^{-1} \circ H(z) = z + h(z) - h(z + h(z)) + g(z + h(z)) \quad .$$

Thus, $g(z + h(z)) = h(z + h(z)) - h(z) = \int_0^1 h'(z + sh(z))h(z)ds$. Setting $\xi = H(z)$, this becomes

$$g(\xi) = \int_0^1 h'(H^{-1}(\xi) + sh(H^{-1}(\xi)))h(H^{-1}(\xi))ds \quad .$$

In a fashion similar to that in the remark just above, $Range(H|_{S_{\sigma-3\delta}}) \supset S_{\sigma-4\delta}$, so if $\xi \in S_{\sigma-4\delta}$, $H^{-1}(\xi) \in S_{\sigma-3\delta}$, and hence $H^{-1}(\xi) + sh(H^{-1}(\xi)) \in S_{\sigma-2\delta}$, so applying the estimates on h and h' from above we obtain

$$\|g\|_{\sigma-4\delta} \leq 2\pi\big(\frac{\Gamma(\nu)}{K}\big)^2\frac{\|\eta\|_\sigma^2}{(2\pi\delta)^{2\nu+1}}$$

□

Let us now examine the transformed diffeomorphism $\tilde{\phi}(x) = H^{-1} \circ \phi \circ H(x)$. Since h is only an approximate solution of (2), $\tilde{\phi}$ will not be exactly a rotation, but since h did solve the **linearized** equation (6), we hope that $\tilde{\phi}$ will differ from a rotation only by terms that are of second order in the small quantities h and η. Using the form of H^{-1} given by Proposition 2.3, we find

$$\begin{aligned}
\tilde{\phi}(x) &= H^{-1} \circ \phi \circ H(x) = H^{-1}(x + h(x) + \rho + \eta(x + h(x))) \\
&= x + h(x) + \rho + \eta(x + h(x)) - h(x + \rho + h(x) + \eta(x + h(x))) + \\
&\qquad + g(x + h(x) + \rho + \eta(x + h(x))) \\
&= x + \rho + \{h(x) - h(x + \rho) + \eta(x)\} + \{\eta(x + h(x)) - \eta(x)\} + \\
&\qquad + \{h(x + \rho) - h(x + \rho + h(x) + \eta(x + h(x)))\} + \\
&\qquad\quad + g(x + h(x) + \rho + \eta(x + h(x))) \quad .
\end{aligned}$$

We first note that because h solves (6), the first expression in braces in the second to last line of this sequence of equalities is equal to $\hat{\eta}_0$. The next expression in braces equals $\int_0^1 \eta'(x+sh(x))h(x)ds$. If $2\pi\Gamma(\nu)\|\eta\|_\sigma < K(2\pi\delta)^{\nu+1}$, $4\pi\delta < 1$, and $x \in S_{\sigma-4\delta}$, we can bound the norm of this expression on $B_{\sigma-4\delta}$ by $2\pi\|\eta\|_\sigma \frac{\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma$. Similarly, the quantity in braces in the next to last line may be rewritten as $\int_0^1 h'(x + \rho + s(h(x) + \eta(x + h(x)))(h(x) + \eta(x + h(x))ds$. Once again, assuming that the conditions on $\|\eta\|_\sigma$ and δ described above hold, and that $x \in S_{\sigma-4\delta}$, then we can bound the norm of this expression on $B_{\sigma-4\delta}$ by

$$\frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\Big\{\frac{\Gamma(\nu)}{K(2\pi\delta)^{\nu}}\|\eta\|_\sigma + \|\eta\|_\sigma\Big\}\|\eta\|_\sigma < \frac{4\pi(\Gamma(\nu))^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2 \quad ,$$

where the last inequality used the fact that $2\pi\delta K < 1$. Finally, if $|Imx| < \sigma - 6\delta$, then $|Im(x + h(x) + \rho + \eta(x + h(x)))| < \sigma - 4\delta$, (since $\|\eta\|_\sigma < \delta$), so that the last term in this expression is bounded by Proposition 2.3.

Define $\tilde{\eta}(x)$ by $\tilde{\phi}(x) = x + \rho + \tilde{\eta}(x)$. By Remark 2.3, $\tilde{\phi}$ has rotation number ρ, so by Lemma 2.1 there exists x_0 such that $\tilde{\eta}(x_0) = 0$. Combining this remark with the expression for $\tilde{\phi}$ just above, we find

$$\begin{aligned}
\hat{\eta}(0) &= -\{\eta(x_0 + h(x_0)) - \eta(x_0)\} \\
&\qquad - \{h(x_0 + \rho) - h(x_0 + \rho + h(x_0) + \eta(x_0 + h(x_0)))\} \\
&\qquad\quad - g(x_0 + h(x_0) + \rho + \eta(x_0 + h(x_0))) \quad .
\end{aligned}$$

In the previous paragraph we bounded the norm of each of the expressions on the right hand side of this equality, so we conclude that

$$|\hat{\eta}(0)| \le 2\pi\|\eta\|_\sigma^2 \frac{\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}} + \frac{4\pi(\Gamma(\nu))^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2 + \frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2 \quad .$$

Combining this estimate with the bounds above, we have proven,

Proposition 2.4 *If* $2\pi\Gamma(\nu)\|\eta\|_\sigma < K(2\pi\delta)^{(\nu+1)}$*, and* $4\pi\delta < 1$*, then* $\tilde{\phi}(x) = H^{-1}\circ\phi\circ H(x) = x+\rho+\tilde{\eta}(x)$*, where*

$$\|\tilde{\eta}\|_{\sigma-6\delta} \leq \frac{16\pi(\Gamma(\nu))^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2 \quad .$$

Remark 2.7 *The important thing to note about the estimate of* $\tilde{\eta}$ *is that in spite of the mess, it is second order in the small quantity* $\|\eta\|_\sigma$ *as we had hoped. That is, there exists a constant* $C(K,\delta,\nu)$ *such that* $\|\tilde{\eta}\|_{\sigma-6\delta} \leq C(K,\delta,\nu)\|\eta\|_\sigma^2$*. This is what makes the Newton's method argument work.*

The proof of Arnold's Theorem is completed by inductively repeating the above procedure. The principal point which we must check is that we don't lose all of our domain of analyticity as we go through the argument − note that $\tilde{\phi}$ is analytic on a narrower strip than was our original diffeomorphism ϕ. The essential reason that there is a nonvanishing domain of analyticity at the completion of the argument is that the amount by which the analyticity strip shrinks at the n^{th} step in the induction will be proportional to the amount by which our diffeomorphism differs from a rotation at the n^{th} iterative step, and thanks to the extremely fast convergence of Newton's method, this is very small.

The Inductive Argument:

Let $\phi_0(x) \equiv \phi(x)$, be our original diffeomorphism, and set $\eta_0(x) = \eta(x)$. Define $\phi_1(x) = H_0^{-1}\circ\phi_0\circ H_0(x)$, and by induction, $\phi_{n+1}(x) = H_n^{-1}\circ\phi_n\circ H_n(x) = x+\rho+\eta_{n+1}(x)$ where $H_n(x) = x + h_n(x)$, and h_n solves

$$h_n(x+\rho) - h_n(x) = \eta_n(x) - \hat{\eta}(0) \quad .$$

Also define the sequence of inductive constants:

- $\delta_n = \frac{\sigma}{36(1+n^2)}$, $n \geq 0$ (Note that this insures that $4\pi\delta_0 < 1$.)
- $\sigma_0 = \sigma$, and $\sigma_{n+1} = \sigma_n - 6\delta_n$, if $n \geq 0$.
- $\epsilon_0 = \|\eta\|_\sigma$, and $\epsilon_n = \epsilon_0^{(3/2)^n}$, if $n \geq 0$.

Note that $\sigma^* = \lim_{n\to\infty}\sigma_n > \sigma/2$. We now have:

Lemma 2.4 (Inductive Lemma) *If*

$$\epsilon_0 < \left(\frac{K}{16\pi\Gamma(\nu)}(\frac{\sigma}{36})^{(\nu+1)}\right)^8 \quad ,$$

then $\phi_{n+1}(x) = x+\rho+\eta_{n+1}(x)$*, with* $\eta_{n+1}\in B_{\sigma_{n+1}}$*, and* $\|\eta_{n+1}\|_{\sigma_{n+1}} \leq \epsilon_{n+1}$*.*

Furthermore, $H_n(x) = x + h_n(x)$ *satisfies*

$$\|h_n\|_{\sigma_n-\delta_n} \le \frac{\Gamma(\nu)\epsilon_n}{K(2\pi\delta_n)^\nu} \quad ,$$

while $H_n^{-1}(x) = x - h_n(x) + g_n(x)$*, where*

$$\|g_n\|_{\sigma-4\delta_n} \le \frac{2\pi\Gamma(\nu)^2\epsilon_n^2}{K^2(2\pi\delta_n)^{2\nu+1}} \quad .$$

Proof: Note that Proposition 2.2 and Proposition 2.3 imply that the estimates on h_n and g_n hold for $n = 0$. The estimate on $\|\eta_1\|_{\sigma_1}$ follows by noting that from Proposition 2.4,

$$\|\eta_1\|_{\sigma_1} \le \frac{16\pi(\Gamma(\nu))^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2$$

and the hypothesis on the inductive constants in the Inductive Lemma was chosen so that this last expression is less than $\epsilon_0^{(3/2)} = \epsilon_1$. This completes the first induction step.

Now suppose that the induction holds for $n = 0, 1, \ldots, N-1$, so that we know that $\|\eta_N\|_{\sigma_N} \le \epsilon_N$. To prove that it holds for $n = N$, first choose h_N to solve $h_N(x+\rho) - h_N(x) = \eta_N(x) - \hat{\eta}_N(0)$. By Proposition 2.2, and the inductive estimate on η_N, we will have $\|h_N\|_{\sigma_N-\delta_N} \le \frac{\Gamma(\nu)\epsilon_N}{K(2\pi\delta_N)^\nu}$, while Proposition 2.3 implies that $H_N^{-1}(x) = x - h_N(x) + g_N(x)$, with $\|g_N\|_{\sigma_N-4\delta_N} \le \frac{2\pi\Gamma(\nu)^2\epsilon_N^2}{K^2(2\pi\delta_N)^{2\nu+1}}$. Finally, if we define $\phi_{N+1} = H_N^{-1}\circ\phi_N\circ H_N = x+\rho+\eta_{N+1}$, with η_{N+1} defined in analogy with $\tilde{\eta}$ in Proposition 2.4, then we see that

$$\|\eta_{N+1}\|_{\sigma_{N+1}} \le \frac{16\pi\Gamma(\nu)^2}{K^2(2\pi\delta_N)^{2\nu+1}}\epsilon_N^2 \quad .$$

Once again, if use the hypotheses on the inductive constants we see that this expression is bounded by $\epsilon_N^{(3/2)} = \epsilon_{N+1}$, which completes the proof of the lemma.

□

With the aid of the Inductive Lemma it is easy to complete the proof of Arnold's Theorem. Define

$$\begin{aligned}\mathcal{H}_N(x) &= H_0\circ H_1\circ\ldots\circ H_N(x)\\ &= x + h_N(x) + h_{N-1}(x+h_n(x) + h_{N-2}(x+h_N(x)+h_{N-1}(x+h_N(x)))\\ &\quad + \ldots + h_0(x+h_1(x+\ldots)\ldots)\end{aligned}$$

By the Induction Lemma, $\mathcal{H}_N$ is analytic on $S_{\sigma_N - 2\delta_N}$ and $\mathcal{H}_N(z) - z$ is bounded by $\sum_{n=0}^{\infty} \frac{\Gamma(\nu)\epsilon_n}{K(2\pi\delta_n)^\nu} \equiv \Delta$. (This sum converges as a consequence of the hypotheses on the induction constants in the Induction Lemma.) Furthermore,

$$\mathcal{H}_{N+1}(z) - \mathcal{H}_N(z) = \mathcal{H}_N \circ H_N(z) - \mathcal{H}_N(z) = \int_0^1 \mathcal{H}'_N(z + th_N(z))h_N(z)ds \quad ,$$

so $\|\mathcal{H}_{N+1} - \mathcal{H}_N\|_{\sigma_{N+1}} \leq (\frac{4\Delta}{\delta_N} + 1)\frac{\Gamma(\nu)\epsilon_N}{K(2\pi\delta_N)^\nu}$. Note that by the definition of the inductive constants, the right hand side of this inequality converges if summed over N. Thus, $\mathcal{H}_N$ converges uniformly to some limit $\mathcal{H}$ on S_{σ^*}, and $\mathcal{H}$ is analytic. Furthermore, $\mathcal{H}(z) = z + \tilde{h}(z)$, where the estimates on $\mathcal{H}_N(z) - z$, plus Cauchy's Theorem imply that if $\delta^* = \sigma^*/16$, then $\|\tilde{h}'\|_{\sigma^* - \delta^*} \leq \Delta/\delta^* < \delta^*$, again using the definition of the inductive constants. By an argument similar to that following Lemma 2.3, we see that $\mathcal{H}$ is invertible on the image of $S_{\sigma^* - \delta^*}$ and that this image contains $S_{\sigma^* - 2\delta^*}$.

Finally, note that $\phi \circ \mathcal{H}_N(z) = \mathcal{H}_N \circ \phi_N(z) = \mathcal{H}_N(z + \rho + \eta_N(z))$. As $N \to \infty$, we see that

$$\phi \circ \mathcal{H}(z) = \lim_{N\to\infty} \mathcal{H}_N \circ \phi_N(z) = \lim_{N\to\infty} \mathcal{H}_N(z + \rho + \eta_N(z)) = \mathcal{H} \circ R_\rho(z) \quad ,$$

for all $z \in S_{\sigma^* - 2\delta^*}$. (The last equality used the inductive estimate on η_N.) Since $\mathcal{H}$ is invertible on this domain, this implies $\mathcal{H}^{-1} \circ \phi \circ \mathcal{H} = R_\rho$, so $\mathcal{H}$ is the diffeomorphism whose existence was asserted in Arnold's Theorem.

□

Remark 2.8 *Suppose that in Arnold's Theorem we were given a diffeomorphism of the (apparently) more general form*

$$\psi(x) = x + \alpha + \mu(x) \quad .$$

but still with rotation number ρ *of type* (K, ν) *(where* $\alpha \neq \rho$*.) We can rewrite* $\psi(x) = x + \rho + (\alpha - \rho + \mu(x)) \equiv x + \rho + \eta(x)$*. If* $\|\eta\|_\sigma = \|(\alpha - \rho) + \mu\|_\sigma \leq \epsilon(K, \nu, \sigma)$*, then Theorem 2.1 implies that* ψ *is analytically conjugate to* R_ρ

Remark 2.9 *In examples it may be difficult to determine from inspection of the initial diffeomorphism what the rotation number is. In such cases there is often a parameter in the problem which can be varied and which allows one to show that the conjugacy in Arnold's Theorem exists at least for most parameter values. For instance, the following result can be proven by easy modifications of the previous methods: (See, [1], page 271.)*

Theorem 2.2 *Consider the family of diffeomophisms:*

$$\phi_{\alpha,\epsilon}(x) = x + \alpha + \epsilon\eta(x) \quad , \tag{7}$$

for $\alpha \in [0,1]$*. For every* $\delta > 0$*, there exists* $\epsilon_0 > 0$ *such that if* $|\epsilon| < \epsilon_0$*, there exists a set* $\mathcal{A}(\epsilon) \subset [0,1]$ *such that for* $\alpha \in \mathcal{A}(\epsilon)$*,* $\phi_{\epsilon,\alpha}$ *is analytically conjugate to a rotation of the circle, and* $|\text{Lebesgue measure } (A(\epsilon)) - 1| < \delta$*.*

Remark 2.10 *It is not necessary to work with analytic functions. For instance, Moser [10] showed that if the original diffeomorphism is* C^k*, and if the rotation number is of type* (K,ν)*, then if* k *is sufficiently large (depending on* ν*), and the diffeomorphism is a sufficiently small perturbation of a rotation, the diffeomorphism is conjugate to a rotation, with a* $C^{k'}$ *conjugacy function, for some* $1 \leq k' < k$*. Note that this theorem is still "local" in that it demands that the diffeomorphism which we start with be "close" to a pure rotation. More recent work of Hermann [7] and Yoccoz [12], has lead to a remarkably complete understanding of the* **global** *picture of the dynamics of circle diffeomorphisms. For instance (see [12]), one can write the real numbers as a union of two disjoint sets* A *and* B*, and prove that any analytic circle diffeomorphism,* ϕ*, with rotation number* $\rho(\phi) \in B$ *is analytically conjugate to the rotation* $R_{\rho(\phi)}$*, while for any* $\alpha \in A$*, there exists an analytic circle diffeomorphism with rotation number* α *which is* **not** *analytically conjugate to* R_α*.*

3 Nearly Integrable Hamiltonian Systems

In this section we address the KAM theory in its original setting, namely nearly integrable Hamiltonian systems. Recall that a Hamiltonian system (in Euclidean space) is a system of $2N$ differential equations whose form is given by

$$\dot{p_j} = -\frac{\partial H}{\partial q_j} \quad ; \quad j = 1, \ldots, N \quad ,$$
$$\dot{q_j} = \frac{\partial H}{\partial p_j} \quad ; \quad j = 1, \ldots, N \quad ,$$

for some function $H(\mathbf{p}, \mathbf{q})$. (Here $\mathbf{p} = (p_1, \ldots, p_N)$ and $\mathbf{q} = (q_1, \ldots, q_N)$.)

In general these equations are just as hard to solve as any other system of $2N$ coupled, nonlinear, ordinary differential equations, but in special circumstances (the **integrable** Hamiltonian systems) there exists a special set of variables known as the **action-angle** variables, $(\mathbf{I}, \phi)$, $\mathbf{I} \in \mathbf{R}^N$ and $\phi \in \mathbf{T}^N$, such that in terms of these variables, $H(\mathbf{I}, \phi) = h(\mathbf{I})$. Since the Hamiltonian does not depend on the angle variables ϕ, the equations of motion are very simple – they become

$$\begin{aligned} \dot{I_j} &= -\frac{\partial H}{\partial \phi_j} = 0 \ ; \quad j = 1, \ldots, N \quad , \\ \dot{\phi_j} &= \frac{\partial H}{\partial I_j} \equiv \omega_j(\mathbf{I}) \ ; \quad j = 1, \ldots, N \quad . \end{aligned}$$

We can solve these equations immediately, and we find that $\mathbf{I}(t) = \mathbf{I}(0)$, and $\phi(t) = \omega(I)t + \phi(0)$. Thus, for an integrable system with bounded trajectories, the action variables $\mathbf{I}$ are constants of the motion, while the angle variables ϕ just precess around an N-dimensional torus with angular frequencies ω given by the gradient of the Hamiltonian with respect to the actions. (In particular, if the components of $\omega(\mathbf{I})$ are irrationally related to one another, $\phi(t)$ is a quasi-periodic function.)

Remark 3.1 *The three-body (or N-body) problem, in which we ignore the mutual interaction between the planets is an integrable system.*

Now suppose that we start with an integrable Hamiltonian $h(\mathbf{I})$ and make a small perturbation which depends on both the action and the angle variables – as in the case of the solar system when we consider the gravitational interactions between the planets. Then the Hamiltonian takes the form:

$$H(\mathbf{I}, \phi) = h(\mathbf{I}) + f(\mathbf{I}, \phi) \quad . \tag{8}$$

As before we will assume that the Hamiltonian function is analytic in order to avoid complications. More precisely, if we think of $f(\mathbf{I}, \phi)$ as a function on $\mathbf{R}^N \times \mathbf{R}^N$, which is periodic in ϕ, then we assume that there exists some $\mathbf{I}^* \in \mathbf{R}^N$ such that H can be extended to an analytic function on the set $\mathcal{A}_{\sigma,\rho}(\mathbf{I}^*) = \{(\mathbf{I},\phi) \in \mathbf{C}^N \times \mathbf{C}^N \mid |\mathbf{I}-\mathbf{I}^*| < \rho\ ,\ \ |Im(\phi_j)| < \sigma\ ,\ \ j = 1,\ldots,N\ \}$. (I will always use the ℓ^1 norm for N-vectors –*i.e.* $|\mathbf{I}| = \sum_{j=1}^N |I_j|$.) We define the norm of a function f, analytic on $\mathcal{A}_{\sigma,\rho}(\mathbf{I}^*)$ by $\|f\|_{\sigma,\rho} = \sup_{(\mathbf{I},\phi)\in\mathcal{A}_{\sigma,\rho}(\mathbf{I}^*)} |f(\mathbf{I},\phi)|$. (As in the previous section, one can assume without loss of generality that $\sigma < 1$.)

In addition, since we are interested in nearly integrable Hamiltonian systems, we will assume that f has small norm. (Just as we assumed that η was small in the previous section.) Furthermore, we can assume that $\int_{\mathbf{T}^N} f(\mathbf{I},\phi) d^N\phi = 0$, since if this were nonzero it could be absorbed by redefining $h(\mathbf{I})$.

Question: *Do the trajectories of this perturbed Hamiltonian system still lie on invariant tori, at least for f sufficiently small?*

To state the answer of this question more precisely, we need an analogue of the numbers of type (K,ν) introduced in the previous section.

Definition 3.1 *We say that a vector $\omega \in \mathbf{R}^N$ is of type (L,γ) if*

$$|\langle\omega,\mathbf{n}\rangle| = |\sum_{j=1}^N \omega_j n_j| \geq L|n|^{-\gamma}\ \ ,\ \text{for all } \mathbf{n} \in \mathbf{Z}^N\backslash\mathbf{0}\ \ .$$

Remark 3.2 *Note that if ρ is of type (K,ν), then the vector $(\rho,-1)$ is of type (L,γ) with $K = L$ and $\gamma = \nu - 1$. Also, we again assume without loss of generality that $L \leq 1$.*

Given this remark, and the fact that we know that the numbers of type (K,ν) are a subset of the real line of full Lebesgue measure, the following result (whose proof we omit) is not surprising.

Proposition 3.1 *If $\gamma > N$, almost every $\omega \in \mathbf{R}^N$ is of type (L,γ) for some $L < 1$.*

We are now in a position to state the KAM theorem.

Theorem 3.1 (KAM) *Suppose that $\omega(\mathbf{I}^*) \equiv \omega^*$ is of type (L,γ), and that the the Hessian matrix $\frac{\partial^2 h}{\partial I^2}$ is invertible at $\mathbf{I}^*$. (And hence on some neighborhood of $\mathbf{I}^*$.) Then there exists $\epsilon_0 > 0$ such that if $\|f\|_{\sigma,\rho} < \epsilon_0$, the Hamiltonian system (8) has a quasi-periodic solution with frequencies $\omega(\mathbf{I}^*)$.*

Remark 3.3 *Although we have claimed in the theorem only that at least one quasi-periodic solution exists in the perturbed hamiltonian system, we will see in the course of the proof that the whole torus, $\mathbf{I} = \mathbf{I}^*$, survives.*

Remark 3.4 *One might wonder why we study quasi-periodic orbits rather than the apparently simpler periodic orbits. If one considers values of the action variables for which the frequencies $\omega_j(\mathbf{I})$ are all rationally related, then the integrable hamiltonian will have an invariant torus, filled with* **periodic** *orbits. However, under a typical perturbation, all but finitely many of these periodic orbits will disappear. Hence, the quasi-periodic orbits are, in this sense, more stable than the periodic ones.*

As we will see, the proof follows very closely the outline of the previous section. In particular, we begin with:

Step 1: Analysis of the Linearized equation

The basic idea is to find new variables $(\tilde{\mathbf{I}}, \tilde{\phi})$ such that in terms of these new variables (8) will be integrable. However, not just any change of variables is allowed, because most changes of variables will not preserve the Hamiltonian form of the equations of motion. We will admit only those changes of variables which do preserve the Hamiltonian form of the equations. Such transformations are known as **canonical** changes of variables. There is a large literature on canonical transformations, (for a nice introduction see [2]), but pursuing it would take us too far afield. In order to come to the point in as expeditious a fashion as possible, let us just note the following:

Proposition 3.2 *Suppose that there exists a smooth function $\Sigma(\tilde{\mathbf{I}}, \phi)$ such that the equations:*

$$\mathbf{I} = \frac{\partial \Sigma}{\partial \phi} \ , \quad \tilde{\phi} = \frac{\partial \Sigma}{\partial \tilde{\mathbf{I}}} \quad ,$$

can be inverted to find $(\mathbf{I}, \phi) = \Phi(\tilde{\mathbf{I}}, \tilde{\phi})$. Then Φ is a canonical transformation, and Σ is called its generating function.

Proof: See [2], section 48.

Remark 3.5 *Note that $\Sigma(\tilde{\mathbf{I}}, \phi) = \langle \tilde{\mathbf{I}}, \phi \rangle$ is the generating function for the identity transformation. (Here, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^N$.)*

Remark 3.6 *There are other ways of generating canonical transformations. In particular, the Lie transform method has proven to be very convenient for computational purposes [5]. However, the generating function method offers a simple and direct way to prove the KAM theorem and for that reason I have chosen it here.*

We would like to find a canonical transformation $(\mathbf{I}, \phi) = \Phi(\tilde{\mathbf{I}}, \tilde{\phi})$ such that $\tilde{H}(\tilde{\mathbf{I}}, \tilde{\phi}) = H \circ \Phi(\tilde{\mathbf{I}}, \tilde{\phi}) = \tilde{h}(\tilde{\mathbf{I}})$, or

$$H(\frac{\partial \Sigma}{\partial \phi}(\tilde{\mathbf{I}}, \phi), \phi) = \tilde{h}(\tilde{\mathbf{I}}) \quad . \tag{9}$$

(This, by the way, is the Hamilton-Jacobi equation. In the last century, Jacobi proved the integrability of a number of physical systems by finding solutions of this equation.) In our example, (9) can be written as:

$$h((\frac{\partial \Sigma}{\partial \phi}(\tilde{\mathbf{I}}, \phi)) + f((\frac{\partial \Sigma}{\partial \phi}(\tilde{\mathbf{I}}, \phi), \phi) = \tilde{h}(\tilde{\mathbf{I}}) \quad . \tag{10}$$

Since H is "close" to an integrable Hamiltonian for f small, we can hope that the canonical transformation is "close" to the identity transformation. Using the fact that we know the generating function for the identity transformation, we will look for canonical transformations whose generating functions are of the form $\Sigma(\tilde{\mathbf{I}}, \phi) = \langle \tilde{\mathbf{I}}, \phi \rangle + S(\tilde{\mathbf{I}}, \phi)$, where S is $\mathcal{O}(\|f\|_{\sigma,\rho})$, the amount by which our Hamiltonian differs from an integrable one. If we substitute this form for Σ into (10) and expand, retaining only terms that are formally of first order in the small quantities $\|f\|_{\sigma,\rho}$ and $\|S\|_{\sigma,\rho}$, we obtain the linearized Hamilton-Jacobi equation:

$$\langle \omega(\tilde{\mathbf{I}}), \frac{\partial S}{\partial \phi}(\tilde{\mathbf{I}}, \phi) \rangle + f(\tilde{\mathbf{I}}, \phi) = \tilde{h}(\tilde{\mathbf{I}}) - h(\tilde{\mathbf{I}}) \quad . \tag{11}$$

Once again, we now have a linear equation involving periodic functions, so if we expand $f(\mathbf{I}, \phi) = \sum_{\mathbf{n} \in \mathbf{Z}^N} \hat{f}(\mathbf{I}, \mathbf{n}) e^{i2\pi \langle \mathbf{n}, \phi \rangle}$, we can solve (11) and we find

$$S(\tilde{\mathbf{I}}, \phi) = \frac{i}{2\pi} \sum_{\mathbf{n} \in \mathbf{Z}^N \setminus \mathbf{0}} \frac{\hat{f}(\tilde{\mathbf{I}}, \mathbf{n}) e^{i2\pi \langle \mathbf{n}, \phi \rangle}}{\langle \omega(\tilde{\mathbf{I}}), \mathbf{n} \rangle} \tag{12}$$

Remark 3.7 *Once again, as in (6), the function S defined (formally) by (12) does not satisfy (11), but rather*

$$\langle \omega(\tilde{\mathbf{I}}), \frac{\partial S}{\partial \phi}(\tilde{\mathbf{I}}, \phi) \rangle + f(\tilde{\mathbf{I}}, \phi) = 0 \quad , \tag{13}$$

and we will be forced to estimate the difference between these two equations below.

Note that once again, we will face small denominators. Indeed, for a dense set of points $\mathbf{I}$, the denominators in (12) will vanish for infinitely many choices of $\mathbf{n}$. This is the reason that many people (including Poincaré) at the end of the last century believed that these series diverged. Nonetheless, the results of Kolmogorov, Arnold and Moser show that "most" (in the sense of Lebesgue measure) points $\mathbf{I}$ give rise to a convergent series. Having S be defined only on the complement of a dense set of points $\tilde{\mathbf{I}}$ would be a problem, since we would be hard pressed to take the derivatives we need in order to compute the canonical transformation in Proposition 3.2. To proceed, we take advantage of the fact that because of the analyticity of f, the Fourier coefficients $\hat{f}(\tilde{\mathbf{I}}, \mathbf{n})$ are decaying to zero exponentially fast as $|\mathbf{n}|$ becomes large. Thus, if we truncate the sum defining S to consider only $|\mathbf{n}| < M$, for some large M we will make only a relatively small error in the solution of (11). On the other hand, since there are now only finitely many terms in the sum defining S, we can find open sets of action-variables on which the generating function is defined. Before stating the precise estimate on S, we introduce a few preliminaries.

First, define $\Omega \geq 1$, such that

$$\max \left((\sup_{|\mathbf{I}-\mathbf{I}^*|\leq\rho} \| \frac{\partial^2 h}{\partial I^2} \|), (\sup_{|\mathbf{I}-\mathbf{I}^*|\leq\rho} \| (\frac{\partial^2 h}{\partial I^2})^{-1} \|) \right) < \Omega \quad .$$

(Here, $\|\cdot\|$ is the norm of the matrix considered as an operator from $\mathbf{C}^N \to \mathbf{C}^N$ with the ℓ^1 norm.) Analogously, define $\tilde{\Omega}$ such that $\sup_{|\mathbf{I}-\mathbf{I}^*|\leq\rho} \|(\frac{\partial^3 h}{\partial I^3})\| < \tilde{\Omega}$. (In this case, $\|\cdot\|$ is the norm of $(\frac{\partial^3 h}{\partial I^3})$ considered as a bilinear operator from $\mathbf{C}^N \times \mathbf{C}^N \to \mathbf{C}^N$.) Next note that if we define

$$S^{<}(\tilde{\mathbf{I}}, \phi) = \frac{i}{2\pi} \sum_{\substack{\mathbf{n}\in\mathbf{Z}^N\setminus\mathbf{0} \\ |\mathbf{n}|\leq M}} \frac{\hat{f}(\tilde{\mathbf{I}}, \mathbf{n}) e^{i2\pi\langle\mathbf{n},\phi\rangle}}{\langle\omega(\tilde{\mathbf{I}}), \mathbf{n}\rangle} \quad ,$$

$S^{<}$ will no longer be a solution of (13), but rather will solve

$$\langle\omega(\tilde{\mathbf{I}}), \frac{\partial S}{\partial\phi}(\tilde{\mathbf{I}}, \phi)\rangle + f^{<}(\tilde{\mathbf{I}}, \phi) = 0 \quad ,$$

where $f^{<}(\tilde{\mathbf{I}}, \phi) \equiv \sum_{|\mathbf{n}|\leq M} \hat{f}(\tilde{\mathbf{I}}, \mathbf{n}) e^{i2\pi\langle\mathbf{n},\phi\rangle}$. Note that we have already discarded all terms that were formally of more than first order in $\|f\|_{\sigma,\rho}$ in order to derive (11). Thus, if in deriving this equation for $S^{<}$, we change (11) only by amounts of this order, we won't have qualitatively worsened our approximation. We will choose M in order to insure that this is the case.

Proposition 3.3 *Choose $0 < \delta < \sigma$, and set $M = |\log(\|f\|_{\sigma,\rho})|/(\pi\delta)$. If $\rho < L/(2\Omega M^{\gamma+1})$ and $4\pi\delta < 1$, then $S^{<}$ is analytic on $\mathcal{A}_{\sigma-\delta,\rho}(I^*)$, and*

$$\|S^{<}\|_{\sigma-\delta,\rho} \leq \left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}} \right)^N \frac{(2N^{\gamma})\|f\|_{\sigma,\rho}}{2\pi L}$$

Proof: Recall that we chose our domain $\mathcal{A}_{\sigma,\rho}(I^*)$ so that it was centered (in the $\mathbf{I}$ variables) at a point with $\omega(\mathbf{I}^*) = \omega^*$. Now suppose that we choose $|\mathbf{n}| < M$, and consider $\langle\omega(\mathbf{I}), \mathbf{n}\rangle$ for some other point $\mathbf{I}$ in our domain. Writing $\mathbf{I} = \mathbf{I}^* + (\mathbf{I} - \mathbf{I}^*)$, we see that $|\langle\omega(\mathbf{I}), \mathbf{n}\rangle - \langle\omega(\mathbf{I}^*), \mathbf{n}\rangle| \leq \Omega|\mathbf{n}|\rho$. If we then use the fact that ω^* is of type (L, γ), we find that for $|\mathbf{n}| \leq M$ and all $|\mathbf{I} - \mathbf{I}^*| < \rho$,

$$|\langle\omega(\mathbf{I}), \mathbf{n}\rangle| = |\langle\omega^*, \mathbf{n}\rangle + (\langle\omega(\mathbf{I}), \mathbf{n}\rangle - \langle\omega^*, \mathbf{n}\rangle)| \geq \frac{L}{|\mathbf{n}|^{\gamma}} - \Omega|\mathbf{n}|\rho \geq \frac{L}{2|\mathbf{n}|^{\gamma}} \quad ,$$

where the last inequality used the hypothesis on ρ and the fact that $|\mathbf{n}| \leq M$. If we combine this observation with the fact that $|\hat{f}(\tilde{\mathbf{I}}, \mathbf{n})| \leq \|f\|_{\sigma,\rho} e^{-2\pi\sigma|\mathbf{n}|}$, by Cauchy's theorem, we find

$$\begin{aligned} \|S^{<}\|_{\sigma-\delta,\rho} &\leq \sum_{|\mathbf{n}|\leq M} \frac{2|\mathbf{n}|^{\gamma}}{2\pi L} \|f\|_{\sigma,\rho} e^{-2\pi\delta|\mathbf{n}|} \\ &\leq \frac{2\|f\|_{\sigma,\rho}}{2\pi L} N^{\gamma} (1 + 2\sum_{m=0}^{M} m^{\gamma} e^{-2\pi\delta|\mathbf{m}|})^{N} \\ &\leq \left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}}\right)^{N} \frac{2N^{\gamma}\|f\|_{\sigma,\rho}}{2\pi L} \quad . \end{aligned}$$

In going from the first to second line of this inequality, we used the fact that

$$|\mathbf{n}|^{\gamma} e^{-2\pi\delta|\mathbf{n}|} \leq N^{\gamma} (\max_{j} |n_j|)^{\gamma} e^{-2\pi\delta|\mathbf{n}|} \leq N^{\gamma} \prod_{j=1}^{N} \max(1, |n_j|) e^{-2\pi\delta|\mathbf{n}_n|} \quad ,$$

so that $\sum_{|\mathbf{n}|\leq M} |\mathbf{n}|^{\gamma} e^{-2\pi\delta|\mathbf{n}|} \leq N^{\gamma}(1 + 2\sum_{m=1}^{M} m e^{-2\pi\delta m})^{N}$.

□

Now that we know that the generating function is well-defined, we can proceed to check that the canonical transformation is defined and analytic, just as we did in Proposition 2.3 in the previous section.

Proposition 3.4 *If*

$$\left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}}\right)^{N} \frac{16N^{\gamma+1}\|f\|_{\sigma,\rho}}{2\pi\delta\rho L} < 1 \quad ,$$

$\rho < L/(2\Omega M^{\gamma+1})$ *and* $4\pi\delta < 1$, *then the equations*

$$\mathbf{I} = \tilde{\mathbf{I}} + \frac{\partial S^{<}}{\partial \phi} \quad , \quad \text{and} \quad \tilde{\phi} = \phi + \frac{\partial S^{<}}{\partial \tilde{\mathbf{I}}} \quad , \tag{14}$$

define an analytic and invertible canonical transformation $(\mathbf{I}, \phi) = \Phi(\tilde{\mathbf{I}}, \tilde{\phi})$ *on the set* $\mathcal{A}_{\sigma-3\delta,\rho/4}$.

Proof: Just as in the proof of Lemma 2.3 we begin by using the analytic inverse function theorem to check that (14) can be inverted. In both of the expressions in this equation, the inverse function theorem can be applied provided $\|\frac{\partial^2 S^{<}}{\partial \mathbf{I} \partial \phi}\|_{\sigma-2\delta,\rho/2} < 1$. This in turn, follows immediately from the estimate in Proposition 3.3 and Cauchy's Theorem.

The remainder of the proposition follows if we check that the transformation is onto the domain $\mathcal{A}_{\sigma-3\delta,\rho/4}$. (This is analogous to the proof of Proposition 2.3.) Note that if $(\tilde{\mathbf{I}},\phi) \in \mathcal{A}_{\sigma-2\delta,\rho/2}$,

$$\|\frac{\partial S^{<}}{\partial \phi}\|_{\sigma-2\delta,\rho/2} \leq \left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}}\right)^{N} \frac{2N^{\gamma+1}\|f\|_{\sigma,\rho}}{2\pi\delta L} < \rho/8 \quad ,$$

by the hypothesis of the Proposition, while

$$\|\frac{\partial S^{<}}{\partial \tilde{\mathbf{I}}}\|_{\sigma-\delta,\rho/4} \leq \left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}}\right)^{N} \frac{8N^{\gamma+1}\|f\|_{\sigma,\rho}}{2\pi\rho L} < \delta/2 \quad ,$$

again by the hypotheses of the Proposition. Thus, $|\mathbf{I}-\tilde{\mathbf{I}}| < \rho/8$, while $|\phi-\tilde{\phi}| < \delta/2$. This implies that the canonical transformation maps the set $\mathcal{A}_{\sigma-2\delta,\rho/2}$ onto $\mathcal{A}_{\sigma-3\delta,\rho/4}$, and hence that $(\mathbf{I},\phi) = \Phi(\tilde{\mathbf{I}},\tilde{\phi})$ on this set.

□

Remark 3.8 *For a vector valued function like* $\frac{\partial S^{<}}{\partial \phi}$ *on a domain* $\mathcal{A}_{\sigma,\rho}$, $\|\frac{\partial S^{<}}{\partial \phi}\|_{\sigma,\rho} \equiv \sup_{\mathcal{A}_{\sigma,\rho}} |\frac{\partial S^{<}}{\partial \phi}|$, *where we recall that* $|\frac{\partial S^{<}}{\partial \phi}|$ *is the* ℓ^1 *norm of* $\frac{\partial S^{<}}{\partial \phi}$. *This is the origin of the extra factor of* N *in these estimates.*

Step 2: The Newton Step

Now, just as we did in the case of circle diffeomorphisms, where we transformed our original diffeomorphism with the approximate conjugacy function obtained by solving the linearized conjugacy equation, we transform our original Hamiltonian with the approximate canonical transformation, whose generating function is $S^{<}$, and show that the difference between the transformed Hamiltonian and an integrable Hamiltonian is of second order in the small quantity $\|f\|_{\sigma,\rho}$. As before, we will use this fact as the basis for a Newton's method argument.

Proposition 3.5 *Define* $\tilde{H}(\tilde{\mathbf{I}},\tilde{\phi}) = H \circ \Phi(\tilde{\mathbf{I}},\tilde{\phi}) \equiv \tilde{h}(\tilde{\mathbf{I}}) + \tilde{f}(\tilde{\mathbf{I}},\tilde{\phi})$. *If*

$$\left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}}\right)^{N} \frac{16N^{\gamma+1}\|f\|_{\sigma,\rho}}{2\pi\delta\rho L} < 1 \quad ,$$

$\rho < L/(2\Omega M^{\gamma+1})$ *and* $4\pi\delta < 1$, *then* $\tilde{H}$ *is analytic on* $\mathcal{A}_{\sigma-3\delta,\rho/4}$, *and one has the estimates,*

$$\|h-\tilde{h}\|_{\sigma-3\delta,\rho/4} \leq (\Omega+2)\left(\left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}}\right)^{N} \frac{2N^{\gamma+1}\|f\|_{\sigma,\rho}}{2\pi\delta\rho L}\right)^{2} \quad ,$$

and

$$\|\tilde{f}\|_{\sigma-3\delta,\rho/4} \leq 2(\Omega+2)\left(\left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}}\right)^N \frac{2N^{\gamma+1}\|f\|_{\sigma,\rho}}{2\pi\delta\rho L}\right)^2 \quad .$$

Remark 3.9 *The important thing to note is that $\tilde{f}$, the amount by which our transformed Hamiltonian fails to be integrable is quadratic in the small quantity $\|f\|^2_{\sigma,\rho}$. Just as in Proposition 2.4 in the previous section, this will form the basis of a Newton's method argument, which will allow us to prove the existence of a quasi-periodic solution with frequencies ω^*.*

Proof: Using Taylor's Theorem, we can rewrite

$$\begin{aligned}
\tilde{H}(\tilde{\mathbf{I}},\tilde{\phi}) &= H(\tilde{\mathbf{I}}+\frac{\partial S^<}{\partial\phi},\phi(\tilde{\mathbf{I}},\tilde{\phi})) \\
&= h(\tilde{\mathbf{I}}+\frac{\partial S^<}{\partial\phi}) + f(\tilde{\mathbf{I}}+\frac{\partial S^<}{\partial\phi},\phi(\tilde{\mathbf{I}},\tilde{\phi})) \\
&= h(\tilde{\mathbf{I}}) + \langle\omega(\tilde{\mathbf{I}}),\frac{\partial S^<}{\partial\phi}\rangle + \int_0^1\int_0^t \langle(\frac{\partial\omega}{\partial I}(\tilde{\mathbf{I}}+v\frac{\partial S^<}{\partial\phi})\frac{\partial S^<}{\partial\phi}),\frac{\partial S^<}{\partial\phi}\rangle dvdt \\
&\quad + f(\tilde{\mathbf{I}},\phi) + \int_0^1 \langle\frac{\partial f}{\partial\mathbf{I}}(\tilde{\mathbf{I}}+t\frac{\partial S^<}{\partial\phi},\phi),\frac{\partial S^<}{\partial\phi}\rangle dt \quad .
\end{aligned}$$

From the definition of $S^<$, we know that $\langle\omega(\tilde{\mathbf{I}}),\frac{\partial S^<}{\partial\phi}\rangle + f(\tilde{\mathbf{I}},\phi) = f^{\geq}(\tilde{\mathbf{I}},\phi) \equiv \sum_{|\mathbf{n}|\geq M}\hat{f}(\tilde{\mathbf{I}},\mathbf{n})e^{2\pi i\langle\phi,\mathbf{n}\rangle}$. Thus, we can define

$$\begin{aligned}
\tilde{h}(\tilde{\mathbf{I}}) &= h(\tilde{\mathbf{I}}) + average\{\int_0^1\int_0^t \langle(\frac{\partial\omega}{\partial I}(\tilde{\mathbf{I}}+v\frac{\partial S}{\partial\phi})\frac{\partial S}{\partial\phi}),\frac{\partial S}{\partial\phi}\rangle dvdt\} \\
&\quad + average\{\int_0^1 \langle\frac{\partial f}{\partial\mathbf{I}}(\tilde{\mathbf{I}}+t\frac{\partial S}{\partial\phi},\phi),\frac{\partial S}{\partial\phi}\rangle dt\} + average f(\tilde{\mathbf{I}},\phi(\tilde{\mathbf{I}},\tilde{\phi})) \quad ,
\end{aligned}$$

where $average\{g(\tilde{\mathbf{I}},\tilde{\phi})\} \equiv \int_{\mathbf{T}^N} g(\tilde{\mathbf{I}},\tilde{\phi})d\tilde{\phi}$, and

$$\begin{aligned}
\tilde{f}(\tilde{\mathbf{I}},\tilde{\phi}) &= f^{\geq}(\tilde{\mathbf{I}},\phi(\tilde{\mathbf{I}},\tilde{\phi})) \\
&\quad + \int_0^1\int_0^t \langle(\frac{\partial\omega}{\partial I}(\tilde{\mathbf{I}}+v\frac{\partial S^<}{\partial\phi})\frac{\partial S^<}{\partial\phi}),\frac{\partial S^<}{\partial\phi}\rangle dvdt \\
&\quad + \int_0^1 \langle\frac{\partial f}{\partial\mathbf{I}}(\tilde{\mathbf{I}}+t\frac{\partial S^<}{\partial\phi},\phi(\tilde{\mathbf{I}},\phi),\frac{\partial S^<}{\partial\phi}\rangle dt \\
&\quad - average\{\int_0^1\int_0^t \langle(\frac{\partial\omega}{\partial I}(\tilde{\mathbf{I}}+v\frac{\partial S^<}{\partial\phi})\frac{\partial S^<}{\partial\phi}),\frac{\partial S^<}{\partial\phi}\rangle dvdt\} \\
&\quad - average\{\int_0^1 \langle\frac{\partial f}{\partial\mathbf{I}}(\tilde{\mathbf{I}}+t\frac{\partial S^<}{\partial\phi},\phi),\frac{\partial S^<}{\partial\phi})\rangle dt\} \\
&\quad - average f(\tilde{\mathbf{I}},\phi(\tilde{\mathbf{I}},\tilde{\phi})).
\end{aligned}$$

Remark 3.10 *Subtracting the average of the three quantities in $\tilde{f}$ insures that when we expand $\tilde{f}$ in a Fourier series, there will be no $\mathbf{n} = \mathbf{0}$ coefficient – this was used in solving (11).*

Both $\tilde{h}$ and $\tilde{f}$ are easy to estimate using the estimates of Proposition 3.3 and Cauchy's Theorem. For instance,

$$\| \int_0^1 \langle \frac{\partial f}{\partial \mathbf{I}}(\tilde{\mathbf{I}} + t\frac{\partial S^{<}}{\partial \phi}, \phi), \frac{\partial S^{<}}{\partial \phi} \rangle dt \|_{\sigma-3\delta,\rho/4}$$
$$\leq \frac{2\|f\|_{\sigma,\rho}}{\rho} \left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}} \right)^N \frac{2N^{\gamma+2}\|f\|_{\sigma,\rho}}{2\pi\delta L} \quad ,$$

while,

$$\| \int_0^1 \int_0^t \langle (\frac{\partial \omega}{\partial I}(\tilde{\mathbf{I}} + v\frac{\partial S^{<}}{\partial \phi}) \frac{\partial S^{<}}{\partial \phi}), \frac{\partial S^{<}}{\partial \phi} \rangle dv dt \|_{\sigma-3\delta,\rho/4}$$
$$\leq \Omega \left(\left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta)^{\gamma+1}} \right)^N \frac{2N^{\gamma+1}\|f\|_{\sigma,\rho}}{2\pi\delta L} \right)^2 \quad .$$

Finally, we have the estimate

$$\|f^{\geq}\|_{\sigma-3\delta,\rho/4} \quad \leq \quad \sum_{|\mathbf{n}|\geq M} \|f\|_{\sigma,\rho} e^{-2\pi\delta|\mathbf{n}|} \leq \|f\|_{\sigma,\rho} e^{-\pi\delta M} \sum_{|\mathbf{n}|\geq M} e^{-\pi\delta|\mathbf{n}|}$$
$$\leq (\frac{4}{\pi\delta})^N \|f\|_{\sigma,\rho}^2 \quad ,$$

where the last of these inequalities came from using the definition of M in Proposition 3.3.

If we combine these remarks, we immediately obtain the estimates stated in the Proposition.

□

The Induction Argument:

The induction follows closely the lines of the induction step in the case of the circle diffeomorphisms. We have to keep track of two more inductive constants – ρ_n to control the size of the domain of the action variables, and M_n to control how we cut off the sum defining $S^{<}$ at the n^{th} stage of the iteration. Thus, we define our original Hamiltonian $H(\mathbf{I}, \phi) = H_0(\mathbf{I}, \phi)$ and set $h(\mathbf{I}) = h_0(\mathbf{I})$ and $f(\mathbf{I}, \phi) = f_0(\mathbf{I}, \phi)$. Also define

- $\delta_n = \frac{\sigma}{36(1+n^2)}$, $n \geq 0$.

- $\sigma_0 = \sigma$, and $\sigma_{n+1} = \sigma_n - 4\delta_n$, if $n \geq 0$.
- $\rho_0 \leq \rho$, and $\rho_{n+1} = \rho_n/8$, with ρ_0 chosen to satisfy the hypothesis of the following Lemma.
- $\epsilon_0 = \|f\|_{\sigma,\rho}$, and $\epsilon_n = \epsilon_0^{(3/2)^{(n/\gamma)}}$, if $n \geq 0$.
- $M_n = |\log \epsilon_n|/(\pi\delta_n)$.

We set $H_{n+1} = H_n \circ \Phi_n = h_{n+1} + f_{n+1}$, with $\hat{f}_{n+1}(\mathbf{I}, 0) = 0$, where Φ_n is the canonical transformation whose generating function $S_n^<$ solves the equation

$$\langle \omega_n(\tilde{\mathbf{I}}), \frac{\partial S_n^<}{\partial \phi}(\tilde{\mathbf{I}}, \phi) \rangle + f_n^<(\tilde{\mathbf{I}}, \phi) = 0 \quad ,$$

with $f_n^<(\tilde{\mathbf{I}}, \phi) \equiv \sum_{|\mathbf{n}| \leq M_n} \hat{f}_n(\tilde{\mathbf{I}}, \mathbf{n}) e^{i2\pi\langle \mathbf{n}, \phi \rangle}$, and $\omega_n(\tilde{\mathbf{I}}) = \frac{\partial h_n}{\partial \mathbf{I}}(\tilde{\mathbf{I}})$. At the n^{th} stage of the iteration we will work on the domain $\mathcal{A}_{\sigma_n,\rho_n}(I_n) = \{(\mathbf{I}, \phi) \in \mathbf{C}^N \times \mathbf{C}^N \mid |\mathbf{I} - \mathbf{I}_n| < \rho_n \ , \ \ |Im(\phi_j)| < \sigma_n \ , \ \ j = 1, \ldots, N \ \}$, where $\mathbf{I}_n$ is chosen so that $\omega_n(\mathbf{I}_n) = \omega^*$, and we define $\Omega_n = \max(1, \sup \| \frac{\partial^2 h_n}{\partial I^2} \|, \|(\frac{\partial^2 h_n}{\partial I^2})^{-1}\|)$, with the supremum in these expressions running over all $\mathbf{I}$ with $|\mathbf{I} - \mathbf{I}_n| < \rho_n$.

We then have

Lemma 3.1 (KAM Induction Lemma) *There exists a positive constant c_1 such that if*

$$\epsilon_0 < 2^{-c_1 N(\gamma+1)} \frac{\sigma^{8N(4\gamma+1)} \rho_0^8 L^{16}}{\Gamma(\gamma+1)^{16N} \Omega^8} \ , \quad \text{and} \quad \rho_0 < \frac{2^{-c_1} L}{\Omega M_0^{\gamma+1}} \quad .$$

then

- *The generating function $S_n^<$ satisfies*

$$\|S_n^<\|_{\sigma_n - \delta_n, \rho_n} \leq \left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta_n)^{\gamma+1}} \right)^N \frac{2N^\gamma \epsilon_n}{2\pi L} \quad .$$

- *Φ_n is defined and analytic on $\mathcal{A}_{\sigma_n - 3\delta_n, \rho_n/4}(\mathbf{I}_n)$ and maps this set into $\mathcal{A}_{\sigma_n - 2\delta_n, \rho_n/2}(\mathbf{I}_n)$.*
- $\|f_{n+1}\|_{\sigma_{n+1}, \rho_{n+1}} \leq \epsilon_{n+1}$.
- $\|h_{n+1} - h_n\|_{\sigma_{n+1}, \rho_{n+1}} \leq \epsilon_{n+1}$.
- $|\mathbf{I}_{n+1} - \mathbf{I}_n| < \rho_n/8$.

Before proving this lemma, we show how the KAM theorem follows from it. If the perturbation f in our Hamiltonian is sufficiently small, the hypotheses of the Induction Lemma will be satisfied, and roughly speaking, the idea is that as $n \to \infty$, $H_n(\mathbf{I}, \phi) \to h^\infty(\mathbf{I})$, an integrable system, since $f_n \to 0$. Since all of the orbits of an integrable system are quasiperiodic, this would complete the proof. However, as n becomes larger and larger, the size of the domain in the action variables on which H_n is defined goes to zero. Thus, we must be a little careful with this limit.

Begin by defining $\Psi_n = \Phi_0 \circ \Phi_1 \circ \ldots \Phi_n$. By the induction lemma, $\Psi_n : \mathcal{A}_{\sigma_n - 3\delta_n, \rho_n/4}(\mathbf{I}_n) \to \mathcal{A}_{\sigma_0,\rho_0}(\mathbf{I}_0)$, and $H_n = H_0 \circ \Psi_{n-1}$. In particular, if $(\mathbf{I}^n(t), \phi^n(t))$ is a solution of Hamilton's equations with Hamiltonian H_n, then $\Psi_{n-1}(\mathbf{I}^n(t), \phi^n(t))$ is a solution of Hamilton's equations with Hamiltonian H_0.

Consider the equations of motion of H_n:

$$\dot{\mathbf{I}} = -\frac{\partial f_n}{\partial \phi} \ , \quad \dot{\phi} = \omega_n(\mathbf{I}) + \frac{\partial f_n}{\partial \mathbf{I}} \quad .$$

Since $\|\frac{\partial f_n}{\partial \mathbf{I}}\|_{\sigma_n, \rho_n/2} \le 2\epsilon_n N/\rho_n$, and $\|\frac{\partial f_n}{\partial \phi}\|_{\sigma_n - \delta_n, \rho_n} \le \epsilon_n N/\delta_n$, the trajectory with initial conditions $(\mathbf{I}_n, \phi_0)$ (for any $\phi_0 \in \mathbf{T}^N$), will remain in $\mathcal{A}_{\sigma_n - 3\delta_n, \rho_n/4}(\mathbf{I}_n)$ for all times $|t| \le T_n = 2^n$, by our hypothesis on ϵ_0, and the definition of the induction constants. Furthermore, if $(\mathbf{I}^n(t), \phi^n(t))$ is the solution with these initial conditions, we have

$$\max\left(\sup_{|t| \le T_n} |\mathbf{I}^n(t) - \mathbf{I}_n|, \sup_{|t| \le T_n} |\phi^n(t) - (\omega^* t + \phi_0)| \right) \le 2^{2n+2} \Omega \epsilon_n N / \rho_n \delta_n \quad .$$

Noting that the inductive estimates on $\mathbf{I}_n$ imply that there exists $\mathbf{I}^\infty$ with $\lim_{n\to\infty} \mathbf{I}_n = \mathbf{I}^\infty$, we see that for t in any compact subset of the real line, $(\mathbf{I}^n(t), \phi^n(t)) \to (\mathbf{I}^\infty, \omega^* t + \phi_0)$ (again using the definition of the inductive constants). Using the inductive bounds on the canonical transformation one can readily establish that

$$\|\Psi_n(\mathbf{I}, \phi) - (\mathbf{I}, \phi)\|_{\sigma_{n+1}, \rho_{n+1}} \le \sum_{j=0}^{\infty} 2N \left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta_j)^{\gamma+1}} \right)^N \left(\frac{8N^\gamma \epsilon_j}{2\pi \delta_j \rho_j L} \right) \equiv \Delta \quad ,$$

while

$$\begin{aligned} \|\Psi_n(\mathbf{I}, \phi) \quad - \quad & \Psi_{n-1}(\mathbf{I}, \phi)\|_{\sigma_{n+1}, \rho_{n+1}} \\ = \quad & \|\Psi_{n-1} \circ \Phi_n(\mathbf{I}, \phi) - \Psi_{n-1}(\mathbf{I}, \phi)\|_{\sigma_{n+1}, \rho_{n+1}} \\ & \le (2N + \frac{16\tilde{\Delta}}{\rho_n \delta_n}) \left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta_n)^{\gamma+1}} \right)^N \left(\frac{8N^\gamma \epsilon_n}{2\pi \delta_n \rho_n L} \right) \quad . \end{aligned}$$

Using the definition of the inductive constants, we see that the sum over n of this last expression converges and hence $\lim_{n\to\infty} \Psi_n(\mathbf{I}^\infty, \omega^* t + \phi_0) = (\mathbf{I}^*(t), \phi^*(t))$

exists and is a quasi-periodic function with frequency ω^*. Similarly, $\lim_{n\to\infty} |\Psi_n(\mathbf{I}^\infty, \omega^* t + \phi_0) - \Psi_n(\mathbf{I}^n(t), \phi^n(t))| = 0$, for t in any compact subset of the real line.

Combining these two remarks, find that $\lim_{n\to\infty} \Psi_n(\mathbf{I}^n(t), \phi^n(t)) = (\mathbf{I}^*(t), \phi^*(t))$, so $(\mathbf{I}^*(t), \phi^*(t))$ is a quasi-periodic solution of Hamilton's equations for the system with Hamiltonian H_0 as claimed.

□

Remark 3.11 *Note that this argument is independent of the point ϕ_0 that we take on the original torus. Thus it shows that* **every** *trajectory on the unperturbed torus is preserved.*

Proof: (of Lemma 3.1.) Note that Propositions 3.3, 3.4, and 3.5, plus the assumption on the induction constants imply that we can start the induction, provided $\mathcal{A}_{\sigma_0-3\delta_0,\rho_0/4}(\mathbf{I}_0) \supset \mathcal{A}_{\sigma_1,\rho_1}(\mathbf{I}_1)$. From the definitions of the domains and the inductive constants, we see that this will follow provided $|\mathbf{I}_0 - \mathbf{I}_1| < \rho_0/8$. To see that this is so we note that $\omega_0(\mathbf{I}_0) = \omega_1(\mathbf{I}_1)$. Thus, $\omega_0(\mathbf{I}_0) - \omega_0(\mathbf{I}_1) = \frac{\partial(h_1-h_0)}{\partial \mathbf{I}}(\mathbf{I}_1)$. But, $\|\frac{\partial(h_1-h_0)}{\partial \mathbf{I}}(\mathbf{I}_1)\|_{\sigma_0-3\delta_0,\rho_0/6} \leq 12\epsilon_1/\rho_0$, while

$$\begin{aligned}\omega_0(\mathbf{I}_0) - \omega_0(\mathbf{I}_1) &= \frac{\partial\omega_0}{\partial\mathbf{I}}(\mathbf{I}_0)(\mathbf{I}_0 - \mathbf{I}_1) \\ &\quad + \int_0^1 \int_0^t (\frac{\partial^2\omega_0}{\partial\mathbf{I}^2}(I_0 + sI_1)(\mathbf{I}_1 - \mathbf{I}_0))^2 ds dt \quad .\end{aligned}$$

Since $\| \left(\frac{\partial\omega_0}{\partial\mathbf{I}}\right)^{-1} \| \leq \Omega$ and $\|\frac{\partial^2\omega_0}{\partial\mathbf{I}^2}\| \leq \tilde{\Omega}$, this implies that $|\mathbf{I}_0 - \mathbf{I}_1| < \rho_0/8$ by the definition of the induction constants, provided $\Omega\tilde{\Omega}\rho_0 < 1/2$, which will follow if the constant c_1 in the Lemma is sufficiently large. This completes the first induction step.

Suppose that the induction argument holds for $n = 0, 1, \ldots, K-1$. To prove it for $n = K$ we first note if $S_K^<$ is defined by:

$$S_K^<(\tilde{\mathbf{I}}, \phi) = \frac{i}{2\pi} \sum_{\substack{\mathbf{n}\in\mathbf{Z}^N\setminus\mathbf{0} \\ |\mathbf{n}|\leq M_K}} \frac{\hat{f}_K(\tilde{\mathbf{I}}, \mathbf{n}) e^{i2\pi\langle\mathbf{n},\phi\rangle}}{\langle\omega_K(\tilde{\mathbf{I}}), \mathbf{n}\rangle} \quad ,$$

then by Proposition 3.3, we have

$$\|S_K^<\|_{\rho_K, \sigma_K-\delta_K} \leq \left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta_K)^{\gamma+1}}\right)^N \frac{2N^\gamma\epsilon_K}{2\pi L} \quad .$$

Note that the hypothesis in Proposition 3.3 becomes $\rho_K < L/(2\Omega_K M_K^{\gamma+1})$.

where,

$$\begin{aligned}\Omega_K &= \max(1, \sup_{|\mathbf{I}-\mathbf{I}_K|<\rho_K} \|\frac{\partial^2 h_K}{\partial I^2}\|, \sup_{|\mathbf{I}-\mathbf{I}_K|<\rho_K} \|(\frac{\partial^2 h_K}{\partial \mathbf{I}^2})^{-1}\|) \\ &\leq \Omega \max(1+\sum_{j=1}^{K} \frac{64N\epsilon_j}{\rho_j^2}, (1-\sum_{j=1}^{K} \frac{64\Omega N\epsilon_j}{\rho_j^2})^{-1}) \leq 2\Omega \quad ,\end{aligned}$$

using the definition of the inductive constants. This observation, plus the hypothesis on ρ_0 in the inductive lemma, guarantees that the hypothesis of Proposition 3.3 is satisfied. Thus, by Proposition 3.4, the canonical transformation Φ_K defined by

$$\mathbf{I} = \tilde{\mathbf{I}} + \frac{\partial S_K^<}{\partial \phi} \quad , \quad \text{and} \quad \tilde{\phi} = \phi + \frac{\partial S_K^<}{\partial \tilde{\mathbf{I}}} \quad , \tag{15}$$

is analytic and invertible on the set $\mathcal{A}_{\sigma_K - 3\delta_K, \rho_K/4}(\mathbf{I}_K)$, and maps this set into $\mathcal{A}_{\sigma_K,\rho_K}(\mathbf{I}_K)$.

If we then define f_{K+1} and h_{K+1}, as we defined $\tilde{f}$ and $\tilde{h}$ in Proposition 3.5 we see that

$$\|f_{K+1}\|_{\sigma_K - 3\delta_K, \rho_K/4} \leq 2(\Omega_K + 2)\left(\left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta_K)^{\gamma+1}}\right)^N \frac{2N^{\gamma+1}\epsilon_K}{2\pi\delta_K\rho_K L}\right)^2$$

while

$$\|h_K - h_{K+1}\|_{\sigma_K - 3\delta_K, \rho_K/4} \leq (\Omega_K + 2)\left(\left(\frac{8\Gamma(\gamma+1)}{(2\pi\delta_K)^{\gamma+1}}\right)^N \frac{2N^{\gamma+1}\epsilon_K}{2\pi\delta_K\rho_K L}\right)^2$$

If we use the bound on ϵ_0, and the definitions of the inductive constants, we see that the quantities on the right hand sides of both of these inequalities are less than ϵ_{K+1}. The proof of the inductive lemma will be completed if we can show that $\mathcal{A}_{\sigma_{K+1},\rho_{K+1}}(\mathbf{I}_{K+1}) \subset \mathcal{A}_{\sigma_K - 3\delta_K, \rho_K/4}(\mathbf{I}_K)$. This follows in a fashion very similar to the proof that $\mathcal{A}_{\sigma_1,\rho_1}(\mathbf{I}_1) \subset \mathcal{A}_{\sigma_0 - 3\delta_0, \rho_0/4}(\mathbf{I}_0)$, which we demonstrated above, so we omit the details.

□

Remark 3.12 *From the point of view of applications of this theory it is often convenient to know not just what happens to a single trajectory, but rather the behavior of whole sets of trajectories. Simple modifications of the preceding argument allow one to demonstrate the following variant of the KAM theorem. (See [4].) Consider the family of Hamiltonian systems*

$$H_\epsilon = h(\mathbf{I}) + \epsilon f(\mathbf{I}, \phi) \quad . \tag{16}$$

Suppose that there exists a bounded set $V \subset \mathbf{R}^N$ such that $\frac{\partial^2 h}{\partial \mathbf{I}^2}(\mathbf{I})$ is invertible for all $\mathbf{I} \in V$, and that for every ϵ in some neighborhood of zero H_ϵ is analytic on a set of the form $\mathcal{A}_{\sigma,\rho}(V) = \{(\mathbf{I}, \phi) \in \mathbf{C}^N \times \mathbf{C}^N \mid |\mathbf{I} - \tilde{\mathbf{I}}| < \rho \ , \ \text{for some} \ \ \tilde{\mathbf{I}} \in V \ \ , \text{and} \ \ |Im(\phi_j)| < \sigma \ , \ \ j = 1, \ldots, N \ \}$.

Theorem 3.2 *For every $\delta > 0$, there exists $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$, there exists a set $P_\epsilon \subset V \times \mathbf{T}^N$, such that the Lebesgue measure of $(V \times \mathbf{T}^N) \backslash P_\epsilon$ is less than δ and for any point $(\mathbf{I}_0, \phi_0) \in P_\epsilon$, the trajectory of (16) with initial conditions $(\mathbf{I}_0, \phi_0)$ is quasi-periodic.*

Thus an informal way of stating the KAM theorem is to say that"most" trajectories of a nearly integrable Hamiltonian systems remain quasi-periodic.

Remark 3.13 *Just as in the case of Arnold's theorem about circle diffeomorphisms, the KAM theorem also remains true when the Hamiltonian is only finitely differentiable, rather than analytic. For a nice exposition of this theory, see [11].*

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