Itineraries of entire functions

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Abstract

We use symbolic dynamics to describe the set of allowable itineraries of orbits of the complex exponential family $\lambda \exp(z)$. First, we show that the set of allowable itineraries is the same for every member of this family. We then show that the set of itineraries is the same for every map of finite exponential order. In addition, we study other transcendental entire functions, and show that they also have the same set of itineraries. Finally, we give an example of a class of functions with a different set of itineraries.

Keywords. complex exponential, Julia set, symbolic dynamics, complex dynamics

1 Introduction

The study of complex analytic maps as dynamical systems has a long history. Fatou [10] and Julia [11] studied the iteration of these maps in the early part of the twentieth century. The field laid relatively dormant for several decades, but there has been a large amount of work on complex dynamics in the last 30 years.

Most of the work in complex dynamics has centered around rational maps of the complex plane and the structure of their Julia sets. Recall that the Julia set of a map can be defined as the closure of the repelling periodic points of the map, or, equivalently, the points at which the family of iterates of the map fails to be normal. A good summary of the properties of Julia sets can be found in [2]. Various authors have studied the dynamics of the exponential family of maps. The pioneering work on the Julia sets of exponential maps was done by Devaney [5], [6], [7] and Misiurewicz [13]. See [1] for a good summary of the properties of the Julia sets of the complex exponential. The dynamics of the complex exponential are useful in other ways, such as understanding the standard family [9]. A common technique to study Julia sets is to use symbolic dynamics, for example, see [3].

Our goal in this paper is to study the symbolic dynamics of entire functions which are of finite exponential order, namely $E_\lambda(z) = \lambda e^z$,

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$E_{\alpha,\beta}(z) = \alpha e^z + \beta e^{-z}$, and $E_p(z) = e^{p(z)}$. (We use Conway's [4] definition of exponential order throughout this paper, simply that f is of exponential order α if $|f(z)| < \exp(|z|^\alpha)$ for z large.)

For the map $E_\lambda(z)$, the complex plane splits naturally into fundamental domains whose images cover $\mathbb{C} \setminus \{0\}$ under one iteration of the map. To each orbit we assign an infinite sequence of integers, and this sequence corresponds to the itinerary of the orbit, i.e., those fundamental domains in which the points on the orbit lie. Devaney and Krych, in [7], studied these itineraries and, for each λ, gave necessary and sufficient conditions for a sequence of integers to be an allowable sequence of the map E_λ, i.e., that there is an orbit in $\mathbb{C}$ with that sequence as its itinerary under E_λ.

In this paper, we do two things. First, we show that the set of allowable sequences for E_λ is independent of λ, answering a question posed in [7]. Second, we extend the results in two directions: to functions of the form $\alpha e^z + \beta e^{-z}$, and to functions of the form $e^{p(z)}$. In the first case, we can use a simple conjugacy to obtain results for $e^{\lambda z}$ and $e^{\alpha z} + e^{-\beta z}$. This last class of functions is useful, since it contains many common entire functions, such as $\sin(\lambda z)$, $\cos(\lambda z)$, $\cosh(\lambda z)$, $\sinh(\lambda z)$, and others. The second generalization gives us functions which are less familiar, but this allows us to completely characterize functions of finite exponential order.

In [7] and [1], it is shown that the Julia set of E_λ is homeomorphic to a set of half-lines indexed by the itineraries of E_λ. It is also shown that the map E_λ, restricted to the Julia set, is conjugate to the well-known shift map on itineraries. This, together with the result that the itineraries of E_λ and E_μ are always the same, suggests that E_λ and E_μ are topologically conjugate, since they have exactly the same symbolic dynamics. But in [8] it is shown that if λ and μ are distinct real numbers greater than $1/e$, then E_λ and E_μ are not conjugate. Thus this is an example of a dynamical system in which the natural symbolic dynamics of two different maps are the same, but the maps are not conjugate.

2 Itineraries of λe^z

Definition 2.1. *For $n \in \mathbb{Z}$, let*

$$R(n) = \{z \in \mathbb{C} \mid \pi(2n-1) \leq \operatorname{Im} z < \pi(2n+1)\}$$

be the strip of height 2π, centered about $2n\pi i$. If $E_\lambda^j(z) \in R(n)$, we define $s_j(z) = n$. The sequence

$$\{s_0(z), s_1(z), \dots, s_n(z), \dots\} = \mathbf{s}(z)$$

is the **itinerary of z under E_λ**.

Definition 2.2. *Let $\mathbf{s} = (s_0, s_1, s_2, \dots)$ be a sequence of integers. The set of allowable sequences with coefficient $\lambda \in \mathbb{R}$, $\lambda > 0$, is the set*

$$A_\lambda = \{\mathbf{s} \mid \textit{there is an } x > 0 \textit{ such that } |s_j| \leq E_\lambda^j(x)\}.$$

We also say that $\mathbf{s}$ *is* **exponential of order λ**.

Note: If we modify the definition of A_λ so that we have fixed constants C, D, and

$$A_\lambda = \{\mathbf{s} \mid \text{there is an } x \text{ such that } |Cs_j + D| \leq E_\lambda^j(x)\},$$

then A_λ is clearly the same set, which we can see by replacing x with $Cx + D$.

Lemma 2.1. *If* $\operatorname{Re} z \leq \rho$, *then*

$$\left|E_\lambda^j(z)\right| \leq E_{|\lambda|}^j(\rho).$$

Proof. We will prove this by induction. First, since $\operatorname{Re} z \leq \rho$, we have

$$|E_\lambda(z)| = |\lambda|\,|e^z| = |\lambda|\, e^{\operatorname{Re} z} \leq |\lambda|\, e^\rho.$$

Now, assume that

$$\left|E_\lambda^j(z)\right| \leq E_{|\lambda|}^j(\rho).$$

Then

$$\operatorname{Re}(E_\lambda^j(z)) \leq \left|E_\lambda^j(z)\right| \leq E_{|\lambda|}^j(\rho),$$

so that

$$\begin{aligned}\left|E_\lambda(E_\lambda^j(z))\right| &= |\lambda| \left|\exp(E_\lambda^j(z))\right| \\ &= |\lambda| \exp(\operatorname{Re}(E_\lambda^j(z))) \\ &\leq |\lambda| \exp(E_{|\lambda|}^j(\rho)) \\ &= E_{|\lambda|}^{j+1}(\rho).\end{aligned}$$

□

Lemma 2.2. *Choose* $c > 0$. *Let* $\zeta = \max(|\operatorname{Re} z|, -\log(c))$, *and let* $x = \zeta + 2\pi$. *Then*

$$E_c^j(x) \geq E_c^j(\zeta) + \pi.$$

Proof. Choose $\zeta = |\operatorname{Re} z|$, $x = \zeta + \pi$, and $c \in \mathbb{R}$. Since $\zeta > 0$, we have

$$E_c(x) = ce^\zeta e^\pi \geq ce^\zeta + \pi,$$

and

$$E_c^j(x) \geq E_c^j(\zeta) + \pi.$$

□

Theorem 2.3. *The set of possible itineraries for* E_λ *is the set* $A_{|\lambda|}$, *i.e., the set of possible itineraries depends only on the modulus of* λ.

Note: Much of the following proof is directly from [7], but we include the argument here for completeness.

Proof. First we assume that $\mathbf{s}$ is the itinerary of some z under the map E_λ, and we show that there is an x such that

$$|s_j| \leq E^j_{|\lambda|}(x).$$

We know that $\left|\operatorname{Im} E^j_\lambda(z)\right| \geq \pi\,|2s_j - 1| = 2\pi\,|s_j - 1/2|$, from the definition of the s_j. So

$$\begin{aligned} 2\pi\,|s_j| &\leq \left|\operatorname{Im} E^j_\lambda(z)\right| + \pi \\ &\leq \left|E^j_\lambda(z)\right| + \pi \\ &\leq E^j_{|\lambda|}(\zeta) + \pi \qquad \text{(by Lemma 2.1)} \\ &\leq E^j_{|\lambda|}(x) \qquad \text{(by Lemma 2.2)}. \end{aligned}$$

Hence $\mathbf{s}$ is of the correct exponential order, i.e., we have produced an x which guarantees that $\mathbf{s} \in A_{|\lambda|}$.

Now we want to show the converse. Choose $\lambda \in \mathbb{C}$, and assume that we have an itinerary $\mathbf{s}$ such that there is an x with $|s_j| \leq E^j_{|\lambda|}(x)$. We will show that there is a $z \in \mathbb{C}$ such that z has itinerary $\mathbf{s}$ under E_λ.

We choose x so that

$$\pi\,|2s_j + 1| \leq E^j_{|\lambda|}(x).$$

Define

$$S(n, x) = \{z \in \mathbb{C} | x \leq \operatorname{Re} z \leq x + 2\pi, \pi(2n-1) \leq \operatorname{Im} z \leq \pi(2n+1)\}.$$

To find z with the desired itinerary, $\mathbf{s}$, define

$$B_j = S(s_j, E^j_{|\lambda|}(x)).$$

Note that each B_j lies in the sector $|\operatorname{Im} z| \leq \operatorname{Re} z$. This is because the upper (resp. lower) left corner of B_j is given by $E^j_{|\lambda|}(x) + i\pi\,|2s_j + 1|$ if s_j is positive (resp. negative), and x has been chosen so that

$$\pi\,|2s_j + 1| \leq E^j_{|\lambda|}(x).$$

(See Figure 1.)

Lemma 2.4. *If* $E^{j+1}_{|\lambda|}(x) > 1$, *then* $E_\lambda(B_j) \supset B_{j+1}$.

Proof. The image of B_j is the annulus with inner radius $E^{j+1}_{|\lambda|}(x)$ and outer radius $E_{|\lambda|}(E^j_{|\lambda|}(x) + \pi)$. The lines $y = \pm x$ intersect the outer circle at points with real part $(\sqrt{2}/2)e^\pi E^{j+1}_{|\lambda|}(x)$. This is larger than $E^{j+1}_{|\lambda|}(x) + \pi$, which is the real coordinate of the right hand side of the box B_{j+1}, as long as $E^{j+1}_{|\lambda|}(x) > 1$. Incidentally, the hypothesis is assured without loss of generality, since we can always make our original x a little larger. $\square$

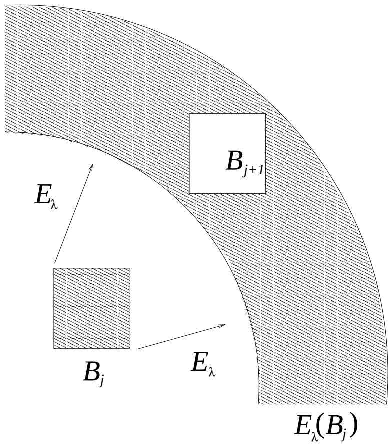

Figure 1: The choice of B_j and B_{j+1}.

Now, define

$$V_n = \{z \in B_0 \mid E_\lambda^j(z) \in B_j, j \le n\}.$$

Each V_n is closed, and clearly $V_{n+1} \subset V_n$, so we define

$$V = \bigcap_{n \ge 0} V_n \ne \emptyset.$$

We claim that if $z \in V$, then z has itinerary $\mathbf{s}$. This seems obvious, but there is the minor point that the bottom boundary of every B_j actually has the wrong itinerary, since the strips $R(n)$ were defined to be half-open and not closed. This is not a problem, however, since B_{j+1} always maps strictly inside $E_\lambda(B_j)$ (see Figure 1), and the boundary of V_{n+1} doesn't touch the boundary of V_n for any n.

□

To summarize, we have shown that for the complex exponential function E_λ, the possible itineraries depend only on the modulus of λ.

We will now show that for any $\lambda > \mu > 0$, the sets A_λ and A_μ are the same, by showing that if a sequence $\mathbf{s} = \{s_i\}$ growing at exponential rate λ, we can choose an x so that $|s_i| \le E_\mu^i(x)$, so that $\mathbf{s}$ is also in A_μ.

Lemma 2.5. *Fix $\lambda > \mu > 0$. If*

1. $x > 1$,
2. $y - x > 2\lambda$, *and*
3. $y - x > \log(y-x) + \log\left(\frac{3}{2\mu}\right)$,

then $\mu e^y - \lambda e^x > y - x$, i.e.

$$E_\mu(y) - E_\lambda(x) > y - x.$$

Proof.

$$\begin{aligned} y - x &> \log\left[(y-x)\frac{3}{2\mu}\right] \\ &= \log\left[\frac{(y-x)}{\mu} + \frac{(y-x)}{2\mu}\right] \\ &> \log\left[\frac{(y-x)}{\mu} + \frac{\lambda}{\mu}\right]. \\ e^{(y-x)} &> \frac{y-x}{\mu} + \frac{\lambda}{\mu}, \\ \mu e^{(y-x)} - \lambda &> y - x, \end{aligned}$$

so

$$\mu e^y - \lambda e^x > e^x(y-x) > (y-x).$$

□

Note: We could obtain the same result with much weaker assumptions.

The proof of the next lemma is completely straightforward.

Lemma 2.6. *Fix* $c > 0$. *If* $d_1 > \log(d_1) + c$, *then* $d_2 > \log(d_2) + c$ *for all* $d_2 > d_1 > 1$.

Lemma 2.7. *Fix* $\lambda > \mu > 0$, *with the same assumptions on* x *and* y *as in Lemma 2.5. Then* $E_\mu^j(y) > E_\lambda^j(x)$ *for all* j.

Proof. We know that $E_\mu(y) - E_\lambda(x) > y - x$ from Lemma 2.5. Clearly $E_\mu(y) - E_\lambda(x) > 2\lambda$, and $E_\lambda(x) > 0$. Also, by Lemma 2.6

$$E_\mu(y) - E_\lambda(x) > \log(E_\mu(y) - E_\lambda(x)) + \log(3/2\mu),$$

so that $E_\mu(y), E_\lambda(x)$ satisfy the hypotheses of Lemma 2.5. Thus

$$E_\mu^2(y) - E_\lambda^2(x) > E_\mu(y) - E_\lambda(x) > y - x.$$

By the same logic, if the hypotheses hold for the n-th iterates, the hypotheses will hold for the $(n+1)$-th iterates also. □

Theorem 2.8. $A_\lambda = A_\mu$.

Proof. Assume that $\lambda > \mu$. The inclusion $A_\lambda \subseteq A_\mu$ is more difficult.

Let $\mathbf{s} \in A_\lambda$. Assume that there is an x such that $|s_j| \leq E_\lambda^j(x)$ for all j. So, in particular, $\mathbf{s}$ is bounded by the sequence

$$\{x, \lambda e^x, \lambda e^{\lambda e^x}, \lambda e^{\lambda e^{\lambda e^x}}, \dots\}.$$

We need to show that there is a y such that $s_j \leq E_\mu^j(y)$. Choose $y > x$ with the conditions in Lemma 2.5. Then

$$E_\mu^j(y) \geq E_\lambda^j(x) \geq s_j,$$

so that $\mathbf{s}$ is of order μ.

Since $E_\mu^j(x) \leq E_\lambda^j(x)$ for all j, the other inclusion follows. □

3 Itineraries of $\alpha e^z + \beta e^{-z}$

Now we will use a similar idea to understand the itineraries of the map

$$E_{\alpha,\beta}(z) = \alpha e^z + \beta e^{-z}.$$

The techniques used in this section will be analogous to those in the previous section, but there are two main differences. First, the fundamental domains are no longer strips, but half-strips. For example, if we take a strip of height 2π, but only in the right-hand half-plane, e^z will cover the outside of the unit disc, but not the inside. On the other hand e^{-z} will cover the inside of the unit disc, but not the outside. Together, they will cover the entire plane. Second, the image of a box under $\alpha e^z + \beta e^{-z}$ is no longer an annulus. Although we will perform a construction similar to the one in the last section, it will be somewhat more technical.

Definition 3.1. *For $n \in \mathbb{Z}$, let*

$$R^+(n) = R(n) \cap \{\operatorname{Re} z \geq 0\}, R^-(n) = R(n) \cap \{\operatorname{Re} z < 0\}.$$

If $E^j_{\alpha,\beta}(z) \in R^+(n)$, we define $s_j(z) = n_+$, and if $E^j_{\alpha,\beta}(z) \in R^-(n)$, we say $s_j(z) = n_-$. We then say the sequence

$$\{s_0(z), s_1(z), \ldots, s_n(z), \ldots\} = \mathbf{s}(z)$$

is the **itinerary of** z **under** $E_{\alpha,\beta}$. *We say, if $s_j = n_\pm$, that*

$$\begin{aligned} &[s_j] = n \\ &\langle s_j \rangle = \begin{cases} 0 & \text{if } s_j = n_+, \\ 1 & \text{if } s_j = n_-, \end{cases} \\ &\|s_j\| = |n|\,, \text{ the absolute value of } [s_j]. \end{aligned}$$

Definition 3.2. *Let $\mathbf{s} = \{s_0, s_1, \ldots, s_n, \ldots\}$ be a sequence of integers with ± subscripts. We define*

$$\tilde{A}_\lambda = \{\mathbf{s} \mid \text{there is an } x \text{ such that } \|s_j\| \leq E^j_\lambda(x)\}.$$

In other words, a sequence in $\tilde{A}_\lambda$ is just a sequence in A_λ with ± subscripts. Also, since we have shown that A_λ is independent of λ (which implies that $\tilde{A}_\lambda$ is also), we will sometimes use A and $\tilde{A}$, to stress that these sets are independent of λ.

The following lemmas (and their proofs) are analogous to Lemmas 2.1 and 2.2:

Lemma 3.1. *If $\operatorname{Re} z \leq \rho$, then*

$$\left|E^j_{\alpha,\beta}(z)\right| \leq E^j_{|\alpha|+|\beta|}(\rho).$$

Lemma 3.2. *Let $\zeta = \max(|\operatorname{Re} z|, \log(\alpha/\beta))$, and let $x = \zeta + 2\pi$. Then*

$$E^j_{\alpha,\beta}(x) \geq E^j_{\alpha,\beta}(\zeta) + \pi.$$

Theorem 3.3. *Let* $\alpha, \beta \in \mathbb{C}$, *and* $\lambda = \min(|\alpha|, |\beta|)$. *Then if* $\mathcal{E}_{\alpha,\beta}$ *is the set of itineraries of* $E_{\alpha,\beta}$,

$$\tilde{A}_\lambda \subset \mathcal{E}_{\alpha,\beta} \subset \tilde{A}_{|\alpha|+|\beta|}.$$

By definition,

$$\begin{aligned} 2\pi\|s_j\| &\leq \left|\operatorname{Im} E^j_{\alpha,\beta}(z)\right| + \pi \\ &\leq \left|E^j_{\alpha,\beta}(z)\right| + \pi \\ &\leq E^j_{\alpha,\beta}(\zeta) + \pi \\ &\leq E^j_{\alpha,\beta}(x). \end{aligned}$$

Therefore $\mathbf{s} \in \tilde{A}_{|\alpha|+|\beta|}$.

The above argument carried over from the previous section almost verbatim. The converse is more technical. There are two new issues. First, the image of a box under $E_{\alpha,\beta}$ is not an annulus, as before. However, as we show in Lemma 3.4, the image of a box contains an annulus with sufficient width, and this is enough for our construction. Second, we have to choose the boxes more carefully, since now our fundamental domains are half-strips instead of strips. Given an itinerary, we not only have to choose the height of the boxes correctly, but place them in the proper half-plane as well.

Definition 3.3. *We denote the annulus centered at the origin, with inner radius* r *and outer radius* R, *as* $\mathcal{A}(r, R)$.

As mentioned above, the image of a box under $E_{\alpha,\beta}$ is not an annulus as it was in the case of E_λ. The next lemma will show that, under some simple conditions which depend only on α and β, the image of a box under $E_{\alpha,\beta}$ is a set which contains an annulus. (See Figure 2.)

Lemma 3.4. *If* $x > 0, x > \log|\beta|$, *then*

$$E_{\alpha,\beta}(S(n,x)) \supset \mathcal{A}\left(|\alpha| e^x + 1, |\alpha| e^{x+2\pi} - 1\right).$$

If $x < 0, x < -\log|\alpha|$, *then*

$$E_{\alpha,\beta}(S(n,x)) \supset \mathcal{A}\left(|\beta| e^{-x} + 1, |\beta| e^{-x+2\pi} - 1\right).$$

There are several points to note in this lemma. First, the only condition we need for the above lemma is that our box is chosen sufficiently far from the imaginary axis. Second, we can make these annuli as "thick" as we want. For example, consider the case of $x > 0$. We'll prove that if we assume also that $x > -\log|\alpha|$, then the difference between the inner and outer radii is more than 500. Analogously, in the case of $x < 0$, if we assume that $x < \log|\beta|$, the same result applies. This is important in that we want the image of one box to cover the next. Finally, we note the form of the inner and outer radii, and see that when x is positive, these radii tend to grow like αe^z, and for x negative, they tend to grow like βe^{-z}. This reflects the fact that far to the right (resp. left), the map is very close to αe^z (resp. βe^{-z}).

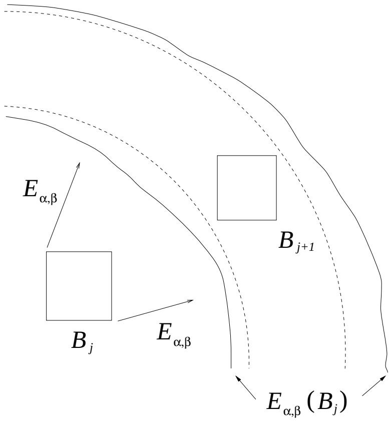

Figure 2: Although the image of B_j is not an annulus, it contains one, and we will construct the B_j's so that the inner annulus contains B_{j+1}.

Proof. Choose $x_0 > 0$ and $x > \log|\beta|$. Let us consider the image of the vertical line $\operatorname{Re} z = x_0$. By the triangle inequality, we have

$$\begin{aligned} |E_{\alpha,\beta}(z)| - |\alpha e^z| &\leq \left|\beta e^{-z}\right|, \text{ or} \\ |E_{\alpha,\beta}(z)| - |\alpha|\, e^{x_0} &\leq |\beta|\, \frac{1}{e^{x_0}} \\ &\leq 1. \qquad \text{(Recall that } x_0 > \log|\beta|.\text{)} \end{aligned}$$

The vertical line $\operatorname{Re} z = x_0$ will correspond to the left edge of the box $S(x_0, \cdot\,)$. What we have shown here is that the image of the left hand side of $S(x_0, \cdot\,)$ has modulus at most 1 more than $|\alpha|\, e^{x_0}$.

Now, consider the set of points z with $\operatorname{Re} z = x_0 + 2\pi$. Using the triangle inequality again, we have

$$\begin{aligned} ||E_{\alpha,\beta}(z)| - |\alpha|\, |e^z|| &\leq |\beta|\, \left|e^{-z}\right|, \text{ or} \\ \left||E_{\alpha,\beta}(z)| - |\alpha|\, e^{x_0+2\pi}\right| &\leq \beta \frac{1}{e^{x_0+2\pi}} \\ &\leq 1. \end{aligned}$$

The vertical line $\operatorname{Re} z = x_0 + 2\pi$ will correspond to the right hand side of the box $S(x_0, \cdot\,)$, which we have shown is at least 1 less than $|\alpha|\, e^{x_0+2\pi}$.

Also, if we assume that $x > -\log|\alpha|$, then

$$\begin{aligned} |\alpha|\, e^{x+2\pi} - 1 - (|\alpha|\, e^x + 1) &= |\alpha|\, e^x (e^{2\pi} - 1) - 2 \\ &> e^{2\pi} - 3 \\ &> 500. \end{aligned}$$

The proof for $x < 0$ is exactly the same. $\square$

Note: To summarize the above theorem, as long as we guarantee

$$|x| > \max(-\log|\alpha|\,, \log|\beta|),$$

we know that the image of our boxes under $E_{\alpha,\beta}$ will contain annuli of extremely large width. Why have we done this? Recall the construction of the last section. We produced a sequence of boxes B_j such that $E_\lambda(B_j) \supseteq B_{j+1}$. We did this by noting that the image of B_j was an annulus, and we placed B_{j+1} inside this annulus. From there, we were able to find a z with a given itinerary. We will do the same thing here. Now that we know that $E_{\alpha,\beta}(B_j)$ will contain an annulus, we will place B_{j+1} inside it.

Now, let $\mathbf{s} \in \bar{A}_\lambda$, with $\lambda = \min(|\alpha|, |\beta|)$, so that we can choose an x with

$$2\pi\|s_j\| + 1 \leq E^j_\lambda(|x|).$$

Assume without loss of generality that $|x| > \max(-\log|\alpha|, \log|\beta|)$. Let the sign of x be positive if $\langle s_0\rangle = 0$, and negative if $\langle s_0\rangle = 1$, i.e., let $x = (-1)^{\langle s_0\rangle}|x|$.

Define the function

$$F(x) = \begin{cases} E_{|\alpha|}(|x|) + 1, & \text{if } x > 0 \\ E_{|\beta|}(|x|) + 1, & \text{if } x < 0. \end{cases}$$

We will define the B_j inductively. First, let $B_0 = S([s_0], x)$. Note that its location is completely determined by s_0, because the sign of x is the subscript of s_0. Now, assume that we have defined all $B_j = S([s_j], x_j)$ up to B_n. Then, let

$$B_{n+1} = S([s_{n+1}], (-1)^{\langle s_{n+1}\rangle} F(x_n)).$$

In other words, we make sure each B_j is in the correct half-strip (right or left), and choose the size of the x-coordinate so that it grows at the correct rate. Note that the definition of F implies that

$$|x_j| \geq E^j_\lambda(x),$$

where, again, $\lambda = \min(|\alpha|, |\beta|)$.

Note, that by an argument similar to that used during the proof of Theorem 2.3, that each B_j lies in the "double sector" $|\mathrm{Im}\, z| \leq |\mathrm{Re}\, z|$.

We are ready to state and prove the main lemma. Using Lemma 3.4 and the following note, we prove it in much the same way as Lemma 2.4.

Lemma 3.5. $E_{\alpha,\beta}(B_j) \supset B_{j+1}$.

Proof. Assume that $x_j > 0$. The proof for $x_j < 0$ is almost entirely the same. We then need to show that

$$B_{j+1} \subseteq \mathcal{A}_{j+1} = \mathcal{A}\left(|\alpha|\, e^{x_j} + 1, |\alpha|\, e^{x_j + 2\pi} - 1\right).$$

By our "double sector" argument above, we need only show that the corner closest to the origin and the corner farthest away are both inside $\mathcal{A}_{j+1}$. The former is true by the definition of F. If, as before, we consider the lines $y = \pm x$, they intersect the outer circle of $\mathcal{A}_{j+1}$ at points with real part

$$\pm\frac{\sqrt{2}}{2}|\alpha|\, e^{2\pi} e^{x_j} - 1,$$

while the right-hand side of B_{j+1} has real part

$$\pm\left(|\alpha|\, e^{x_j}+2\pi\right).$$

Since we have already assumed that $|x_j| > \log|\alpha|$, it is easy to show that

$$\left|\frac{\sqrt{2}}{2}\,|\alpha|\, e^{2\pi} e^{x_j}-1\right| > \left||\alpha|\, e^{x_j}+2\pi\right|,$$

and we are done. □

Just as above, we define

$$V_n=\{z\in B_0 | E_\lambda^j(z)\in B_j, j\leq n\}.$$

Each V_n is closed, and clearly $V_{n+1}\subset V_n$, so we define

$$V:=\bigcap_{n\geq 0} V_n\neq\emptyset.$$

Clearly, $z\in V$ has itinerary $\mathbf{s}$ (see the argument at the end of Theorem 2.3).

This shows that $\tilde{A}_\lambda\subset\mathcal{E}$. Considering the above remarks, we have that $\mathcal{E}=\tilde{A}$.

4 Other transcendental functions

We have classified the itineraries of $E_{\alpha,\beta}$. Consider the case where $\alpha=\beta$, i.e. $\alpha(e^z+e^{-z})$. Using the conjugacy $z\mapsto\alpha z$, $\alpha(e^z+e^{-z})$ is conjugate to $e^{\alpha z}+e^{-\alpha z}$, so we know the itineraries of $e^{\alpha z}+e^{-\alpha z}$ as well. This applies to the functions

$$\begin{aligned}\cos(\alpha z)&=1/2(\exp(i\alpha z)+\exp(-i\alpha z)),\text{ and}\\ \cosh(\alpha z)&=1/2(\exp(\alpha z)+\exp(-\alpha z)),\end{aligned}$$

in particular.

Similarly, we can write down a conjugacy for

$$\begin{aligned}\sin(\alpha z)&=1/2(\exp(i\alpha z)-\exp(-i\alpha z)),\text{ and}\\ \sinh(\alpha z)&=1/2(\exp(\alpha z)-\exp(-\alpha z)).\end{aligned}$$

Since we have used this conjugacy, the fundamental domains are no longer half-strips which are horizontal, but will be half-strips rotated by $\arg(i\alpha)$. For example, consider the functions $\cos(az)$ or $\sin(az)$ with a real. The fundamental domains are now $iR^{\pm}(n)$, so that the half-strips are vertical in this case.

5 Itineraries of $e^{p(z)}$

In this section, we denote $E_p(z) = e^{p(z)}$, where p is a polynomial in z.

Definition 5.1. *Let* $\mathbf{s} = (s_0, s_1, \dots)$ *be a sequence of integers. Given a real-valued polynomial* $p(x)$*, we define the set of itineraries of order* $p(x)$ *to be the set of sequences*

$$A_p = \{\mathbf{s} \mid \textit{there is an } x > 0 \textit{ such that } |s_j| \leq E_p^j(x)\}.$$

Remark: This is a slight shift in notation from previous sections. For example, the set of sequences of order $p(x) = \lambda x$ was written A_λ earlier, and is now written $A_{\lambda x}$. Also, the set we now call A_x was called A_1 earlier, and is the same set referred to in Theorem 2.8.

Our first goal in this section is Theorem 5.4, which is a result analogous to Theorem 2.8.

Definition 5.2. *Let* $p_1(x) = a_n x^n + \cdots + a_0$ *and* $p_2(x) = b_m x^m + \cdots + b_0$*. We define an ordering on polynomials as such:*

1. *If* $n < m$*, or*
2. *if* $n = m$ *and* $a_n < b_m$*, or*
3. *if* $n = m$*,* $a_i = b_i$ *for all* $i > n_0$*, and* $a_{n_0} < b_{n_0}$*, then*

$p_1 \prec p_2$.

Remark: An equivalent definition to the above is that $p_1 \prec p_2$ if

$$\lim_{x \to \infty} (p_2(x) - p_1(x)) = \infty,$$

since $p_1 \prec p_2$ iff the leading order term of $p_2 - p_1$ is positive. Also, it is clear that $\prec$ is transitive, and a linear ordering.

Lemma 5.1. *If* p_1 *and* p_2 *are polynomials with* $p_1 \prec p_2$*, then*

$$A_{p_1} \subseteq A_{p_2}.$$

Proof. Let $\mathbf{s} \in A_{p_1}$. Then there is an x so that $|s_j| \leq E_{p_1}^j(x)$. Clearly, any larger x will also work. So choose an $x^* > 0$ large enough so that $p_1(x) < p_2(x)$ for all $x > x^*$. Then we have

$$|s_j| \leq E_{p_1}^j(x^*) \leq E_{p_2}^j(x^*),$$

so that $\mathbf{s}$ is also in A_{p_2}. □

Lemma 5.2. *If* $F(x) = e^{x^n}$ *for* $n \in \mathbb{Z}^+$*, and* x *is chosen so that* $F(x) > 1$*, then* $\log^k \circ F^k(x) \leq \log(n) + nx^n + 1$*.*

Proof. First, we note a potentially confusing overuse of notation. We have typically represented the kth iterate of a function F as F^k, but we will also use the superscript notation below to refer to the power of a number, so the nth power of the kth iterate of a point x will be denoted $\left(F^k(x)\right)^n$.

We will prove the theorem by induction. For $k = 1$, $\log \circ F(x) = x^n$.

$$\begin{aligned}
\log^k \circ F^k(x) &= \log^{k-1} \circ \log \circ F \circ F^{k-1}(x) \\
&= \log^{k-2} \circ \left(n \log \left(F^{k-1}\right)(x)\right) \\
&= \log^{k-2} \circ \left(n \left(F^{k-2}(x)\right)^n\right) && (**) \\
&= \log^{k-3} \circ \left(\log n + n \left(F^{k-3}(x)\right)^n\right) && (***) \\
&= \log^{k-4} \circ \log \left(\log n + n \left(F^{k-3}(x)\right)^n\right) \\
&\leq \log^{k-4} \left(\log n + n \log \left(F^{k-3}(x)\right) + \frac{\log n}{n\left(F^{k-3}(x)\right)^n}\right) \\
& && \text{(Taylor's Theorem)} \\
&\leq \log^{k-4} \left(\log n + n \log \left(F^{k-3}(x)\right) + 1\right) \\
&= \log^{k-4} \left(\log n + n \left(F^{k-4}(x)\right)^n + 1\right) && (****)
\end{aligned}$$

For $k \leq 4$, the proof is simply to end the above string of inequalities at the correct place (e.g. for $k = 2$, stop at $(**)$). Assume that $q \geq 5$ and consider the last line above.

$$\begin{aligned}
&\log^{k-q} \left(\log n + n \left(F^{k-q}(x)\right)^n + 1\right) \\
&\quad = \log^{k-(q+1)} \circ \log \left(\log n + n \left(F^{k-q}(x)\right)^n + 1\right) \\
&\quad \leq \log^{k-(q+1)} \left(\log n + n \log \left(F^{k-q}(x)\right)^n + \frac{1 + \log n}{n\left(F^{k-q}(x)\right)^n}\right) \\
&\quad \leq \log^{k-(q+1)} \left(\log n + n \log \left(F^{k-q}(x)\right) + 1\right) \\
&\quad = \log^{k-(q+1)} \left(\log n + n \log \left(F^{k-(q+1)}(x)\right)^n + 1\right)
\end{aligned}$$

and we are done. ☐

Lemma 5.3. *If $p_1(x) = x^m$ and $p_2(x) = x^n$, then*

$$A_{p_1} = A_{p_2}.$$

Proof. Assume that $m < n$. Since $x^m \prec x^n$, we know that $A_{x^m} \subseteq A_{x^n}$. So we only need to show that $A_{x^m} \supseteq A_{x^n}$, i.e., if we have a sequence $\mathbf{s} \in A_{x^n}$, then $\mathbf{s} \in A_{x^m}$ also. We will actually do better than this; we will show that $A_{x^n} \subseteq A_x$, i.e., for any x, we can choose a y such that

$$E^k_{x^n}(x) \leq E^k(y).$$

If $F(x) = e^{x^n}$, we need to choose y so that

$$F^k(x) \leq E^k(y)$$

This is equivalent to asking if we can satisfy:

$$\log^k \circ F^k(x) \leq \log^k \circ E^k(y),$$

or

$$1 + \log n + nx^n \leq y.$$

But for a given x and n, we can certainly chose y so. Thus, if $\mathbf{s} \in A_{x^n}$, then

$$|s_j| \leq E^j_{x^n}(x) \leq E^k(y),$$

so that

$$A_{x^n} \subseteq A_x.$$

□

Note that this is actually the same set referred to in Theorem 2.8.

Theorem 5.4. *If* p_1 *and* p_2 *are any real-valued polynomials, then*

$$A_{p_1} = A_{p_2}.$$

Proof. Let

$$\begin{aligned} p_1(x) &= a_n x^n + \cdots + a_0, \\ p_2(x) &= b_m x^m + \cdots + b_0, \end{aligned}$$

so that $p_1 \prec p_2$. Clearly, $p_2 \prec x^{m+1}$ and $x^{n-1} \prec p_1$, or

$$x^{n-1} \prec p_1 \prec p_2 \prec x^{m+1}.$$

Lemma 5.1 gives us

$$A_{x^{n-1}} \subseteq A_{p_1} \subseteq A_{p_2} \subseteq A_{x^{m+1}}.$$

Since by Lemma 5.3 the leftmost and rightmost sets are equal, then

$$A_{p_1} = A_{p_2}.$$

□

The result we expect to have, in analogy with earlier sections, is that if we consider the itineraries of the function $e^{p(z)}$, we would expect the itineraries to be exactly those sequences which belong to the set A. It turns out that this is true. Note that for the rest of the section, polynomials are allowed to have complex domains and complex coefficients.

Given a polynomial $p(z) = a_n z^n + \cdots + a_0$, we define

$$|p|(z) = |a_n| z^n + \cdots + |a_0|.$$

Lemma 5.5. *If* $p(z) = a_n z^n + \cdots + a_0$, *and* $\operatorname{Re} z \leq \rho$ *(with* ρ *large), then*

$$\left| E^j_p(z) \right| \leq E^j_{|p|}(\rho).$$

Proof. The proof is exactly the same as in the case of Lemma 2.1. □

The same argument as used in the forward direction of Theorems 2.3 and 3.3 will guarantee that if we consider the itinerary $\mathbf{s}$ of a point z under E_p, then $\mathbf{s} \in A_{|p|}$.

Now we need to show that given polynomial $p(z)$ and an itinerary $\mathbf{s} \in E_{|p|}$, that there is a point z with that itinerary under E_p.

Choose x so that $\pi\,|2s_j + 1| \leq E^j_{|p|}(x)$, and denote $x_j = E^j_{|p|}(x)$.

Consider the sequence $z_j = x_j + i2\pi s_j$. We will now define B_j for every j. Take the image $p(z_j)$, and construct the box

$$\hat{B}_j = S(\operatorname{Im} p(z_j), \operatorname{Re} p(z_j)).$$

Now, define B_j to be the preimage of $\hat{B}_j$ under p.

So in contrast to the constructions in earlier sections, the B_j are not actually boxes themselves, but they are constructed so that $p(B_j)$ are boxes. Now, $E(\hat{B}_j) = E_p(B_j)$ is the familiar annulus.

As we did in the earlier constructions, we want to show that $B_{j+1} \subseteq E_p(B_j)$. First, we will show that $z_{j+1} \in E_p(B_j)$. Then we note that the B_j are small, and the larger x_j is, the smaller B_j are. This is because the area of $\hat{B}_j$ is always the same, but for complex numbers with large real part, $p(z)$ is very expansive. Thus the B_j will be as small as we like, and we can guarantee that $B_{j+1} \subseteq E_p(B_j)$ if $z_{j+1} \in E_p(B_j)$.

We know that $E_p(B_j)$ is an annulus. Its inner radius is

$$\left| e^{p(z_j)} \right| = e^{\operatorname{Re} p(z_j)},$$

and its outer radius is

$$e^{\operatorname{Re} p(z_j) + 2\pi}.$$

We know that

$$\begin{aligned} \operatorname{Re} p(z_j) &\leq |p(z_j)| \\ &\leq |p|\,(|z_j|). \end{aligned}$$

The inner radius of $E_p(B_j)$ is $\exp(\operatorname{Re} p(z_j))$, and

$$\exp(\operatorname{Re} p(z_j)) \leq E_{|p|}(|z_j|).$$

Thus the inner radius of $E_p(B_j)$ will intersect the real axis at a point to the left of $x_{j+1} = \operatorname{Re} z_{j+1}$. If x_j is large enough, $\operatorname{Re} z_{j+1}$ will be much less than $e^{p(x_j)+2\pi}$, we will have $z_{j+1} \in E_p(B_j)$. And since we can choose our B_j to be as small as we like, we will have $B_{j+1} \subseteq E_p(B_j)$.

The rest is the same as before. Again, define

$$V_n = \{z \in B_0 \mid E^j_p(z) \in B_j, j \leq n\}.$$

Each V_n is closed, and clearly $V_{n+1} \subset V_n$, so we define

$$V = \bigcap_{n \geq 0} V_n \neq \emptyset.$$

Clearly, $z \in V$ has itinerary $\mathbf{s}$ (see the arguments at the end of Theorem 2.3), and this concludes the proof of the theorem. $\square$

6 A different set of itineraries

In the previous sections, we have shown that the sets of allowable itineraries are the same, whether we are considering the family E_λ, the family $E_{\alpha,\beta}$, or the family E_p. The last family is especially surprising, since intuitively, the "growth" of E_p depends heavily on the polynomial $p(z)$. But Theorem 5.4 says that the itineraries do not depend on p, i.e., the itineraries for every function of finite exponential order come from this same set A. Since the same set A worked for so many diverse functions, this might lead one to suspect that the itineraries of functions of infinite exponential order also come from A, but we show that this is not the case.

The simplest function of infinite exponential order is $F(z) = \exp(\exp(z))$. We can restrict this to the real axis to get $F(x) = \exp(\exp(x))$. We will show that there are orbits of F which are not in A, by showing that there is an x such that we cannot pick a y with $F^k(x) \leq E^k(y)$.

Lemma 6.1.

$$\log^k \circ F^k(x) = E^k(x).$$

Proof. The equality is clear for $k = 1$. Assume that it holds true for k. Then

$$\begin{aligned}\log^{k+1} \circ F^{k+1}(x) &= \log^k \circ \log \circ F \circ F^k(x) \\ &= \log^k \circ \exp\left(F^k(x)\right) \\ &= \log^{k-1} \circ F^k(x).\end{aligned}$$

Since by the induction hypothesis, $\log^k \circ F^k(x) = E^k(x)$, then

$$\begin{aligned}\log^{k-1} \circ F^k(x) &= \exp\left(\log^k \circ F^k(x)\right) \\ &= E\left(E^k(x)\right) \\ &= E^{k+1}(x),\end{aligned}$$

and we are done. □

Now assume that we can choose a y with $F^k(x) \leq E^k(y)$. Then we have

$$\begin{aligned}\log^k \circ F^k(x) &\leq \log^k \circ E^k(y) \\ E^k(x) &\leq y.\end{aligned}$$

Obviously, if we choose $x = 1$, there is no y that works for all k. Therefore the orbit of 1 under F is not in A. By a similar construction to that done in previous sections, we can show that there are orbits under $F(z)$ whose itineraries do not lie in A.

This leads to a counterintuitive picture. The Julia sets of these functions (as seen in [1],[5],[6],[7],[12]) contain of "hairs" which are indexed by their itineraries. We have shown that the functions E_{p_1} and E_{p_2} have the same itineraries, and while we do not expect their Julia sets to look alike, the hairs in them can be indexed by the same set of numbers. On the other hand, E and $F = E^2$ have exactly the same Julia sets, yet the indices come from different sets of numbers entirely.

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