CAS MA 575 – Linear Models.

Boston University, Fall 2013

HOMEWORK 6

1. Recall that in Lecture 6.2, we have defined a weighted least squares (WLS) regression problem, with the following mean and variance functions:

$$\mathbb{E}[Y|X = \mathbf{x}_i] = \mathbf{x}_i^T\boldsymbol{\beta}, \qquad \text{and} \qquad \mathbb{V}\text{ar}[Y|X = \mathbf{x}_i] = \frac{\sigma^2}{w_i},$$

where the w_i's are known positive numbers, such that $w_i > 0$, for every $i = 1, \ldots, n$. Alternatively, this can be expressed in matrix notation, $\mathbb{V}\text{ar}[\mathbf{y}|\mathbf{X}] = \sigma^2\mathbf{W}^{-1}$, where $\mathbf{W}$ is a diagonal matrix, whose $\mathbf{W}_{ii}$ element is w_i.

   (a) Compute the *variance/covariance matrix* of the WLS estimator

   $$\widehat{\boldsymbol{\beta}} := (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}\mathbf{y},$$

   using the fact that $\mathbf{X}^T\mathbf{W}\mathbf{X}$ is symmetric.

   (b) Compute the *hat matrix* for this model, which can be defined through the transformations that we have previously seen with $\mathbf{W} = \mathbf{W}^{1/2}\mathbf{W}^{1/2}$,

   $$\mathbf{z} := \mathbf{W}^{1/2}\mathbf{y}, \qquad \mathbf{M} := \mathbf{W}^{1/2}\mathbf{X}, \qquad \text{and} \qquad \mathbf{d} := \mathbf{W}^{1/2}\mathbf{e}.$$

   Note that, in this case, the hat matrix projects $\mathbf{z}$ to

   $$\widehat{\mathbf{z}} := \mathbf{M}\widehat{\boldsymbol{\beta}}.$$

   Therefore, you should find an expression of the form $\widehat{\mathbf{z}} = \mathbf{H}\mathbf{z}$.