MA 575, Linear Models : Homework 4

Problem 2.3

2.3.1

$$
\begin{aligned}
\mathbb{E}[Y_i|x_i = \bar{x}] &= \mathbb{E}\left[\alpha + \beta_1(\bar{x} - \bar{x}) + e_i\right] \\
&= \alpha + \mathbb{E}[e_i] \\
&= \alpha
\end{aligned}
$$

Therefore, α can be interpreted as the expected value of Y_i when x_i is equal to the average values of x_i.

2.3.2

$$
RSS(\alpha, \beta_1) = \sum_{i=1}^{n} \left(y_i - \alpha - \beta_1(x_i - \bar{x})\right)^2
$$

Therefore,

$$
\begin{aligned}
\frac{\partial RSS(\alpha, \beta_1)}{\partial \alpha} &= -2\sum_{i=1}^{n} \left(y_i - \alpha - \beta_1(x_i - \bar{x})\right) \\
&= -2\sum_{i=1}^{n}(y_i - \alpha) + 2\sum_{i=1}^{n} \beta_1(x_i - \bar{x}) \\
&= -2n(\bar{y} - \alpha)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial RSS(\alpha, \beta_1)}{\partial \beta_1} &= -2\sum_{i=1}^{n} \left(y_i - \alpha - \beta_1(x_i - \bar{x})\right)(x_i - \bar{x}) \\
&= -2\sum_{i=1}^{n}(y_i - \alpha)(x_i - \bar{x}) + 2\beta_1 SXX
\end{aligned}
$$

Setting the partial derivatives to zero, we obtain :

$$\hat{\alpha} = \bar{y}$$
$$\hat{\beta}_1 = \frac{SXY}{SXX}$$

2.3.3

$$\begin{aligned} Var(\hat{\alpha}) &= Var(\bar{y}) \\ &= Var\left(\sum_{i=1}^{n} \frac{y_i}{n}\right) \\ &= \frac{\sigma^2}{n} && \text{(The } y_i \text{ are i.i.d.)} \end{aligned}$$

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{SXX} \qquad \text{(See Equation (2.11) in book)}$$

$$\begin{aligned} Cov(\hat{\alpha}, \hat{\beta}_1) &= Cov(\hat{\beta}_0 + \hat{\beta}_1\bar{x}, \hat{\beta}_1) \\ &= Cov(\hat{\beta}_0, \hat{\beta}_1) + \bar{x}Var(\hat{\beta}_1) \\ &= -\frac{\bar{x}\sigma^2}{SXX} + \frac{\bar{x}\sigma^2}{SXX} && \text{(See Appendix A.4 for } Cov(\hat{\beta}_0, \hat{\beta}_1)\text{)} \\ &= 0 \end{aligned}$$

Problem 2.8

2.8.1

Let's imagine that one fits a linear regression to a sample $(x_i, y_i)_{i=1..n}$ and obtain the estimators for β_0 and β_1. Let's now imagine this person wants to fit a linear regression to the sample $(cx_i, y_i)_{i=1..n}$ (where $c \neq 0$).

2.8.1.a) What are the expressions of the estimators of β_0^*, β_1^* and σ_*^2 parameters of this new linear regression?

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + e_i \\ &= \beta_0 + \frac{\beta_1}{c} cX_i + e_i \end{aligned}$$

Therefore,

$$\hat{\beta}_0^* = \hat{\beta}_0 \qquad ; \qquad \hat{\sigma}_*^2 = \hat{\sigma}^2 \qquad ; \qquad \hat{\beta}_1^* = \frac{\hat{\beta}_1}{c}$$

2.8.1.b) Is the expression of the R^2 affected by the change of the predictor value X?

$$
\begin{aligned}
R^2(\hat{\beta}_0^*, \hat{\beta}_1^*) &= \frac{\sum_{i=1}^n (\hat{y}_i - \bar{y})^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n \left(\hat{\beta}_0^* + \hat{\beta}_1^* c x_i - \bar{y}\right)^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n \left(\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y}\right)^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= R^2(\hat{\beta}_0, \hat{\beta}_1)
\end{aligned}
$$

The R^2 value is unaffected by the change of the valu of the predictor X.

2.8.1.c) Will the t-test statistic be affected by the change of the predictor value X?

$$
\begin{aligned}
\frac{\hat{\beta}_1^*}{se(\hat{\beta}_1^*)} &= \frac{\hat{\beta}_1 \big/ c}{\hat{\sigma}^* \Big/ \sqrt{\sum_{i=1}^n (c x_i - c\bar{x})^2}} \\
&= \frac{\hat{\beta}_1}{\hat{\sigma} \big/ \sqrt{SXX}} \\
&= \frac{\hat{\beta}_1}{se(\hat{\beta}_1)}
\end{aligned}
$$

The t-test statistic is unaffected by the change of the predictor value X

2.8.2

We now consider the case when Y is replaced by dY where $d \neq 0$.

2.8.2.a What are the expressions of the estimators of β_0^*, β_1^* and σ_*^2 parameters of this new linear regression?

$$dY_i = d\beta_0 + d\beta_1 X_i + de_i$$

In that case,

$$\hat{\beta}_0^* = d\hat{\beta}_0 \qquad ; \qquad \hat{\beta}_1^* = d\hat{\beta}_1 \qquad ; \qquad \hat{\sigma}_*^2 = d^2 \hat{\sigma}^2$$

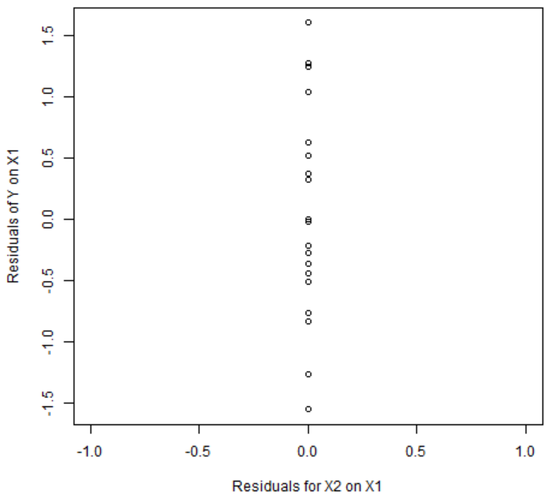
2.8.2.b) Are the expression of the R^2 and the t-test statistic affected by the change of the predicted value Y?

Using the same logic that for questions (2.8.1.b) and (2.8.1.c), it easily shown that the expressions of the R^2 and the t-test statistic are not affected by the change of the predicted value Y.

Problem 3.3

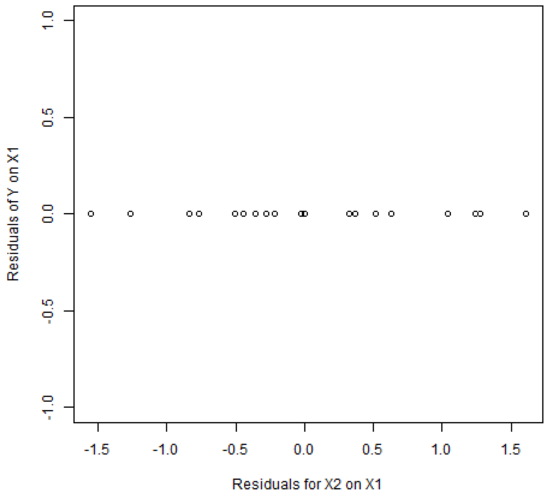
Question 3.3.1

Since X_2 is exactly linear with respect to X_1 the residuals of the regression of X_2 on X_1 will all be nulls. Therefore, the plot will look as follow :



Question 3.3.2

As Y is perfectly linear with respect to X_1 all the residuals of that regression will be null. Therefore, the plot will look as follow :



Question 3.3.3

The plots will have the same shape if X_1 is uncorrolated to X_2 and Y.

Question 3.3.4

In the added plot of X_2 after X_1, the residuals of the regression Y over X_1 are along the y-axis. These are necesseraly smaller than Y and therefore, the vertical variation in an added-variable plot for X_2 after X_1 is always less than or equal to the vertical variation in a plot of Y versus X_2.

Problem 3.4

Question 3.4.1

Let's denote :

$$\mathcal{Y} = \begin{pmatrix} y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix} \quad ; \quad \mathcal{X} = \begin{pmatrix} x_{11} - \bar{x}_1 & x_{21} - \bar{x}_2 \\ \vdots & \vdots \\ x_{1n} - \bar{x}_1 & x_{21} - \bar{x}_2 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \quad ; \quad \bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

Under the previous notations, it is known that

$$\hat{\beta} = (\mathcal{X}'\mathcal{X})^{-1} \mathcal{X}'\mathcal{Y} \qquad \text{and} \qquad \hat{\beta}_0 = 0$$

It is easy to show that :

$$\mathcal{X}'\mathcal{X} = \begin{pmatrix} SX_1X_1 & SX_1X_2 \\ SX_1X_2 & SX_2X_2 \end{pmatrix} \quad ; \quad X'Y = \begin{pmatrix} SX_1Y \\ SX_2Y \end{pmatrix}$$

As X_1 and X_2 are uncorrolated, we obtain :

$$\mathcal{X}'\mathcal{X} = \begin{pmatrix} SX_1X_1 & 0 \\ 0 & SX_2X_2 \end{pmatrix}$$

Therefore,

$$\hat{\beta}_1 = SX_1\mathcal{Y}/SX_1X_1$$
$$\hat{\beta}_2 = SX_2\mathcal{Y}/SX_2X_2$$

Also, the slope of X_2 on X_1 is null as X_2 and X_1 are uncorrolated.

Question 3.4.2

Residuals of Y on X_1 : $\epsilon_{Y_i} = (y_i - \bar{y}) - \hat{\beta}_1(x_{1i} - \bar{x}_1)$
Residuals of X_2 on X_1 : $\epsilon_{X_{2i}} = x_{2i} - \bar{x}_2$

Question 3.4.3

Let

$$\epsilon_{Y_i} = \gamma + \alpha\epsilon_{X_{2i}} + e_i$$

Then :

3.4.2.a Let's derive theformulae of $\hat{\alpha}$

$$\hat{\alpha} = \frac{S\epsilon_Y\epsilon_{X_2}}{S\epsilon_{X_2}\epsilon_{X_2}}$$
$$\hat{\gamma} = \bar{\epsilon}_Y - \hat{\alpha}\bar{\epsilon}_{X_2}$$

Developping the expression of $\hat{\alpha}$, we obtain :

$$\begin{aligned}\hat{\alpha} &= \frac{SX_2\mathcal{Y} - \hat{\beta}_1 SX_1X_2}{SX_2X_2} \\ &= \frac{SX_2\mathcal{Y}}{SX_2X_2} \qquad \text{(Reminder: } SX_1X_2 = 0\text{)} \\ &= \hat{\beta}_2\end{aligned}$$

3.4.2.b Let's derive theformulae of $\hat{\gamma}$

$$\begin{aligned}\hat{\gamma} &= \sum_{i=1}^{n}(y_i - \bar{y}) - \hat{\beta}_1\sum_{i=1}^{n}(x_{1i} - \bar{x}_1) - \hat{\alpha}\sum_{i=1}^{n}(x_{2i} - \bar{x}_2) \\ &= 0\end{aligned}$$