A Sierpinski Mandelbrot Spiral for the Rational Map $z^4 + \lambda / z^3$

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Abstract

We identify three structures that lie in the parameter plane of the rational map $F(z) = z^4 + \lambda / z^3$. There exists a “Sierpinski spiral” of infinitely many alternating Mandelbrot sets and Sierpinski holes that accumulate to the parameter at the end of the arc. There exists as well another Sierpinski spiral arc of infinitely many alternating Mandelbrot sets and Sierpinski holes that accumulate to the parameter at the “center” of the arc. One can picture a parameter traveling along infinitely many arcs of the first type in a spiraling fashion and converging to the unique parameter value at the center of the other arc. These infinitely many arcs comprise the “Sierpinski Mandelbrot spiral.”

In this paper we consider only the map $F_\lambda(z) = z^4 + \lambda / z^3$ where $\lambda$ is a complex parameter. It is known that there are several intriguing geometric structures surrounding the negative real axis in the parameter plane for rational maps of the form $F_\lambda(z) = z^n + \lambda / z^d$. In the case where $n$ and $d$ are even, it has been shown in [5] that there is a “Cantor necklace” that lies along the negative real axis in the parameter plane and a “principal” Mandelbrot set along the positive axis. A Cantor necklace is a set that is a continuous image of the Cantor middle-thirds set to which is adjoined countably many open disks in the plane in place of the removed open intervals along the real line. For parameters inside these open disks (which we call Sierpinski holes), the Julia set of $F_\lambda$ is known to be a Sierpinski curve (i.e., is homeomorphic to the Sierpinski carpet fractal), and the different dynamical behaviors on these Julia sets is completely understood [12]. In the case where $n$ is odd and $d$ is even, there is no such Cantor necklace; rather there are now two “principal” Mandelbrot sets, one along the positive real axis and the other along the negative real axis. As a consequence, the dynamical behavior for these parameters is very different from the behavior when $n$ and $d$ are both even. Thus the remaining case is when $n$ is even and $d$ is odd; we will look at a specific instance of this.

As when $d$ is even, we again have a principal Mandelbrot set straddling the positive real axis. But the structure on and around the negative real axis is very different. When $n \geq 2$ is even and $d \geq 3$ is odd, [1] proves the existence of a “Mandelbrot arc” (an MS arc) in a neighborhood of the negative real axis in the parameter plane. Phase one of the construction of the MS arc is proving the existence of a “Sierpinski arc” (an SM arc). This paper uses many of the same ideas and proof techniques, for $n = 4$ and $d = 3$, by finding the SM arc and infinitely many other arcs of the same type that can be found inductively. Each of these infinitely many arcs tend to a $\lambda$ in the parameter plane such that, in dynamical space, some iterate of the critical value is a fixed point. As a sequence, these arcs limit to the arc that tends to the $\lambda$ in the parameter plane such that, in dynamical space, the critical value is a fixed point. Each “preimage” arc exists in a successively smaller region as the arcs tend to
the aforementioned $\lambda$ in a spiraling pattern, comprising the “Sierpinski Mandelbrot spiral” (an SM spiral). For simplicity, we will restrict to the case $n = 4$ and $d = 3$, but the spiral exists for most members of the family when $n \geq 4$ is even and $d \geq 3$ is odd. The other cases involve different techniques and will be described in a future paper.

1 Preliminaries

In this paper we consider

$$F_\lambda(z) = z^4 + \frac{\lambda}{z^3}$$

where $\lambda \in \mathbb{C}$ is nonzero.

When $|z|$ is large, we have that $|F_\lambda(z)| > |z|$, so the point at $\infty$ is an attracting fixed point in the Riemann sphere. We denote the immediate basin of attraction of $\infty$ by $B_\lambda$. There is also a pole at the origin for this map, and so there is a neighborhood of the origin that is mapped into $B_\lambda$. If the preimage of $B_\lambda$ surrounding the origin is disjoint from $B_\lambda$, we call this region the trap door and denote it by $T_\lambda$.

The Julia set of $F_\lambda$, $J(F_\lambda)$, has several equivalent definitions. $J(F_\lambda)$ is the set of all points at which the family of iterates of $F_\lambda$ fails to be a normal family in the sense of Montel. Equivalently, $J(F_\lambda)$ is the closure of the set of repelling periodic points of $F_\lambda$, and it is also the boundary of the set of all points whose orbits tend to $\infty$ under iteration of $F_\lambda$, not just those in the boundary of $B_\lambda$. See [11].

One checks that there are 7 critical points that are given by

$$c_\lambda = \left(\frac{3\lambda}{4}\right)^{\frac{1}{7}}$$

with the corresponding critical values given by

$$v_\lambda = \frac{(7)\lambda^{\frac{6}{7}}}{3^{\frac{1}{7}}4^{\frac{4}{7}}}.$$ 

There are also 7 prepoles given by

$$p_\lambda = (-\lambda)^{\frac{1}{7}}.$$ 

We denote the critical point that lies in $\mathbb{R}^-$ when $\lambda \in \mathbb{R}^-$ by $c_0 = c_0^\lambda$ (and then $c_0^\lambda$ varies analytically with $\lambda$). We denote the prepole that lies in $\mathbb{R}^+$ when $\lambda \in \mathbb{R}^-$ by $p_3 = p_3^\lambda$. The other prepoles are denoted by $p_j = p_j^\lambda$ where again $-3 \leq j \leq 3$ and the $p_j$ are arranged in the clockwise order as $j$ increases.

The straight ray extending from the origin to $\infty$ and passing through the critical point $c_\lambda$ is called the critical point ray. This ray is mapped two-to-one onto the portion of the straight ray from the origin to $\infty$ that starts at the critical value $F_\lambda(c_\lambda)$ and extends to $\infty$ beyond this critical value. A similar straight line extending from 0 to $\infty$ and passing through a prepole $p_\lambda$ is a prepole ray, and this ray is mapped one-to-one onto the entire straight line passing through both the origin and the point $(-\lambda)^{4/(7)}$. 

2
Let $\omega$ be an $(7)^{th}$ root of unity. Then we have $F_\lambda(\omega z) = \omega^4 F_\lambda(z)$, and so it follows that the dynamical plane is symmetric under the rotation $z \mapsto \omega z$. In particular, all of the critical orbits have “similar” fates. If one critical orbit tends to $\infty$, then all must do so. If one critical orbit tends to an attracting cycle of some period, then all other critical orbits also tend to an attracting cycle, though these other cycles may have different periods. Nonetheless, the points on these attracting cycles are all symmetrically located with respect to the rotation by $\omega$. As a consequence, each of $B_\lambda, T_\lambda,$ and $J(F_\lambda)$ are symmetric under rotation by $\omega$. Similarly, one checks easily that the parameter plane is symmetric under the rotation $\lambda \mapsto \nu \lambda$ where $\nu$ is an $(3)^{rd}$ root of unity. The parameter plane is also symmetric under complex conjugation $\lambda \mapsto \overline{\lambda}$.

The Escape Trichotomy [7] holds for this rational map. The first scenario in this trichotomy occurs when one and hence, by symmetry, all of the critical values lie in $B_\lambda$. In this case it is known that $J(F_\lambda)$ is a Cantor set. The corresponding set of $\lambda$-values in the parameter plane is denoted by $C$ and called the Cantor set locus. The second scenario is that the critical values all lie in $T_\lambda$ (which we assume is disjoint from $B_\lambda$). In this case the Julia set is a Cantor set of simple closed curves surrounding the origin. This can only happen when $n, d \geq 2$ but not both equal to 2 [10]. We call the region $E_1$ in the parameter plane where this occurs the “McMullen domain”; it is known that $E_1$ is an open disk surrounding the origin [3]. The third scenario is that the orbit of a critical point enters $T_\lambda$ at iteration 2 or higher. Then, by the above symmetry, all such critical orbits do the same. In this case, it is known that the Julia set is a Sierpinski curve [6], i.e., a set that is homeomorphic to the well known Sierpinski carpet fractal. The regions in the parameter plane for which this happens are the open disks that we call Sierpinski holes [13]. If the critical orbits do not escape to $\infty$, then it is known [8] that the Julia set is a connected set. Thus we call the set of parameters for which the critical orbits either do not escape or else enter the trap door at iteration 2 or higher the connectedness locus. This is the complement of $C \cup E_1$.

Figure 1: Parameter plane for the family $z^4 + \lambda/z^3$ with three symmetrically located Mandelbrot sets. The central disk is $E_1$, the McMullen domain. The outer region is $C$, the Cantor Set Locus. Everything else is the connectedness locus.
In [2] it has been shown that there are 3 principal Mandelbrot sets in the parameter plane for these maps. These are symmetrically located by the rotation $\nu z$ around the origin and extend from the Cantor set locus down to the McMullen domain.

For more details about the dynamical properties of these maps and the structure of the parameter plane, see [4].

2 Phase One: the $\bar{0}$ arc

There are three symmetrically located Mandelbrot sets in the parameter plane. Because of the $\lambda \mapsto \nu \lambda$ symmetry in the parameter plane, we need only be concerned with $2\pi/3 \leq \text{Arg } \lambda \leq 4\pi/3$.

As stated earlier, there are seven critical points. As $\lambda$ rotates from $\text{Arg } \lambda = \pi$ to $\text{Arg } \lambda = 2\pi/3$, $c_0$ rotates exactly one-seventh as much in the corresponding direction (i.e., $2\pi(1/42)$). We denote the other critical points by $c_j = c_0^j$ for $-3 \leq j \leq 3$ where the $c_j$ are arranged in the clockwise order as $j$ increases. Note that, when $\text{Arg } \lambda = 2\pi/3$, $c_3$ lies on the ray through $\exp(2\pi i (2/42))$. When $\text{Arg } \lambda = 4\pi/3$, $c_{-3}$ lies on the ray through $\exp(2\pi (-2/42))$. The critical values of $F_\lambda$ are given by $v^\lambda = \kappa \lambda^{4/7}$ where $\kappa$ is the constant given by $7/(4^{1/7}3^{3/7})$. One computes that $\kappa \approx 1.98$. We denote by $v_j = v^\lambda_j$ the critical value that is the image of $c_j$.

Recall that there are also seven prepoles. Note that, when $\lambda \in \mathbb{R}^-$, the critical point $c_0$ lies between the two prepole rays passing through $p_0$ and $p_{-1}$.

As in [1], we will first find an arc in dynamical space consisting of infinitely many alternating preimages of portions of the trap door and a certain open disk, then use these to prove the existence of an arc in the parameter plane consisting of infinitely many alternating Sierpinski holes and Mandelbrot sets. We will refer to the dynamical arc as the $TL$ arc. The motivation for that name will be made clear later. In contrast, the arc in the parameter plane is known as a “Sierpindelbrot” arc, or SM arc for short. An SM arc is an arc in the parameter plane that passes alternately along the spines of infinitely many baby Mandelbrot sets and through the centers of the same number of Sierpinski holes. By the spine of the Mandelbrot set we mean the analogue of the portion of the real axis lying in the usual Mandelbrot set associated with the quadratic family $z^2 + c$. This particular arc to be constructed will be called the $\bar{0} TL$ arc in dynamical space, and its existence will lead to the existence of the $\bar{0}$ SM arc in the parameter plane.

The $\bar{0}$ SM arc includes infinitely Mandelbrot sets $M^k$ with $k \geq 2$ where $k$ is the period of the attracting cycle for parameters drawn from the main cardioid of $M^k$, i.e., the base period of $M^k$. This arc also includes infinitely many Sierpinski holes $E^k$ with $k \geq 1$ where $k$ is the escape time in $E^k$, i.e., the number of iterations it takes for the orbit of the critical points to enter $T_\lambda$. This arc is the portion of the negative real axis in the parameter plane extending from the McMullen domain out to the endpoint on the boundary of the connectedness locus. Then the Mandelbrot sets and Sierpinski holes are arranged along this arc as follows:

$$ \cdots < M^4 < E^3 < M^3 < E^2 < M^2 < E^1 $$
where, as earlier, $\mathcal{E}_1$ denotes the McMullen domain. In each case there will be an interval of nonzero length between any adjacent Mandelbrot set and Sierpinski hole lying along this arc.

To construct the $\overline{0}TL$ arc and its corresponding parameter arc, we will restrict attention at first to the set of parameters in the annular region $\mathcal{O}$ given by $10^{-10} \leq |\lambda| \leq 2$. Also, let $\mathcal{A}$ be the annulus in the dynamical plane given by $\kappa 10^{-4} \leq |z| \leq \kappa 4^{4/7}$.

Figure 2: A sense of the scale of $\mathcal{O}$ and $\mathcal{A}$

Proposition 2.1.

1. For any $\lambda \in \mathcal{O}$, all points on the outer circular boundary of $\mathcal{A}$ lie in $B_\lambda$, while all points on the inner circular boundary of $\mathcal{A}$ lie in $T_\lambda$. Moreover, $F_\lambda$ maps each of these boundaries strictly outside the boundary of $\mathcal{A}$.

2. If $\lambda$ lies on the inner circular boundary of $\mathcal{O}$, then $v^\lambda$ lies on the inner circular boundary of $\mathcal{A}$ and so $\lambda$ lies in the McMullen domain.

3. If $\lambda$ lies on the outer circular boundary of $\mathcal{O}$, then $v^\lambda$ lies on the outer circular boundary of $\mathcal{A}$ and so $\lambda$ lies in the Cantor set locus in the parameter plane.
Proof. First, if \(|z| = \tau \kappa 4^{1/7}\) for any \(\tau \geq 1\), we have for each \(\lambda \in \mathcal{O}\):

\[
|F_\lambda(z)| \geq |\tau^4 \kappa 4^{16/7}| - \frac{\lambda}{\tau^3 \kappa 3^{1/2}/7} \\
\geq \tau^4 1.94^{16/7} - \frac{2}{\tau^3 \kappa 3^{1/2}/7} \\
\geq 300 \tau^4 - \frac{1}{35 \tau^3} \\
> 299 \tau \\
> \tau \kappa 4^{1/7} = |z|
\]

So all points outside the circle \(|z| = \kappa 4^{1/7}\) lie in \(B_\lambda\) when \(\lambda \in \mathcal{O}\).

Similarly, if \(|z| = \kappa 10^{-4}\), then we have

\[
|F_\lambda(z)| \geq \frac{|\lambda|}{\kappa 3^{10-12}} - \kappa 4^{10-16} \geq \frac{10^{-10}}{\kappa 3^{10-12}} - \kappa 4^{10-16} \geq 100 / \kappa^2 - \epsilon
\]

where \(\epsilon \approx 16(10^{-16})\). So this inner boundary is mapped into \(B_\lambda\) and so are all smaller circles around the origin. Hence this circle lies in \(T_\lambda\) (when \(\lambda\) lies in the connectedness locus).

Now if \(\lambda\) lies on the inner circular boundary of \(\mathcal{O}\), then \(|\lambda| = 10^{-10}\) so that \(|v^\lambda| = \kappa 10^{-40/7}\). Hence, for these \(\lambda\)-values, \(v^\lambda \in T_\lambda\) and \(\lambda\) therefore lies in the McMullen domain. If \(\lambda\) lies on the outer circular boundary of \(\mathcal{O}\), then \(|\lambda| = 2\) so that \(|v^\lambda| = \kappa 4^{1/7}\) and thus this boundary circle lies in the Cantor set locus in the parameter plane.

We now restrict attention to a “smaller” subset of \(\mathcal{O}\). Let \(\mathcal{O}'\) be the subset of \(\mathcal{O}\) containing parameters \(\lambda\) for which \(2\pi/3 \leq \arg \lambda \leq 4\pi/3\). This region \(\mathcal{O}'\) can be thought of as a closed sector in the parameter plane.

For any parameter in \(\mathcal{O}'\), let \(L^\lambda\) be the “closed portion of the wedge” in the annulus \(\mathcal{A}\) in dynamical space that is bounded by the two prepole rays through \(p_0\) and \(p_{-1}\). When \(\lambda \in \mathbb{R}^-\), \(L^\lambda\) is thus bounded by the rays extending from 0 and passing through \(\exp(2\pi i(6/14))\) and \(\exp(2\pi i(8/14))\). So the critical point \(c_0\) lies in the interior of \(L^\lambda\).

Next, let \(R_0^\lambda\) be the closed portion of the wedge in \(\mathcal{A}\) that is bounded by the critical point rays passing through \(c_3\) and \(c_{-3}\). When \(\lambda \in \mathbb{R}^-\), this wedge is bounded by the critical point rays extending from 0 and passing through \(\exp(\pm(2\pi i(1/14)))\). It is also bounded by the inner and outer portions of \(\partial \mathcal{A}\) between \(\exp(\pm(2\pi i(1/14)))\). Note that this is distinct from the portion of the actual boundary of the trap door and basin in dynamical space, which are all contained inside the wedge. We will refer to the portion of the boundary of the trap door inside \(R_0^\lambda\) as \(T_0^\lambda\) and the portion of the boundary of the basin inside \(R_0^\lambda\) as \(B_0^\lambda\). See figure 3b. Note that \(R_0^\lambda\) is the symmetric image of \(L^\lambda\) under \(z \mapsto -z\).

Finally, let \(R_1^\lambda\) be the closed portion of the wedge in \(\mathcal{A}\) that is bounded by the critical point rays passing through \(c_2\) and \(c_3\). When \(\lambda \in \mathbb{R}^-\), this wedge is bounded by the critical point rays extending from 0 and passing through \(\exp(2\pi i(1/14))\) and \(\exp(2\pi i(3/14))\). Similar to \(R_0^\lambda\), the portion of the boundary of the trap door inside \(R_1^\lambda\) is \(T_1^\lambda\) and the portion of the boundary of the basin inside \(R_1^\lambda\) is \(B_1^\lambda\).
$T_\lambda$ is already defined to be the trap door, and the inner boundary circle of the annulus lies inside $T_\lambda$. We will refer to the open portion of the trap door bounded by the annulus, $|z| < \kappa 10^{-4}$, as $T_A$. Note that $T_A \subset T_\lambda$. $\partial T_A$ (resp. $\partial B_\lambda$) in $R_0^\lambda$ is a construction and is a nice arc, while $T_0^\lambda$ (resp. $B_0^\lambda$) is part of the Julia set and is not as nice, as seen in figure 3a.

(a) A stylized depiction of the wedge construction, consisting of $L^\lambda, T_A, R_0^\lambda$, and $R_1^\lambda$ with the prepoles and critical points labeled. (b) The (not to scale) wedge construction in the dynamical space for $\lambda = -0.29$. $T_0^\lambda$ and $B_0^\lambda$ are traced in the figure.

Figure 3: The wedge construction.

**Proposition 2.2.** For each $\lambda \in \mathcal{O}'$

1. $F_\lambda$ maps $R_0^\lambda$ in one-to-one fashion onto a region that contains the interior of $R_0^\lambda \cup R_1^\lambda \cup L^\lambda \cup T_A$;

2. $F_\lambda$ maps $R_1^\lambda$ in one-to-one fashion onto a region that contains the interior of $R_0^\lambda \cup R_1^\lambda \cup L^\lambda \cup T_A$;

3. $F_\lambda$ maps $L^\lambda$ two-to-one over a region that contains the interior of $R_0^\lambda$;

4. As $\lambda$ winds once around the boundary of $\mathcal{O}'$, the critical value $F_\lambda(c_0^\lambda)$ winds once around the boundary of $R_0^\lambda$, (i.e., the winding index of the vector connecting this critical value to the prepole $p_3^\lambda$ lying in the interior of $R_0^\lambda$ is one).

Proof. For the first case, note that the straightline boundaries of $R_0^\lambda$ are mapped two-to-one onto the critical point rays passing through $v^\lambda_3$ and $v^\lambda_{-3}$. When $2\pi/3 < \text{Arg} \lambda \leq 4\pi/3$, these rays are disjoint from all of $L^\lambda, R_0^\lambda, R_1^\lambda$, and $T_A$. When $\lambda$ rotates clockwise to $\text{Arg} \lambda = 2\pi/3$, the sectors $L^\lambda, R_0^\lambda, R_1^\lambda$, and $T_A$ rotate clockwise $2\pi(1/42)$ radians. The critical value ray $v_3^\lambda$ rotates clockwise $2\pi(4/42)$ radians, and lies on the $c_2^\lambda$ critical ray upper boundary of $R_1^\lambda$. 

7
at \( \exp(2\pi i(8/42)) \). When \( \lambda \) rotates counter-clockwise to \( \text{Arg } \lambda = 4\pi/3 \), the sectors \( L^\lambda, R_0^\lambda, R_1^\lambda \), and \( T_A \) rotate counter-clockwise \( 2\pi(1/42) \) radians. The critical value ray \( v_{-3}^\lambda \) rotates counter-clockwise \( 2\pi(4/42) \) radians, and still lies below the \( c_{-3}^\lambda \) critical ray boundary of \( R_0^\lambda \). By the previous proposition, the outer boundary curve of \( R_0^\lambda \) is mapped to an arc that lies in the basin and also lies outside the circular boundaries of \( L^\lambda, R_0^\lambda, R_1^\lambda \), and \( T_A \). This image arc connects the two critical value rays in the basin, and lies to the right of these rays in the basin. The inner boundary is mapped to a similar arc connecting these rays but now lying to the left. Consequently, the image of \( R_0^\lambda \) properly contains the interiors of \( L^\lambda, R_0^\lambda, R_1^\lambda \), and \( T_A \).

For the second case, note that the straightline boundaries of \( R_1^\lambda \) are mapped two-to-one onto the critical point rays passing through \( v_3^\lambda \) and \( v_2^\lambda \). When \( 2\pi/3 < \text{Arg } \lambda < 4\pi/3 \), these rays are disjoint from all of \( L^\lambda, R_0^\lambda, R_1^\lambda \), and \( T_A \). As above, when \( \lambda \) rotates clockwise to \( \text{Arg } \lambda = 2\pi/3 \), the sectors \( L^\lambda, R_0^\lambda, R_1^\lambda \), and \( T_A \) rotate clockwise \( 2\pi(1/42) \) radians. The critical value ray \( v_3^\lambda \) has already been covered in the first case, as the lower boundary of \( R_1^\lambda \) and the upper boundary of \( R_0^\lambda \) are the same. When \( \lambda \) rotates counter-clockwise to \( \text{Arg } \lambda = 4\pi/3 \), the sectors \( L^\lambda, R_0^\lambda, R_1^\lambda \), and \( T_A \) rotate counter-clockwise \( 2\pi(1/42) \) radians. The critical value ray \( v_2^\lambda \) rotates counter-clockwise \( 2\pi(4/42) \) radians, and lies on the \( c_{-3}^\lambda \) critical ray lower boundary of \( R_1^\lambda \) at \( \exp(2\pi i (-2/42)) \). By the previous Proposition, the outer boundary curve of \( R_1^\lambda \) is mapped to an arc that lies in the basin and also lies outside the circular boundaries of \( L^\lambda, R_0^\lambda, R_1^\lambda \), and \( T_A \). This image arc connects the two critical value rays in the basin, and lies to the left of these rays in the basin. The inner boundary is mapped to a similar arc connecting these rays but now lying to the right. Consequently, the image of \( R_1^\lambda \) properly contains the interiors of \( L^\lambda, R_0^\lambda, R_1^\lambda \), and \( T_A \).

For the third case, we have that the straightline boundaries of \( L^\lambda \) contain the prepoles \( p_0^\lambda \) and \( p_{-1}^\lambda \), which are both mapped to straight lines passing through the origin. In the case of \( p_0^\lambda \), this straight line passes through \( \exp(2\pi i(4/14)) \) when \( \lambda \in \mathbb{R}^- \). Then as \( \lambda \) increases or decreases by at most \( \pi/3 \), the argument of this image line rotates by at most one-seventh of \( \pi/3 \) in the corresponding direction. Hence this line lies strictly outside \( R_0^\lambda \). Note that the argument cannot be applied to \( R_1^\lambda \), as the \( p_0^\lambda \) prepole ray and the \( c_2^\lambda \) critical point ray (the upper boundary of \( R_1^\lambda \)) both intersect \( \exp(2\pi i(3/14)) \) at \( \lambda \in \mathbb{R}^- \). Similar arguments to the \( p_0^\lambda \) prepole ray work for the image of the \( p_{-1}^\lambda \) prepole ray. For the circular boundaries of \( L^\lambda \), by the previous Proposition, they are both mapped to curves in \( B_\lambda \) that lie outside the outer boundary of \( A \), but now these curves are arcs that connect the image of the prepole rays passing to the right of these lines. Hence the image of \( L^\lambda \) covers \( R_0^\lambda \) two-to-one.

For the fourth case, when \( \text{Arg } \lambda = 2\pi/3 \), the image of \( c_0^\lambda \) lies on the ray passing through \( \exp(2\pi i(-4/42)) \), and when \( \text{Arg } \lambda = 4\pi/3 \), the critical value lies on the complex conjugate ray. So, for these parameters, the critical value lies on the line that includes the straight line boundary of \( R_0^\lambda \). For the circular boundaries of \( O' \), the previous Proposition shows that the critical value now rotates around the corresponding circular boundary of \( R_0^\lambda \). So the critical value does indeed wind once around \( R_0^\lambda \).

Part 1 of proposition 8 states that \( R_0^\lambda \) contains a preimage of itself, and so there must be a fixed point in \( R_0^\lambda \). One checks easily that this fixed point is on \( B_0^\lambda \). We can now prove the
existence of an arc of infinitely many preimages of $L^\lambda$ and $T_A$ beginning at $T_0^\lambda$ and extending
to that fixed point. Inside $R_0^\lambda$ there is a preimage of the region that contains the interior of
$L^\lambda \cup T_A \cup R_0^\lambda$. The preimage of $L^\lambda$ is connected to $T_0^\lambda$. The preimage of $R_0^\lambda$ is connected to $B_0^\lambda$.
The prepole $p_3^\lambda$ is inside the preimage of $T_A$, so that we have the preimages of $L^\lambda \cup T_A \cup R_0^\lambda$ in
order of “increasing $|z|$.” Increasing $|z|$ is true when $\lambda$ lies on the real axis in the parameter
plane, but not necessarily so for $\lambda$ with imaginary component. It is more precise to say that
the preimages are arranged along the arc extending from $T_0^\lambda$ to the fixed point on $B_0^\lambda$.

(a) The construction $L^\lambda, T_A$, and $R_0^\lambda$.

(b) $0L^\lambda, 0T_A$, and $0R_0^\lambda$ inside $R_0^\lambda$.

(c) $0_2L^\lambda, 0_2T_A$, and $0_2R_0^\lambda$ inside $R_0^\lambda$ are hard to see.

(d) Zoomed in to more clearly depict $0_2L^\lambda, 0_2T_A$, and $0_2R_0^\lambda$.

Figure 4: More and more preimages of the right wedge.

Continuing, inside the preimage of $R_0^\lambda$ there is another preimage of the region that con-
tains the interior of $L^\lambda \cup T_A \cup R_0^\lambda$ in order along the arc. Inside the preimage of the preimage
of $R_0^\lambda$ there is another preimage of the region that contains the interior of $L^\lambda \cup T_A \cup R_0^\lambda$ in
order along the arc. And so on . . .

It is useful to to refer to the preimages by their itineraries. In other words, inside $R^\lambda_0$ is $0L^\lambda$, $0T_A$, and $0R^\lambda_0$. A point in $0L^\lambda$ is in $R^\lambda_0$, and after one iteration is in $L^\lambda$. Similarly, a point in $0T_A$ is in $R^\lambda_0$, and after one iteration is in $T_A$. And a point in $0R^\lambda_0$ is in $R^\lambda_0$, and after one iteration is still in $R^\lambda_0$. Inside $0R^\lambda_0$ is $00L^\lambda$, $00T_A$, and $00R^\lambda_0$. A point inside these preimages is in $R^\lambda_0$, then $R^\lambda_0$, then $L^\lambda$, $T_A$, or still $R^\lambda_0$ respectively. We will use the notation $0_k$ to represent a sequence of $k$ 0’s (i.e., $00L^\lambda = 0_2L^\lambda$). Continuing, inside $00R^\lambda_0$, or $0_2R^\lambda_0$, is $0_3L^\lambda$, $0_3T_A$, and $0_3R^\lambda_0$.

This process continues iteratively as successive preimages of the region that contains the interior of $L^\lambda$, $T_A$, and $R^\lambda_0$ accumulate at the fixed point on $B^\lambda_0$ inside $R^\lambda_0$. Informally, we say that the arc “buds” from $T^\lambda_0$ and “accumulates” at the fixed point $B^\lambda_0$. As this arc consists of preimages named with only 0’s that accumulate at the fixed point corresponding to the set with itinerary 000..., it is called the $0TL$ arc. This arc in dynamical space consists of $T_A$ and infinitely many alternating preimages of $L^\lambda$ and $T_A$ (we may drop the $\lambda$ when naming preimages and in labeling diagrams):

$$T_A < 0L < 0T < 0_2L < 0_2T < 0_3L < 0_3T < \cdots < \text{f.p. in } R^\lambda_0 \text{ at } B^\lambda_0$$

The $0TL$ arc in dynamical space quickly become hard to view due to the difference in scale of each successive preimage. It is helpful to visualize this construction not to scale, with each preimage of $L^\lambda$ represented by a line segment, and each preimage of $T_A$ by a circle. Figure 5 is a stylized depiction of the $0TL$ arc in dynamical space.

![Figure 5: $0L^\lambda$ is connected to $T^\lambda_0$. The arrow points to the fixed point in $R^\lambda_0$ on $B^\lambda_0$, so each successive preimage is a pair of smaller copies of $L^\lambda$ and $T_A$, and these preimages accumulate on the fixed point.](image)
Before we can use the existence of the $\mathcal{O} TL$ arc in $R_0^\lambda$ to show the existence of the SM arc in the parameter plane, we recall the concept of a polynomial-like map. Let $G_\mu$ be a family of holomorphic maps that depends analytically on the parameter $\mu$ lying in some open disk $\mathcal{D}$. Suppose each $G_\mu : U_\mu \rightarrow V_\mu$ where both $U_\mu$ and $V_\mu$ are open disks that also depend analytically on $\mu$. $G_\mu$ is then said to be polynomial-like of degree 2 if, for each $\mu$:

- $G_\mu$ maps $U_\mu$ two-to-one onto $V_\mu$ and so there is a unique critical point in $U_\mu$;
- $V_\mu$ contains $U_\mu$;
- As $\mu$ winds once around the boundary of $\mathcal{D}$, the critical value winds once around $U_\mu$ in the region $V_\mu - U_\mu$.

As shown in [9], for such a family of polynomial-like maps, there is a homeomorphic copy of the Mandelbrot set in the disk $\mathcal{D}$. Moreover, for $\mu$-values in this Mandelbrot set, $G_\mu | U_\mu$ is conjugate to the corresponding quadratic map given by this homeomorphism.

We have an arc of infinitely many alternating preimages of $T_A$ and $L^\lambda$ in the dynamical space. We will show that the center of each preimage of $T_A$ is the critical value for a specific $\lambda$ that is the center of a Sierpinski hole. Then we use the machinery of polynomial-like maps on some open sets to prove that each preimage of $L^\lambda$ corresponds to a Mandelbrot set. Thus, the existence of the dynamical $TL$ arc proves the existence of the corresponding arc of infinitely many Sierpinski holes and Mandelbrot sets along the negative real axis in the parameter plane.

**Theorem 2.1.** There exists the $\mathcal{O} SM$ arc along the negative real axis in the parameter plane that consists of infinitely many Mandelbrot sets $\mathcal{M}^k$ with $k \geq 2$ and infinitely many Sierpinski holes $\mathcal{E}^k$ with $k \geq 1$. Here $k$ denotes the base period of $\mathcal{M}^k$ and the escape time of $\mathcal{E}^k$. These sets are arranged along the negative real axis in this manner:

$$\text{Cantor set locus} < \cdots < \mathcal{M}^4 < \mathcal{E}^3 < \mathcal{M}^3 < \mathcal{E}^2 < \mathcal{M}^2 < \mathcal{E}^1.$$  

**Proof.** We will first prove that each preimage of $T_A$ in dynamical space corresponds to a Sierpinski hole in the parameter plane (the trap door itself corresponds to the McMullen domain). By construction, for each $\lambda \in \mathcal{O}'$, there is a unique prepole $p_3^\lambda$ in the interior of $R_0^\lambda$. Since $F_\lambda$ maps $R_0^\lambda$ one-to-one over itself, there is a unique preimage of this prepole, $z_3^\lambda$, in $R_0^\lambda$, so $F^2_\lambda(z_3^\lambda) = 0$. Continuing, for each $\lambda \in \mathcal{O}'$, there is a unique point $z_k^\lambda$ in $R_0^\lambda$ for which we have $F_\lambda(z_k^\lambda) = z_{k-1}^\lambda$ and so $F^{k-1}_\lambda(z_k^\lambda) = 0$. This holds true for the $(k + 1)^{st}$ case as well.

The points $z_k^\lambda$ vary analytically with $\lambda$ and are strictly contained in the interior of $R_0^\lambda$. So we may consider the function $H^k(\lambda)$ defined on $\mathcal{O'}$ by $H^k(\lambda) = v_0^\lambda - z_k^\lambda$ where $v_0^\lambda = F_\lambda(c_0^\lambda)$. When $\lambda$ rotates once around the boundary of $\mathcal{O}'$, $v_0^\lambda$ rotates once around the boundary of $R_0^\lambda$ while $z_k^\lambda$ remains in the interior of $R_0^\lambda$. Hence $H^k(\lambda)$ has winding number one around the boundary of $\mathcal{O}'$ and so there must be a unique zero in $\mathcal{O}'$ for each $H^k$. This $\lambda$ is then the parameter that lies at the center of the escape time region $\mathcal{E}^k$. Using the technique from [13], it then follows that $\mathcal{E}^k$ is an open disk in the parameter plane.
The \( \lambda \) in the parameter plane such that \( v_0^\lambda = p_3^\lambda \) (with \( p_3^\lambda \) being the center of 0T in dynamical space) is the center of \( \mathcal{E}^2 \), so 2 is equivalently the escape time of \( c_0^\lambda \) for \( \lambda \in \mathcal{E}^2 \) and the length of the sequence 0T.

The \( \lambda \) in the parameter plane such that \( v_0^\lambda = z_3^\lambda \) (with \( z_3^\lambda \) being the center of 00T in dynamical space) is the center of \( \mathcal{E}^3 \), so 3 is equivalently the escape time of \( c_0^\lambda \) for \( \lambda \in \mathcal{E}^3 \) and the length of the sequence 00T.

Continuing in this fashion, the \( \lambda \) in the parameter plane such that \( v_0^\lambda = z_k^\lambda \) (with \( z_k^\lambda \) being the center of 0\(_{k-1}\)T in dynamical space) is the center of \( \mathcal{E}^k \), so \( k \) is equivalently the escape time of \( c_0^\lambda \) for \( \lambda \in \mathcal{E}^k \) and the length of the sequence 0\(_{k-1}\)T.

Note that, as \( \lambda \) decreases along \( \mathbb{R}^- \), both \( v_0^\lambda \) and \( z_k^\lambda \) increase along \( \mathbb{R}^+ \). It follows that the portion of \( \mathcal{E}^{k+1} \) in \( \mathbb{R}^- \) lies to the left of \( \mathcal{E}^k \) in the parameter plane.

To prove the existence of the Mandelbrot sets \( \mathcal{M}^k \), recall that the orbit of the point \( z_k^\lambda \) under \( F_\lambda \) remains in \( R_0^\lambda \) before entering \( T_A \) and landing at 0 at iteration \( k-1 \) (here \( z_2^\lambda = p_3^\lambda \)). For each \( k \geq 2 \), let \( E_k^\lambda \) be the open set surrounding \( z_k^\lambda \) in \( R_0^\lambda \) that is mapped onto \( T_A \) by \( F_{\lambda-1}^k \). Let \( D_k^\lambda \) be the set in \( R_0^\lambda \) consisting of points whose first \( k-2 \) iterations lie in \( R_0^\lambda \) but whose \((k-1)\)st iterate lies in the interior of \( L^\lambda \). Since \( F_\lambda \) is univalent on \( R_0^\lambda \), each \( D_k^\lambda \) is an open disk. Furthermore, the boundary of \( D_k^\lambda \) meets a portion of the boundaries of both \( E_{k-1}^\lambda \) and \( E_k^\lambda \) (where \( E_1^\lambda = T_A \)). Since \( F_{\lambda-1}^k \) maps \( D_k^\lambda \) one-to-one over the interior of \( L^\lambda \) and then \( F_\lambda \) maps \( L^\lambda \) two-to-one over a region that contains \( R_0^\lambda \), we have that \( F_k^\lambda \) maps \( D_k^\lambda \) two-to-one over a region that completely contains \( R_0^\lambda \). Moreover, the critical value for \( F_k^\lambda \) is just \( v_0^\lambda \), which, by the preceding Proposition, winds once around the exterior of \( R_0^\lambda \) as \( \lambda \) winds once around the boundary of \( \mathcal{O}' \). Hence, \( F_k^\lambda \) is a polynomial–like map of degree two on \( D_k^\lambda \) and this proves the existence of a baby Mandelbrot set \( \mathcal{M}^k \) lying in \( \mathcal{O}' \) for each \( k \geq 2 \). When \( \lambda \) is real and negative, we have that the centers of the escape regions \( \mathcal{E}^k \) lie along \( \mathbb{R}^- \) and, since the real line is invariant under \( F_\lambda \) when \( \lambda \in \mathbb{R}^- \), both \( c_0^\lambda \) and \( v_0^\lambda \) also lie on the real axis. Then, by the \( \lambda \mapsto -\lambda \) symmetry in the parameter plane, the spines of those Mandelbrot sets also lie in \( \mathbb{R}^- \).

Next, since the \( E_k^\lambda \) and \( D_k^\lambda \) are arranged along the TL arc from the trap door out to the fixed point in the following fashion:

\[
T_A = E_1^\lambda < D_2^\lambda < E_2^\lambda < D_3^\lambda < E_3^\lambda < \ldots
\]

we have that the \( \mathcal{E}^k \) and \( \mathcal{M}^k \) are arranged along the negative real axis in the parameter plane in the opposite manner:

\[
\ldots < \mathcal{M}_4 < \mathcal{E}_3 < \mathcal{M}_3 < \mathcal{E}_2 < \mathcal{M}_2 < \mathcal{E}_1
\]

as shown in figure 6.
Finally, when $\lambda \in \mathbb{R}^-$, there is a non-empty interval lying between each adjacent $M^k$ and $E^j$ (where $j = k$ or $k - 1$). This interval contains parameters for which $F^k(\lambda_0^1)$ lies in $L^\lambda$, but then $F^{k+1}(\lambda_0^1)$ is back in $R^\lambda_0$ and close to $\partial B^\lambda$. As a consequence, it takes more than $k$ additional iterations for this critical orbit to reach $T_A$ or return to $L^\lambda$.

3 Phase Two: the $\bar{T} TL$ arc in dynamical space

Theorem 2.1 proves the existence of alternating Mandelbrot sets and Sierpinski holes along the negative real axis in the parameter plane by using the existence of alternating open sets $E^k$ and open disks $D^k$, which are created from alternating preimages of $T_A$ and $L^\lambda$ along the positive real axis in the dynamical plane. This theorem also applies to the family $z^n + \lambda/z^d$ for $n \geq 2$ and even, and $d \geq 3$ and odd, and is first described in [1].

Recall that there is a preimage of $R^\lambda_0$ inside $R^\lambda_0$, and so there is a fixed point inside $R^\lambda_0$ that lies on $B^\lambda_0$. This preimage together with this fixed point imply that successive preimages of the region that contains the interiors of $L^\lambda, T_\lambda$, and $R^\lambda_0$ accumulate at the fixed point on $B^\lambda_0$. Recall that proposition 8 refers to two right wedges, and according to that proposition, there is a preimage of $R^\lambda_1$ inside $R^\lambda_1$. Note that, since $F(R^\lambda_1) \supset R^\lambda_1$, there exists a unique fixed point in $R^\lambda_1$ as well. However, unlike in $R^\lambda_0$, that fixed point does not lie on the boundary of either the basin or the trap door. One checks easily that this fixed point lies in the interior of $R^\lambda_1$.

The natural question arises: is there an arc in $R^\lambda_1$ analogous to the arc in $R^\lambda_0$? Is there some kind of "$\bar{T} TL$ arc" of alternating preimages of $L^\lambda, T_\lambda$, and $R^\lambda_1$ that accumulate at the fixed point in $R^\lambda_1$?
The answer is yes, but the construction of the $\overline{TL}$ arc is not as straightforward as that of the $\overline{UL}$ arc.

In the proof of part 2 of Proposition 8, the outer boundary curve of $R^\lambda_1$ is mapped to an arc to the left of the critical value rays in the basin, while the inner boundary curve of $R^\lambda_1$ is mapped to an arc to the right of the critical value rays. So the preimage of $L^\lambda$ inside $R^\lambda_1$ must share a boundary with the basin while the preimages of $R^\lambda_0$ and $R^\lambda_1$ inside $R^\lambda_1$ must share their boundaries with $T^\lambda$. This is because the image of $c^\lambda_2$ is $p^\lambda_{-3}$, and as $z$ travels clockwise around the boundary of $R^\lambda_1$ from $c^\lambda_{-3}$, $F^\lambda(z)$ travels clockwise around $A$ from $p^\lambda_{-3}$. Thus the orientation remains the same for preimages of $R^\lambda_1$, and the set of preimages is rotated inside $R^\lambda_1$. This applies to all preimages for any number of iterations back. Thus, inside $R^\lambda_1$, each set of preimages that contains the interior of $R^\lambda_0 \cup R^\lambda_1 \cup L^\lambda$ is rotated in their corresponding preimage of $R^\lambda_1$. See figure 7b.

![Diagram](image)

(a) The preimages in $R^\lambda_0$ have the same orientation. Part of $\partial \partial L^\lambda$ is on the inner $\partial A$ and part of $\partial \partial R^\lambda_0$ is on the outer $\partial A$.

(b) the preimages in $R^\lambda_1$ have the same orientation but have been "rotated." Part of $\partial \partial L^\lambda$ is on the outer $\partial A$ and part of $\partial \partial R^\lambda_1$ is on the inner $\partial A$.

Figure 7: mappings for $R^\lambda_0$ vs $R^\lambda_1$

Using proposition 2, we can now prove the existence of an arc of infinitely many preimages of $L^\lambda$ and $T_A$ inside $R^\lambda_1$ beginning at both $B^\lambda_1$ and $T^\lambda_1$ and accumulating at the fixed point in the interior of $R^\lambda_1$. Inside $R^\lambda_1$ there is a preimage of the region that contains the interior of $R^\lambda_1 \cup L^\lambda$, with the preimage of $L^\lambda$ connected to $B^\lambda_1$ and the preimage of $R^\lambda_1$ connected to $T^\lambda_1$. There is also the prepole $p^\lambda_2$, which is inside the preimage of $T_A$. Note that in contrast to the placement of the preimages in $R^\lambda_0$, these preimages are rotated, so that we have the preimages of $R^\lambda_1 \cup T_A \cup L^\lambda$ in order along the arc from $T^\lambda_1$ to $B^\lambda_1$. Inside the preimage of $R^\lambda_1$ there is another rotated preimage of the region that contains the interiors of $L^\lambda \cup T_A \cup R^\lambda_1$ in order along the arc. Inside the preimage of the preimage of $R^\lambda_1$ there is another rotated preimage of the region that contains the interiors of $R^\lambda_1 \cup T_A \cup L^\lambda$ in order along the arc.
Thus, $1L^\lambda$ and $1T_A$ are located inside $R_1^\lambda$, with $1L^\lambda$ connected to $B_\lambda$ at $B_1^\lambda$. Place their preimages $12L^\lambda$ and $12T_A$ inside $R_1^\lambda$. As $1L^\lambda$ is connected to $B_1^\lambda$, their respective preimages $1_2L^\lambda$ and $1_2T_A$ are connected. The sets are ordered $T_A, 1L^\lambda, 1T_A, \ldots, 1_2T_A, 1_2L^\lambda, B_\lambda$ along the arc. Place their preimages $1_3L^\lambda$ and $1_3T_A$ inside $R_1^\lambda$. As $1_2L^\lambda$ is connected to $T_\lambda$ at $T_1^\lambda$, their respective preimages $1_3L^\lambda$ and $1_1T_A$ are connected. The sets are ordered $T_A, 1L^\lambda, 1T_A, 1_3L^\lambda, 1_3T_A, \ldots, 1_4T_A, 1_4L^\lambda, 1_2T_A, 1_2L^\lambda, B_\lambda$ along the arc.

Figure 8: More and more preimages of the construction inside the upper right wedge, analogous to figure 4.

This process continues iteratively as successive preimages of the region that contains the interiors of $L^\lambda, T_A$, and $R_1^\lambda$ “meet in the middle,” i.e., accumulate at the fixed point in the interior of $R_1^\lambda$. The preimages with an odd number of 1’s in their name (i.e., even escape time or base period) are closer to the basin, and the preimages with an even number of 1’s in their name (i.e., odd escape time or base period) are closer to the trap door. Informally, we say that the $\bar{1}$TL arc “buds” from $B_1^\lambda$ and $T_1^\lambda$ and “accumulates” at the fixed point inside $R_1^\lambda$. As this arc consists of preimages named with only 1’s that accumulate at the fixed point corresponding to the set with itinerary $111\ldots$, it is called the $\bar{1}$TL arc. This arc in dynamical space consists of infinitely many alternating preimages of $L^\lambda$ and $T_A$ between $B_1^\lambda$ and $T_1^\lambda$:

$$T_1^\lambda < 1_2L < 1_2T < 1_4L < 1_4T < \cdots < \text{f.p. in } R_1^\lambda < \cdots < 1_3T < 1_3L < 1T < 1L < B_1^\lambda.$$ 

See figure 9 for a depiction of the $\bar{1}$TL arc in a style similar to the $\bar{0}$TL arc.
Figure 9: $1_2 L^\lambda$ is connected to $T_1^\lambda$ and $1 L^\lambda$ is connected to $B_1^\lambda$. The X represents the fixed point in the interior of $R_1^\lambda$, so each successive preimage is a pair of smaller copies of $L^\lambda$ and $T_A$ on alternating sides of the X, and these preimages accumulate on the fixed point.

4 Phase Three: The $\bar{T} TL$ spiral in dynamical space

Up to this point, we have used Proposition 8 to find preimages of $R_0^\lambda$ inside $R_0^\lambda$, and preimages of $R_1^\lambda$ inside $R_1^\lambda$. In actuality, there is a preimage of the interiors of $L^\lambda \cup T_A \cup R_0^\lambda \cup R_1^\lambda$ in $R_0^\lambda$, and a (rotated) preimage of the interiors of $L^\lambda \cup T_A \cup R_0^\lambda \cup R_1^\lambda$ in $R_1^\lambda$. This means that the $\bar{0} TL$ arc, contained entirely inside $R_0^\lambda$, has a preimage contained entirely inside $R_1^\lambda$.

This preimage is the $1\bar{0} TL$ arc. $10L^\lambda$ and $10T_A$ are the preimages of $0L^\lambda$ and $0T_A$. As $0L^\lambda$ and $T_0^\lambda$ are connected, their respective preimages $10L^\lambda$ and $1T_A$ are connected. Recall that the $\bar{0} TL$ arc buds from $T_0^\lambda$. Then the $1\bar{0} TL$ arc buds from $1T_A$. The $\bar{0} TL$ arc accumulates at the fixed point on $B_0^\lambda$. Thus, the $1\bar{0} TL$ arc accumulates at the preimage of the fixed point on $B_0^\lambda$, which is on $T_1^\lambda$. Because of the rotation of the preimages inside $R_1^\lambda$, this preimage of the fixed point of all $0'$s is above the $\bar{T} TL$ arc, and so the entire $1\bar{0} TL$ arc is above the $\bar{T} TL$ arc.

Next, consider the preimage in $R_1^\lambda$ of the $1\bar{0} TL$ arc that we just constructed. This is the $1_2\bar{0} TL$ arc. As the $1\bar{0} TL$ arc buds from $1T_A$ and accumulates at $T_1^\lambda$, the $1_2\bar{0}$ arc buds from $1_2 T_A$ and accumulates at $1T_A$. Because of the rotation of the preimages inside $R_1^\lambda$, the $1_2\bar{0} TL$ arc is below the $\bar{T} TL$ arc.

Continuing, the preimage in $R_1^\lambda$ of the $1_2\bar{0} TL$ arc is the $1_3\bar{0} TL$ arc. As the $1_2\bar{0} TL$ arc buds from $1_2 T_A$ and accumulates at $1T_A$, the $1_3\bar{0} TL$ arc buds from $1_3 T_A$ and accumulates at $1_2 T_A$. Because of the rotation of the preimages inside $R_1^\lambda$, the $1_3\bar{0} TL$ arc is above the $\bar{T} TL$ arc.
This process continues iteratively and we have infinitely many successive preimages of the $\overline{0} TL$ arc, lying alternately above and below the $\overline{1} TL$ arc, accumulating at the fixed point inside $R_1^\lambda$. The $\overline{0} TL$ arc and its infinitely many preimages each consist of infinitely many preimages of $T_A$ and $L^\lambda$.

If we imagine a complex number $z$ traveling along this construction, its path would be a "spiral" that begins at the accumulation point of the $\overline{0} TL$ arc, along the $\overline{0} TL$ arc to the trap door, along the $1\overline{0} TL$ arc to $1T_A$, along the $1_2\overline{0} TL$ arc to $1_2T_A$, along the $1_3\overline{0} TL$ arc to $1_3T_A$, and so on, passing through every preimage of $T_A$ in the $\overline{1} TL$ arc and limiting to the accumulation point of the $\overline{1} TL$ arc. See figure 10.

![Figure 10: The $\overline{1} TL$ spiral in $R_1^\lambda$. Not shown is the beginning of the spiral on $B_0^\lambda$, the $\overline{0} TL$ arc, and the trap door, before $z$ “enters” this picture at the accumulation point of the $1\overline{0} TL$ arc. From the accumulation point of the $1\overline{0} TL$ arc, $z$ travels through infinitely many preimages of $T_A$ and $L^\lambda$ to $1T$. It then passes through the accumulation point of the $1_2\overline{0} TL$ arc, through infinitely many preimages of $T_A$ and $L^\lambda$ to $1_2T$. It then passes through the accumulation point of the $1_3\overline{0} TL$ arc, through infinitely many preimages of $T_A$ and $L^\lambda$ to $1_3T$. And so on.]

We have found a structure consisting of the $1\overline{0} TL$ arc and infinitely many of its preimages, where each arc consists of infinitely many alternating preimages of $T_A$ and $L^\lambda$, in which the arcs tend to the fixed point of all $1$'s. Each iteration takes place in a smaller and smaller region as the arcs tend to the fixed point in a spiral pattern. As it passes through every preimage of $T_A$ in the $\overline{1}$ arc, we will call this the $\overline{1} TL$ spiral.
5 The 0TL SM spiral in the parameter plane

The 0TL spiral in dynamical space consists of preimages of $T_A$ and $L^\lambda$. As each preimage of $L^\lambda$ in dynamical space proves the existence of a Mandelbrot set in the parameter plane, and each preimage of $T_A$ in dynamical space proves the existence of a Sierpinski hole in the parameter plane, the dynamical spiral suggests the existence of a parameter spiral of Mandelbrot sets and Sierpinski holes. However, for all $\pi/3 < \arg \lambda < 2\pi/3$, the critical value lies in $R_0^\lambda$. As a consequence, there are no such Mandelbrot sets or Sierpinski holes in the parameter plane that correspond to the preimages of $T_A$ and $L^\lambda$ in $R_1^\lambda$.

But the 0TL spiral is in $R_1^\lambda$, and there is a preimage of $R_1^\lambda$ inside $R_0^\lambda$. Therefore, we consider the 0TL TL spiral in dynamical space, and use that to prove the existence of the 0TL SM spiral in the parameter plane.

**Theorem 5.1.** There exists a 0TL SM arc below the negative real axis in the parameter plane that consists of infinitely many Mandelbrot sets $M^k$ and infinitely many Sierpinski holes $E^k$ both with $k \geq 3$. As before, $k$ denotes the base period of $M^k$ and the escape time of $E^k$. These sets are arranged in from the Cantor set locus to the McMullen domain in this manner:

- Cantor set locus, $M^3, E^3, M^5, E^5, M^7, E^7, \ldots, E^8, M^8, E^6, M^6, E^4, M^4, E^1$.

Note that the Cantor set locus and the McMullen domain are not included in the 0TL arc.

Furthermore, there exists the 0TL SM spiral below the negative real axis in the parameter plane that “spirals” from the Cantor set locus along the 0 arc to the 0 hole (Sierpinski hole with itinerary 0T), along the 010 arc to the 01 hole, along the 0120 arc to the 012 hole, along the 0130 arc to the 013 hole, and so on, passing through each Sierpinski hole in the 0TL arc and limiting to the Sierpinski hole with center $\lambda$ such that $F_\lambda^2(c_0^\lambda)$ is the fixed point in $R_1$.

**Proof.** We will again prove that each preimage of $T_A$ in dynamical space corresponds to a Sierpinski hole in the parameter plane. By construction, for each $\lambda \in O'$, there is a unique prepole $p_0^\lambda$ in the interior of $R_1^\lambda$. Since $F_\lambda$ maps $R_0^\lambda$ one-to-one onto a region containing the interior of $R_1^\lambda$, there is a unique preimage of this prepole, $w_4^\lambda$, in $R_1^\lambda$, so $F_\lambda^3(w_4^\lambda) = 0$. Continuing, for each $\lambda \in O'$, there is a unique point $w_k^\lambda$ in $R_0^\lambda$ for which we have $F_\lambda(w_k^\lambda) = w_{k-1}^\lambda$ and so $F_\lambda^{k-1}(w_k^\lambda) = 0$. This holds true for the $(k+1)^{st}$ case as well.

The points $w_k^\lambda$ vary analytically with $\lambda$ and are strictly contained in the interior of $R_0^\lambda$. So we may consider the function $H^k(\lambda)$ defined on $O'$ by $H^k(\lambda) = v_0^\lambda - w_k^\lambda$ where $v_0^\lambda = F_\lambda(c_0^\lambda)$. When $\lambda$ rotates once around the boundary of $O'$, $v_0^\lambda$ rotates once around the boundary of $R_0^\lambda$ while $w_k^\lambda$ remains in the interior of $R_0^\lambda$. Hence $H^k(\lambda)$ has winding number one around the boundary of $O'$ and so there must be a unique zero in $O'$ for each $H^k$. This $\lambda$ is then the parameter that lies at the center of the escape time region $E^k$. $E^k$ is an open disk in the parameter plane.

The $\lambda$ in the parameter plane such that $v_0^\lambda = w_3^\lambda$ (with $w_3^\lambda$ being the center of 01T in dynamical space) is the center of $E^3$, so 3 is equivalently the escape time of $v_0^\lambda$ for $\lambda \in E^3$ and the length of the sequence 01T. Under iteration by $F_\lambda$, $c_0^\lambda \in L^\lambda \rightarrow w_3^\lambda \in R_0^\lambda \rightarrow p_2^\lambda \in R_1^\lambda \rightarrow 0 \in T_A$. 

18
The $\lambda$ in the parameter plane such that $v_0^\lambda = w_4^\lambda$ (with $w_4^\lambda$ being the center of 01$_2T$ in dynamical space) is the center of $\mathcal{E}^4$, so 4 is equivalently the escape time of $v_0^\lambda$ for $\lambda \in \mathcal{E}^4$ and the length of the sequence 01$_2T_A$. Under iteration by $F_\lambda$, $c_0^\lambda \in L^\lambda \rightarrow w_4^\lambda \in R_0^\lambda \rightarrow w_3^\lambda \in R_0^\lambda \rightarrow p_3^\lambda \in \mathcal{T}_A$.

Continuing in this fashion, the $\lambda$ in the parameter plane such that $v_0^\lambda = w_k^\lambda$ (with $w_k^\lambda$ being the center of 01$_{k-2}T_A$ in dynamical space) is the center of $\mathcal{E}^k$, so $k$ is equivalently the escape time of $v_0^\lambda$ for $\lambda \in \mathcal{E}^k$ and the length of the sequence 01$_{k-2}T_A$.

Note that, as $k$ increases from an odd value to an even value, $|\lambda|$ decreases, while both $|v_0^\lambda|$ and $|w_k^\lambda|$ increase. As $k$ increases from an even value to an odd value, $|\lambda|$ increases, while both $|v_0^\lambda|$ and $|w_k^\lambda|$ decrease. Successively higher odd values of $k$ have lower $|\lambda|$ in the parameter plane and higher $|v_0^\lambda|$ and $|w_k^\lambda|$ in dynamical space, while successively higher even values of $k$ have higher $|\lambda|$ in the parameter plane and lower $|v_0^\lambda|$ and $|w_k^\lambda|$ in dynamical space.

It follows that, in the parameter plane, in order of decreasing $|\lambda|$, the sets are arranged $\mathcal{E}^k, \mathcal{E}^{k+1}$ for $k$ odd, $\mathcal{E}^{k+1}, \mathcal{E}^k$ for $k$ even, with all odd escape time Sierpinski holes having higher center $|\lambda|$ than all even escape time Sierpinski holes.

To prove the existence of the Mandelbrot sets $\mathcal{M}^k$, recall that the point $w_4^\lambda \in R_0^\lambda$, has an orbit under $F_\lambda$ in $R_1^\lambda$ for iterates 1 through $k-2$, and enters $\mathcal{T}_A$ at iteration $k-1$ (here $w_2^\lambda = p_2^\lambda$). For each $k \geq 3$, let $E^k_{\lambda}$ be the open set surrounding $w_k^\lambda$ in $R_0^\lambda$ that is mapped onto $\mathcal{T}_A$ by $F_{\lambda}^{k-1}$. Let $D^k_\lambda$ be the set in $R_0^\lambda$ consisting of points whose first $k-2$ iterates lie in $R_1^\lambda$ but whose $(k-1)^{st}$ iterate lies in the interior of $L_\lambda$. Since $F_\lambda$ is univalent on $R_0^\lambda$ and $R_1^\lambda$, each $D^k_\lambda$ is an open disk. Furthermore, the boundary of $D^k_\lambda$ meets a portion of the boundaries of both $E_\lambda^{k-2}$ and $E^k_{\lambda}$. Since $F_{\lambda}^{k-1}$ maps $D^k_{\lambda}$ one-to-one over the interior of $L_\lambda$ and then $F_\lambda$ maps $L_\lambda$ two-to-one over a region that contains $R_0^\lambda$, we have that $F_{\lambda}^k$ maps $D^k_{\lambda}$ two-to-one over a region that completely contains $R_0^\lambda$. Moreover, the critical value for $F_\lambda$ is just $v_0^\lambda$, which, by Proposition 8, winds once around the exterior of $R_0^\lambda$ as $\lambda$ winds once around the boundary of $\mathcal{O}'$. Hence, $F_{\lambda}^k$ is a polynomial-like map of degree two on $D^k_{\lambda}$ and this proves the existence of a baby Mandelbrot set $\mathcal{M}^k$ lying in $\mathcal{O}'$ for each $k \geq 3$.

As in the above proof, the location of each Sierpinski hole in the 01 SM spiral can be computed by solving $F^k(v^\lambda) = p_2^\lambda$, where $p_2^\lambda$ is the prepole in $R_1^\lambda$ and $k$ corresponds to the escape time of the Sierpinski hole $\mathcal{E}^k$. They can also be verified by selecting $\lambda$ in that Sierpinski hole and observing the itinerary of the critical value. A straightforward computation shows that the 01$T_A$ region corresponds to the 01 Sierpinski hole $\mathcal{E}^3$ located below the negative real axis in the parameter plane. This means: the $\lambda$ parameter value at the center of the 01 Sierpinski hole results in the map $F_\lambda$ such that $c_0^\lambda$ maps to $v_0^\lambda$ inside 01$T_A$, which maps to 1$T_A$, which maps to exactly the center of the trap door (the origin). Similarly, the 01$2T_A$ region, which is the preimage of the 01$T_A$ region, corresponds to the 01$2$ Sierpinski hole $\mathcal{E}^4$. Then we can adopt the convention of referring to the 01$2$ Sierpinski hole as the “preimage” of the 01 Sierpinski hole because their corresponding regions in dynamical space have that relationship. We will use the same convention for “preimage” SM arcs overall because their corresponding $TL$ arcs have that relationship. For example, the 1$20$ $TL$ arc is
the preimage of the 10 TL arc, therefore we will refer to the 120 SM arc as a “preimage” of the 10 SM arc.

Figure 11: The 01 SM spiral in the parameter plane. The parameter λ “begins” on the negative real axis at the parameter such that the critical value is the fixed point in \( R_0^1 \). λ travels through infinitely many Sierpinski holes and Mandelbrot sets comprising the 0 SM arc to the 0 Sierpinski hole. λ then passes through infinitely many Sierpinski holes and Mandelbrot sets comprising the 010 SM arc to the 01 Sierpinski hole. λ then passes through infinitely many Sierpinski holes and Mandelbrot sets comprising the 0120 SM arc to the 012 Sierpinski hole. And so on, ending at the parameter such that the critical value is the fixed point in \( R_1^\lambda \). Note that the parameter travels in the opposite direction of the arrows.

Figure 11 shows the 01 SM spiral as λ enters the connectedness locus from the Cantor set locus, travels along the 0 SM arc into the 0 Sierpinski hole, then along the 010 arc into the 01 Sierpinski hole, then along the 0120 arc into the 012 Sierpinski hole, and so on. The 013 Sierpinski hole is a speck at this scale.
6 Infinitely many $TL$ spirals in dynamical space and infinitely many SM spirals in the parameter plane

In dynamical space (meaning all spirals are $TL$ spirals), we have used Proposition 8 to find the preimage of the $\bar{T}$ spiral: the $0\bar{T}$ spiral in $R_0^\lambda$. This $0\bar{T}$ spiral has its preimage in $R_0^\lambda$, the 00$\bar{T}$ spiral, and its preimage in $R_1^\lambda$, the 10$\bar{T}$ spiral. As expected, the 00$\bar{T}$ spiral passes through every preimage of $T_\Lambda$ in the 00$\bar{T}$ arc, and the similar statement can be made about the 10$\bar{T}$ spiral. The 00$\bar{T}$ spiral has its preimages the 000$\bar{T}$ arc and the 100$\bar{T}$ spiral.

As for the 10$\bar{T}$ spiral, its preimage in $R_1^\lambda$ must pass through every preimage of $T_\Lambda$ in the 110$\bar{T}$ arc, is connected to $110T_\Lambda$, which is part of the 11$\bar{0}$ arc, which is part of the path of the $\bar{T}$ spiral. Its preimage in $R_1^\lambda$ is connected to $010T_\Lambda$ which is part of the 01$\bar{0}$ arc.

It suffices to find the successive preimages in $R_1^\lambda$ of the $0\bar{T}$ spiral to show that there are infinitely SM many spirals in the parameter plane of the form $0_k\bar{T}$. In dynamical space, we also find their preimages in $R_1^\lambda$, spirals of the form $10_k\bar{T}$, without difficulty.

A stylized picture in dynamical space with the original $\bar{T}$ $TL$ spiral and 3 iterations of preimages, in a similar fashion to figure 10:
Figure 12: The upper wedge is $R^\lambda_1$ and the lower wedge is $R^\lambda_0$. Depicted here are $\overline{1} TL$ spiral, its preimage the $0\overline{1} TL$ spiral, its preimages the $0\overline{2} TL$ and $10\overline{1} TL$ spirals, and the $0\overline{2} TL$ spiral’s preimages, the $0\overline{3} TL$ and $10\overline{2} TL$ spirals.
In parameter space, all SM spirals of the form $0_k \overline{1}$ can be found and verified as well. A more complete version of figure 11 including some of these infinitely many spirals in the parameter plane would look like:

![Figure 13](image)

Figure 13: Shown here is the $0 \overline{1}$ SM spiral as in figure 11, as well as its preimage the $0_2 \overline{1}$ SM spiral, and the $0_2 \overline{1}$ SM spiral’s preimage the (barely visible) $0_3 \overline{1}$ SM spiral. Whereas the $0 \overline{1}$ SM spiral passes through the $\overline{0}$ SM arc to the $0$ Sierpinski hole and then the $01\overline{0}$ SM arc, the $0_2 \overline{1}$ SM spiral passes through the $\overline{0}$ SM arc to the $0_2$ Sierpinski hole and then the $0_2 \overline{1}$ SM arc.

We have shown that infinitely many SM spirals exist in the $\pi \leq \text{Arg} \lambda \leq 4\pi/3$ region of the parameter plane. However, the choice of $R^\lambda_1$ being above $R^\lambda_0$ was arbitrary. If we had chosen the other right wedge to be below $R^\lambda_0$ - "the $R-1^\lambda$ wedge" - we would have the symmetric "$\overline{1}$" $TL$ spiral in dynamical space and the corresponding counterclockwise "$0(\overline{1})$" SM spiral in the parameter plane, along with their infinitely many preimages.

Furthermore, by the 3-fold symmetry of the parameter plane, symmetric copies of these infinitely many spirals also exist in the $0 \leq \text{Arg} \lambda \leq 2\pi/3$ and $4\pi/3 \leq \text{Arg} \lambda \leq 2\pi$ regions.
of the parameter plane. Together with the “lower wedge,” this means that for each spiral in the $\pi \leq \arg \lambda \leq 4\pi/3$ region of the parameter plane, there are actually six spirals in the parameter plane, as in figure 14.

![Figure 14: Looks pretty.](image)

7 The general case with $n \geq 4$ is even and $d \geq 3$ is odd.

In a subsequent paper we will address the general case. The same construction applies to some values of $n$ and $d$, but there are some very different cases for certain combinations.

References


