Local Structures in QFT

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Acknowledgment: Spencer Bloch, David Broadhurst, Francis Brown, Alain Connes, Walter van Suijlekom, Karen Yeats,$\cdots$

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The polylog as a Hodge structure

Iterated integrals: obvious Hopf algebra structure

$$\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2\pi i & 0 \\ -Li_2(z) & 2\pi i \ln(z) & (2\pi i)^2 \end{pmatrix} = (C_1, C_2, C_3) \tag{1}$$

$$\mathrm{Var}(\Im Li_2(z) - \ln|z| \, \Im Li_1(z)) = 0 \tag{2}$$

Hodge sructure from Hopf algebra structure: branch cut ambiguities columnwise

Griffith transversality $\Leftrightarrow$ differential equation

Hopf algebra of graphs $H = \mathbb{Q}1 \oplus \bigoplus_{j=1}^{\infty} H^j$

- The coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \overbrace{\sum_{\gamma = \cup_i \gamma_i, \omega_4(\gamma_i) \geq 0} \gamma \otimes \Gamma/\gamma}^{\Delta'(\Gamma)} \tag{3}$$

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- The antipode

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$$\begin{aligned} S_R^{\Phi}(\Gamma) &= -R\left(\Phi(h) - \sum S_R^{\Phi}(\gamma)\Phi(\Gamma/\gamma)\right) \\ &= -R\,\Phi\left(m(S_R^{\Phi} \otimes \Phi\ P)\Delta(\Gamma)\right) \end{aligned} \tag{6}$$

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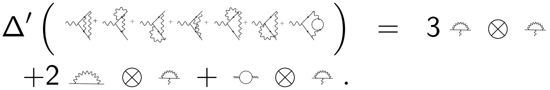
- The renormalized Feynman rules

$$\Phi_R = m(S_R^{\Phi} \otimes \Phi)\Delta \qquad (7)$$

An Example

- The co-product



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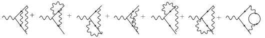
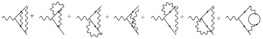
$\Delta'\Big($  $\Big) = 3$  $\otimes$ 

$+2$  $\otimes$  $+$  $\otimes$  .

- The counterterm

$S_R^{\Phi}($  $) = -Rm\left[S_R^{\Phi}\otimes\Phi P\right]\times$

$\times\Delta\Big($  $\Big)$

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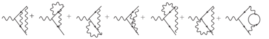
$\Delta'\Big($  $\Big) = 3$  $\otimes$ 

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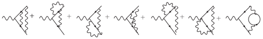
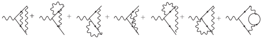
$\times \Delta\Big($  $\Big)$

$= -R\Big\{\Phi\Big($  $\Big) +$

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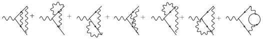
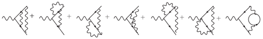
- The renormalized result

$\Phi_R = (\mathrm{id} - R)m(S_R^{\Phi} \otimes \Phi P)\Delta\Big($  $\Big)$

$= (\mathrm{id} - R)\Big\{\Phi\Big($  $\Big)$

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sub-Hopf algebras

- summing order by order

$$c_k^r = \sum_{|\Gamma|=k, \mathrm{res}(\Gamma)=r} \frac{1}{|Aut(\Gamma)|}\Gamma \Rightarrow \Delta(c_k^r) = \sum_j \mathrm{Pol}_j(c_m^s) \otimes c_{k-j}^r. \quad (8)$$

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- Hochschild closedness

$$X^r = 1 \pm \sum_j c_j^r \alpha^j = 1 \pm \sum_j \alpha^j B_+^{r;j}(X^r Q^j(\alpha)), \quad (9)$$

$Q^j = \frac{X^v}{\sqrt{\prod_{\text{edges e at v}} X^e}}$. Evaluates to invariant charge.

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- $bB_+^{r;j} = 0$.

$$\Delta B_+^{r;j}(X) = B_+^{r;j}(X) \otimes 1 + (id \otimes B_+^{r;j})\Delta(X). \quad (10)$$

Implies locality of counterterms upon application of Feynman rules $\Phi B_+^{r;j}(X) = \int d\mu_{r;j}\Phi(X)$:

$$\bar{R}(\Gamma) = m(S_\Phi^R \otimes \Phi P))\Delta B_+^{r;j} = \int d\mu_{r;j}\Phi^R(X). \quad (11)$$

Symmetry

- Ward and Slavnov–Taylor ids

$$i_k := c_k^{\bar{\psi}\psi} + c_k^{\bar{\psi}\not{A}\psi} \tag{12}$$

span Hopf (co-)ideal I:

$$\Delta(I) \subseteq H \otimes I + I \otimes H. \tag{13}$$

$$\Delta(i_2) = i_2 \otimes 1 + 1 \otimes i_2 + (c_1^{\frac{1}{4}F^2} + c_1^{\bar{\psi}\not{A}\psi} + i_1) \otimes i_1 + i_1 \otimes c_1^{\bar{\psi}\not{A}\psi}.$$

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- Ideals for Slavnov–Taylor ids generated by equality of renormalized charges, also for the master equation in Batalin-Vilkovisky (see Walter van Suijlekom's lectures)
- Similar ideals for the core Hopf algebra are respected by the BCFW recursion, and fit naturally with the structure of perturbative quantum gravity

Kinematics and Cohomology

- Exact co-cycles

$$[B_+^{r,j}] = B_+^{r;j} + b\phi^{r;j} \tag{14}$$

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- Variation of momenta

$$G^R(\{g\}, \ln s, \{\Theta\}) = 1 \pm \Phi^R_{\ln s, \{\Theta\}}(X^r(\{g\})) \tag{15}$$

with $X^r = 1 \pm \sum_j g^j B_+^{r;j}(X^r Q^j(g))$, $bB_+^{r;j} = 0$. Also,

$$G^r = \left[\sum_{j=1}^{\infty} \gamma_j(\{g\}, \{\Theta\}) \ln^j s\right] + \overbrace{G_0^r}^{\text{abelian factor}} \tag{16}$$

Then, for MOM and similar schemes (not MS!):
$\{\Theta\} \to \{\Theta'\} \Leftrightarrow B_+^{r;j} \to B_+^{r;j} + b\phi^{r;j}$.
$\Phi^R_{L_1+L_2,\{\Theta\}} = \Phi^R_{L_1,\{\Theta\}} \star \Phi^R_{L_2,\{\Theta\}}$.

Leading log expansions and the RGE

- The invariant charge Q^v
  For each vertex v, a charge Q^v:

$$Q^v(g) = \frac{X^v(g)}{\prod_e \sqrt{X^e}}, \tag{17}$$

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- $$\left(\partial_L + \beta(g)\partial_g - \sum_{e\ adj\ r} \gamma_1^e\right) G^r(g,L) = 0 \tag{18}$$
  rewrites in terms of the Dynkin operator
  $(\gamma_1^r(g) = S \star Y(X^r(g)))$:
  $$\gamma_k^r(g) = \frac{1}{k}\left(\gamma_1^r(g) - \sum_{j\in R} s_j \gamma_1^j g \partial_g\right)\gamma_{k-1}^r(g) \tag{19}$$

Ordinary differential equations vs DSE

- RGE+DSE
  the iterated integral structure

$$\Phi^R(B_+^{r;j}(X)) = \int \Phi^R(X) d\mu_{r;j} \tag{20}$$

  allows to combine $X^r = 1 \pm \sum_j B_+(X^r Q^j)$ with RGE to

$$\gamma_1^r = P(g) - [\gamma_1^r(g)]^2 + \sum_{j \in R} s_j \gamma_1^j g \partial_g \gamma_1^r(g). \tag{21}$$

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- massless gauge theories
  $\beta(g) = g\gamma_1(g)/2$ for γ_1 anomalous dim of gauge propagator

$$\gamma_1(g) = \overbrace{P(g)}^{\textit{existence assumed}} - \gamma_1(g)(1 - g\partial_g)\gamma_1(g) \tag{22}$$

  (Ward Id QED, background field gauge (Abbott) QCD)

Limiting mixed Hodge structures

- Hopf algebra from flags

$$f := \gamma_1 \subset \gamma_2 \subset \ldots \subset \Gamma, \; \Delta'(\gamma_{i+1}/\gamma_i) = 0 \tag{23}$$

The set of all such flags $F_\Gamma \ni f$ determines Hopf algebra structure, $|F_\Gamma|$ is the length of the flag.

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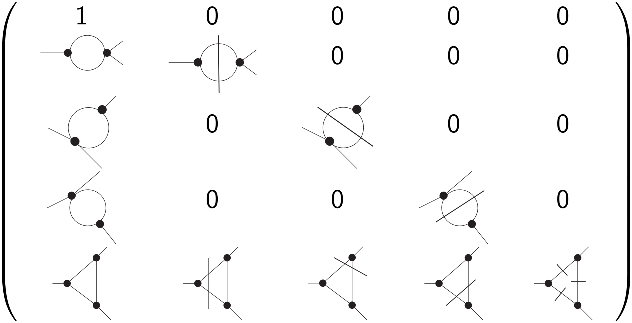
- It also determines a column vector $v = v(F_\Gamma)$ and a nilpotent matrix $(N) = (N(|F_\Gamma|))$, $(N)^{k+1} = 0$, $k = corad(\Gamma)$ such that

$$\lim_{t\to 0}(e^{-\ln t(N)})\Phi_R(v(F_\Gamma)) = (c_1^\Gamma(\Theta)\ln s, c_2^\Gamma(\Theta), c_k^\Gamma(\Theta)\ln^k s)^T \tag{24}$$

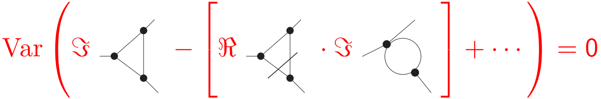
where k is determined from the co-radical filtration and t is a regulator say for the lower boundary in the parametric representation.

The Feynman graph as a Hodge structure

Hopf algebra structure as above



$= (C_1, C_2, C_3, C_4, C_5)$



Hodge sructure: cut-reconstructability: from Hopf algebra structure: branch cut ambiguities columnwise

Griffith transversality $\Leftrightarrow$ differential equation?

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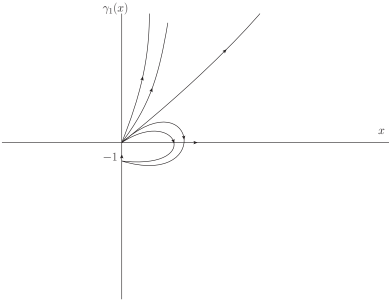
- $\gamma_1(x) = P(x) - \gamma_1(x)^2 + \gamma_1(x) x \partial_x \gamma_1(x)$ with $P(x) > 0$

QED

- sub Hopf algebra for vacuum polarization suffices
- $\gamma_1(x) = P(x) - \gamma_1(x)^2 + \gamma_1(x) x \partial_x \gamma_1(x)$ with $P(x) > 0$
  $P(x)$ twice differentiable
  $\gamma_1(x_0) = \gamma_0 > 0$
  different solutions distinguished by $e^{-\frac{1}{x}}$ behaviour
  $\frac{d\gamma_1}{dx} = \gamma_1 - \gamma_1^2 - P$,
  $\frac{dx}{dL} = x\gamma_1$
  $L = \int_{x_0}^{x(L)} \frac{dz}{z\gamma_1(z)}$



- separatrix exists and might have no Landau pole:
  $D(P) = \int_{x_0}^{\infty} \frac{P(z)dz}{z^3} < \infty, \int_{x_0}^{\infty} \frac{2dz}{z\sqrt{1+4P(z)-1}} < \infty$

QCD

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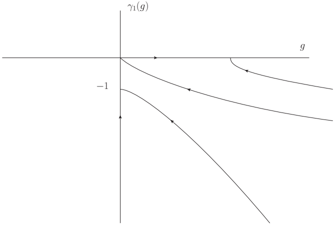
- sub Hopf algebra for gluon polarization suffices in background field gauge
- $\gamma_1(g) = P(g) - \gamma_1(g)^2 + \gamma_1(g)g\partial_g\gamma_1(g)$ with $P(g) < 0$
  $P(g)$ twice differentiable and concave near 0
  unique solution which flows into $(0,0)$ at large Q^2
  $L = \int_{g_0}^{g(L)} \frac{dz}{z\gamma_1(z)} \rightarrow$
  $L_\Lambda = -\int_{g(L_\Lambda)}^{\infty} \frac{dz}{z\gamma_1(z)}$,
  $L_\Lambda = \ln Q^2/\Lambda_{QCD}$
  $f_{disp}(Q^2) = \int_0^\infty \frac{\Im(f(\sigma))d\sigma}{\sigma+Q^2-i\eta}$
  and ODE



- separatrix exists and gives asymptotic free solution, with finite mass gap for inverse propagator iff $\gamma_1(x) < -1$ for some $x > 0$.
  $|D(P)| < \infty \rightarrow \gamma_1(x) \sim sx,\ x \rightarrow \infty$. That allows for dispersive methods as introduced by Shirkov et.al. in field theory.