2
Perturbative quantum field theory

2.1 pQFT

In this chapter we repeat the main steps towards a derivation of the Feynman rules, following the well-known path of canonical quantization. This is standard material, and readers who are not acquainted with such topics are referred to [Bjorken and Drell 1965, Bogoliubov and Shirkov 1980, Itzykson and Zuber 1980, Kaku 1993, Weinberg 1995, Peskin and Schroeder 1995, Teller 1997]. We hope that the short summary given here, similar to that in [Kreimer 1997a], is helpful for readers who want to refresh their memory.

Having introduced Feynman rules, we next introduce Schwinger--Dyson equations as a motivation for the introduction of Z-factors. We remark on dimensional regularization and give a short expository overview of the basic notions of renormalization, again meant to be a reminder more than a thorough exposition, which, whenever necessary, can be found in the literature on the subject [Bogoliubov and Shirkov 1980, Collins 1984].

As far as our notation and conventions are concerned, we follow [Kaku 1993, Kreimer 1997a].

2.1.1 Canonical quantization

Free propagating particles are specified by various types of lines (propagators) and interact at vertices, points in spacetime where three or more lines merge together. These vertices describe the coupling of particles, and are specified by a coupling constant which determines the strength of the interaction at the vertex. In this way, each graph built out of propagators and vertices describes a process in pQFT. For a specified set of initial and final particles in pQFT one considers all possible graphs which allow for these initial and final states. If there are no closed loops in the graphs, we speak of tree-level graphs.

Higher order contributions in the coupling constant demand more interaction vertices. The specified set of external particles is kept fixed and thus the expansion in the coupling constant becomes a loop expansion. Now, in the presence of closed loops in the graph, one has to take into account all possible momenta for internal particles. This demands the integration over such internal loop momenta. For the most interesting theories, these integrations are ill-defined. They diverge for large internal momenta. The cure for this difficulty, the presence of UV-divergences and their handling by the theory of renormalization, lies at the heart of any pQFT, and is the core topic of this book.

Let us now introduce the Feynman rules. Our starting point is the consideration of a scalar field theory: there is only one self-interacting scalar particle present which transforms trivially under the Lorentz group. More realistic examples involve particles sitting in more complicated representations of the Lorentz group (spin), but we omit these complications in this elementary exposition.

We have the usual free Lagrangian given by

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2. \tag{2.1}$$

Later we will add interaction terms of the form $g\phi^n, n \geq 3$, to this Lagrangian. But we first consider quantization of the free theory. The classical calculus of variations gives as the equation of motion

$$\partial_\mu \frac{\delta L}{\delta \partial_\mu \phi} - \frac{\delta L}{\delta \phi} = 0 \Rightarrow \ (\Box + m^2)\phi = 0. \tag{2.2}$$

With the field $\phi(x)$ acting as the coordinate, parametrized by the spacetime point $x = (\mathbf{x}, t)$, we find as its canonical conjugate momentum ($\dot{\phi} \equiv \frac{\partial}{\partial t}\phi(\mathbf{x}, t)$)

$$\pi(\mathbf{x}, t) \equiv \frac{\delta L}{\delta \dot{\phi}} = \dot{\phi}(\mathbf{x}, t), \tag{2.3}$$

and, accordingly, a Hamiltonian density

$$H = \frac{1}{2}[\pi^2 + (\nabla \phi)^2 + m^2\phi^2]. \tag{2.4}$$

We recall that the transition to quantum theory emerges when we impose non-vanishing commutators between coordinates and momenta. In one-particle non-relativistic quantum mechanics, one has the commutator $[p, q] = i$ (we employ units in which $\hbar = 1$).

In field theory, we consider ϕ as an operator-valued distribution. At each spacetime point, we have to impose a commutator between ϕ and the canonical conjugate momentum π. Let us consider commutators at equal time. We demand that commutators between expressions separated by spacelike distances vanish.† At equal time, this becomes

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{2.5}$$

Note that this separation is Lorentz invariant. Lorentz transformations map spacelike distances to spacelike distances.

We are now looking for fields $\phi(x)$ fulfilling the Klein–Gordon equation of motion and the commutation relations. One readily verifies that the *ansatz*

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^4k \delta(k^2 - m^2)\Theta(k_0)[A(k)e^{-ikx} + A^\dagger e^{ikx}], \tag{2.6}$$

has the required properties. Here, we use

$$\begin{aligned} \omega_k &:= \sqrt{\mathbf{k}^2 + m^2}, \qquad \mathrm{e}^{ikx} = \exp\left(i\omega_k t - \mathbf{k}\cdot\mathbf{x}\right), \\ \int \mathrm{d}^4k \delta(k^2 - m_2)\Theta(k_0) &= \int \frac{\mathrm{d}^3k}{2\omega_k}, \qquad a(k) := \frac{A(k)}{\sqrt{2\omega_k}}. \end{aligned} \tag{2.7}$$

Upon Fourier transformation, we obtain expressions for ϕ and π in which the operator-valued Fourier coefficients $a, a^\dagger$ have to obey commutation relations as well:

$$[a(k), a^\dagger(k')] = \delta^3(\mathbf{k} - \mathbf{k}'). \tag{2.8}$$

Then we obtain the Hamiltonian as

$$\begin{aligned} H &= \int \mathrm{d}^3k\ \omega_k \left[a^\dagger(k)a(k) + \frac{1}{2}\right] \\ &= \frac{1}{2}\int \mathrm{d}^3k\ \omega_k \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right]. \end{aligned} \tag{2.9}$$

The Hamiltonian is the generator of the time evolution and accordingly we find the generator of spacelike translations by replacing ω_k by the spacelike components of the four-vector $k_\mu = (\omega_k, \mathbf{k})$

$$P_i = \int \mathrm{d}^3k\ k_i \left[a^\dagger(k)a(k) + \frac{1}{2}\right]. \tag{2.10}$$

† This incorporates a notion of causality. Fields with spacelike separation do not influence each other.

By comparison with quantum mechanics we recognize the Hamiltonian as a sum (integral) over harmonic oscillators, one for each mode k. We are led to assume the existence of a ground-state $|0\rangle$ annihilated by $a(k)$ for all k:

$$a(k)|0\rangle = 0. \tag{2.11}$$

Next we want to define normalized states

$$a^\dagger(k)|0\rangle = |k\rangle, \tag{2.12}$$

via creation operators $a^\dagger(k)$ which ought to create one-particle states of momentum k from the ground-state. There is a well-known problem here: from quantum mechanics we know that the harmonic oscillator has a ground-state with non-vanishing energy. Imposing such a ground-state at every spacetime point and integrating over the whole of spacetime results in an infinite energy for the ground-state, the vacuum. We discard the unobservable ground-state energy by introducing a normal ordering of creation operators $a^\dagger(k)$ and annihilators $a(k)$. We write all creation operators to the left and all annihilators to the right:

$$\begin{aligned} &:a(k)a^\dagger(k'): \; = \; :a^\dagger(k)a(k'): \; = a^\dagger(k)a(k'), \\ &\Rightarrow H = \int \mathrm{d}^3k\, \omega_k \; :a^\dagger(k)a(k): \; . \end{aligned} \tag{2.13}$$

Now we can introduce states $|k_1, \ldots, k_n\rangle$ as

$$\begin{aligned} |k_1, \ldots, k_n\rangle &= a^\dagger(k_1), \ldots, a^\dagger(k_n)|0\rangle, \\ \langle k_1, \ldots, k_n| &= \langle 0|a(k_1), \ldots, a(k_n), \end{aligned} \tag{2.14}$$

with normalization

$$\begin{aligned} \langle k'|k\rangle &= \langle 0|a(k')a^\dagger(k)|0\rangle = \langle 0|[a(k')a^\dagger(k) - a^\dagger(k')a(k)]|0\rangle \\ &= \delta(\mathbf{k} - \mathbf{k}'). \end{aligned} \tag{2.15}$$

We interpret the states above as multiparticle states for particles with momenta k_i. There are no interactions present, cf. Fig. 2.1. This finishes the quantization of the free Klein–Gordon field. What has to be done next is to introduce interactions.

2.1.2 Perturbation theory

To proceed, we introduce a source term in the Klein–Gordon equation:

$$(\Box + m^2)\phi = J(x). \tag{2.16}$$

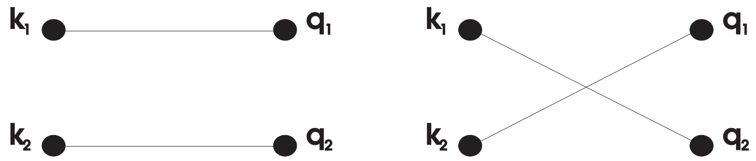

Fig. 2.1 Free-particle states like $\langle k_1, k_2|q_1, q_2\rangle$ can be graphically represented as above. Their normalization demands either $k_1 = q_1,\ k_2 = q_2$ or $k_1 = q_2,\ k_2 = q_1$. In general, free multiparticle states $\langle k_1, \ldots, k_n|q_1, \ldots, q_m\rangle$ vanish for $n \neq m$, and for $n = m$ their normalization delivers all pairings of the k_i with the q_j. There is no interaction present, which is indicated by straight lines in the figure, without any vertices. A straight line represents an incoming particle with momentum k_i propagating undisturbed to become an outgoing particle with the same momentum $q_j = k_i$.

We look for a Green function (the propagator) $\Delta_{\mathrm{F}}(x-y)$. It is a solution of

$$(\Box + m^2)\Delta_{\mathrm{F}}(x-y) = -\delta^4(x-y). \tag{2.17}$$

One easily sees that by inversion of the equation of motion

$$\begin{aligned} \Delta_{\mathrm{F}}(x-y) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik(x-y)} \Delta_{\mathrm{F}}(k), \\ \Delta_{\mathrm{F}}(k) &= \frac{1}{k^2 - m^2 + i\eta}. \end{aligned} \tag{2.18}$$

Note that in a slight abuse of notation, we used the same symbol Δ_{F} for the propagator in coordinate and momentum space. The presence of the small imaginary part $i\eta$ specifies boundary conditions.† This Feynman propagator $\Delta_{\mathrm{F}}(k)$ is the factor which we assign to the edges in the Feynman graphs. Using our ansatz for the fields ϕ, one finds that the propagator can be written with the help of a time-ordered product T:

$$\Delta_{\mathrm{F}}(x-y) = \langle 0|T[\phi(x)\phi(y)]|0\rangle,$$

† One can show that these boundary conditions are in accord with causality: positive energy solutions propagate forwards in time. Nevertheless, in a relativistic covariant theory we also have negative energy solutions propagating backwards in time, corresponding to antiparticles propagating forwards in time.

$$T[\phi(x)\phi(y)] \;:=\; \phi(x)\phi(y) \Leftrightarrow x_0 > y_0,$$
$$T[\phi(x)\phi(y)] \;:=\; \phi(y)\phi(x) \Leftrightarrow y_0 > x_0. \tag{2.19}$$

What remains is to determine the factor which goes with the vertices, and the rules as how to evaluate the resulting expressions. At this point we have to introduce perturbation theory. We will see that perturbation theory still allows for a graphical representation, adjoining vertices to the straight lines of Fig. 2.1 and we will now sketch the remaining steps leading us to the celebrated Feynman rules.

We are interested in the S-matrix. It describes transitions between incoming particles and final particles. This is the generic situation in particle physics: one has a set of well-defined incoming particles $|f\rangle_{\rm in}$ (prepared in some beam, say) which we assume to be non-interacting, one further has an interaction regime (a collision, say of two beams, or a beam and a target) and detects a set of outgoing particles $|f\rangle_{\rm out}$, which we again assume to be non-interacting.

We define the S-matrix, which describes the probability of a transition from an initial state $|i\rangle_{\rm in}$ to a final state $|f\rangle_{\rm out}$.

$$S_{f,i} := {}_{\rm out}\langle f|i\rangle_{\rm in}. \tag{2.20}$$

The asymptotic states $|f\rangle_{\rm out}$ and $|i\rangle_{\rm in}$ are related by a basis transformation:

$$|f\rangle_{\rm in} = R|f\rangle_{\rm out}, \Rightarrow S_{f,i} = {}_{\rm in}\langle f|R|i\rangle_{\rm in} = {}_{\rm out}\langle f|R|i\rangle_{\rm out}. \tag{2.21}$$

Asymptotic states are assumed to fulfil the free Klein–Gordon equation, and thus allow for a spectral decomposition similar to a free field:

$$|q\rangle_{\rm in} = a^\dagger(q)|0\rangle, \tag{2.22}$$

where $a^\dagger(q)$ is the usual generator of a free one-particle state with momentum q.

Our goal is to replace the fully interacting field ϕ everywhere by asymptotic fields.† This is achieved with help of the LSZ formalism [Lehmann et al. 1955]. One can express the abstract S-matrix element between n outgoing particles specified by n momenta p_i

† Asymptotically, we assume that the free fields and the full interacting fields are connected in the weak sense: $\lim_{t\to\infty}\langle r|\phi|s\rangle = Z^{\frac{1}{2}}\langle r|\phi_{\rm in}|s\rangle$. At asymptotic times before any interaction takes place the matrix elements of the full field approach the matrix elements of the free field.

and m incoming particles specified by m momenta q_j as a vacuum expectation value of fully interacting fields:

$$\begin{aligned} &{}_{\text{out}}\langle p_1,\dots,p_n|q_1,\dots,q_m\rangle_{\text{in}} = \\ &(iZ^{-\frac{1}{2}})^{n+m}\int \mathrm{d}^4y_1\cdots\mathrm{d}^4x_m \prod_{i=1}^{n}\prod_{j=1}^{m} e^*_{p_i}(y_i)e_{q_j}(x_j) \qquad (2.23)\\ &\times(\Box+m^2)_{y_1}\cdots(\Box+m^2)_{x_m}\langle 0|T[\phi(y_1)\cdots\phi(x_m)]|0\rangle, \end{aligned}$$

where $e_q(x) := \mathrm{e}^{iqx}/\sqrt{2\omega_k(2\pi)^3}$. To find such expectation values we now have to resort to perturbation theory.

As an example we consider an interaction Hamiltonian

$$H_{\mathrm{I}} = \int \mathrm{d}^3x\,\mathcal{H}_{\mathrm{I}} = \int \mathrm{d}^3x\,\frac{g}{4!}\phi^4(x). \qquad (2.24)$$

Assuming the existence of a unitary operator $U(t)$ such that

$$\phi(\mathbf{x},t) = U(t)\phi_{\mathrm{in}}(\mathbf{x},t)U^{-1}(t), \qquad (2.25)$$

one can show that $U(t) \equiv U(t,-\infty)$ fulfils the requirements of a time evolution operator

$$\begin{aligned} U(t_1,t_1) &= 1,\\ U^{-1}(t_1,t_2) &= U(t_2,t_1),\\ U(t_1,t_2)U(t_2,t_3) &= U(t_1,t_3), \end{aligned} \qquad (2.26)$$

with boundary condition $\lim_{t\to\infty}U(-t)|0\rangle = |0\rangle$.† As $U(t)$ is a time evolution operator, it satisfies a differential equation:

$$\partial_t U(t) = -iH_{\mathrm{I}}(t)U(t). \qquad (2.27)$$

Solving this equation as an exponential one finally obtains a perturbative expansion in terms of a Dyson series for $U(t)$:

$$\begin{aligned} U(t) &= 1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{t}\mathrm{d}^4x_1\int_{-\infty}^{t}\mathrm{d}^4x_2\cdots\\ &\quad\int_{-\infty}^{t}\mathrm{d}^4x_n T[H_{\mathrm{I}}(x_1)\cdots H_{\mathrm{I}}(x_n)], \end{aligned} \qquad (2.28)$$

† The assumption that such a unitary operator connecting the full field ϕ and the asymptotic field ϕ_{in} exists cannot be maintained in a rigorous approach to QFT and has to be relaxed somewhat. A good account of the problem can be found in [Haag 1992]. Nevertheless, it allows a quick derivation of perturbation theory. It is not understood why, then, perturbation theory works so well.

which eventually serves to give the perturbation expansion for the Green function

$$G(x_1, \ldots, x_n) := \langle 0|T[\phi(x_1) \cdots \phi(x_n)]|0\rangle \tag{2.29}$$

as

$$G(x_1, \ldots, x_n) = \sum_{m=1}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} \mathrm{d}^4 y_1 \cdots \mathrm{d}^4 y_m \times \langle 0|T[\phi_{\mathrm{in}}(x_1) \cdots \phi_{\mathrm{in}}(x_n) H_{\mathrm{I}}(y_1) \cdots H_{\mathrm{I}}(y_m)]|0\rangle. \tag{2.30}$$

From these vacuum correlators one can obtain all S-matrix elements with the help of Eq.(2.23).

We see that we obtain the Green function as a time-ordered product of free fields, with interactions H_{I} inserted at various points.

To proceed, we use Wick's theorem [Wick 1950]. This theorem is a combinatorial result which allows us to express the time-ordered product of fields as a normal-ordered product [which has vanishing vacuum expectation value by definition Eq.(2.13)] and products of free propagators. Using the spectral properties of free fields, Eq.(2.6), one easily establishes

$$T[\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2)] =: \phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2) : + \langle 0|T[\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2)]|0\rangle,$$

and derives the vacuum expectation of free fields as a sum over all pairings of two fields in propagators

$$\begin{aligned} &\langle 0|T[\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2) \cdots \phi_{\mathrm{in}}(x_n)]|0\rangle \\ &= \sum \langle 0|T[\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2)]|0\rangle \cdots \langle 0|T[\phi_{\mathrm{in}}(x_{n-1})\phi_{\mathrm{in}}(x_n)]|0\rangle. \end{aligned}$$

Using Eq.(2.17), one can discard the Klein–Gordon factors in Eq.(2.23), as they become delta functions.

Summarizing, we obtain to first order in the interaction Hamiltonian H_{I} for our example of $H_{\mathrm{I}} \sim \phi^4$

$$\begin{aligned} &G(x_1, x_2, x_3, x_4) \\ &= -\frac{ig}{4!} \int \mathrm{d}^4 y \langle 0|T[\phi_{\mathrm{in}}(x_1) \cdots \phi_{\mathrm{in}}(x_4)\phi_{\mathrm{in}}^4(y)]|0\rangle + \cdots \\ &= (-ig) \int \mathrm{d}^4 y \prod_{i=1}^{4} (-\Delta_{\mathrm{F}}(x_i - y)) + \cdots, \end{aligned} \tag{2.31}$$

which is described in Fig. 2.2. The ellipsis represents higher order terms in the interaction. We see that the net effect of the presence of the interaction term is to allow for propagators which merge in

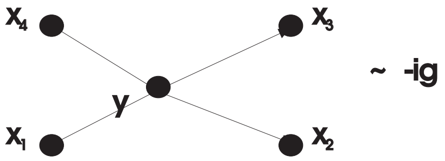

Fig. 2.2 A graphical illustration of the first term in Eq.(2.31). In scalar ϕ^4-theory, the coefficient of four propagators merging at one vertex is $-ig$.

one point, the vertex. Such a vertex is to be multiplied with a coupling constant g.

It is elementary to explain how momentum conservation at such a vertex comes about. At each vertex located at y, say, we have a factor

$$\int \prod_i \mathrm{d}^D p_i \int \mathrm{d}^D y \mathrm{e}^{p_i\cdot(x_i-y)} \Delta_{\mathrm{F}}(p_i)$$
$$= \int \prod_i \mathrm{d}^D p_i \mathrm{e}^{p_i\cdot x_i} \int \mathrm{d}^D y \mathrm{e}^{(p_1+p_2+p_3+p_4)\cdot y} \Delta_{\mathrm{F}}(p_i).$$

Integrating out y, we obtain, in a field theory in D dimensions,

$$\int \prod_i \mathrm{d}^D p_i \delta^{(D)}(p_1+p_2+p_3+p_4) \int \prod_i \mathrm{d}^D x_i \mathrm{e}^{p_i\cdot x_i} \Delta_F(p_i), \quad (2.32)$$

a δ-function which guarantees that the sum of all incoming momenta vanishes.

2.1.3 Feynman rules

We are now prepared to introduce the Feynman rules. We give them in momentum space for Fourier-transformed Green functions and Fourier-transformed propagators $\Delta_{\mathrm{F}}(q_i)$.

In momentum space, the remaining integrations amount to integrations over internal loop momenta, which appear when we have closed loops in the graph, due to the fact that the momentum conservation at each vertex is not sufficient to fix all momenta in such circumstances. We start with the Feynman rules for ϕ^4-theory in four dimensions.

2.1.3.1 ϕ^4-theory

- For each internal scalar field line, associate a propagator given by $i\Delta_{\mathrm{F}}(p) = \frac{i}{p^2-m^2+i\eta}$.
- At each vertex respect momentum conservation and place a factor of $-ig$.
- For each closed loop in the graph, integrate over $\int \frac{d^4q}{(2\pi)^4}$, where q is the momentum associated with the loop.
- For any Feynman graph G built from these propagators and vertices divide by the symmetry factor of the graph G, which is assumed to contain r vertices. The symmetry factor is then given by the number of possibilities to connect r vertices to give G, divided by $(4!)^r$.

It should be noted that the free propagator is a solution of the free Klein–Gordon equation, and the vertex is derived from the interaction term in the Lagrangian by a functional derivative with respect to the appropriate external fields.

Next, we turn to the Feynman rules for some other theories.

2.1.3.2 ϕ_6^3

For a scalar field theory with a cubic interaction in six dimensions, ϕ_6^3, there is a modification in the symmetry factor. The Lagrangian comes now with a cubic coupling $g\phi^3/3!$. The vertex $-ig$ connects to three propagators. The loop integration is with respect to a measure

$$\int \frac{d^6k}{(2\pi)^6}, \tag{2.33}$$

and the symmetry factor demands a final division by 3!.

2.1.3.3 Yukawa theory

This is the coupling of fermions with scalar particles. We have an interaction term $g\phi\bar{\psi}\psi$ which couples a fermion to a scalar particle. The propagator for the scalar particle is similar to the previous cases, while the propagator for the fermion is

$$\frac{1}{\not{p}-m+i\eta} = \frac{\not{p}+m}{p^2-m^2+i\eta}. \tag{2.34}$$

We employ the standard Feynman dagger notation here, with $\not{p} = \sum_{\mu=0,\ldots,D-1} p_\mu \gamma^\mu$, the γ_μ being the elements of a Clifford algebra with metric tensor $g_{\mu\nu}$, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, g^\mu_\mu = \gamma_\mu\gamma^\mu = D$.

2.1.3.4 Quantum electrodynamics

In quantum electrodynamics (QED) one finds the following set of rules.

• For each internal electron line, associate a propagator given by $iS_{\mathrm{F}}(p) = \frac{i}{\not{p}-m+i\eta} = \frac{i(\not{p}+m)}{p^2-m^2+i\eta}$.
• For each internal photon line (in the Feynman gauge), associate a propagator $iD_{\mathrm{F}}(p)_{\mu\nu} = -\frac{ig_{\mu\nu}}{p^2+i\eta}$.
• At each vertex place a factor of $-ie\gamma_\mu$.
• For each internal loop, integrate over $\int \frac{\mathrm{d}^4 q}{(2\pi)^4}$.
• Incoming electrons have momenta entering the diagram, while incoming positrons have momenta leaving the diagram. Internal fermion lines appear with arrows in both clockwise and counter-clockwise directions. However, topologically equivalent diagrams are counted only once.

We can now use these Feynman rules to construct the perturbative expansion of the Green functions. Figure 2.3 gives examples for a theory with a cubic self-interaction.

2.1.4 Schwinger–Dyson equations

Having introduced Feynman rules and hence Feynman graphs, the next structure we want to introduce are Schwinger–Dyson equations. The loop expansion is controlled by such equations. They establish an iteration of Green functions in terms of themselves.

For the moment, let us consider the Schwinger–Dyson equations for a theory with cubic interactions, and let us concentrate on the vertex functions.

To simplify matters even further, we discard loop corrections at the propagator and internal vertices. The Schwinger–Dyson equation is then an equation which gives the (renormalized) vertex function in terms of itself and some kernel function K:

$$\mathbf{\Gamma}(p, p+q) \quad = \quad Z\Gamma_0 + \int \mathrm{d}^4 k \mathbf{\Gamma}(k+q, k) P(k) P(k+q)$$

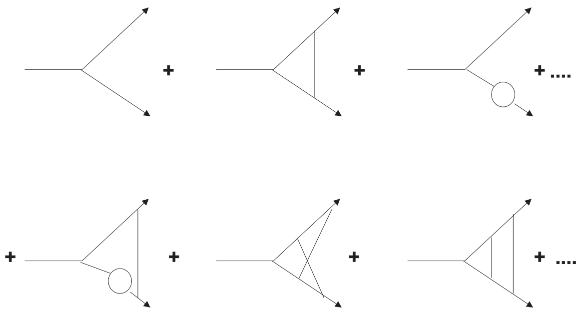

Fig. 2.3 Increasing the number of interaction terms amounts to a loop expansion of the theory (here, ϕ^3 in six dimensions). For example, through an interaction of the form $g\phi^3$, three free propagators merge at a vertex. Each extra propagator provides a factor g^2, so that a n-loop graph provides a correction of order g^{2n}. These loop graphs have to be integrated over internal loop momenta. This corresponds to a sum over all possibilities for the momenta of the internal particles. Momentum conservation at each vertex, and the resulting overall momentum conservation for external particles, are not able to fix these internal loop momenta. They have to be integrated over the full six-dimensional momentum space.

$$\begin{aligned} & & \times\, K(k, k+q, p+q, p), \\ Z &=& 1 + \sum_{i=1}^{\infty} \hbar^i Z^{[i]}, \\ \mathbf{\Gamma}(p, p+q) &=& \Gamma_0 + \sum_{i=1}^{\infty} \hbar^i \mathbf{\Gamma}^{[i]}(p, p+q), \\ K(k, k+q, p+q, p) &=& \sum_{i=1}^{\infty} \hbar^i K^{[i]}(k, k+q, p+q, p). \end{aligned} \tag{2.35}$$

Both the vertex function $\mathbf{\Gamma}$ and the kernel K are formal series in $\hbar$. The lowest order in the vertex function is the tree-level vertex Γ_0. Note that the kernel K is at least of order $\hbar$ and it is for the moment assumed that the kernels $K^{[i]}$ and propagators P are explicitly known. Figures 2.4 and 2.5 explain these notions.

If we set $Z^{[i]} = 0$, $\forall i \geq 1$, we find the following perturbation expansion for the bare (unrenormalized) function Γ (where we simply write $K(k)$ for $K(k,k+q,p+q,p)$):

$$
\begin{aligned}
\Gamma(p,p+q) &= \Gamma_0 + \int \mathrm{d}^4 k \Gamma(k,k+q) P(k) P(k+q) K(k), \\
\Gamma(p,p+q) &= \Gamma_0 + \sum_{i=1}^{\infty} \hbar^i \Gamma^{[i]}(p,p+q), \\
\Gamma &= \Gamma_0 + \hbar \int \mathrm{d}^4 k \Gamma_0 P(k) P(k+q) K^{[1]}(k) \\
&\quad + \hbar^2 \int \mathrm{d}^4 k \Big[\Gamma_0 P(k) P(k+q) K^{[2]}(k) \\
&\quad + \Gamma^{[1]}(k,k+q) P(k) P(k+q) K^{[1]}(k) \Big] + \cdots .
\end{aligned}
$$

It usually so happens that the above integrals are ill-defined. In such cases it might be possible to give meaning to them through the introduction of a regularization. Let us assume that there is a parameter ϵ such that the limit $\epsilon \to 0$ reproduces the above integrals, but such that for $\epsilon \neq 0$ all integrals exist. The evaluation of the above integrals will then result in Laurent series in this parameter. The proper pole terms in these Laurent series prohibit the desired limit $\epsilon \to 0$.

Now let us actually repeat the above series expansion, this time allowing for non-vanishing $Z^{[i]}$, and thus coming back to Eqs.(2.35). We obtain for the renormalized vertex function $\mathbf{\Gamma} = \Gamma_0 + \hbar \mathbf{\Gamma}^{[1]} + \hbar^2 \mathbf{\Gamma}^{[2]} + \cdots$:

$$
\begin{aligned}
\mathbf{\Gamma} &= \Gamma_0 + \hbar Z^{[1]} + \hbar \int \mathrm{d}^4 k \Gamma_0 P(k) P(k+q) K^{[1]}(k) \\
&\quad + \hbar^2 Z^{[2]} + \hbar^2 \int \mathrm{d}^4 k \Big[\Gamma_0 P(k) P(k+q) K^{[2]}(k) \\
&\quad + \Gamma^{[1]}(k,k+q) P(k) P(k+q) K^{[1]}(k) \Big] \\
&\quad + \hbar^2 Z^{[1]} \int \mathrm{d}^4 k \Gamma_0 P(k) P(k+q) K^{[1]}(k) + \cdots .
\end{aligned}
$$

Assume that we determine the $Z^{[i]}$s by the following condition:

$$
Z^{[1]}_{K^{[1]}} = -\mathrm{PP}\left(\int \mathrm{d}^4 k P(k) P(k+q) K^{[1]}(k) \right) \tag{2.36}
$$

$$
Z^{[2]}_{K^{[2]}} = -\mathrm{PP}\left(\int \mathrm{d}^4 k P(k) P(k+q) K^{[2]}(k) \right) \tag{2.37}
$$

$$\begin{aligned} Z^{[2]}_{K^{[1]}K^{[1]}} &= -\mathrm{PP}\left(\int \mathrm{d}^4k \int \mathrm{d}^4 l P(l)P(l+k)K^{[1]}(l)P(k)\right. \\ &\quad \times P(k+q)K^{[1]}(k) \\ &\quad \left.+ Z^{[1]}_{K^{[1]}} \int \mathrm{d}^4k P(k)P(k+q)K^{[1]}(k)\right), \qquad (2.38) \\ Z^{[2]} &= Z^{[2]}_{K^{[2]}} + Z^{[2]}_{K^{[1]}K^{[1]}}, \qquad (2.39) \end{aligned}$$

where PP refers to the projection onto the proper pole part of its argument. Then, it is easy to see that, with the so-determined coefficients of the Laurent series $Z^{[i]}$, the corresponding $\mathbf{\Gamma}^{[i]}$ are free of poles in ϵ, and thus correspond to finite quantities.

Here, we see for the first time the process of renormalization at work. First, by the help of a regularization we replaced the ill-defined integrals by a Laurent series. We then introduced a further series Z which compensates the pole parts such that the remaining series is finite. Later, we will see that Z obtains an algebraic meaning: it is derived from an antipode in a Hopf algebra which we will introduce in Chapter 3. It is a generalized inverse of the series for the vertex function insofar as it removes all the UV-infinite divergences from its perturbative expansion. It thus removes almost all of it, apart from the 'small' finite part. The introduction of a regularization is convenient but not necessary. One could use a Taylor expansion of the integrand and proceed in the BPHZ spirit [Bogoliubov and Shirkov 1980], still realizing the underlying Hopf algebra structure.

In Fig. 2.4 we explain the Schwinger–Dyson equations graphically. They nicely organize the perturbative expansion. Each Feynman diagram consists of propagators and vertices which are Green functions which allow for a perturbative expansion and a corresponding Schwinger–Dyson equation in their own right. Knowing the full solution of the Schwinger–Dyson equations, incorporating full propagators and internal vertex corrections, would correspond to knowing the full non-perturbative Green functions.

Of special interest are the four-point functions which serve as the kernel K in the Schwinger–Dyson equation for the vertex. Their skeleton diagrams are Feynman diagrams which are the basic blocks of iteration in the equations. Any proper skeleton contribution is not two-line reducible. When we close two of its lines we again get a vertex function. These notions are explained in

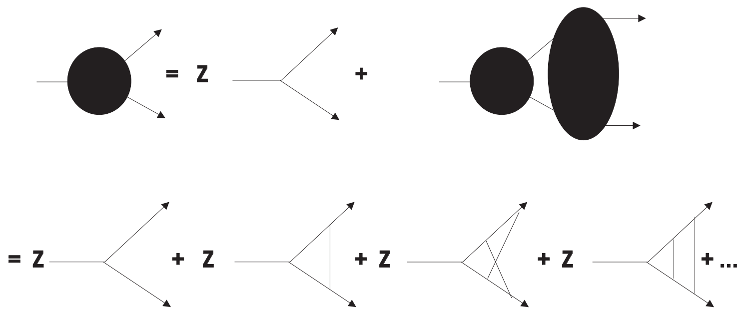

Fig. 2.4 We can generate the three-point Green function $\mathbf{\Gamma}$ iteratively. An integration over the internal momentum in the loop on the rhs is understood. The kernel K, a four-point function given as the big black oval blob, allows a skeleton expansion, as in Fig. 2.5. In general, all bare internal propagators and internal vertices have to be replaced by full Green functions, and the coupled system of equations for these functions is hard to solve [Bjorken and Drell 1965].

Fig. 2.5.

2.2 Regularization

Feynman integrals will usually suffer from the fact that they are ill-defined at large internal momenta. We have noted already that the first step is to introduce a regulator, to gain control over these divergences in a well-defined way. A very simple idea to control an ill-defined integral which is divergent at the upper boundary is to introduce a cut-off Λ, $\Lambda \ll q^2, m^2$, as in the following scalar two-point function:

$$\int \mathrm{d}^4 k \frac{\Theta[\Lambda - k^2]}{[k^2 - m^2 + i\eta][(k+q)^2 - m^2 + i\eta]}. \tag{2.40}$$

Evaluating the integral, one finds terms $\sim \log(\Lambda)$, indicating the presence of a divergence when $\Lambda \to \infty$. One could now evaluate Feynman graphs by introducing such a cut-off for each internal loop integration, and afterwards apply some rule on how to handle the terms which diverge with Λ. For example, we can introduce

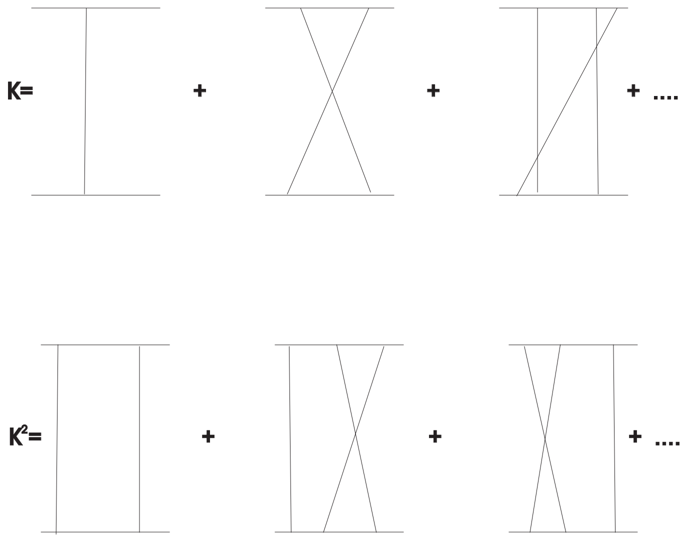

Fig. 2.5 The skeleton expansion given for bare internal propagators and vertices. In its dressed form (reinserting blobs for every internal vertex or propagator) it generates the kernel K of the full Schwinger–Dyson equations. Note that the lowest order skeleton is the one-particle exchange graph of order g^2. Iteration of skeletons delivers products of kernel functions. Iterated graphs separate into various pieces when we cut at two propagators, and thus are two-line reducible.

a Z-factor as indicated before which then would have to absorb the terms which diverge with the cut-off. Such an approach using a cut-off is usually not very useful in particle physics. It happens that the cut-off interferes with gauge symmetry. Thus, one loses gauge invariance at least in intermediate steps, which is a very unwelcome feature of such an approach.

We now introduce dimensional regularization(DR), the regularization prescription which has become the standard choice in particle physics. This is mainly due to the fact that it respects all symmetries of the theory under consideration.† As the chosen

† There is an objection to be raised here: the chiral anomaly is present in DR

regularization prescription is irrelevant for the future considerations, and is only important insofar as it delivers examples, we will be succinct. More thorough treatments can be found in [Collins 1984]. Note also that DR is well-defined even from a mathematicians' viewpoint [Speer 1964, Etingof 1999]. Furthermore, DR is distinguished by its use of a dimensionless regularization parameter $(4-D)/2$, which allowed us to identify the process of renormalization with a suitable Birkhoff decomposition in very recent work [Connes and Kreimer 1999].

2.2.1 Dimensional regularization

Assume you are given a Feynman integral which depends on a couple of external parameters, say, momenta and masses. Internal loop momenta must be integrated. The integration occurs in D dimensions. For integer D, the integral will diverge for large internal loop momenta. Dimensional regularization (DR) is a regularization which obeys the following properties:

- it maintains Poincaré invariance;
- formally, it is defined for complex D. The difference between D and the integer dimension $\mathbf{D}$ in which we want to evaluate the integrals is $\mathbf{D}-D =: 2\epsilon \in \mathbb{C}$;
- if the integral exists, it agrees with the usual Riemann integral in the limit $\epsilon \to 0$;
- otherwise, it delivers a Laurent series in ϵ.

This axiomatizes dimensional regularization [Collins 1984]. A definition is obtained as follows. First, we separate a subspace which is spanned by the external momenta in the process under consideration. As an example, let the Feynman integrand depend solely on one external momentum q. Then each internal loop momentum l allows for a covariant separation $l = l_{||} + l_{\perp}$, where we have $l_{\perp} \cdot q = 0$, which defines $l_{\perp}$. Accordingly, the D-dimensional space to be integrated separates into a parallel and orthogonal space in a straightforward manner. Turning to spherical coordinates in the $(D-1)$-dimensional orthogonal space, the angular integrations in

as well as in any other regularization scheme. As nature seems to favour theories distinguished by the absence of anomalies, this is of no further concern for our purposes.

this space are well-defined and do not depend on any external vectors. They thus give the volume of the $(D-1)$-dimensional unit sphere,

$$\frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}. \tag{2.41}$$

What remains is an integration over the modulus $l_\perp$ of the loop momentum in the orthogonal space, $\int \mathrm{d}l_\perp l_\perp^{D-2} f(l_\perp^2)$. Dimensional regularization effectively happens in this orthogonal space. There, the analytic expressions can now be promoted by analytic continuation to non-integer dimensions D, in which case the deviation from the nearest integer $\mathbf{D}$ serves as the regularization parameter, $\epsilon = (\mathbf{D} - D)/2$, as defined before. For almost all our applications it is sufficient to rely on the following formulae, which are easily derived from the previous considerations [Collins 1984].

$$\begin{aligned} \int \mathrm{d}^D k \frac{[k^2]^a}{[k^2 - m^2]^b} &= \pi^{D/2} [-m^2]^{D/2+a-b} \\ &\quad \times \frac{\Gamma(a + \frac{D}{2})\Gamma(b - a - \frac{D}{2})}{\Gamma(\frac{D}{2})\Gamma(b)}, \end{aligned} \tag{2.42}$$

$$\begin{aligned} \int \mathrm{d}^D k \frac{1}{[k^2]^a\ [(k+q)^2]^b} &= \pi^{D/2} [q^2]^{\frac{D}{2}-a-b} \\ &\quad \times \frac{\Gamma(a+b-\frac{D}{2})\Gamma(\frac{D}{2}-a)\Gamma(\frac{D}{2}-b)}{\Gamma(a)\Gamma(b)\Gamma(D-a-b)}, \end{aligned} \tag{2.43}$$

$$\int \mathrm{d}^D k [k^2]^a = 0,\ \forall a, b\ \in \mathbb{C}. \tag{2.44}$$

Hence it is a property of DR that whenever the parameter D equals the spacetime dimension of the field theory under consideration, we recover the original integrals. For small deviations from this integer, we obtain a Laurent series in the regularization parameter ϵ. For example, the one-loop vertex graph in ϕ_6^3-theory in Fig. 2.6 delivers $(D = 6 - 2\epsilon)$†

$$\begin{aligned} \Gamma(p_1, p_2, m^2) &= \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k^2 - m^2)((k+p_1)^2 - m^2)((k-p_2)^2 - m^2)} \\ &= g^3 \frac{1}{2^D \pi^{D/2} 2\epsilon} + g^3 F(p_1, p_2, m^2) + \mathcal{O}(\epsilon), \end{aligned} \tag{2.45}$$

† Conventions as in [Collins 1984]. In our conventions, we would drop the $\frac{1}{2^D \pi^{D/2}}$-term. Also, we usually omit to give the DR scale parameter μ explicitly, which can be reinserted by replacing $\mathrm{d}^{DD}k \to [\mu^2]^\epsilon \mathrm{d}^D k$.

where $F(p_1,p_2,m^2)$ is independent of $\epsilon = (6-D)/2$, while the pole term is independent of masses and momenta p_1, p_2. In general, one obtains a proper Laurent series in ϵ of degree at most n in an n-loop calculation, which means that we have at most poles of order $1/\epsilon^n$ in the regularized Feynman integrals from a n-loop calculation. The pole terms reflect the UV-divergences of the Feynman graphs. Renormalization theory dictates how to handle these divergences.

As almost nowhere in this book will the reader need to apply DR, we will be content with these short remarks. The message to be kept in mind is that regularization schemes do exist, and that they deliver Laurent series in regularization parameters which parametrize the ill-definedness of the integrals.

2.3 Basic facts about renormalization

By now, we have Feynman rules and a regularization procedure which turns the Feynman integrals into some Laurent series in a parameter ϵ. But we still do not know what to do with the pole terms. To handle them in an intelligent manner we need renormalization theory. We have introduced a Z-factor previously, in a rather ad hoc manner. We want to discuss how this comes about naturally in field theory, and what sort of limitations exist to such an approach. Consider a one-loop graph as in Fig. 2.6 for ϕ^3-theory in six dimensions. According to our Feynman rules we have to integrate the internal momentum. The analytic expression is of the form given already in Eq.(2.45) and thus contains the obvious logarithmic divergence for large internal momentum:

$$\Gamma(p_1,p_2,m^2) \quad \sim \quad \int \mathrm{d}^6k \frac{1}{k^6} \sim \infty. \qquad (2.46)$$

It is apparent that the integration is ill-defined. Such UV divergences appear in all pQFT which are of interest in particle physics. The name stems from the fact that they appear at large internal momenta, corresponding to large internal (ultraviolet) energies. We want to understand how a theory plagued by such infinities can still be well-defined and able to deliver sensible predictions, i.e. how it can be renormalized. Let us see how our brave introduction of Z in the Schwinger–Dyson equation fits into field theory.

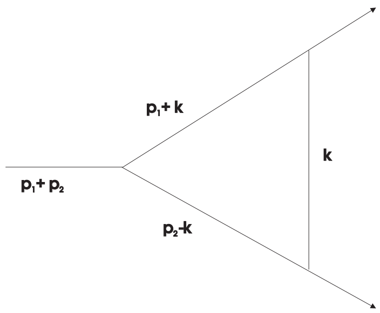

Fig. 2.6 Such a one-loop graph demands an integration of internal momenta. As already noted, it turns out to be ill-defined.

2.3.1 Multiplicative renormalization

The theory we want to explore is ϕ^3-theory in six dimensions (ϕ^3_6). We will soon see that precisely in six dimensions it is a renormalizable theory. We recall the Lagrangian of ϕ^3_6:

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3. \tag{2.47}$$

For the moment we want to have a first look at renormalization theory in action by handling the divergences present up to order g^2 in the coupling. This is the one-loop case. Let us rewrite the Lagrangian

$$L = Z_2\frac{1}{2}(\partial_\mu \phi)^2 - Z_{\mathrm{m}}\frac{1}{2}m^2\phi^2 + Z_1\frac{g}{3!}\phi^3. \tag{2.48}$$

It is the fact that these multiplicative factors can render the theory finite which justifies the name multiplicative renormalization. It is a crucial property for our Z-factors that they are independent of external momenta, as otherwise such a dependence would obstruct the equation of motion derived from the Lagrangian.

We now want to derive conditions for these Z-factors so that they can render the theory finite. First we note that they correspond to a change in the Feynman rules:

- vertex: $ig \to iZ_1 g$;
- field: $\phi \to Z_2^{1/2}\phi$;
- mass: $m^2 \to m^2 + \delta m^2$, $(m^2 + \delta m^2)Z_2 = m^2 Z_m$.

Now, for simplification, we restrict our attention to the divergence in the vertex corretion. Accordingly, we set $Z_{\mathrm{m}} = Z_2 = 1$ and check if we can determine Z_1 so as to render the theory finite. At this loop order, this means that we mainly have to render one Feynman graph finite. The new Feynman rules change the result for this one-loop graph, Fig. 2.6

$$\mathbf{\Gamma}^{[1]}(p_1, p_2, m^2) = Z_1^3 \left[g^3 \frac{1}{2^D \pi^{D/2} 2\epsilon} + g^3 F(p_1, p_2, m^2) + \mathcal{O}(\epsilon) \right].$$

Now, obviously, the one-loop vertex function $\Gamma^{(1)}(p_1, p_2, m^2)$ is of order g^3, while the tree-level vertex is $\sim ig$.

We want to render the theory finite at the one-loop level, which means that we want to have a finite result to order g^3 for the vertex function. With our modified Feynman rules, to order g^3 we thus have

$$\mathbf{\Gamma}(p_1, p_2, m^2) = Z_1 g + Z_1^3 g^3 \left(\frac{1}{2^D \pi^{D/2} 2\epsilon} + \text{finite terms} \right). \quad (2.49)$$

We see that Z_1 must have the following form

$$Z_1 = 1 - \frac{g^2}{2^6 \pi^3 2\epsilon}. \quad (2.50)$$

This renders $\mathbf{\Gamma}$ finite to order g^3.

To make the one-loop renormalization complete, we would also have to include the other Z-factors, all of them of the form

$$Z_i = 1 - \sum_{j=1}^{\infty} \sum_{k=-j}^{\infty} (g^2)^j c_{jk}^{[i]} \epsilon^k. \quad (2.51)$$

So, a Z-factor which renormalizes the monomial specified by the subscript i in the Lagrangian can have at most a pole in ϵ of degree j if we calculate radiative corrections to the order $(g^2)^j$.

Each monomial in the Lagrangian obtains such a Z-factor.† Theories which can be rendered finite by introducing one Z-factor for each monomial in the original Lagrangian are called renormalizable. We more or less focus on these theories for our purposes.‡

† In our notation, $c_{11}^{[1]} = \frac{1}{2(2^6 \pi^3)}$.

‡ One can relax this restriction somewhat. Effective theories may have an increasing number of terms in the Lagrangian. At each loop order the need to compensate divergences introduces new monomials. As long as the number of these monomials remains finite at each loop order, one still can give these theories some sense, as their divergences at any loop order can still

Higher order corrections – loop corrections – are obtained by considering the Green functions for each of these monomials, with

$$\langle 0|T[\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)|0\rangle \tag{2.52}$$

generating divergences to be absorbed by Z_2, the wave function renormalization, and $Z_{\rm m}$, the mass renormalization, while the divergences in the three-point function

$$\langle 0|T[\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3)|0\rangle \tag{2.53}$$

are compensated by a Z-factor called Z_1, the vertex renormalization, from now on.

Let us come back to our example. Assume that we also have determined $Z_{\rm m}, Z_2$ so that the two- and three-point Green functions are finite at the one-loop level. But these finite values are not determined yet. They depend on our choice for the finite parts of the Z-factors, according to Eq.(2.51).

Usually, one determines the Z-factors so that the values for masses and coupling constants inferred from the Green functions $G(x_1,\ x_2)$ and $G(x_1, x_2, x_3)$ coincide with the experimental values for masses and coupling constants. Specifying such a set of *renormalization conditions* determines the finite part in the Z-factors. Renormalization conditions as just described are known as on-shell conditions. Though they are useful in practice, we do not use them here. We are solely interested in the divergences of the theory. This suggests that we use minimal subtraction (MS) conditions, for which all $c_{jk}^{[i]} = 0, k \geq 0$, by definition. With these conditions, one only removes the pole parts in ϵ, so that MS Z-factors provide a bookkeeping of the UV singular structure of the theory.†

We can make such a choice only for the Green functions which suffer from divergences, as they are the only ones accompanied

be absorbed by local counterterms. The Hopf algebra approach presented in the next chapter is applicable in such cases.

† The MS renormalization scheme is the most natural one from a mathematician's viewpoint: it renders the process of renormalization equal to the determination of the Birkhoff decomposition of a Green function into a part holomorphic at $D = 4$ and a counterterm part holomorphic at $(4 - D)/2 = \infty$. The underlying group is the one associated to the Lie algebra whose universal enveloping algebra is dual to the Hopf algebra of renormalization to be introduced in Chapter 3. For details on these aspects we refer the reader to [Connes and Kreimer 1999].

by Z-factors. Their finite values cannot be predicted by a renormalizable theory. They have to be taken from experiment, if one wants to test the theory. Fortunately, all other Green functions can then be calculated in a unique manner. If we now calculate some n-point Green function in ϕ_6^3, $n \geq 4$, its value must be and is determined.

Success or failure of the theory then depends on the answer to the following question: do the S-matrix elements for processes involving n-point Green functions correctly describe experimentally observed cross-sections for these processes? Will a pQFT be predictive for general S-matrix elements, once a finite set of parameters is taken from experiment? The answer is affirmative; any observed particle scattering process is successfully described by a renormalizable pQFT. This is the motivation for pQFT, and the motivation for the enormous effort put into loop calculations these days.

We now summarize the most important notions of the classical textbook approach to renormalization.

2.3.2 Power counting

We will introduce these notions by simply considering a two-loop example. Consider Fig. 2.7. It is a two-loop graph. This is obvious by inspection. We can also infer this from counting the number of propagators. The one-loop vertex correction has three internal propagators. In the next loop order we have to specify two points somewhere in the one-loop graph, which become internal vertices and are to be connected by a propagator. Altogether this generates three more propagators. In general, for a theory with a cubic interaction, each loop order produces three more propagators. So the number of propagators determines the loop order.

In our example, we have realized a very special two-loop graph. We see that the previously considered one-loop graph is nested in the whole graph as a subgraph. So what are our expectations for the structure of the divergences of such a graph?

We have two internal loop momenta. We route them as indicated in the figure. The loop momentum l runs solely through the one-loop subgraph. We do expect to see a divergence when $l \to \infty$ simply because the one-loop subgraph is the divergent one-loop

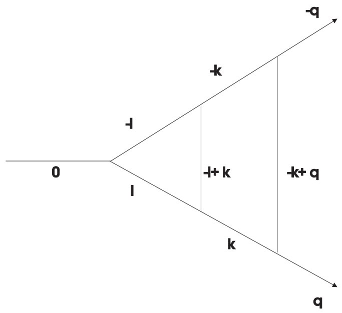

Fig. 2.7 Momentum flow in the two-loop graph. We recognize the one-loop vertex as a subgraph. The graph considered here is actually the one-loop graph, iterated by the Schwinger–Dyson equation Eq.(2.35), as indicated by its two-line reducibility.

vertex function of Fig. 2.6, only this time with momenta $(0,k,k)$ at its external lines. So for any finite value of k we expect a divergence from large l. This subdivergence is logarithmic, as it behaves as $\int d^6l/l^6$ for large l.

Now assume we keep l fixed, with $k \to \infty$. We observe that k runs through a four-point function, and thus through four propagators. It converges: $\int d^6k/k^8$ is finite for large k. So we expect no divergence from the sector (l fixed, $k \to \infty$).

Let us next consider the sector where $(l \to \infty, k \to \infty)$ jointly, e.g., $k \sim l \sim r$. Altogether our two-loop graph has six internal propagators. Each propagator behaves as $1/r^2$. In the region $l \sim k$ the whole integrand behaves as $1/r^{12}$, and the whole graph as $\sim \int d^{12}r/r^{12} \sim \int dr/r$. So we expect a divergence for large $r \sim k \sim l$. We call it a logarithmic degree of divergence for obvious reasons.

That this somewhat naive analysis is correct and the determination of divergent sectors reduces to a counting of propagators (*edge counting*) is due to Weinberg's theorem [Weinberg 1960]. The resulting method which we now describe is known as power counting.

In general, any graph consists of a product of propagators. Let us assume that our Feynman rules are such that all vertices are dimensionless.† Each propagator P_i has a weight c_i. We determine this weight as the degree of the inverse propagator, for example for a scalar propagator $\sim 1/(p^2 - m^2)$ the degree is two, because the inverse propagator behaves as p^2 for large p.

Now, consider an n-loop Feynman graph. Each loop contributes a $\mathbf{D}$-dimensional integration measure, where $\mathbf{D} = 6$ in our ϕ^3_6 example. Assume we have n_i propagators of weight c_i in the graph, and r different sorts of propagators altogether. Then we call a graph overall divergent if

$$\omega := n\mathbf{D} - \sum_{j:=1}^{r} c_j n_j \geq 0. \tag{2.54}$$

For Figs. 2.6 and 2.7 we have $\omega = 0$. Such graphs we call logarithmically divergent, while $\omega = 1$ or 2 refers to linearly or quadratically divergent graphs.

A few remarks are in order. Note that an addition of extra loops will not change the degree of divergence in our ϕ^3_6-theory. Any new loop generates three more propagators, while on the other hand we have one more integration to do. Hence, the net change vanishes: $6 - 2 \times 3 = 0$. So in six dimensions any three-point vertex function in ϕ^3-theory is overall logarithmically divergent, while any two-point function is overall quadratically divergent, and any Green function with four or more external particles remains overall convergent. This depends on the dimension. In $\mathbf{D} < 6$ dimensions for example even the vertex function in ϕ^3 is overall convergent, and addition of k further loops reduces the degree of divergence by $k(\mathbf{D} - 6)$. On the other hand, in $\mathbf{D} > 6$ dimensions any r-point Green function becomes divergent for a sufficiently high loop number, as $\mathbf{D} - 6$ now worsens the degree of divergence with each loop number.

We call $\mathbf{D} = 6$ the critical dimension of ϕ^3-theory. It is renormalizable in this dimension, which means that we have the possibility of using a finite number of Z-factors in the Lagrangian to render the theory finite. Precisely in this critical dimension the coupling constant is dimensionless. Only the Green functions corresponding to fundamental monomials in the Lagrangian suffer from divergences in loop integrations. In $\mathbf{D} > 6$ dimensions the

† Otherwise we have to count weighted vertices as well in the power counting. The weight of the vertices considered here is zero.

theory becomes non-renormalizable and any Green function diverges at a sufficiently high loop number. The coupling constant has negative dimension. In $\mathbf{D} < 6$ dimensions the theory is super-renormalizable. Divergences appear only at a sufficiently low loop number and the coupling constant has positive dimension.

These notions of criticality and renormalizability extend to other pQFTs in an obvious manner. All other theories we refer to in this book are critical in four dimensions. Particle physics seem to favour pQFTs which are critical in four dimensions.

Let us return to our two-loop example in Fig. 2.7. Power counting tells us that it has a logarithmic overall degree of divergence. We further know that there is a divergence from its one-loop subgraph. We can indeed extend the notion of power counting in an obvious manner. Let us in our minds put a little box around any divergent subgraph γ_i which we detect in a Feynman graph Γ. We do a power counting for the Feynman graph contained in this box, and say that the degree of divergence of the subgraph γ_i is its overall degree of divergence ω_i. We then say that Γ has an overall degree of divergence ω and subdivergences contributed by subgraphs γ_i of degree of divergence ω_i.

Figure 2.8 provides examples of various different ways in which subdivergences can appear. They can come in nested form and as disjoint subdivergences, and they can be overlapping. This last form is the most complicated one. In Chapter 3 we will see that one can disentangle overlapping subdivergences to be expressible by the other two forms and renormalize arbitrary Feynman graphs according to rules prescribed by a Hopf algebra.

Renormalization theory is an iterative procedure to eliminate the divergences of the theory. One first eliminates the subdivergences, and then hopes to control the remaining overall degree of divergence. This iteration procedure can be summarized in a forest formula [Zimmermann 1969, Bogoliubov and Shirkov 1980, Itzykson and Zuber 1980] but for now we will study our two-loop example to see how the iteration works. We will derive the forest formula as resulting from an antipode later, in Sections 3.5 to 3.7, when we have more experience with the different types of divergences.

Let us finish our two-loop example. We have already determined the Z_1-factor to order g^2, to achieve the one-loop renormalization.

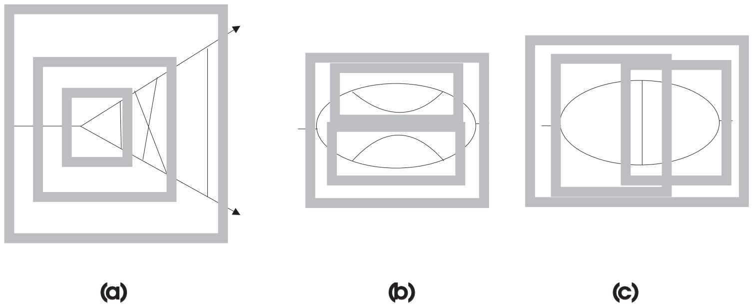

Fig. 2.8 There are various ways subdivergences can appear. They can be nested as in (a), disjoint as in (b) or overlapping as in (c). We put every subdivergence in a grey box. These boxes are called forests.

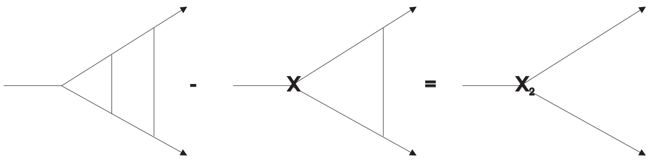

Fig. 2.9 We add a counterterm graph. It absorbs the subdivergence in the subgraph, present for $l \to \infty$, k fixed. After subtraction of the one-loop counterterm graph we can determine the two-loop Z-factor, indicated by X_2.

We are interested now in obtaining a finite result at order g^5. Thus, we need Z_1 to order g^4. At this order we not only have the two-loop graph, but also a one-loop graph with a counterterm insertion from the renormalization of the one-loop divergence, which is precisely the product $Z_1^{[1]}\Gamma^{[1]}$ appearing on the rhs of Eq.(2.38). It appears due to our replacement $g\phi^3 \to Z_1 g\phi^3 = g\phi^3 - g^3 \frac{c_{11}^{[1]}}{\epsilon}\phi^3 + \cdots$ resulting from the very presence of Z-factors in the Lagrangian. We have to add this counterterm graph to the two-loop graph, as is done in Fig. 2.9. The counterterm was determined to render the one-loop vertex function finite. This function appears as a subgraph in the full two-loop graph, and it is indeed true that for any *fixed* k the difference of the full two-loop contribution and the counterterm contribution is finite.

The only remaining divergence in the sum of the two terms of Fig. 2.9 appears when both loop momenta tend to infinity jointly. This is the overall divergence and this is true in general. After elimination of all subdivergences by counterterms the remaining divergences appear when all loop momenta tend to infinity jointly.

Finally, our renormalization program continues as for the one-loop case. The sought-for difference of the full two-loop graph and counterterm contribution has the form

$$g^5\left[\frac{c_2}{\epsilon^2}+\frac{c_1}{\epsilon}+\cdots\right]\Rightarrow c_{21}^{[1]}=-c_1, c_{22}^{[1]}=-c_2 \tag{2.55}$$

and the finite part $c_{20}^{[1]}$ must again be determined by the chosen renormalization conditions, and does vanish in the MS scheme.†

It is only in this combination of the full two-loop graph with the counterterm contribution that the $c_{2i}^{[1]}$, $i\in\{1,2\}$, come out as pure numbers as they should. The two-loop graph itself will contain divergences of the form $\log q^2/\epsilon$, where the divergence comes from the divergent subgraph, and multiplies a $\log q^2$ which appears in the finite part of the second loop integral. Such non-local contributions are compensated by terms coming from the renormalization of subdivergences. We will see soon how this mechanism works in detail.

In fact, much more complicated non-polynomial functions of q^2 and masses could be present in a real Green function. Fortunately, after addition of the counterterm contribution, these non-local contributions cancel, and the $c_{2i}^{[1]}$ appear as pure numbers. This is a general property as well: after adding the appropriate counterterms for subdivergences the overall divergences remain local, demanding only polynomials in fields and their derivatives as counterterms in the Lagrangian.

In this chapter we studied a nested subdivergence to explain the basic mechanism of renormalization. We will learn to handle more general disjoint or overlapping cases in the next chapter.

† For ϕ_6^3 one finds explicitly $c_1=\frac{1}{16(2^{2D}\pi^D)}$ and $c_2=\frac{-1}{8(2^{2D}\pi^D)}$.

The whole renormalization procedure has been remarkably successful in QED as well as in the standard model of elementary particle physics and still provides the only successful mechanism for calculations in particle physics.