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1 MA 557 HW Problem 1

Given:

$$\phi(\Gamma) = \prod_{e \in \Gamma^{[1]}_{int}} \prod_{v \in \Gamma^{[0]}} \int \text{Prop}(k_e)\, \delta^{(D)}(\sum_{j \text{ adj. } v} k_j)\, d^D k_e$$

Prove:

$$\phi(\Gamma) = \delta^{[D]}(\sum_{j \in \Gamma^{[1]}_{ext}} k_j) \int d^D k_1 \dots d^D k_r \cdot \text{Int}(\Gamma),$$

where $r = b_1(\Gamma)$ is the first betti number of Γ and $\text{Int}(\Gamma)$ is a function of the external momenta and r internal momenta. (The external momenta are considered fixed.)

Proof: In order for a path to contribute to ϕ, momentum must be conserved at each vertex. (This is what the δ's express.) This amounts to a system of n_v linear equations in n_{int} variables, where n_v is the number of vertices and n_{int} is the number of internal edges.

However, this system will be inconsistent if external momentum is not conserved. Therefore, if external momentum *is* conserved, at least one of the equations will be redundant. So we can transform our integral to

$$\phi(\Gamma) = \delta^{(D)}(\sum_{j \in \Gamma^{[1]}_{ext}} k_j) \prod_{e \in \Gamma^{[1]}_{int}} \prod_{i=1}^{n_v - 1} \int \text{Prop}(k_e)\, \delta^{(D)}(f_i)\, d^D k_e,$$

where the f_i are linear functions of the internal momenta. This formula amounts to an integral over the vector space of solutions to the system $\{f_1, \dots, f_{n_v-1}\}$. We now show that the dimension of this space is equal to r.

First suppose $r = 0$. Then if $n_v > 1$, Γ will have at least one vertex adjacent to 3 external edges. These 3 external edges uniquely determine the momentum of the internal edge. Thus the dimension of the solution space will not change if we replace this vertex with a single external edge in place of the one internal edge. We call this process "pruning" a vertex.

By repeated pruning, we can reduce any graph with $b_1 = 0$ to a graph with one vertex and four external edges. This graph corresponds to a system of zero equations in zero variables, and hence the dimension of the solution space is zero. This proves that the dimension of solution space is zero for any simply-connected graph.

Now consider a general graph Γ. Pick an internal edge which lies in a cycle, so that its removal will not disconnect the graph. We can "snip" this edge by replacing it with two external edges, whose momenta is such that external momentum is still conserved. "Snipping" a cycle reduces the betti number by one, and also reduces the dimension of solution space by one, since one internal edge is removed. By snipping all r cycles, we are left with a simply-connected graph, and hence a zero-dimenional solution space. This means that the dimension of the solution space for the original Γ was r. This completes the proof.

nice.

See Bloch - Esnault - Kreimer

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MA 557 cont'd

2. $\underline{\underline{\varphi_4^4}}$ Claim: $w(\Gamma) = 4 - e$ $e = \#$ of external edges

Pf. The claim is clearly true for the graph $+$.

A general graph Γ can be reduced to $+$ by a process of "Snipping" cycles and "pruning" vertices with 3 external edges, as in problem 1. We will prove that the quantity $w + e$ is invariant under these two operations

"Snipping" a cycle
- Adds two external edges
- Subtracts one external edge
- Decreases b_1 by 1
- $w \to w - 2$, $e \to e + 2$ ✓

"Pruning" a vertex with 3 external edges
- Subtracts two external edges
- Subtracts one vertex
- Subtracts one internal edge
- $w \to w + 2$, $e \to e - 2$ ✓

QED   Claim: $w(\Gamma) = 4 - e_1 - \frac{3}{2} e_2$ $e_1 = \#$ external $\sim$
$e_2 = \#$ external $\rightarrow$

The claim is clearly true for the graph $\sim\!\!\!\nwarrow$. A general graph Γ can be reduced to $\sim\!\!\!\nwarrow$ by four operations: –

1. Snipping a $\sim$ in a cycle
2. Snipping a $\rightarrow$ in a cycle
3. Pruning Γ' to Γ'
4. Pruning Γ' to Γ'

We prove that $w + e_1 + \frac{3}{2} e_2$ is invariant under these operations.

<u>Snipping $\rightsquigarrow$ from a cycle</u>

- Decreases b_1 by 1
- Adds two external $\rightsquigarrow$
- Subtracts one internal $\rightsquigarrow$
- $w \to w - 2$, $e_1 \to e_1 + 2$

<u>Snipping $\to$ from a cycle</u>

- Decreases b_1 by 1
- Adds two external $\to$
- Subtracts one internal $\to$
- $w \to w - 3$, $e_2 \to e_2 + 2$

<u>Pruning $\Gamma' \rightsquigarrow \xi$ to $\Gamma' \rightsquigarrow$</u>

- Decreases external $\to$ by 2
- Adds one external $\rightsquigarrow$
- Subtracts one internal $\rightsquigarrow$
- Subtracts one vertex
- $w \to w + 2$, $e_1 \to e_1 + 1$, $e_2 \to e_2 - 2$

<u>Pruning $\Gamma' \to \xi$ to $\Gamma' \to$</u>

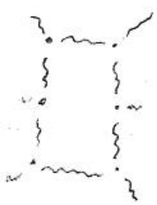
- Subtracts one external $\rightsquigarrow$
- Subtracts one vertex
- Subtracts one internal $\to$
- $w \to w + 1$, $e_1 \to e_1 - 1$

3. $\Gamma =$  Find $\Delta(\Gamma)$

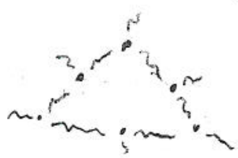
<u>Possible sub graphs</u>

 $w = 4 \cdot 1 - (8 + -4) = 0$ ✓

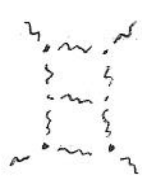
 $w = 4 \cdot 2 - (14 + -6) = 0$ ✓

 $w = 4 \cdot 1 - (12 + -6) = -2$ X

 $w = 4 \cdot 2 - (16 + -7) = -1$ X

 $w = 4 \cdot 1 - (12 + -6) = -2$ X

$\underline{\Delta}(\Gamma) = 3$  $\otimes$ 

$+ 3$  $\otimes$  $+ \delta \otimes \underline{\Gamma} + \underline{\Gamma} \otimes \delta$

very nice!