# A K3 IN $\phi^4$

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Abstract. The problem of determining the number of points of graph hypersurfaces over a finite field $\mathbb{F}_q$ goes back to a question raised by Kontsevich in 1997, and is related to Feynman integral computations in quantum field theory. Stembridge showed that the point-count is polynomial in $q$ for all graphs with $\leq 12$ edges, but Belkale and Brosnan showed in 2003 that the counting functions are of general type for very large graphs. In this paper, we give a sufficient combinatorial criterion (vertex width $\leq 3$) for polynomial point-counts, and compute this polynomial explicitly for some infinite families of physical graphs. We then exhibit some small counter-examples with vertex width 4, whose point counts are related to a singular K3 surface, and expressible in terms of a certain modular form of weight 3 and CM by $\mathbb{Q}(\sqrt{-7})$.

## 1. Introduction

We first recall the definition of graph hypersurfaces and the history of the point-counting problem, before explaining its relevance to Feynman integral calculations in perturbative Quantum Field theory.

1.1. **Points on graph hypersurfaces.** Let $G$ be a connected graph, which may have multiple edges. The graph polynomial of $G$ is defined by associating a variable $\alpha_e$ to every edge $e$ of $G$ and setting

$$\Psi_G = \sum_{T \subseteq G} \prod_{e \notin T} \alpha_e \in \mathbb{Z}[\alpha_e] \tag{1}$$

where the sum is over all spanning trees $T$ of $G$ (connected subgraphs meeting every vertex of $G$ but which have no loops). These polynomials go back to the work of Kirchhoff in relation to the resistance of electrical circuits.

The graph hypersurface $X_G$ is defined to be the zero locus of $\Psi_G$ in projective space $\mathbb{P}^{N_G-1}$, where $N_G$ is the number of edges of $G$. It is highly singular in general. For any prime power $q$, let $\mathbb{F}_q$ denote the field with $q$ elements, and consider the point-counting function:

$$[X_G] : q \mapsto |X_G(\mathbb{F}_q)| \in \mathbb{N} \ .$$

In 1997, Kontsevich informally conjectured that this function might be polynomial in $q$ for all graphs. This question was studied by Stanley, Stembridge and others, and in particular was proved for all graphs with at most twelve edges [21], and various families of graphs obtained by deleting trees in complete graphs [20], [8]. But in [2], Belkale and Brosnan used Mnëv's universality theorem to prove the surprising result that the $[X_G]$ are very general in the following precise sense.

**Theorem 1.** *(Belkale-Brosnan). For every scheme $Y$ of finite type over* Spec $\mathbb{Z}$*, there exist finitely many polynomials $p_i \in \mathbb{Z}[q]$, and graphs $G_i$ such that*

$$s[Y] = \sum_i p_i[X_{G_i}] \ ,$$

*where $Y$ denotes the point-counting function on $Y$, and $s$ is a product of terms of the form $q^n - q$, where $n > 1$. In particular, $[X_G]$ is not always polynomial.*

This does not imply, however, that the point-counting functions $[X_{G_i}]$ themselves are arbitrary. The methods of [4] §4 in particular, imply strong constraints on $[X_G]$. Unfortunately, Belkale and Brosnan's method constructs graphs $G_i$ with very large numbers of edges, and no explicit counter-example was known until recently, when Doryn [11] and Schnetz [15] independently constructed graphs which are quasi-polynomial (i.e., which become polynomial only after a finite extension of the base field). It was also hoped that various 'physicality' constraints on $G$ might be sufficient to ensure the validity of Kontsevich's conjecture in this weaker sense.

However, the counter-examples we construct below show that this hope is false, in the strongest possible way.

1.2. **Feynman integrals and motives.** The point-counting problem has its origin in the question of determining the arithmetic content of perturbative quantum field theories. For this, some convergency conditions are required on the graphs. A connected graph $G$ is said to be primitively divergent if:

$$\begin{array}{rcll} N_G & = & 2h_G & \\ N_\gamma & > & 2h_\gamma & \text{for all strict subgraphs } \gamma \subsetneq G \ , \end{array}$$

where $h_\gamma$ denotes the number of loops (first Betti number) and $N_\gamma$ the number of edges in a graph. In this case, the residue of $G$ is defined by the absolutely convergent projective integral [5]

$$I_G = \int_\sigma \frac{\Omega_N}{\Psi_G^2} \ , \tag{2}$$

where $\sigma = \{(\alpha_1 : \ldots : \alpha_N) \subset \mathbb{P}^{N-1}(\mathbb{R}) : \alpha_i \geq 0\}$ is the real coordinate simplex in projective space, and $\Omega_N = \sum_{i=1}^N (-1)^i d\alpha_1 \ldots \widehat{d\alpha_i} \ldots d\alpha_N$. This defines a map from the set of primitive divergent graphs to positive real numbers. It is important to note that the quantities $I_G$ are renormalization-scheme independent. We say that $G$ is in $\phi^4$ theory if every vertex of $G$ has degree at most four. Even in this case, the numbers $I_G$ are very hard to evaluate, and only known analytically for a handful of graphs. Despite the difficulties in computation, the remarkable fact has been observed by Broadhurst, Kreimer, and Schnetz that every graph whose period is computable (either analytically or numerically) is consistent with being a multiple zeta value. This was the original motivation for Kontsevich's question.

The algebraic approach to this problem comes from the observation that the numbers $I_G$ are periods in the sense of algebraic geometry. To make this precise, the integrand of (2) defines a cohomology class in $H^{N_G-1}(\mathbb{P}^{N_G-1}\backslash X_G)$, and the integrand a relative homology class in $H_{N_G-1}(\mathbb{P}^{N_G-1}, B)$ where $B = V(\prod_{i=1}^{N_G} \alpha_i)$, which contains the boundary of the simplex $\sigma$. Thus as a first approximation, one could consider the relative mixed Hodge structure

$$H^{N-1}(\mathbb{P}^N \backslash X_G, B\backslash(B \cap X_G)) \ . \tag{3}$$

For technical reasons related to the fact that $\sigma$ meets $X_G$ non-trivially, the integral $I_G$ is not in fact a period of (3). However, one of the main constructions of [5] is to blow up boundary components of $B$ to obtain a slightly different relative mixed Hodge structure called the graph motive $M_G$. The integral $I_G$ is now a period of $M_G$. The general conjectures on mixed Tate motives would then say that if $M_G$ is of mixed Tate type (its weight graded pieces are of type $(p,p)$) and satisfies some ramification conditions, then conjecturally it should follow that the period $I_G$ is a multiple zeta value.

Although not explicitly stated in [5], it follows from the geometry underlying their construction and the relative cohomology spectral sequence that $M_G$ is controlled by the absolute mixed Hodge structures $H^i(\mathbb{P}^i\backslash X_\gamma)$, where $\gamma$ ranges over all minors (subquotients) of $G$. Thus the simplest way in which the period $I_G$ could be a multiple zeta value is if the mixed Hodge structure $M_\gamma$ were entirely of Tate type, or, even stronger, if $H^\bullet(\mathbb{P}^i\backslash X_\gamma)$ were of Tate type in all cohomological dimensions, for all minors $\gamma$ of $G$. To simplify matters further, one can ask the somewhat easier question of whether the Euler characteristics of the $X_\gamma$'s are of Tate type. In this way, one is led to consider the class of $X_G$ in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ and ask if it is a polynomial in the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1_k]$. This is surely the reasoning behind Kontsevich's original question, although it was formulated almost ten years before $M_G$ was defined. Note, however, that there is *a priori* no way to construe information about $I_G$ from the Grothendieck class $[X_G]$.

1.3. **Results and contents of the paper.** We begin in §2 by reviewing some algebraic and combinatorial properties of graph and related polynomials and invariants. In §3, we discuss implications for the class of the affine graph hypersurface $[X_G]$ in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$. The first observation is that for primitive-divergent graphs, there is an invariant $c_2(G)$ of a graph $G$ such that

$$[X_G] \equiv c_2(G)\mathbb{L}^2 \mod \mathbb{L}^3 ,$$

and $c_2(G)$ is given explicitly by the class of an intersection of two affine hypersurfaces whenever $G$ has a 3-valent vertex. This intersection satisfies a Calabi-Yau property in the sense that, after projectifying, the total degree is exactly one greater than the dimension of the ambient projective space.

The class $c_2(G) \mod \mathbb{L}$ has many combinatorial properties not satisfied by the full class $[X_G]$, and is therefore much more tractable. However, at some point in §3 we are forced to use the Chevalley-Warning theorem on the point-counts modulo $q$ of polynomials of small degree. Since this result is apparently not known on the level of the Grothendieck ring we make the following conjecture:

**Conjecture 1.** *Consider $\ell$ polynomials $P_1, \ldots, P_\ell \in \mathbb{Z}[\alpha_1, \ldots, \alpha_N]$ which satisfy $\sum_{i=1}^{\ell} \deg P_i < N$. Then $[V(P_1, \ldots, P_\ell)] \equiv 0 \mod \mathbb{L}$ in $K_0(Var_k)$ for any field $k$.*

Because this conjecture is unavailable, we are then forced to pass to point-counts modulo $q$. Denoting the corresponding counting functions by $[.]_q$, we write

$$[X_G]_q \equiv c_2(G)q^2 \mod q^3 ,$$

and view $c_2(G)$ as a map from prime powers $q$ to $\mathbb{F}_q$. The main observation of §3, proved in [15] under certain conditions, is that

$$[X_G]_q \equiv [{}^5\Psi_G(e_1, \ldots, e_5)]_q q^2 \mod q^3$$

where ${}^5\Psi_G(e_1,\ldots,e_5)$ is the 5-invariant of any set of five edges $e_1,\ldots,e_5$ of a graph, defined in [7]. Using the Chevalley-Warning theorem to cut out parasite terms, this enables one to compute the point-counts of $c_2(G)$ modulo $q$ by taking iterated resultants (the 'denominator reduction' of [7, 9]), which reduces the problem to counting points on hypersurfaces of smaller and smaller dimension. This is the key to constructing non-Tate counter-examples.

Some of the known properties of $c_2(G) \mod q$ are given in §4. We summarize some of these properties for primitive-divergent graphs in $\phi^4$ theory here:

- If $G$ is two-vertex reducible then $c_2(G) \equiv 0 \mod q$.
- If $G$ has weight-drop (in the sense of [9]), then $c_2(G) \equiv 0 \mod q$.
- If $G$ has vertex-width $\leq 3$, then $c_2(G) \equiv -1 \mod q$.
- $c_2(G)$ is invariant under double triangle reduction.

A further property is conjectural:

**Conjecture 2.** *$c_2(G)$ is invariant under the completion relation (*[14], [15]*).*

In short, the invariant $c_2(G)$ detects many of the main qualitative features of the residue $I_G$ that one is interested in, but is more malleable. We expect $c_2(G)$ to be invariant under many more combinatorial operations than those listed above. Intuitively, $c_2(G)$ should be closely related to the action of Frobenius on the smallest subquotient motive of $M_G$ which is spanned by the Feynman differential form $\frac{\Omega_N}{\Psi_G^2}$.

In §5 we review the notion of vertex-width and prove the first positive result. Note that this result is valid in the Grothendieck ring $K_0(\mathrm{Var}_k)$.

**Theorem 2.** *Let $G$ have vertex-width at most 3. Then $[\Psi_G]$ is a polynomial in $\mathbb{L}$.*

It was proved in [7] that a variant of the motive $M_G$ is mixed Tate in this case, but the proof we give here is totally elementary and gives an effective way to compute the polynomial $[\Psi_G]$ by induction over the minors of $G$. It also enables one to compute the Grothendieck classes of any infinite family of graphs obtained by inserting triangles into a known graph (see §13.2 of [5]). In §5.4 and §5.5, we carry this out for the wheels and zig-zag graphs, which have vertex width 3.

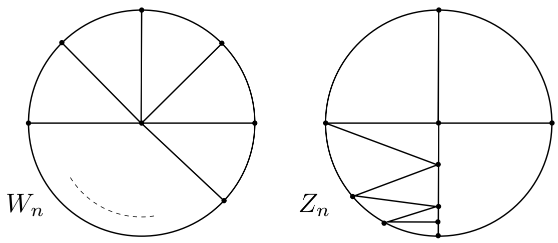


FIGURE 1. The wheels with spokes (left), and zig-zags (right).

These two families are interesting because they are the only families for which a conjectural formula for the residue $I_G$ is known. The family of wheels was obtained independently by Doryn in [11]. To our knowledge, the zig-zags give the first explicit computation of $[\Psi_G]$ for a non-trivial family of physically interesting (i.e., primitively divergent in $\phi^4$ theory) graphs. The recurrence relations used to deduce the polynomial for the zig-zags are surprisingly involved (as expected since the conjectural formula for their residue depends on the parity of $n$), and require solving simultaneous equations for counting functions of generating series of some associated infinite families of graphs. We hope that this will lead to insights for computing the full relative cohomology group $M_G$, and ultimately the motivic coproduct, which is currently not known for a single non-trivial example.

In §6 we turn to non-Tate counter-examples, which are accessed via their $c_2$ invariants. The idea is to find a graph $G = G_1 \cup G_2$, where the subgraph $G_1$ has small vertex width, and $G_2$ has vertex width $> 3$ but few edges. The denominator reduction for $G$ then reduces $c_2(G)$ down to a determinant of graph polynomials of small degree derived from $G_2$. By a series of manipulations one can extract a polynomial which defines a surface of degree 4 in $\mathbb{P}^3$, whose minimal desingularization $X$ is a K3 surface. In §7, we study this surface in more detail, and exhibit sufficiently many lines in $X$ to show that its Néron-Severi group is of maximal rank 20, and that its Picard lattice has discriminant $-7$. This proves that $X$ is a singular $K3$ surface, which have been classified by Shioda and Inose [18]. The modularity of such surfaces is known by [12], and in this case $H^2_{tr}(X)$ is a submotive of the symmetric square of the first cohomology group of the elliptic curve:

$$E_{49A1}: \qquad y^2 + xy = x^3 - x^2 - 2x - 1 \ ,$$

which has complex-multiplication by $\mathbb{Q}(\sqrt{-7})$. In §7.2 we write down the modular form of weight 2 and level 49 whose coefficients give the point counts on $E_{49A1}$.

**Theorem 3.** *There exists a non-planar primitive-divergent graph in $\phi^4$ with 8 loops and vertex width 4 such that, for some constant $c_0 \in \mathbb{N}$,*

$$c_2(G) \equiv c_0 + a_q^2 \mod q$$

*where $q + 1 - a_q = [E_{49A1}]_q$ is the number of points on $E_{49A1}$. In particular,*

$$[X_G] \equiv (c_0 + a_q^2)q^2 \mod q^3$$

*cannot be a polynomial in $q$. There exists a* ***planar*** *primitive-divergent graph in $\phi^4$ theory with 9 loops with the same property.*

The fact that this counter-example has vertex width 4 shows that theorem 2 cannot be improved. Finally, in §7.4 we give a second 8-loop example which experimentally yields a singular K3 with CM by $\mathbb{Q}(\sqrt{-8})$ by similar methods.

1.4. **Discussion.** Despite Belkale and Brosnan's cautionary result, for a while there remained several optimistic hopes about the nature of $\phi^4$ theory:

(1) Even though general graphs have non-Tate Euler characteristics, it could be that graphs coming from physically relevant theories are still of Tate type (the counter-examples have unphysical numbers of edges).
(2) It could be that the counter-examples occur at such high loop orders as to be irrelevant from the point of view of resummable series.
(3) Failing (1) and (2), it could still be the case that planar graphs have Tate Euler characteristics, i.e., all non-Tate counter-examples can be characterized by having a high genus or crossing number.
(4) Even though the Euler characteristics are non-Tate, it could be that the piece of the graph motive which carries the period is always mixed Tate.

The previous theorem shows that (1), (2) and (3) are false. Point (4) is more subtle. However, it follows from the original interpretation of the denominator reduction in [7] that the $c_2$-invariant of a graph should correspond to the 'framing' on $M_G$, i.e., the smallest submotive of $M_G$ which is spanned by the integrand of (2). In any case, this makes it very probable that (4) is false too. A likely candidate for the periods of the graphs of the previous theorem might therefore come from the periods of the motivic fundamental group of the punctured elliptic curve $E$.

It should be emphasized that the residues $I_G$ of primitive graphs in $\phi^4$ are renormalization-scheme independent, and universal in the sense that any quantum field theory in 4 space-time dimensions will only affect the numerator, and not the denominator, of the corresponding parametric integral representation (barring infra-red divergences). Since the motive $M_G$ only depends on the denominators, one can reasonably expect that such non-mixed Tate phenomena will propagate into most, if not all, renormalizable massless quantum field theories at sufficiently high loop orders.

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## 2. Graph polynomials

Throughout this paper, $G$ will denote a connected graph. In this section, we make no assumptions about the primitive-divergence or otherwise of $G$.

### 2.1. **Matrix representation.**

We recall some basic results from [7]. We will use the following matrix representation for the graph polynomial.

**Definition 4.** Choose an orientation on the edges of $G$, and for every edge $e$ and vertex $v$ of $G$, define the incidence matrix:

$$(\mathcal{E}_G)_{e,v} = \begin{cases} 1, & \text{if the edge } e \text{ begins at } v, \\ -1, & \text{if the edge } e \text{ ends at } v, \\ 0, & \text{otherwise.} \end{cases}$$

Let $A$ be the diagonal matrix with entries $\alpha_e$, for $e \in E(G)$, and set

$$\widetilde{M}_G = \left( \begin{array}{c|c} A & \mathcal{E}_G \\ \hline -\mathcal{E}_G^T & 0 \end{array} \right)$$

where the first $e_G$ rows and columns are indexed by the set of edges of $G$, and the remaining $v_G$ rows and columns are indexed by the set of vertices of $G$, in some order. The matrix $\widetilde{M}_G$ has zero determinant. Choose any vertex of $G$ and let $M_G$ denote the square $(e_G + v_G - 1) \times (e_G + v_G - 1)$ matrix obtained from it by deleting the row and column indexed by this vertex.

It follows from the matrix-tree theorem that the graph polynomial satisfies

$$\Psi_G = \det(M_G) \ .$$

**Definition 5.** Let $I, J, K$ be subsets of the set of edges of $G$ which satisfy $|I| = |J|$. Let $M_G(I, J)_K$ denote the matrix obtained from $M_G$ by removing the rows (resp. columns) indexed by the set $I$ (resp. $J$) and setting $\alpha_e = 0$ for all $e \in K$. Let

$$\Psi^{I,J}_{G,K} = \det M_G(I, J)_K \ .$$

It is clear that $\Psi^{\emptyset,\emptyset}_{G,\emptyset} = \Psi_G$, and $\Psi^{I,J}_{G,K} = \Psi^{J,I}_{G,K}$. If $K = \emptyset$, we will often drop it from the notation. We also write $\Psi^{I}_{G,K}$ as a shorthand for $\Psi^{I,I}_{G,K}$.

Since the matrix $M_G$ depends on various choices, the polynomials $\Psi^{I,J}_{G,K}$ are only well-defined up to sign. In what follows, for any graph $G$, we shall fix a particular matrix $M_G$ and this will fix all the signs in the polynomials $\Psi^{I,J}_{G,K}$ too.

**Proposition 6.** *The monomials which occur in* $\Psi^{I,J}_{G,K}$ *have coefficient* $\pm 1$, *and are precisely the monomials which occur in both* $\Psi^{I,I}_{G,J\cup K}$ *and* $\Psi^{J,J}_{G,I\cup K}$.

**Definition 7.** If $f = f_0 + f_x x$ and $g = g_0 + g_x x$ are polynomials of degree one in $x$, recall that their resultant is defined by:

$$[f,g]_x = f_x g_0 - g_0 f_x \ . \tag{4}$$

We now state some identites between Dodgson polynomials which will be used in the sequel. The proofs can be found in ([7], §2.4-2.6).

2.2. **General identities.** The first set of identities only use the fact that $\Psi_G$ is the determinant of a symmetric matrix, and therefore hold for any graph $G$.

(1) *The Contraction-Deletion formula.* It is clear from its definition that $\Psi^{I,J}_{G,K}$ is linear in every Schwinger variable $\alpha_e$, and can be written:
$$\Psi^{I,J}_{G,K} = \Psi^{Ie,Je}_{G,K}\alpha_e + \Psi^{I,J}_{G,Ke} \ .$$
The contraction-deletion relations state that
$$\Psi^{Ie,Je}_{G,K} = \Psi^{I,J}_{G\backslash e,K} \text{ and } \Psi^{I,J}_{G,Ke} = \Psi^{I,J}_{G/\!\!/ e,K} \ ,$$
where $G\backslash e$ is the graph obtained by deleting the edge $e$, and $G/\!\!/e$ denotes the graph obtained by contracting the edge $e$.

(2) *Dodgson-type identities.* Let $I, J$ be two subsets of edges of $G$ such that $|I| = |J|$ and let $a, b, x \notin I \cup J$. Then the first Dodgson identity is:
$$\left[\Psi^{I,J}_{G,K}, \Psi^{Ia,Jb}_{G,K}\right]_x = \Psi^{Ix,Jb}_{G,K}\Psi^{Ia,Jx}_{G,K}$$
Let $I, J$ be two subsets of edges of $G$ such that $|J| = |I| + 1$ and let $a, b, x \notin I \cup J$. Then the second identity is:
$$\left[\Psi^{Ia,J}_{G,K}, \Psi^{I,Jb}_{G,K}\right]_x = \Psi^{Ix,J}_{G,K}\Psi^{Iab,Jx}_{G,K}$$

(3) *Plücker formula.* Let $i, j, k, l$ denote any 4 distinct edges of $G$. Then
$$\Psi^{ij,kl}_{G} - \Psi^{ik,jl}_{G} + \Psi^{il,jk}_{G} = 0 \ .$$

2.3. **Graph-specific identities.** The second set of identities depend on the particular combinatorics of a graph $G$, and mostly follow from the fact that $\Psi_{G\backslash I} = 0$ if $I$ contains all the edges surrounding a vertex, and $\Psi_{G/\!\!/K} = 0$ if $h_1(K) > 0$.

(1) *Vanishing property for vertices.* Suppose that $E = \{e_1, \ldots, e_k\}$ are the set of edges which are adjacent to a given vertex of $G$. Then
$$\Psi^{I,J}_{G,K} = 0 \qquad \text{if} \qquad E \subset I \quad \text{or} \quad E \subset J \ .$$

(2) *Vanishing property for loops.* Suppose that $E = \{e_1, \ldots, e_k\}$ are a set of edges in $G$ which contain a loop. Then
$$\Psi^{I,J}_{G,K} = 0 \quad \text{if} \quad (E \subset I \cup K \text{ or } E \subset J \cup K) \quad \text{and} \quad E \cap I \cap J = \emptyset \ .$$

2.4. **Local structure.** We use these to deduce the local structure of $\Psi_G$ in some simple circumstances. Many more identities are derived in [7].

(1) *Local 2-valent vertex.* Suppose that $G$ contains a 2-valent vertex, whose neighbouring edges are labelled $1,2$. Then

$$\Psi^{12}_G = 0 \quad \text{and} \quad \Psi^{1,2}_G = \Psi^1_{G,2} = \Psi^2_{G,1}$$

which imply that $\Psi_G = \Psi_{G\backslash 1/\!\!/2}(\alpha_1+\alpha_2) + \Psi_{G/\!\!/12}$. In general, if $|I| = |J|$ are sets of edges such that $\{1,2\} \notin I \cup J \cup K$, then

$$\Psi^{1I,2J}_{G,K} = \Psi^{I,J}_{G\backslash 1/\!\!/2,K} = \Psi^{I,J}_{G\backslash 2/\!\!/1,K}\ .$$

(2) *Doubled edge.* Suppose that $G$ contains doubled edges $1,2$. Then

$$\Psi_{G,12} = 0 \quad \text{and} \quad \Psi^{1,2}_G = \Psi^1_{G,2} = \Psi^2_{G,1}$$

which imply that $\Psi_G = \Psi_{G\backslash\{1,2\}}\alpha_1\alpha_2 + \Psi_{G\backslash 1/\!\!/2}(\alpha_1+\alpha_2)$. In general, if $|I| = |J|$ are sets of edges such that $\{1,2\} \notin I \cup J \cup K$, then

$$\Psi^{1I,2J}_{G,K} = \Psi^{I,J}_{G\backslash 1/\!\!/2,K} = \Psi^{I,J}_{G\backslash 2/\!\!/1,K}\ .$$

(3) *Local star.* Suppose that $G$ contains a 3-valent vertex, whose neighbouring edges are labelled $1,2,3$. Then we have ([7], Example 32)

$$\Psi^{123}_G = 0 \quad \text{and} \quad \Psi^{12}_{G,3} = \Psi^{13}_{G,2} = \Psi^{23}_{G,1}$$

which follow from contraction-deletion. Furthermore, for $\{a,b,c\} = \{1,2,3\}$ we have the identities

$$\Psi^{ab,bc} = \Psi^{ab}_c = \ldots = \Psi^{bc}_a \quad \text{and} \quad \Psi^a_{bc} = \Psi^{a,c}_b + \Psi^{a,b}_c$$

These identities propagate to higher order Dodgson polynomials. Let $i,j \notin \{1,2,3\}$. Then for all $\{a,b,c\} = \{a',b',c'\} = \{1,2,3\}$, we have ([7], §7.4):

$$\Psi^{abc,aij} = 0 \quad \text{and} \quad \Psi^{aci,bcj} = \pm\Psi^{i,j}_{G\backslash\{a',b'\}/\!\!/c'}\ .$$

(4) *Local triangle.* Suppose that $G$ contains a triangle, with edges $1,2,3$. Then

$$\Psi_{123} = 0 \quad \text{and} \quad \Psi^1_{23} = \Psi^2_{13} = \Psi^3_{12}$$

which follow from contraction-deletion. Furthermore, for $\{a,b,c\} = \{1,2,3\}$ we have the identities

$$\Psi^{a,b}_c = \Psi^a_{bc} = \ldots = \Psi^b_{ac} \quad \text{and} \quad \Psi^{ab}_c = \Psi^{ab,ac} + \Psi^{ab,bc}$$

Now let $i,j \notin \{1,2,3\}$. For all $\{a,b,c\} = \{a',b',c'\} = \{1,2,3\}$, we have

$$\Psi^{ab,ij}_c = 0 \quad \text{and} \quad \Psi^{ai,bj}_c = \pm\Psi^{i,j}_{G\backslash a'/\!\!/\{b',c'\}}\ .$$

2.5. **The five-invariant.**

**Definition 8.** Let $i,j,k,l,m$ denote any five distinct edges in a graph $G$. The five-invariant of these edges, denoted ${}^5\Psi_G(i,j,k,l,m)$ is defined to be the determinant

$${}^5\Psi_G(i,j,k,l,m) = \pm\det\begin{pmatrix} \Psi^{ij,kl}_{G,m} & \Psi^{ik,jl}_{G,m} \\ \Psi^{ijm,klm}_G & \Psi^{ikm,jlm}_G \end{pmatrix}$$

It can be shown that the five-invariant is well-defined, i.e., permuting the five indices $i, j, k, l, m$ only modifies the right-hand determinant by a sign. In general, the 5-invariant is irreducible of degree 2 in each Schwinger variable. However, in the case when three of the five edges $i, j, k, l, m$ form a star or a triangle, it splits, i.e., factorizes into a product of Dodgson polynomials.

**Example 9.** Suppose that $G$ contains a triangle $a, b, c$. Then

$$^5\Psi_G(a,b,c,i,j) = \pm \det \begin{pmatrix} \Psi^{ab,ij}_{G,c} & \Psi^{ai,bj}_{G,c} \\ \Psi^{abc,cij}_{G} & \Psi^{aci,bcj}_{G} \end{pmatrix} = \pm \Psi^{i,j}_{G\backslash a /\!\!/ \{b,c\}} \Psi^{abc,cij}_{G} \ .$$

It factorizes because $\Psi^{ab,ij}_{G,c} = 0$ by the vanishing property for loops. By contraction-deletion, $\Psi^{ai,bj}_{G,c} = \Psi^{ai,bj}_{G/\!\!/ c}$, and this is $\Psi^{i,j}_{G\backslash a/\!\!/\{b,c\}}$, by the last equation of §2.3, (4), since the edges $a, b$ now form a 2-loop in the quotient graph $G/\!\!/c$.

2.6. **Denominator reduction.** Given a graph $G$ and an ordering on its edges, we can extract a sequence of higher invariants, as follows.

**Definition 10.** Define $D^5_G(e_1, \ldots, e_5) = {}^5\Psi_G(e_1, \ldots, e_5)$. Let $n \geq 5$ and suppose that we have defined $D^n_G(e_1, \ldots, e_n)$. Suppose furthermore that $D^n_G(e_1, \ldots, e_n)$ factorizes into a product of linear factors in $\alpha_{n+1}$, i.e., it is of the form $(a\alpha_{n+1} + b)(c\alpha_{n+1} + d)$. Then we define

$$D^{n+1}_G(e_1, \ldots, e_{n+1}) = \pm(ad - bc) \ ,$$

to be the resultant of the two factors of $D^n_G(e_1, \ldots, e_n)$. A graph $G$ for which the polynomials $D^n_G(e_1, \ldots, e_n)$ can be defined for all $n$ is called *denominator-reducible*. It can happen that $D^n_G(e_1, \ldots, e_n)$ vanishes. Then $G$ is said to have *weight-drop*.

For general graphs above a certain loop order and any ordering on their edges, there will come a point where $D^n_G(e_1, \ldots, e_n)$ is irreducible (typically for $n = 5$). Thus the generic graph is not denominator reducible. One can prove, as for the 5-invariant, that $D^n_G(e_1, \ldots, e_n)$ does not depend on the order of reduction of the variables, although it may happen that the intermediate terms $D^k_G(e_{i_1}, \ldots, e_{i_k})$ may factorize for some choices of orderings and not others.

## 3. The class of $X_G$ in the Grothendieck Ring of Varieties

Let $k$ be a field. The Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ is the free abelian group generated by isomorphism classes $[X]$, where $X$ is a separated scheme of finite type over $k$, modulo the inclusion-exclusion relation $[X] = [X\backslash Z] + [Z]$, where $Z \subset X$ is a closed subscheme. It has the structure of a commutative ring induced by the product relation $[X \times_k Y] = [X] \times [Y]$, with unit $1 = [\mathrm{Spec}\, k]$. One defines the Lefschetz motive $[\mathbb{L}]$ to be the class of the affine line $[\mathbb{A}^1]$.

**Remark 11.** *We only consider affine varieties here. If $f_1, \ldots, f_\ell \in k[\alpha_1, \ldots, \alpha_n]$ are polynomials, we denote by $[f_1, \ldots, f_\ell]$ the class in $K_0(\mathrm{Var}_k)$ of the intersection of the hypersurfaces $V(f_1) \cap \ldots \cap V(f_\ell)$ in affine space $\mathbb{A}^n(k)$. The dimension of the ambient affine space will usually be clear from the context.*

Let $G$ be a graph. Since the graph polynomial $\Psi_G$ (and, more generally, all Dodgson polynomials $\Psi^{I,J}_{G,K}$) is defined over $\mathbb{Z}$, we can view the element $[\Psi_G]$ in $K_0(\mathrm{Var}_k)$ for any field $k$. Most of the results below are valid in this generality. But at a certain point, we are obliged to switch to point-counting functions since we require the use of the Chevalley-Warning theorem (theorem 22). Recall that if $k$ is a finite field, the point-counting map:

$$\begin{array}{rcl} \# : K_0(\mathrm{Var}_k) & \rightarrow & \mathbb{Z} \\ [X] & \mapsto & \#X(k) \end{array}$$

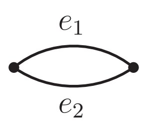
is well-defined, so results about the point-counts can be deduced from results in the Grothendieck ring, but not conversely (for example, it is not known if $\mathbb{L}$ is a zero-divisor). In this case, we shall denote by $[X]_q$ the point-counting function which associates to all prime powers $q$ the integers $\#X(\mathbb{F}_q)$.

3.1. **Linear reductions.** The main observation of [21] is that the class in the Grothendieck ring of polynomials which are linear in many of their variables can be computed inductively by some simple reductions.

**Lemma 12.** *Let $f^1, f_1, g^1, g_1 \in k[\alpha_2, \ldots, \alpha_n]$ denote polynomials of degree $\geq 1$.*
*i). $[f^1\alpha_1 + f_1] = [f^1, f_1]\,\mathbb{L} + \mathbb{L}^{n-1} - [f^1]$*
*ii). $[f^1\alpha_1 + f_1, g^1\alpha_1 + g_1] = [f^1, f_1, g^1, g_1]\,\mathbb{L} + [f^1 g_1 - g^1 f_1] - [f^1, g^1]$*

Various proofs of this lemma can be found in ([15], [5] §8, [21] lemma 2.3, or §3.4 of [1]). Note that the quantity $f^1 g_1 - g^1 f_1$ is nothing other than the resultant with respect to $\alpha_1$ of the polynomials $f^1\alpha_1 + f_1$ and $g^1\alpha_1 + g_1$.

Henceforth we assume that $G$ is a connected graph which is one-particle irreducible, and has no tadpoles (self-edges). We call a graph simple if it has no two-valent vertices (below left) or multiple edges (below right).


**Lemma 13.** *Let $G$ be a graph with a subdivided edge $e_1, e_2$ (left). Then*

$$[\Psi_G] = \mathbb{L}[\Psi_{G/\!\!/e_1}] \ . \tag{5}$$

*Let $G$ be a graph with a doubled edge $e_1, e_2$ (right). Then*

$$[\Psi_G] = (\mathbb{L}-2)[\Psi_{G\backslash e_1}] + (\mathbb{L}-1)[\Psi_{G\backslash \{e_1,e_2\}}] + \mathbb{L}[\Psi_{G\backslash e_1/\!\!/e_2}] + \mathbb{L}^{|N_G|-2} \ . \tag{6}$$

*Proof.* These identities follow from the determination of the corresponding graph polynomials §2.4 (1), (2) and two applications of lemma 12 (see also [1], §4). □

In [1], Aluffi and Marcolli give some general formulae for the classes of graphs obtained by successively subdividing or multiplying edges by (5) and (6).

**Lemma 14.** *Let $G$ be a connected graph such that $h_G \leq N_G - 2$. Then*

$$[\Psi_G] \equiv 0 \mod \mathbb{L}^2 \ .$$

*Proof.* First observe that if $F \in k[\alpha_1, \ldots, \alpha_n]$ is of degree $< n$ and is linear in every variable $\alpha_i$, then $[F] \equiv 0 \mod \mathbb{L}$. This follows immediately from lemma 12 *(i)* and induction on the degree, since $[F] \equiv [\frac{\partial}{\partial\alpha_1}F] \mod \mathbb{L}$ (compare theorem 22).

Now lemma 12 implies that $[\Psi_G] \equiv [\Psi^1_G, \Psi_{G,1}]\mathbb{L} - [\Psi^1_G] \mod \mathbb{L}^2$, and

$$[\Psi^1_G, \Psi_{G,1}] \equiv [\Psi^{1,2}_G] - [\Psi^1_{G\backslash 2}, \Psi_{G\backslash 2,1}] \mod \mathbb{L} \ ,$$

which follows from the Dodgson identity $\Psi^1_{G,2}\Psi^2_{G,1} - \Psi^{12}_G \Psi_{G,12} = (\Psi^{1,2}_G)^2$. By the remark above, $[\Psi^{1,2}_G]$ vanishes mod $\mathbb{L}$ since it is of degree $h_G - 1$ in $\mathbb{A}^{N_G-2}$, and so we can successively remove edges to deduce that

$$[\Psi^1_G, \Psi_{G,1}] \equiv \ldots \equiv [\Psi^1_{G\backslash\{2,\ldots,k\}}, \Psi_{G\backslash\{2,\ldots,k\},1}] \equiv \ldots \equiv 0 \mod \mathbb{L} \ . \tag{7}$$

Thus we have shown that $[\Psi_G] \equiv [\Psi_{G\backslash 1}] \mod \mathbb{L}^2$. Again by an induction removing edges we conclude that $[\Psi_G] \equiv 0 \mod \mathbb{L}^2$. □

**Definition 15.** Let $G$ be as above. It follows from the *proof* of the previous lemma that there exists $c_2(G) \in K_0(\mathrm{Var}_k)$ which is well-defined modulo $\mathbb{L}$, such that

$$[\Psi_G] \equiv c_2(G)\mathbb{L}^2 \mod \mathbb{L}^3 \ .$$

Below we give some simple formulae for $c_2(G)$ under various assumptions on $G$.

**Corollary 16.** *Suppose that $G$ has a 2-valent vertex. Then $c_2(G) \equiv 0 \mod \mathbb{L}$.*

*Proof.* By lemma 13, we can write $[\Psi_G] \equiv \mathbb{L}[\Psi_{G\backslash e}] \equiv 0 \mod \mathbb{L}^3$. □

3.2. **3-valent vertices.** Our approach to studying $[\Psi_G]$ uses the existence of a 3-valent vertex to simplify the calculations. If $G$ is simple (it suffices that $G$ has no two-valent vertices), then such a vertex exists whenever

$$N_G > 2h_G - 2 \ , \tag{8}$$

which holds for all primitively-divergent graphs. To see this, note that Euler's formula for a connected graph implies that $N_G - V_G = h_G - 1$. If $\alpha$ denotes the average degree of the vertices of $G$, then $N_G = \frac{\alpha}{2} V_G$, and (8) implies that $\alpha < 4$.

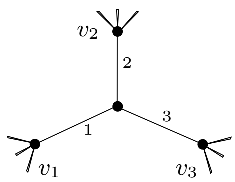



FIGURE 2. A three-valent vertex

The existence of a 3-valent vertex implies that $\Psi_G$ has a simple structure.

**Definition 17.** Let $v_1, v_2, v_3$ be any three vertices in $G$ which form a three-valent vertex as shown above. Following [7], we will use the notation:

$$f_{123} = \Psi_{G/\!\!/\{123\}} \ , \ f_1 = \Psi^{2,3}_{G,1} \ , \ f_2 = \Psi^{1,3}_{G,2} \ , f_3 = \Psi^{1,2}_{G,3} \ , \ f_0 = \Psi_{G\backslash\{1,2\}/\!\!/3} \ .$$

**Lemma 18.** *In this case, the graph polynomial of $G$ has the following structure:*

$$\Psi_G = f_0(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + (f_1 + f_2)\alpha_3 + (f_1 + f_3)\alpha_2 + (f_2 + f_3)\alpha_1 + f_{123}$$

*where the polynomials $f_i$ satisfy the equation*

$$f_0 f_{123} = f_1 f_2 + f_1 f_3 + f_2 f_3 \ . \tag{9}$$

*Proof.* The general shape of the polynomial comes from the contraction-deletion relations, and §2.4 (3) (or ex. 32 in [7]). Equation (9) is merely a restatement of the first Dodgson identity for $G/\!\!/3$ which gives $(\Psi^{1,2}_{G,3})^2 = \Psi^{12,12}_{G,3}\Psi_{G,123} - \Psi^{1,1}_{G,23}\Psi^{2,2}_{G,13}$. Using the definitions of $f_i$ this translates as $f_3^2 = f_0 f_{123} - (f_1 + f_3)(f_2 + f_3)$. □

**Proposition 19.** *Suppose that $G$ contains a three-valent vertex, and let $f_i$ be given by definition 17. Then*

$$[\Psi_G] = \mathbb{L}^{n-1} + \mathbb{L}^3[f_0, f_1, f_2, f_3, f_{123}] - \mathbb{L}^2[f_0, f_1, f_2, f_3]$$

*Proof.* Let $\beta_i = f_0\alpha_i + f_i$, for $i = 1, 2, 3$. It follows from (9) that

$$f_0\Psi_G = \beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 \ .$$

The right-hand side is the graph polynomial of a sunset diagram (graph with 2 vertices and 3 edges connecting both vertices). It defines a singular quadric in $\mathbb{A}^3$ whose class is $[\mathbb{A}^2]$. It follows that if $U$ denotes the open set $f_0 \neq 0$, we have $[X_G \cap U] = \mathbb{L}^2[U]$. On the complement $V(f_0)$, the graph polynomial $\Psi_G$ reduces to the equation

$$(f_1 + f_2)\alpha_3 + (f_1 + f_3)\alpha_2 + (f_2 + f_3)\alpha_1 + f_{123}$$

which defines a family of hyperplanes in $\mathbb{A}^3$. Thus, consider the fiber of the projection $X_G \cap V(f_0) \rightarrow \mathbb{A}^{n-3} \cap V(f_0)$. In the generic case this is a hyperplane whose class is $[\mathbb{A}^2]$. Otherwise, there are only two possibilities: either all the coefficients $f_1, f_2, f_3, f_{123}$ vanish and the fiber is isomorphic to $\mathbb{A}^3$, or $f_{123}$ is non-vanishing but the other coefficients vanish and the fiber is empty. We have

$$[X_G \cap V(f_0)] = \mathbb{L}^2[f_0] + \mathbb{L}^3[f_0, f_1, f_2, f_3, f_{123}] - \mathbb{L}^2[f_0, f_1, f_2, f_3]$$

Writing $[X_G] = [X_G \cap U] + [X_G \cap V(f_0)]$ gives the result. □

**Corollary 20.** *Suppose that $G$ has a 3-valent vertex. Then*

$$c_2(G) \equiv -[f_0, f_1, f_2, f_3] \mod \mathbb{L} \ .$$

**Lemma 21.** *Let $G$ satisfy $2h_G \leq N_G$ and contain a 3-valent vertex as above. Then*

$$c_2(G) \equiv [\Psi^{1,2}_{G,3}, \Psi^{13,23}_G] \mod \mathbb{L} \ . \tag{10}$$

*Proof.* Using the explicit expression for $\Psi_G$ in lemma 18, we have $\Psi^{1,2}_{G,3} = f_3$ and $\Psi^{13,23}_G = f_0$. It follows from (9) and inclusion-exclusion that:

$$[f_0, f_3] = [f_0, f_1f_2, f_3] = [f_0, f_1, f_3] + [f_0, f_2, f_3] - [f_0, f_1, f_2, f_3]$$

On the other hand $[f_0, f_1 + f_3] = [f_0, f_1, f_3]$ as can be seen by writing (9) in the form $f_0f_{123} = (f_1 + f_3)f_2 + f_1f_3$. By definition, $[f_0, f_1 + f_3] = [\Psi^{12}_{G,3}, \Psi^2_{G,13}]$, which by contraction-deletion is just $[\Psi^1_{G'}, \Psi_{G',1}]$, where $G' = G\backslash 2/\!\!/3$. By the identical argument as (7), this vanishes modulo $\mathbb{L}$. The same is true for $[f_0, f_1 + f_2]$ by symmetry. We have therefore shown that

$$[f_0, f_1, f_2, f_3] \equiv [\Psi^{1,2}_{G,3}, \Psi^{13,23}_G] \mod \mathbb{L} \ .$$

□

3.3. **Counting points over finite fields.** For any prime power $q$, let $\mathbb{F}_q$ denote the field with $q$ elements. Given polynomials $P_1, \ldots, P_\ell \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$, let

$$[P_1, \ldots, P_\ell]_q \in \mathbb{N}$$

denote the number of points on the affine variety $V(\overline{P_1}, \ldots, \overline{P_\ell}) \subset \mathbb{F}_q^n$, where $\overline{P}_i$ denotes the reduction of $P_i$ modulo $q$. Recall the Chevalley-Warning theorem (e.g., [19]) on the point counts of polynomials of small degrees.

**Theorem 22.** *Let $P_1, \ldots, P_\ell \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ such that $\sum_{i=1}^{\ell} \deg P_i < n$. Then*

$$[P_1, \ldots, P_\ell]_q \equiv 0 \mod q \ .$$

It is natural to ask if there exists a lifting of the Chevalley-Warning theorem to the Grothendieck ring of varieties. We were unable to find such a result in the literature so we state this as a conjecture.

**Conjecture 3.** *Let $P_1, \ldots, P_\ell$ be polynomials satisfying the above condition on their degrees. Then $[V(P_1, \ldots, P_\ell)] \equiv 0 \mod \mathbb{L}$ in $K_0(Var_k)$ for any field $k$.*

Since this conjecture is unavailable, we henceforth work with point-counting functions rather than elements in the Grothendieck ring of varieties. It turns out that for many of the results below, one can in fact circumvent this conjecture by elementary arguments. In any case, we now set

$$c_2(G)_q = [\Psi_G^{13,23}, \Psi_3^{1,2}]_q \mod q$$

viewed as a map from all prime powers $q$ to $\mathbb{F}_q$, where $1, 2, 3$ forms a 3-valent vertex. The same formula is also valid when $1, 2, 3$ forms a triangle (see remark 25 below). We have $[\Psi_G]_q \equiv q^2 c_2(G)_q \mod q^3$.

**Lemma 23.** *Suppose that $f = f^1 x + f_1$ and $g = g^1 x + g_1$ are polynomials in $\mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ such that $\deg f + \deg g = n$, which are linear in a variable $x$, and such that the resultant has a non-trivial factorization $f^1 g_1 - f_1 g^1 = ab$. Then*

$$[f, g]_q \equiv -[a, b]_q \mod q$$

*Proof.* It follows from lemma 12 that $[f, g]_q = q[f^1, f_1, g^1, g_1]_q + [ab]_q - [f^1, g^1]_q$. Since $f^1, g^1 \in \mathbb{Z}[\alpha_1, \ldots, \alpha_{n-1}]$ of degree $\deg(f^1) + \deg(g^1) = n-2$, this is congruent to $[ab]_q \bmod q$ by theorem 22. By inclusion-exclusion this is $[ab]_q = [a]_q + [b]_q - [a, b]_q$, and again by theorem 22, $[a]_q$ and $[b]_q$ vanish mod $q$, giving the statement. □

**Corollary 24.** *Let $G$ be a graph. If $2h_G < N_G$ then $c_2(G)_q \equiv 0 \mod q$. Otherwise, suppose that $D_G^n(e_1, \ldots, e_n)$ is the result of the denominator reduction after $n$ steps, where the $\{e_i\}$ contain the 3 edges meeting some 3-valent vertex. Then*

$$c_2(G)_q \equiv (-1)^{n-1} [D_G^n(e_1, \ldots, e_n)]_q \mod q \ .$$

*If $G$ has weight drop then $c_2(G)_q \equiv 0 \mod q$.*

*Proof.* Suppose that $2h_G < N_G$. The terms in (10) satisfy $\deg(\Psi_{G,3}^{1,2}) = h_G - 1$ and $\deg \Psi_G^{13,23} = h_G - 2$ giving total degree $2h_G - 3$, whereas the ambient affine space has dimension $N_G - 3$. The first statement therefore follows from theorem 22.

Applying the previous lemma to equation (10) gives

$$c_2(G)_q \equiv [\Psi_{G,3}^{1,2}, \Psi_G^{13,23}]_q \equiv -[\Psi_G^{13,24}, \Psi_G^{14,23}]_q \mod q$$

by the Dodgson identities. Applying the previous lemma one more time gives

$$c_2(G)_q \equiv [{}^5\Psi_G(1, 2, 3, 4, 5)]_q \mod q$$

by definition of the five-invariant as a resultant. The result then follows immediately from the previous lemma and the definition of the denominator reduction, noting that $[f, g]_q$ and $[fg]_q$ are equivalent modulo $q$ by the degrees and theorem 22. □

Notice that the proofs rely on the fact that the terms $D^n_G(e_1,\ldots,e_n)$ in the denominator reduction are of degree exactly equal to the dimension of the ambient space, and therefore lie on the limit of the Chevalley-Warning theorem (the Calabi-Yau condition for the associated projective varieties).

**Remark 25.** *The previous corollary probably holds without any restriction on the edges $\{e_1,\ldots,e_n\}$, but certainly can be proved when three of the edges $e_1,\ldots,e_n$ form a triangle. This follows from star-triangle duality (see* [7]*, example 35) since, with the notations therein, $\Psi_{G_\triangle}$ is obtained from $\Psi_{G_Y}(\alpha_1^{-1},\alpha_2^{-1},\alpha_3^{-1})\alpha_1\alpha_2\alpha_3$ by setting $f_0=f^{123}$ and $f_{123}=f^0$. By inclusion-exclusion (see, e.g.,* [21] *proposition 3.1), we can write $[\Psi_Y]$ as an alternating sum of classes $[\Psi_K]$ where $K$ is a subgraph of $G$ obtained by contracting/deleting edges $1,2,3$. Working modulo $q^3$, and using the assumptions on the degrees, only the term $[\Psi_{G_\triangle}]$ survives.*

## 4. Properties of the $c_2$-invariant

We state some known and conjectural properties of the $c_2$-invariant of a graph.

4.1. **Triviality of $c_2(G)$.** The following results follow from corollary 24.

**Lemma 26.** *If $G$ is not simple then $c_2(G)_q \equiv 0 \mod q$.*

*Proof.* If $G$ has a split edge $e_1,e_2$ or a doubled edge $e_1,e_2$, then any five-invariant ${}^5\Psi_G(i_1,\ldots,i_5)$ where $e_1,e_2\in\{i_1,\ldots,i_5\}$ necessarily vanishes ([7] lemma 90). □

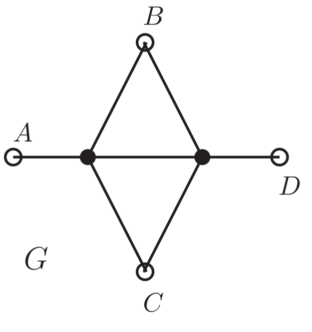
Recall that $G$ is called 2-vertex reducible if there is a pair of distinct vertices such that removing them (and their incident edges) causes the graph to disconnect.

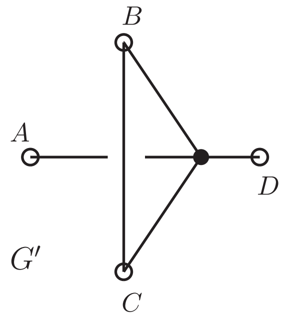
**Proposition 27.** *Let $G$ be 2-vertex reducible. Then $c_2(G)_q \equiv 0 \mod q$.*

*Proof.* It is proved in [9], proposition 36, that such a graph has weight drop. □

**Proposition 28.** *If $G$ is denominator reducible, and non-weight drop, then $c_2(G) \equiv (-1)^{N_G} \mod q$. If $G$ has weight drop then $c_2(G) \equiv 0 \mod q$.*

4.2. **Double triangle reduction.** Consider a graph $G$ which contains seven edges $e_1,\ldots,e_7$ arranged in the configuration shown below on the left (where anything may be attached to vertices $A$-$D$). The double triangle reduction of $G$ is the graph $G'$ obtained by replacing these seven edges with the configuration of five edges $e'_1,\ldots,e'_5$ as shown below on the right. The following theorem was proved in [9].




**Theorem 29.** *Let $G'$ be a double triangle reduction of $G$. Then*

$$D^7_G(e_1,\ldots,e_7) = \pm D^5_{G'}(e'_1,\ldots,e'_5)\ .$$

**Corollary 30.** *Let $G, G'$ be as above. Then $c_2(G)_q \equiv c_2(G')_q \mod q$.*

Since the double-triangle reduction violates planarity, this is the first hint that the genus of a graph is *not* the right invariant for understanding its periods.

4.3. **The completion relation.** It follows from a simple application of Euler's formula that a primitive-divergent graph $G$ in $\phi^4$ with more than 2 loops has exactly four 3-valent vertices $v_1, \ldots, v_4$, and all remaining vertices have valency 4. The completion of $G$ is defined to be the graph $\widehat{G}$ obtained by adding a new vertex $v$ to $G$ and connecting it to $v_1, \ldots, v_4$ [14]. The resulting graph is 4-regular.

**Conjecture 4.** *Let $G_1, G_2$ be two primitive divergent graphs in $\phi^4$ and suppose that $\widehat{G_1} \cong \widehat{G_2}$. Then $c_2(G_1)_q \equiv c_2(G_2)_q \mod q$ .*

The motivation for this conjecture comes from the result [14] that the corresponding residues are the same: $I_{G_1} = I_{G_2}$. Once again, the completion relation does not respect the genus of a graph.

## 5. Tate examples: Graphs of Vertex width 3

In the case when $G$ contains many triangles and three-valent vertices, we can find recurrence relations relating the class $[\Psi_G] \in K_0(\mathrm{Var}_k)$ to that of its minors.

5.1. **The vertex-width of a graph.** Throughout, $G$ is a connected graph.

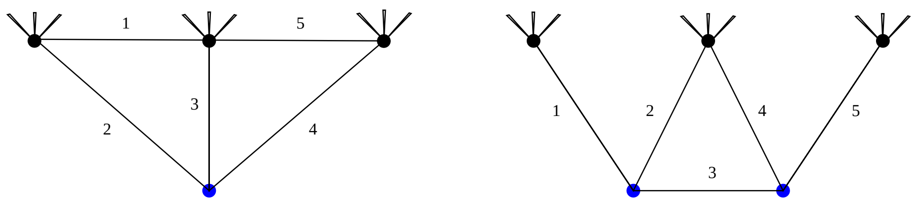
**Definition 31.** Let $\mathcal{O}$ be an ordering on the edges of $G$. It gives rise to a filtration

$$\emptyset = G_0 \subset G_1 \ldots \subset G_{N-1} \subset G_N = G$$

of subgraphs of $G$, where $G_i$ has exactly $i$ edges. To any such sequence we obtain a sequence of integers $v_i^{\mathcal{O}} =$ number of vertices of $G_i \cap (G \backslash G_i)$. We say that $G$ has *vertex-width* at most $n$ if there exists an ordering $\mathcal{O}$ such that $v_i^{\mathcal{O}} \leq n$ for all $i$ [7].

For example, a row of boxes with vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ and edges $\{a_i, b_i\}$, $\{a_i, a_{i+1}\}$, $\{b_i, b_{i+1}\}$, has vertex width two. The wheels and zig-zag graphs have vertex width 3. Bounding the vertex width is a very strong constraint on a graph, and one can show that the set of planar graphs have arbitrarily high vertex width.

Consider the local structure of a graph of vertex width 3 by choosing a pair $G_i$, $G \backslash G_i$, and choosing a minor of $G_i$ which has 5 edges. The trivial case is when the five edges are not simple (i.e., there is a two-valent vertex or doubled edge as in lemma 13). The only two cases which are simple are pictured below:




We refer to the left-hand side as a split triangle, the right as a split vertex. The two situations are dual to each other in a certain sense. The reason that these figures are interesting is that they force the corresponding five-invariants to be 'trivially' split. We claim the following criterion is sufficient for $G$ to be Tate.

**Theorem 32.** *If $G$ has vertex width at most 3, then $[\Psi_G]$ is a polynomial in $\mathbb{L}$.*

In [7] it was shown by a geometric argument that the relative cohomology of the graph hypersurface for such (and slightly more general) classes of graphs gives rise to mixed Tate motives, and that the periods evaluate to multiple polylogarithms. The aim here is to show how the polynomial in $\mathbb{L}$ can be computed via explicit recurrence relations. There are two situations to study: the case of split vertices and split triangles. We shall only consider the former in detail, and this is enough to derive formulae for the wheels and zig-zags explicitly.

5.2. **Split Vertices.** We first recall the terminology and introduce some notations relating to 3-valent vertices §3.2 .

**Definition 33.** Let $e_1, e_2, e_3$ be any three edges in $G$ which form a three-valent vertex. If $f_0, f_1, f_2, f_3, f_{123}$ are given by definition 17, we set

$$\langle G \rangle_{e_1,e_2,e_3} = [f_0, f_1, f_2, f_3, f_{123}] \ . \tag{11}$$

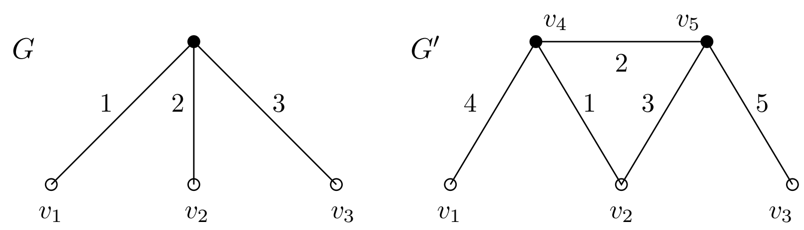



Figure 3. If $G$ is any graph containing a 3-valent vertex (left), $G'$ is the graph obtained by splitting that vertex in two (right).

Now let us consider the graph $G'$ obtained by splitting a 3-valent vertex $G$ in two, with the numbering of its edges as pictured above.

**Theorem 34.** *The class of the graph polynomial of $G'$ can be written explicitly in terms of the invariant $\langle G \rangle_{1,2,3}$, and the classes of minors of $G'$:*

$$[\Psi_{G'}]+(\mathbb{L}-\mathbb{L}^2)\big([\Psi^2_{G',45}]-[\Psi^{13}_{G,2}]\big)+(\mathbb{L}-1)[\Psi_G] = (\mathbb{L}^5-\mathbb{L}^4)\langle G\rangle_{1,2,3}+\mathbb{L}^{|G|-2}(\mathbb{L}^3+\mathbb{L}-1)$$

*Proof.* The structure of the graph polynomial of $G'$ can be obtained as follows. Since $v_4$ is a three-valent vertex in $G'$, it follows that $\Psi_{G'}$ must be of the shape given in lemma 18, for some polynomials $f'_0, f'_1, f'_2, f'_4, f'_{124}$ relative to the edges $1, 2, 4$. By contraction-deletion relations, one easily sees that

$$f'_0 = f_0(\alpha_3+\alpha_5)+(f_2+f_3) \quad , \quad f'_{124} = f_{123}\alpha_3 + (f_1+f_2)\alpha_3\alpha_5$$

$$f'_1 = f_2\alpha_3 \quad , \quad f'_2 = f_3\alpha_3 + f_0\alpha_3\alpha_5 \quad , \quad f'_4 = f_{123} + f_1\alpha_3 + (f_1+f_2)\alpha_5 \ ,$$

where $f_0, f_1, f_2, f_3, f_{123}$ satisfy (9). By proposition 19 we know that $[\Psi_{G'}]$ is given by $\mathbb{L}^3[f'_0, f'_1, f'_2, f'_4, f'_{124}] + \mathbb{L}^{n-1} - \mathbb{L}^2[f'_0, f'_1, f'_2, f'_4]$. The conclusion of the theorem follows by a brute force calculation by exploiting the inclusion-exclusion relations, identities (9), and reducing out the linear variables $\alpha_3, \alpha_5$ using lemma 12 *(ii)*. □

Any inductive procedure to compute the class of a split-vertex graph $G'$ is blocked by the presence of the invariant $\langle G \rangle$. However, it satisfies a recurrence of its own.

**Proposition 35.** *Let $G'$ be as above. Then*

$$\mathbb{L}^3\langle G'\rangle_{1,2,4} = (\mathbb{L}^5 - \mathbb{L}^4)\langle G\rangle_{1,2,3} + \mathbb{L}^2[\Psi^2_{G',45}] + \mathbb{L}^2[\Psi^{12}_{G',45}] - \mathbb{L}^{|G'|-2}$$

*Proof.* The proof is as in the previous theorem: compute the reduction with respect to $\alpha_3, \alpha_5$ of $\mathbb{L}^3[f'_0, f'_1, f'_2, f'_4, f'_{124}]$ in terms of $f_0, f_1, f_2, f_3, f_{123}$, giving:

$$\mathbb{L}^5[f_0, f_1, f_2, f_3, f_{123}] - \mathbb{L}^3[f_0, f_1, f_2] + \mathbb{L}^3[f_2]$$

This is exactly what one obtains from the right-hand side. Alternatively, the proof also follows from lemmas 38 and 13. □

Modulo $\mathbb{L}^4$, the invariant $\langle G\rangle$ drops out altogether.

**Corollary 36.** *Suppose that $|G| \geq 6$, and write $[\Psi_G] \equiv c_3(G)\mathbb{L}^3 + c_2(G)\mathbb{L}^2 \mod \mathbb{L}^4$. Then $c_2(G') \equiv c_2(G) \mod \mathbb{L}$ and*

$$c_3(G') - c_3(G) \equiv c_2(G\backslash\{13\}/\!\!/2) - c_2(G'\backslash 2/\!\!/\{45\}) - c_2(G) \mod \mathbb{L}$$

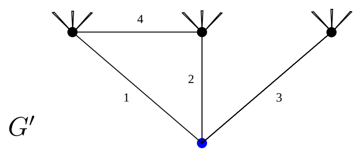
*Proof.* This follows from theorem 34 and lemma 14. □

Iterating this corollary, we see that if the minors $G\backslash\{13\}/\!\!/2$ and $G'\backslash 2/\!\!/\{45\}$ are denominator reducible, then by proposition 28 we deduce a simple recurrence relation relating $c_3(G')$ and $c_3(G)$. Thus, loosely speaking, $c_3$ is related to the number of split vertices (or triangles) for a certain class of denominator-reducible graphs (see in particular the wheels and zig-zag examples below).

5.3. **Split triangles.** By similar considerations one can also write down relations for the case of a split triangle. These can also be deduced from the case of a split triangle above by duality. Since the formulae are rather more complicated[1], we do not write them down.

**Proposition 37.** *Let $G'$ be a split triangle as depicted earlier, with marked edges $1,2,3,4,5$, where $2,3,4$ forms a 3-valent vertex and $1,2,3$ and $3,4,5$ are triangles. Let $G = G'\backslash\{1,5\}$. Then $[\Psi_{G'}]$ is a linear expression in $[\Psi_H]$ where $H$ are non-trivial minors of $G'$, and $\langle G\rangle_{2,3,4}$, with coefficients which are polynomials in $\mathbb{L}$.*

Thus the classes $[\Psi_G]$ can always be computed in terms of smaller graphs, provided that the same is true of $\langle G\rangle$. This is guaranteed by the following lemma, which covers the split triangle and split vertex cases simultaneously.




**Lemma 38.** *Let $G'$ be as indicated above, with edges $1,2,3$ forming a 3-valent vertex and edges $1,2,4$ forming a triangle, and let $G = G'\backslash\{4\}$. Then*

$$\mathbb{L}\langle G'\rangle_{1,2,3} = (\mathbb{L}^2 - \mathbb{L})\langle G\rangle_{1,2,3} + [\Psi^3_{G,2}] + [\Psi^{13}_{G,2}] - \mathbb{L}^{|G|-2}\ .$$

[1]The reason for this apparent asymmetry is due to the choice of ambient space for the point-counts: a more symmetric approach would involve counting in $(\mathbb{A}^1\backslash 0)^n$ since this is invariant under inversions of variables $\alpha \mapsto \alpha^{-1}$ and puts sub and quotient graphs on an equal footing.

*Proof.* Since $G'$ has a 3-valent vertex, $\Psi_{G'}$ has the general shape given by lemma 18 with coefficients $f'_0, f'_1, f'_2, f'_3, f'_{123}$ where, by contraction-deletion:

$$f'_0 = f_0\alpha_4\ , \quad f'_1 = f_1\alpha_4\ , \quad f'_2 = f_2\alpha_4\ , \quad f'_3 = f_0 + f_3\alpha_4\ , \quad f'_{123} = (f_1+f_2)+f_{123}\alpha_4\ ,$$

and $f_0, f_1, f_2, f_3, f_{123}$ are the corresponding structure constants for $G$. Obviously:

$$[f'_0, f'_1, f'_2, f'_3, f'_{123}] = (\mathbb{L}-1)[f_0, f_1, f_2, f_3, f_{123}] + [f_0, f_1 + f_2]$$

which is immediate on considering the two cases $\alpha_4 = 0$ and $\alpha_4 \neq 0$. By definition 17, $[f_0, f_1 + f_2] = [\Psi^1_H, \Psi_{H,1}]$ where $H = G\backslash 3/\!\!/2$. Conclude by lemma 12 *(i)*. □

**Corollary 39.** *For any graph of vertex width $\leq 3$, the class $[\Psi_G]$ is a polynomial in $\mathbb{L}$ which can be computed inductively using the previous results.*

*Proof.* By the discussion in §5.1, the local structure of a 5-edge minor in a graph of vertex width 3 is either non-simple, or is a split-vertex or split-triangle. In the latter cases the invariants $[\Psi_G]$ and $\langle G\rangle_{e_1,e_2,e_3}$, where $e_1, e_2, e_3$ form a 3-valent vertex, are expressible in terms of $[\Psi_H]$ or $\langle H\rangle_{e'_1,e'_2,e'_3}$, where $H$ is a strict minor of $G$. In the non-simple case, the same is true by lemma 13 and §2.4. Since the vertex width is minor-monotone, the result is true by induction. □

5.4. **Example 1: wheels with $n$ spokes.** We use the previous results to compute the classes $[W_n]$ for all $n$, where $W_n$ denotes the wheel with $n$ spokes graph pictured below (left). Let $B_n$ denote the family of graphs obtained by contracting a spoke of $W_n$, which have exactly $n$ vertices on the outer circle (right).

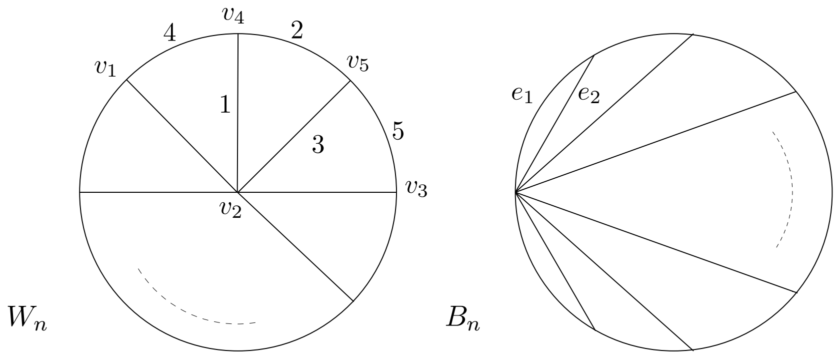



FIGURE 4. The wheels with spokes graphs $W_n$, and a related family $B_n$ of series-parallel graphs.

The graphs $B_n$ are series-parallel reducible, so the classes $[B_n]$ can be computed using lemma 13. This also follows from the results of [1], theorem 5.10.

**Lemma 40.** *Let us set $b_0 = 0$, $b_1 = 1$, and $b_n = [B_n]$ for $n \geq 2$. If $B(t) = \sum_{n\geq 0} b_n t^n$ is the generating series for the family of graphs $B_n$, then we have*

$$B(t) = \frac{t(1 + \frac{\mathbb{L}t}{1-\mathbb{L}^2 t})}{1 - (\mathbb{L}-1)(\mathbb{L}t + \mathbb{L}^2 t^2)}\ . \tag{12}$$

*Proof.* We refer to the two edges $e_1, e_2$ indicated on the diagram above. Since $e_1, e_2$ form a doubled edge, we have by (6):

$$[B_n] = (\mathbb{L}-2)[B_n\backslash e_1] + (\mathbb{L}-1)[B_n\backslash\{e_1, e_2\}] + \mathbb{L}[B_n\backslash e_1/\!\!/e_2] + \mathbb{L}^{2n-3}$$

since $B_n$ has $2n-1$ edges. Now $B_n\backslash e_1$ is isomorphic to the graph obtained from $B_{n-1}$ by subdividing an outer edge, so $[B_n\backslash e_1] = \mathbb{L}[B_{n-1}]$ by (5). The graph $B_n\backslash\{e_1,e_2\}$ has an external leg, which can be supressed, leaving, as before, a copy of $B_{n-2}$ with a subdivided outer edge. Thus $[B_n\backslash\{e_1,e_2\}] = \mathbb{L}^2[B_{n-2}]$. Finally, we have $B_n\backslash e_1/\!\!/e_2 \cong B_{n-1}$, so we obtain

$$[B_n] = \mathbb{L}(\mathbb{L}-2)[B_{n-1}] + \mathbb{L}^2(\mathbb{L}-1)[B_{n-2}] + \mathbb{L}[B_{n-1}] + \mathbb{L}^{2n-3}$$

We deduce that for all $n \geq 4$ we have:

$$b_n = \mathbb{L}(\mathbb{L}-1)b_{n-1} + \mathbb{L}^2(\mathbb{L}-1)b_{n-2} + \mathbb{L}^{2n-3}\ . \tag{13}$$

The constants $b_0, b_1$ are chosen such that the equation is valid for $n = 2, 3$, where $b_2 = \mathbb{L}^2$ and $b_3 = \mathbb{L}^2(\mathbb{L}-1+\mathbb{L}^2)$ by direct computation. The formula for the generating series then follows immediately from the recurrence relation (13). □

One has $b_2 = \mathbb{L}^2$, and

$$b_3 = \mathbb{L}^2(\mathbb{L}^2+\mathbb{L}-1) \quad , \quad b_4 = \mathbb{L}^3(\mathbb{L}^3+2\mathbb{L}^2-3\mathbb{L}+1)$$
$$b_5 = \mathbb{L}^5(\mathbb{L}^3+3\mathbb{L}^2-5\mathbb{L}+2) \quad , \quad b_6 = \mathbb{L}^5(\mathbb{L}^5+4\mathbb{L}^4-7\mathbb{L}^3+2\mathbb{L}^2+2\mathbb{L}-1)\ .$$

Let $v_1, v_2, v_5$ denote any three vertices on $W_n$, joined by a 3-valent vertex ($v_4$) as shown in the diagram above. Let us write $\langle W_n\rangle = \langle W_n\rangle_{v_1,v_2,v_5}$.

**Lemma 41.** *Let $\widehat{w}_n = 0$ for $n \leq 2$ and set $\widehat{w}_n = \langle W_n\rangle$ for $n \geq 3$. Denote the corresponding ordinary generating series by $\widehat{W}(t) = \sum_{n\geq 0}\widehat{w}_n t^n$. Then*

$$\widehat{W}(t) = \frac{t(1+\mathbb{L}t)\,B(t) + \frac{t^2}{\mathbb{L}^2t-1}}{\mathbb{L}-\mathbb{L}^2(\mathbb{L}-1)t}\ . \tag{14}$$

*Proof.* We use proposition 35, applied to the graphs $G' = W_n$ with the edge and vertex labels on $G'$ as shown above. Then $G \cong G'\backslash 1/\!\!/2 \cong W_{n-1}$. We have

$$\mathbb{L}^3\,\widehat{w}_n = (\mathbb{L}^5-\mathbb{L}^4)\,\widehat{w}_{n-1} + \mathbb{L}^2[\Psi^2_{G',45}] + \mathbb{L}^2[\Psi^{12}_{G',45}] - \mathbb{L}^{2n-2}$$

since $W_n$ has $2n$ edges. Now $G'\backslash 2/\!\!/\{4,5\}$ is isomorphic to $B_{n-1}$ and $G'\backslash\{1,2\}/\!\!/\{4,5\}$ gives the graph obtained from $B_{n-2}$ by subdividing one outer edge. Therefore $[\Psi^{12}_{G',45}] = \mathbb{L}[B_{n-2}]$ by lemma 13. We deduce that for all $n \geq 4$,

$$\mathbb{L}^3\,\widehat{w}_n = (\mathbb{L}^5-\mathbb{L}^4)\,\widehat{w}_{n-1} + \mathbb{L}^2 b_{n-1} + \mathbb{L}^3 b_{n-2} - \mathbb{L}^{2n-2}\ . \tag{15}$$

Using the fact that $\widehat{w}_3 = 1$ determines $\widehat{w}_n$ for $n = 0,1,2$. The formula for the generating series $\widehat{W}(t)$ then follows immediately from (15). □

**Proposition 42.** *Let $w_1 = q$, $w_2 = \mathbb{L}^3$, and $w_n = [W_n]$ for $n \geq 3$. Let $W(t) = \sum_{n\geq 0} w_n t^n$ be the generating function for the wheels with spokes graphs. Then*

$$W(t) = \mathbb{L}t\frac{(\mathbb{L}^4-\mathbb{L}^3)\,\widehat{W}(t) + (\mathbb{L}-1)(1-\mathbb{L}^2t^2)B(t) + \frac{(\mathbb{L}^2t^2-\mathbb{L}t^2+1)}{(1-\mathbb{L}^2t)}}{1+(\mathbb{L}-1)t} \tag{16}$$

*where $B(t)$, $\widehat{W}(t)$ are defined above.*

*Proof.* We apply theorem 34 to the graph $G' = W_n$ with the labelling on its edges depicted above. Since $G \cong W_{n-1}$, we deduce for all $n \geq 4$ that

$$[W_n] + (\mathbb{L}-\mathbb{L}^2)\big([\Psi^2_{G',45}] - [\Psi^{123}_{G',45}]\big) + (\mathbb{L}-1)[W_{n-1}] = (\mathbb{L}^5-\mathbb{L}^4)\langle W_{n-1}\rangle + \mathbb{L}^{2n-4}(\mathbb{L}^3+\mathbb{L}-1)$$

As before, $G'\backslash 2/\!\!/\{4,5\} \cong B_{n-1}$, and $G\backslash\{1,3\}/\!\!/2$ is isomorphic to the graph obtained from $B_{n-3}$ by subdividing two outer edges. It follows that $[\Psi^{13}_{G,2}] = \mathbb{L}^2[B_{n-3}]$, giving

$$w_n + (\mathbb{L}-1)w_{n-1} + (\mathbb{L}-\mathbb{L}^2)\big(b_{n-1} - \mathbb{L}^2 b_{n-3}\big) = (\mathbb{L}^5-\mathbb{L}^4)\widehat{w}_{n-1} + \mathbb{L}^{2n-4}(\mathbb{L}^3+\mathbb{L}-1)$$

The formula for the generating function follows from this. □

**Corollary 43.** *Let $c_i(W_n)$ denote the coefficient of $\mathbb{L}^i$ in $[W_n]$. Then $c_2(W_n) = -1$, $c_{2n-1}(W_n) = 1$ and $c_{2n-2}(W_n) = 0$ for all $n$. The outermost non-trivial coefficients are $c_3(W_3) = 1$, and $c_3(W_n) = n$ for all $n \geq 4$, and $c_{2n-3}(W_n) = \binom{n}{2}$ for all $n \geq 3$.*

The following curious identity follows from the explicit description of $W(t)$:

$$[W_n] - [W_n\backslash O] - [W_n/\!\!/O] + [W_n/\!\!/I] = -\mathbb{L}^2(\mathbb{L}-1)^{n-2}\ ,$$

where $O$ denotes any outer edge of $W_n$ (on the rim of the wheel), and $I$ denotes any internal edge or spoke. The combinatorial reason for this is not clear.

**Remark 44.** *The polynomials $w_n$ should have explicit equivariant versions with respect to the symmetry group of $W_n$. This would be relevant to computing the full cohomology of the graph hypersurface complement of $W_n$ and its motivic coproduct.*

The first few values of the polynomials $w_n$ are as follows:

$$\begin{aligned}
w_3 &= \mathbb{L}^2(\mathbb{L}^3+\mathbb{L}-1)\\
w_4 &= \mathbb{L}^2(\mathbb{L}^5+3\mathbb{L}^3-6\mathbb{L}^2+4\mathbb{L}-1)\\
w_5 &= \mathbb{L}^2(\mathbb{L}^7+6\mathbb{L}^5-15\mathbb{L}^4+16\mathbb{L}^3-11\mathbb{L}^2+5\mathbb{L}-1)\\
w_6 &= \mathbb{L}^2(\mathbb{L}^9+10\mathbb{L}^7-29\mathbb{L}^6+37\mathbb{L}^5-33\mathbb{L}^4+26\mathbb{L}^3-16\mathbb{L}^2+6\mathbb{L}-1)\\
w_7 &= \mathbb{L}^2(\mathbb{L}^{11}+15\mathbb{L}^9-49\mathbb{L}^8+71\mathbb{L}^7-70\mathbb{L}^6+64\mathbb{L}^5-57\mathbb{L}^4+42\mathbb{L}^3-22\mathbb{L}^2+7\mathbb{L}-1)
\end{aligned}$$

Note that the wheels $W_n$ are the unique infinitely family of graphs whose residue is known explicitly, namely: $I_{W_n} = \binom{2n-1}{n-1}\zeta(2n-3)$ for $n \geq 3$. One of the main results of [5] is that

$$H^{2n-1}_c(\mathbb{P}^{2n-1}\backslash X_{W_n}) \cong \mathbb{Q}(-2) \tag{17}$$

and that $H^{2n-1}(\mathbb{P}^{2n-1}\backslash X_{W_n})$ is generated by the integrand of $I_G$. It would be interesting to relate their proof to the above computation which gives $c_2(W_n) = -1$.

5.5. **Example 2: Zig-zag graphs.** The second application of the previous results is to compute the classes $[Z_n]$ for all $n$, where $Z_n$ denotes the family of zig-zag graphs with $n$ loops pictured below (left). Let $\overline{Z}_n$ denote the family of graphs obtained by doubling the edge '2' as shown on the right. Note that $Z_3 = W_3$.

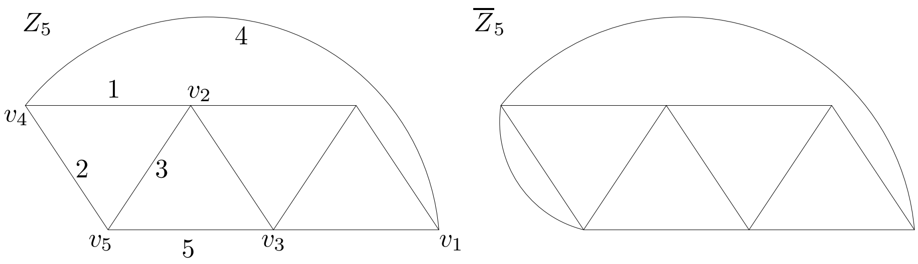



FIGURE 5. The zig-zag graphs $Z_n$.

The graphs $Z_n$ are primitive-divergent graphs in $\phi^4$ theory for all $n \geq 3$. Let $z_0 = 0$, $z_1 = \mathbb{L}+1$, $z_2 = \mathbb{L}^3$, and $z_n = [Z_n]$ for all $n \geq 3$. Likewise, set $\overline{z}_0 = 1$, $\overline{z}_1 = \mathbb{L}^2$, $\overline{z}_2 = \mathbb{L}^4+\mathbb{L}^3-\mathbb{L}^2$, and $\overline{z}_n = [\overline{Z}_n]$ for all $n \geq 3$. Denote the corresponding generating series by $Z(t)$ and $\overline{Z}(t)$.

**Lemma 45.** *A straightforward application of the series-parallel operations gives*

$$\overline{z}_n = (\mathbb{L}-2)z_n + (\mathbb{L}-1)\mathbb{L}^2\overline{z}_{n-2} + \mathbb{L}\overline{z}_{n-1} + \mathbb{L}^{2n-1} \quad n \geq 1 \tag{18}$$

*Proof.* If $e_1, e_2$ denote the two doubled edges, then this follows from the parallel reduction (6) on noting that $\overline{Z}_n \backslash e_1 \cong Z_n$, $\overline{Z}_n \backslash e_1 /\!\!/ e_2 \cong \overline{Z}_{n-1}$, and that $\overline{Z}_n \backslash \{e_1, e_2\}$ is isomorphic to the graph obtained from $Z_{n-2}$ by subdividing two edges, whose class is $\mathbb{L}^2 z_{n-2}$ by two applications of (5). □

We next want to compute recursion relations for the numbers $z_n$ by considering the split vertex shown above in the figure (left). Let us set $\widehat{z}_n = 0$ for $n < 3$, $\widehat{z}_n = \langle Z_n \rangle_{v_1,v_2,v_5}$ for all $n \geq 3$, and let $\widehat{Z}(t)$ be the corresponding generating series. Let $ZB_n$ denote the family of graphs obtained from $Z_{n-1}$ by deleting the edge 4 indicated in the figure above, and doubling edges 1 and 2. A trivial argument along the lines of lemma 40 shows that $\langle ZB_n \rangle = b_n$, with generating series $B$.

**Lemma 46.** *The recurrence relation given in theorem* 34 *translates as:*

$$z_n + (\mathbb{L}-\mathbb{L}^2)(\overline{z}_{n-2} - \mathbb{L}^2 b_{n-3}) + (\mathbb{L}-1)z_{n-1} = (\mathbb{L}^5-\mathbb{L}^4)\widehat{z}_{n-1} + \mathbb{L}^{2n-4}(\mathbb{L}^3+\mathbb{L}-1) \tag{19}$$

*for $n \geq 2$. The recurrence relation of proposition* 35 *yields the relation*

$$\mathbb{L}^3\widehat{z}_n = (\mathbb{L}^5-\mathbb{L}^4)\widehat{z}_{n-1} + \mathbb{L}^2\overline{z}_{n-2} + \mathbb{L}^3 b_{n-2} - \mathbb{L}^{2n-2}\ , \qquad n \geq 2 \tag{20}$$

*Proof.* Let $G' = Z_n$, and apply theorem 34 to $G'$ with the edge numbering shown above. Then $G \cong Z_{n-1}$, $G' \backslash 2 /\!\!/ \{4,5\} \cong \overline{Z}_{n-2}$, and $G \backslash \{1,3\} /\!\!/ 2$ is isomorphic to the graph obtained from $ZB_{n-3}$ by subdividing two edges. It follows from (5) that $[\Psi_{G\backslash\{1,3\}/\!\!/2}] = \mathbb{L}^2 b_{n-3}$, which yields the first equation. The second equation follows from proposition 35, since $G' \backslash \{1,2\} /\!\!/ \{4,5\}$ is isomorphic to the graph obtained from $ZB_{n-2}$ by subdividing one edge, whose polynomial is $\mathbb{L}b_{n-2}$ by (5). □

Equations (18), (19), (20) imply the following identities of generating series:

$$\begin{aligned}
[1-\mathbb{L}t+\mathbb{L}^2(1-\mathbb{L})t^2]\,\overline{Z} - (\mathbb{L}-2)Z - tR \;&=\; 1+(2-\mathbb{L})t \\
[\mathbb{L}^3 - (\mathbb{L}^5-\mathbb{L}^4)t]\,\widehat{Z} - \mathbb{L}^2t^2\big(\overline{Z}+\mathbb{L}B\big) + R \;&=\; t \\
[1+(\mathbb{L}-1)t]Z - (\mathbb{L}^5-\mathbb{L}^4)t\widehat{Z} + (\mathbb{L}-\mathbb{L}^2)t^2(\overline{Z}-\mathbb{L}^2tB) - (\mathbb{L}^3+\mathbb{L}-1)tR \;&=\; (\mathbb{L}+1)t
\end{aligned}$$

in three unknowns, $Z$, $\overline{Z}$, and $\widehat{Z}$, where $R = R(t) = \frac{t}{(1-\mathbb{L}^2t)}$. These equations are easily solved using the expression for $B$ (12). In particular, we obtain an explicit formula for the generating series $Z(t)$ for the zig-zag graphs, which we omit. This is to our knowledge the only explicit formula for the class in the Grothendieck ring of a family of primitive-divergent graphs in $\phi^4$. From this formula one obtains:

**Corollary 47.** *Let $c_i(Z_n)$ denote the coefficient of $\mathbb{L}^i$ in $[Z_n]$. Then $c_2(Z_n) = -1$, $c_{2n-1}(Z_n) = 1$ and $c_{2n-2}(Z_n) = 0$ for all $n$. The outermost non-trivial terms are $c_3(Z_3) = 1$, and $c_3(Z_n) = 8-n$ for all $n \geq 4$, and $c_{2n-3}(W_n) = 2n-5$ for $n \geq 3$.*

In the case of the zig-zags, the analogous result to (17) was proved by Doryn in his thesis [10], and states that

$$H_c^{2n-1}(\mathbb{P}^{2n-1} \backslash X_{Z_n}) \cong \mathbb{Q}(-2)\ .$$

For small $n$, we have:

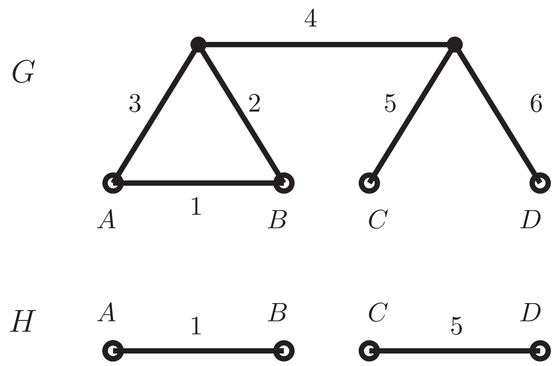
$$\begin{aligned}
z_3 &= \mathbb{L}^2(\mathbb{L}^3+\mathbb{L}-1)\\
z_4 &= \mathbb{L}^2(\mathbb{L}^5+3\mathbb{L}^3-6\mathbb{L}^2+4\mathbb{L}-1)\\
z_5 &= \mathbb{L}^2(\mathbb{L}^7+5\mathbb{L}^5-10\mathbb{L}^4+7\mathbb{L}^3-4\mathbb{L}^2+3\mathbb{L}-1)\\
z_6 &= \mathbb{L}^2(\mathbb{L}^9+7\mathbb{L}^7-12\mathbb{L}^6-2\mathbb{L}^5+16\mathbb{L}^4-12\mathbb{L}^3+2\mathbb{L}^2+2\mathbb{L}-1)\\
z_7 &= \mathbb{L}^2(\mathbb{L}^{11}+9\mathbb{L}^9-13\mathbb{L}^8-18\mathbb{L}^7+55\mathbb{L}^6-58\mathbb{L}^5+41\mathbb{L}^4-23\mathbb{L}^3+7\mathbb{L}^2+\mathbb{L}-1)
\end{aligned}$$

## 6. Non-Tate counter-examples at 8 loops

We use the denominator-reduction method to derive some non-Tate counter-examples to Kontsevich's conjecture at 8 and 9 loops.

6.1. **Combinatorial reductions.** In order to compute the $c_2$ invariant of an 8-loop graph $G$, we proceed in two simpler steps.

Suppose that $G$ is any connected graph with the shape depicted below, where the white vertices $A, B, C, D$ may have anything attached to them. Let $H$ be the minor obtained from $G$ by deleting the edges 2 and 4, and contracting 3 and 6.




**Lemma 48.** *Let $G, H$ be as above. Then $D^6_G(1,2,3,4,5,6) = \pm\Psi^{1,5}_H\Psi^5_{H,1}$.*

*Proof.* The proof is by direct computation of resultants, using the identities between Dodgson polynomials which follow from the existence of local stars and triangles. Since the edges $\{1,2,3\}$ form a triangle, we know from example 9 that

$${}^5\Psi_G(1,2,3,4,5) = \pm\Psi^{123,245}\Psi^{4,5}_{G\backslash 2/\!\!/13}\ .$$

Since $\{2,3,4\}$ forms a 3-valent vertex, we have $\Psi^{123,245} = \Psi^{1,5}_{G\backslash\{2,4\}/\!\!/3}$ by the last equation in §2.3, (3). By contraction-deletion, this last term is also $\Psi^{14,45}_{G\backslash 2/\!\!/3}$, giving

$${}^5\Psi_G(1,2,3,4,5) = \pm\Psi^{14,45}_{G_2}\Psi^{4,5}_{G_2,1}\ ,$$

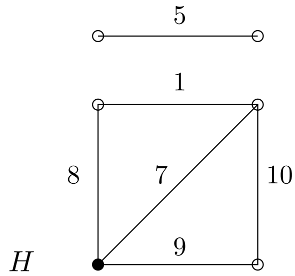
where $G_2$ is the minor $G\backslash 2/\!\!/3$ with the induced numbering of its edges. Now take the resultant with respect to edge 6. Since $\{4,5,6\}$ forms a 3-valent vertex in $G_2$, it follows that $\Psi^{146,456}_{G_2} = 0$ by the vanishing property for vertices. Thus we have

$$[\Psi^{14,45}_{G_2}, \Psi^{4,5}_{G_2,1}]_6 = \pm\Psi^{14,45}_{G_2,6}\Psi^{46,56}_{G_2,1}\ .$$

Again, since $\{4,5,6\}$ is a 3-valent vertex, $\Psi^{46,56}_{G_2,1} = \Psi^{46,56}_{G_2/\!/1} = \Psi^{45}_{G_2/\!/1,6} = \Psi^{5}_{G_2\backslash 4/\!/6,1}$, where the first and third equality are contraction-deletion relations. We have:

$$[\Psi^{14,45}_{G_2}, \Psi^{4,5}_{G_2,1}]_6 = \pm \Psi^{1,5}_{G_2\backslash 4/\!/6} \Psi^{5}_{G_2\backslash 4/\!/6,1}$$

The left-hand side is equal to $\pm D^6_G(1,2,3,4,5)$ by definition, and the minor $G_2\backslash 4/\!/6$ is exactly $H$, which completes the proof. □




**Lemma 49.** *Let $H$ be a graph with the general shape depicted above. The denominator reduction, applied five times to $\Psi^{1,5}_H \Psi^1_{H,5}$ with respect to the edges $7,8,9,10$ is $\Psi^{15,78}_A \Psi_B$, where $A = H\backslash\{10\}/\!/9$ and $B = G\backslash\{1,9,7\}/\!/\{5,8,10\}$.*

*Proof.* By the second Dodgson identity, $[\Psi^{1,5}_H, \Psi^1_{H,5}]_7 = \Psi^{17,15}_H \Psi^{1,7}_{H,5}$. Applying the first Dodgson identity, we then get $[\Psi^{17,15}_H \Psi^{1,7}_{H,5}]_8 = \Psi^{15,78}_H \Psi^{18,17}_{H,5}$. Now,

$$[\Psi^{15,78}_H, \Psi^{18,17}_{H,5}]_9 = \Psi^{15,78}_{H,9} \Psi^{179,189}_{H,5},$$

by definition of the resultant, using the fact that $\Psi^{159,789}_H = 0$, by the vanishing property for vertices applied to the 3-valent vertex 7,8,9. Once more, by the vanishing property applied to the triangle $7,9,10$, we have $(\Psi^{15,78}_{H,9})_{10} = 0$, and therefore

$$[\Psi^{15,78}_{H,9} \Psi^{179,189}_{H,5}]_{10} = \Psi^{15X,78X}_{H,9} \Psi^{179,189}_{H,5X}$$

where $X$ denotes the edge 10. By contraction-deletion, the first factor is $\Psi^{15,78}_A$, and the second is $\Psi^{7,8}_{H'}$ where $H' = H\backslash\{1,9\}/\!/\{5,10\}$. In this latter graph, $7,8$ forms a 2-valent vertex, and so $\Psi^{7,8}_{H'} = \Psi^7_{H,8} = \Psi_{H'\backslash 7/\!/8} = \Psi_B$. □

6.2. **An 8-loop counter-example.** Let $G$ be the eight-loop primitive-divergent $\phi^4$ graph with vertices numbered $1,\ldots,9$ and (ordered) edges $e_1,\ldots,e_{16}$ defined by

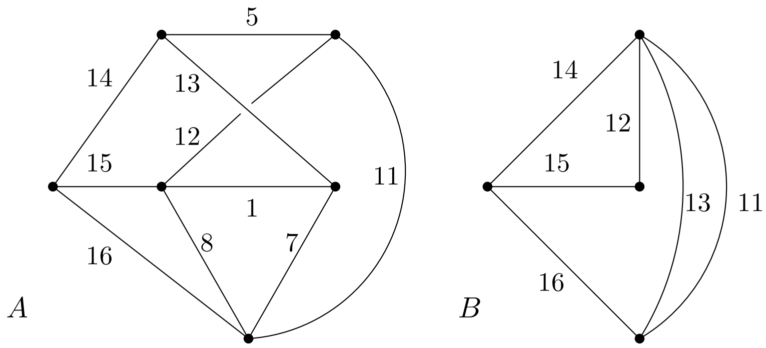
$$12,13,14,25,27,34,58,78,89,59,49,47,35,36,67,69\ , \tag{21}$$

where $ij$ denotes an edge connecting vertices $i$ and $j$. This graph is isomorphic to $P_{8,37}$ minus vertex 3 or 5 in the census [14]. It has 3785 spanning trees. The first six edges form precisely the configuration depicted in lemma 48, and we can subsequently apply lemma 49 to reduce the next four edges. A further reduction with respect to edge 11 gives the following corollary.

**Corollary 50.** *Let $G$ be the 8-loop graph above. Then*

$$D^{11}_G(e_1,\ldots,e_{11}) = \det \begin{pmatrix} \Psi^{15,78}_{A\backslash 11} & \Psi_{B\backslash 11} \\ \Psi^{15,78}_{A/\!/11} & \Psi_{B/\!/11} \end{pmatrix} ,$$

*where $A, B$ are depicted below.*




The polynomial $D_G^{11}(e_1, \ldots, e_{11})$ is irreducible, so to proceed further in the reduction, observe that $A$ and $B$ have a common minor $\gamma = B \backslash \{11\} /\!\!/ \{12, 13\}$ which is the sunset graph on 2 vertices and 3 edges $14, 15, 16$. Its graph polynomial is

$$\Psi_\gamma = \alpha_{14}\alpha_{15} + \alpha_{15}\alpha_{16} + \alpha_{14}\alpha_{16} \ .$$

By direct computation, one verifies that

$$\begin{aligned} \Psi^{15,78}_{A\backslash 11} &= -\alpha_{13}\alpha_{15} \\ \Psi^{15,78}_{A/\!\!/ 11} &= \alpha_{12}(\Psi_\gamma + \alpha_{13}\alpha_{16}) \\ \Psi_{B\backslash 11} &= \Psi_\gamma + \alpha_{12}\alpha_{13} + \alpha_{16}\alpha_{12} + \alpha_{14}\alpha_{12} + \alpha_{15}\alpha_{13} + \alpha_{14}\alpha_{13} \\ \Psi_{B/\!\!/ 11} &= \alpha_{13}(\Psi_\gamma + \alpha_{16}\alpha_{12} + \alpha_{14}\alpha_{12}) \end{aligned} \tag{22}$$

At this point we can eliminate a further variable by exploiting the homogeneity of $D_G^{11}$ (or $\Psi_G$). The affine complement of the zero locus of a homogenous polynomial $F$ admits a $\mathbb{G}_m$ action by scalar diagonal multiplication of the coordinates. For any coordinate $\alpha_e$, we therefore have

$$[F]_q = [F, \alpha_e]_q + (q-1)[F, \alpha_e - 1]_q$$

**Lemma 51.** $[D_G^{11}, \alpha_{16}]_q$ *is a polynomial in* $q$.

*Proof.* By inspection of (22), setting $\alpha_{16} = 0$ in the definition of $D_G^{11}$ causes the terms $\alpha_{14}\alpha_{15}$ to factor out. The other factor is of degree at most one in $\alpha_{14}$ and $\alpha_{15}$, and by a simple application of lemma 12 is therefore Tate. □

We will henceforth work on the hyperplane $\alpha_{16} = 1$. Now we may scale $\alpha_{12}$ and $\alpha_{13}$ by $\Psi_\gamma$, which has the effect of replacing $D_G^{11}$ with $\widetilde{D}$ given by formally setting $\Psi_\gamma$ to be 1 in the previous equations. We have

$$[D_G^{11}]_q \equiv [\widetilde{D}]_q + [\Psi_\gamma, \widetilde{D}]_q \mod q$$

**Lemma 52.** $[\widetilde{D}]_q$ *is constant modulo* $q$.

*Proof.* By inspection of (22), it is clear that the determinant $\widetilde{D}$ is of degree one in the variables $\alpha_{14}$ and $\alpha_{15}$. Applying lemma 12 *(i)* twice, it follows that the class of $\widetilde{D}$ modulo $q$ is equal to the class modulo $q$ of its coefficient of $\alpha_{14}\alpha_{15}$, and this is $\alpha_{12}\alpha_{13}(\alpha_{13}\alpha_{12} + \alpha_{13} + \alpha_{12})$, which is clearly of Tate type. □

It remains to compute $[\Psi_\gamma, \widetilde{D}]_q$, which is given mod $q$ by the resultant $[\Psi_\gamma, \widetilde{D}]_{14}$. Explicitly, it is the polynomial:

$$\begin{aligned}\alpha_{12} + \alpha_{12}\alpha_{15} + \alpha_{13}\alpha_{12}^2 + \alpha_{12}^2 + \alpha_{13}\alpha_{12} + \alpha_{15}\alpha_{13}\alpha_{12} \\ +\alpha_{13}^2\alpha_{15} + \alpha_{13}^2\alpha_{15}^2 + \alpha_{13}^2\alpha_{15}\alpha_{12} + \alpha_{13}^2\alpha_{15}^2\alpha_{12} + \alpha_{15}^2\alpha_{13}\alpha_{12}\end{aligned} \tag{23}$$

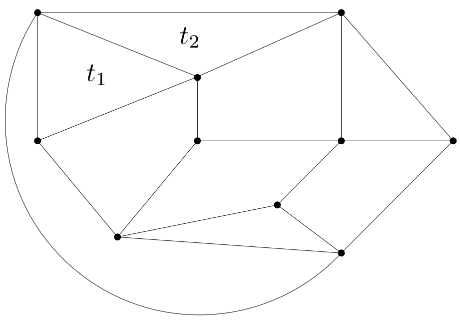
A final innocuous change of variables $\alpha_{13} \mapsto \alpha_{13}/(\alpha_{12}+1)$ reduces this equation to degree 4, and setting $a = \alpha_{13}+1$, $b = \alpha_{12}+1$, $c = \alpha_{15}$ leads to the equation

$$J = a^2bc - ab - ac^2 - ac + b^2c + ab^2 + abc^2 - abc$$

which defines a singular surface in $\mathbb{A}^3$. In conclusion, $c_2(G) \equiv c_0 + [J]_q \mod q$ for some constant $c_0 \in \mathbb{N}$. Chasing the various constant terms in the above gives $c_0 = 3$. Note that this graph has vertex width 4.

6.3. **A planar counter-example.** Consider the planar graph $G_9$ with nine loops and eighteen edges below. It is primitive-divergent and in $\phi^4$ theory.

It contains a double triangle ($t_1$ and $t_2$), bounded by the 4 vertices in the top-left hand corner. By applying a double-triangle reduction, the $c_2$ invariant of this graph is equal to the $c_2$ invariant of a non-planar graph at 8 loops. One verifies that the completion class of this 8 loop graph is the same as the previous example. Thus, accepting the completion conjecture, we have $c_2(G_9)_q \equiv 3 + [J]_q \mod q$ also. In any case, all graphs in the completion class of this 8 loop graph have been calculated by computer [15] which confirms this prediction.




$G_9$

## 7. A singular K3 surface

Consider the homogeneous polynomial of degree four

$$F = b(a+c)(ac+bd) - ad(b+c)(c+d)$$

which satisfies $F|_{d=1} = J$. One easily checks that it has six singular points

$$e_1 = (0:0:0:1) \quad e_2 = (0:1:0:0) \quad e_3 = (1:0:0:0)$$

$$e_4 = (0:0:1:0) \quad e_5 = (0:0:-1:1) \quad e_6 = (1:1:-1:1)$$

which are all of du Val type. Its minimal desingularization is obtained by blowing up the six points $e_1, \ldots, e_6$ and is therefore a K3 surface $X$. Since the Hodge numbers of a K3 satisfy $h^{1,1} = 20$, and $h^{0,2} = h^{2,0} = 1$, both $X$ and $V(F) \subset \mathbb{P}^3$ are not of Tate type and we can already conclude that the graph $G$ is a counter-example to Kontsevich's conjecture.

7.1. **The Picard lattice.** We determine the Picard lattice of $X$ as follows. It follows by inspection of $F$ that the following lines lie on $X$.

$$\begin{array}{ll}
\ell_1:\ c=d=0 & \ell_8:\ c=b+d=0\\
\ell_2:\ b=d=0 & \ell_9:\ b=c+d=0\\
\ell_3:\ a=d=0 & \ell_{10}:\ a-b=c+d=0\\
\ell_4:\ b=c=0 & \ell_{11}:\ a=b=d\\
\ell_5:\ a=c=0 & \ell_{12}:\ a=b=-c\\
\ell_6:\ a=b=0 & \ell_{13}:\ a=-c=d\\
\ell_7:\ a+c=d=0 & \ell_{14}:\ a-d=b+c=0
\end{array} \tag{24}$$

Let $\ell_{15},\ldots,\ell_{20}$ denote the six exceptional divisors lying above the points $e_1,\ldots,e_6$. Since these rational curves have self-intersection $-2$, one easily deduces the following intersection matrix, where the rows and columns correspond to $\ell_1,\ldots,\ell_{20}$.

$$\begin{pmatrix}
-2&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0\\
0&-2&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&-2&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0\\
0&0&0&-2&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0\\
0&0&0&0&-2&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&-2&0&0&0&0&0&0&0&0&1&0&0&1&1&0\\
0&1&0&0&0&0&-2&0&0&0&0&1&0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&-2&0&0&1&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&-2&0&0&0&1&0&0&0&1&0&1&0\\
1&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&1&0&0&-2&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&1&0&0&0&0&-2&0&0&1&0&0&0&0&1\\
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0&0&1&1&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&1\\
0&0&0&1&1&1&0&0&0&0&0&1&0&0&-2&0&0&0&0&0\\
1&0&1&0&1&0&1&0&0&0&0&0&1&0&0&-2&0&0&0&0\\
1&1&0&1&0&0&0&1&1&0&0&0&0&0&0&0&-2&0&0&0\\
0&1&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&-2&0&0\\
0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&-2&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&-2
\end{pmatrix}$$

It has determinant $-7$.

Since 7 is prime, the lines $\ell_1,\ldots,\ell_{20}$ span the full Néron-Severi group. In particular, the rank of $X$ is 20 and so it defines a singular K3 surface. Since $\mathbb{Q}(\sqrt{-7})$ has class number 1, $X$ corresponds to the unique singular K3 in the Shioda-Inose classification [18] with discriminant 7. Now consider the elliptic curve $E=E_{49A1}$ with complex multiplication by $\mathbb{Q}(\sqrt{-7})$ which is given by the affine model

$$y^2+xy=x^3-x^2-2x-1$$

The results of [18] imply that the graph of the complex multiplication in $E\times E$ gives rise to a decomposition of $\mathrm{Sym}^2H^1(E)$ into two pieces, one of which is $H^2_{tr}(X)$. The results of Livné [12] allow one to conclude that the weight 3 modular form corresponding to $H^2_{tr}(X)$ is given by the symmetric square of the modular form of $E$. Alternatively, one sees that $X$ corresponds to the first entry of Table 2 in [16].

7.2. **A modular form.** Consider Ramanujan's double theta function:

$$\theta(r,s) = \sum_{n=-\infty}^{\infty} r^{n(n+1)/2} s^{n(n-1)/2}$$

and write $\theta_{a,b}(q) = \theta(-q^a, -q^b)$. Then, following [13], set

$$\begin{aligned} f_{49}(q) &= \theta_{7,14}(q)^3 \big[ q\,\theta_{21,28}(q) + q^2 \theta_{14,35}(q) - q^4 \theta_{7,42}(q) \big] \\ &= q + q^2 - q^4 - 3q^8 - 3q^9 + 4q^{11} - q^{16} - 3q^{18} + 4q^{22} + 8q^{23} + \ldots \end{aligned}$$

which spans the one-dimensional space of newforms of level 49 and weight 2 (see also [17]). If $a_{p^n}$ denotes the coefficient of $q^{p^n}$ in $f_{49}(q)$, one knows that the number of points of $E$ over $\mathbb{F}_{p^n}$ is $p^n + 1 - a_{p^n}$.

Putting all the previous elements together, we arrive at:

**Theorem 53.** *Let $G$ be the 8-loop non-planar graph defined in §6.2. Then the number of points of the affine graph hypersurface $X_G$ over $\mathbb{F}_{p^n}$ satisfies:*

$$X_G(\mathbb{F}_{p^n}) \equiv 3 + a_{p^n}^2 p^{2n} \pmod{p^{3n}} ,$$

*where $a_{p^n}$ is defined as above. In particular, $X_G(\mathbb{F}_{p^n})$ is not a polynomial in $p^n$.*

Assuming the completion conjecture 4, or by the double-triangle theorem and the computer calculation in [15], the graph of §6.3 has exactly the same property, and yields a planar counter-example at 9 loops.

7.3. **A further counter-example.** Consider the 8-loop primitive-divergent $\phi^4$ graph with vertices numbered $1 \ldots 9$ and edges given by:

$$12, 13, 19, 26, 27, 34, 35, 37, 46, 48, 56, 58, 69, 78, 79, 89$$

Its completion class is the graph $P_{8,39}$ in [14]. This graph can also be treated in a similar manner to the previous example, and experimentally yields the new form of weight 3 and complex multiplication by $\mathbb{Q}(\sqrt{-8})$, corresponding to the second entry of table 2 in [16].

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