Multiple zeta values and other periods in quantum field theory

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In the first talk, I shall discuss the mathematics and physics of multiple zeta values and alternating Euler sums, exposing some of the wonderful mathematical structure of these objects and indicating where they arise in quantum field theory (QFT). In the second, I shall discuss the appearance of polylogarithms of the sixth root of unity, and singular values of elliptic integrals, in QFT, and report the summation of infinite series of diagrams.

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1 Multiple zeta values

1.1 Zeta values

For integer $s > 1$, the zeta values

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

divide themselves into two radically different classes. At *even* integers we have

$$\begin{aligned}
\zeta(2) &= \frac{\pi^2}{6} \\
\zeta(4) &= \frac{\pi^4}{90} \\
\zeta(6) &= \frac{\pi^6}{945} \\
\zeta(8) &= \frac{\pi^8}{9450} \\
\zeta(10) &= \frac{\pi^{10}}{93555}
\end{aligned}$$

and hence integer relations such as

$$5\zeta(4) - 2\zeta^2(2) = 0. \tag{1}$$

Yet no such relations have been found for *odd* arguments.

To prove (1), consider the wonderful formula

$$\frac{\cos(z)}{\sin(z)} = \sum_{n=-\infty}^{\infty} \frac{1}{z - n\pi}$$

in which the cotangent function is given by the sum of its pole terms, each with unit residue. Multiplying by z, to remove the singularity at $z = 0$, and then combining the terms with positive and negative n, we obtain

$$\frac{z\cos(z)}{\sin(z)} = 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - z^2}.$$

Expanding about $z = 0$ we obtain

$$\frac{1 - z^2/2! + z^4/4! + O(z^6)}{1 - z^2/3! + z^4/5! + O(z^6)} = 1 - 2\,\zeta(2)\frac{z^2}{\pi^2} - 2\,\zeta(4)\frac{z^4}{\pi^4} + O(z^6)$$

and easily prove that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$.

1.2 Double sums

For integers $a > 1$ and $b > 0$, let

$$\zeta(a, b) = \sum_{m>n>0} \frac{1}{m^a n^b}$$

which is a multiple zeta value (MZV) with weight $a + b$ and depth 2. Then, when a and b are both greater than 1, the double sum in the product

$$\zeta(a)\zeta(b) = \sum_{m>0} \frac{1}{m^a} \sum_{n>0} \frac{1}{n^b}$$

can be split into 3 terms, with $m > n > 0$, $m = n > 0$ and $n > m > 0$. Hence

$$\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(a + b) + \zeta(b, a). \tag{2}$$

There are further algebraic relations. Consider the sums

$$T(a, b, c) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j + k)^a j^b k^c}.$$

Multiplying the numerator by $(j + k) - j - k = 0$ we obtain

$$0 = T(a - 1, b, c) - T(a, b - 1, c) - T(a, b, c - 1)$$

and hence by repeated application of

$$T(a, b, c) = T(a + 1, b - 1, c) + T(a + 1, b, c - 1)$$

we may reduce these Tornheim double sums to MZVs. For example

$$T(1, 1, 1) = 2\zeta(2, 1).$$

We also have

$$T(1, 1, 1) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j + k)jk} = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{j + k} \right).$$

But now the inner sum has only j terms and hence

$$T(1, 1, 1) = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{n=1}^{j} \frac{1}{n} = \zeta(2, 1) + \zeta(3).$$

Comparing the two results for $T(1, 1, 1)$, we find that

$$\zeta(2, 1) = \zeta(3).$$

More generally, for $a > 1$, Euler found that

$$\zeta(a, 1) = \frac{a}{2}\zeta(a + 1) - \frac{1}{2} \sum_{b=2}^{a-1} \zeta(a + 1 - b)\zeta(b). \tag{3}$$

Moreover, Euler found the evaluation of all MZVs with odd weight and depth 2. For odd $a > 1$ and even $b > 0$ we have

$$\begin{aligned}\zeta(a,b) &= -\frac{1+C(a,b,a+b)}{2}\zeta(a+b)\\ &\quad + \sum_{k=1}^{(a+b-3)/2} C(a,b,2k+1)\zeta(a+b-2k-1)\zeta(2k+1)\end{aligned} \tag{4}$$

where

$$C(a,b,c) = \binom{c-1}{a-1} + \binom{c-1}{b-1}.$$

For example, we obtain

$$\begin{aligned}\zeta(3,2) &= -\frac{11}{2}\zeta(5) + \frac{\pi^2}{2}\zeta(3)\\ \zeta(2,3) &= \zeta(2)\zeta(3) - \zeta(5) - \zeta(3,2)\\ &= \frac{9}{2}\zeta(5) - \frac{\pi^2}{3}\zeta(3)\end{aligned}$$

using (4) and (2).

With weight $w = a + b < 8$ there is only one double sum $\zeta(a,b)$ not covered by Euler's explicit formulas, namely

$$\zeta(4,2) = \zeta^2(3) - \frac{4}{3}\zeta(6)$$

with an evaluation whose proof will be considered later.

To obtain such evaluations by empirical methods, you may use the `EZFace` interface
`http://oldweb.cecm.sfu.ca/cgi-bin/EZFace/zetaform.cgi`
which supports the `lindep` function of `Pari-GP`. For example, the input
`lindep([z(4,2),z(3)^2,z(6)])`
produces the output
`3., -3., 4.`
corresponding to the integer relation

$$3\zeta(4,2) - 3\zeta^2(3) + 4\zeta(6) = 0.$$

At weight $w = 8$, it appears that $\zeta(5,3)$ cannot be reduced to zeta values and their products, though we have no way of proving that such a reduction cannot exist. [We cannot even prove that $\zeta(3)/\pi^3$ is irrational.] I shall take $\zeta(5,3)$ as an (empirically) irreducible MZV of weight 8 and depth 2. Then all other double sums of weight 8 may be reduced to $\zeta(5,3)$ and zeta values. For example,

$$20\zeta(6,2) = 40\zeta(5)\zeta(3) - 8\zeta(5,3) - 49\zeta(8).$$

It is proven that the number of irreducible double sums of even weight $w = 2n$ is no greater than $\lceil n/3 \rceil - 1$. Up to weight $w = 12$, we may take the irreducible double sums to be $\zeta(5,3)$, $\zeta(7,3)$ and $\zeta(9,3)$. Later we shall see that the proven reduction

$$\zeta(7,5) = \frac{14}{9}\zeta(9,3) + \frac{28}{3}\zeta(7)\zeta(5) - \frac{24257\pi^{12}}{2298646350} \tag{5}$$

sets us a puzzle. There is only one irreducible MZV with weight 12 and depth 2.

1.3 Triple sums

The first MZV of depth 3 that has not been reduced to MZVs of lesser depth (and their products) occurs at weight 11. It is proven that

$$\zeta(a,b,c) = \sum_{l>m>n>0} \frac{1}{l^a m^b n^c}$$

is reducible when the weight $w = a+b+c$ is even or less than 11. I conjectured that all MZVs of depth 3 are expressible in terms of $\mathbf{Q}$-linear combinations of the set

$$\mathcal{B}_3 = \{\zeta(2a+1, 2b+1, 2c+1) | a \geq b \geq c,\ a > c\}$$

together with double sums, $\zeta(a,b)$, single sums, $\zeta(c)$, and their products. This was borne out by the investigations by Borwein and Girgensohn in
`http://www.combinatorics.org/Volume_3/PDF/v3i1r23.pdf`
and more recently by Blümlein, Broadhurst and Vermaseren in
`http://arxiv.org/PS_cache/arxiv/pdf/0907/0907.2557v2.pdf`
with the associated MZV `DataMine`
`http://www.nikhef.nl/~form/datamine/`
providing strong evidence for many of the claims made in this talk.

My conjecture implies that the number of irreducible MZVs of weight $w = 2n+3$ and depth 3 is $\lceil n^2/12 \rceil - 1$, with the sequence

$$1, 2, 2, 4, 5, 6, 8, 10, 11, 14, 16, 18, 21, 24, 26, 30$$

giving the numbers for odd weights from 11 to 41.

1.4 A quadruple sum

The mystery of MZVs really begins here. At weight 12 there first appears a quadruple sum that has not been reduced to MZVs with depths less than 4. In the `DataMine` we take this to be

$$\zeta(6,4,1,1) = \sum_{k>l>m>n>0} \frac{1}{k^6 l^4 m n}$$

and prove, by exhaustion, that the following methods are insufficient to reduce it.

1.5 Shuffles, stuffles and duality

For integers $s_j > 0$ and $s_1 > 1$, the MZV

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1 > n_2 > \ldots > n_k > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k}}$$

may be encoded by a word of length $w = \sum_{j=1}^{k} s_j$ in the two letter alphabet (A, B), as follows. We write A, $s_1 - 1$ times, then B, then A, $s_2 - 1$ times, then B, and so on, until we end with B. For example

$$\begin{aligned}\zeta(5,3) &= Z(AAAABAAB)\\ \zeta(6,4,1,1) &= Z(AAAAABAAABBB)\end{aligned}$$

where the function Z takes a word as it argument and evaluates to the corresponding MZV. Note that the word must begin with A and end with B. The weight of the MZV is the length of the word and the depth is the number of B's in the word.

We may evaluate the MZV from an iterated integral defined by its word. For example

$$\zeta(2,1) = Z(ABB) = \int_0^1 \frac{\mathrm{d}x_1}{x_1} \int_0^{x_1} \frac{\mathrm{d}x_2}{1-x_2} \int_0^{x_2} \frac{\mathrm{d}x_3}{1-x_3} \tag{6}$$

where we use the differential form $\mathrm{d}x/x$ whenever we see the letter A and the differential form $\mathrm{d}x/(1-x)$ whenever we see the letter B. Then the equality of the nested sum $\zeta(2,1)$ with the iterated integral $Z(ABB)$ follows from binomial expansion of $1/(1-x_2)$ and $1/(1-x_3)$ in (6).

The shuffle algebra of MZVs is the identity

$$Z(U)Z(V) = \sum_{W\in\mathcal{S}(U,V)} Z(W) \tag{7}$$

where $\mathcal{S}(U,V)$ is the set of words obtained by all permutations of the letters of UV that preserve the order of letters in U and the order of letters in V. For example, suppose that $U = ab$ and $V = xy$. Then $\mathcal{S}(U,V)$ consists of the words

$$\mathcal{S}(ab,xy) = \{abxy, axby, xaby, axyb, xayb, xyab\}\,.$$

The only legal two-letter word is AB. Hence setting $a = x = A$ and $b = y = B$ we obtain

$$Z(AB)Z(AB) = 2Z(ABAB) + 4Z(AABB)$$

which shows that

$$\zeta^2(2) = 2\zeta(2,2) + 4\zeta(3,1).$$

We also have the "stuffle" identity

$$\zeta(2)\zeta(2) = \zeta(2,2) + \zeta(4) + \zeta(2,2)$$

from shuffling the arguments in a product of zetas and adding in the extra "stuff" that originates when summation variables are equal. Hence we conclude that $\zeta(3,1) = \frac{1}{4}\zeta(4)$. The evaluation $\zeta(2,2) = \frac{3}{4}\zeta(4)$ requires the extra piece of information $\zeta^2(2) = \frac{5}{2}\zeta(4)$ obtained from expanding the cotangent function.

Like the shuffle algebra, the stuffle algebra can be used to express any product of MZVs as a sum of MZVs. For example

$$\zeta(3,1)\zeta(2) = \zeta(3,1,2) + \zeta(3,3) + \zeta(3,2,1) + \zeta(5,1) + \zeta(2,3,1).$$

By combining shuffles, stuffles and reductions of $\zeta(2)$, $\zeta(4)$ and $\zeta(6)$ to powers of π^2 we may prove that

$$Z(AAABAB) = \zeta(4,2) = \zeta^2(3) - \frac{4}{3}\zeta(6).$$

Moreover, we obtain the same value for the depth-4 MZV

$$Z(ABABBB) = \zeta(2,2,1,1)$$

since $Z(W) = Z(\widetilde{W})$, where the dual $\widetilde{W}$ of a word W is obtained by writing it backwards and then exchanging A and B. This duality was observed by Zagier. It follows from the transformation $x \to 1-x$ in the iterated integral, which exchanges the differential forms $\mathrm{d}x/x$ and $\mathrm{d}x/(1-x)$ and reverses the ordering of the integrations. Hence

$$\zeta(2,3,1) = Z(ABAABB) = Z(AABBAB) = \zeta(3,1,2).$$

Thus we arrive at a well-defined question: for a given weight $w > 2$ and a given depth $d > 0$, what is rank-deficiency $D_{w,d}$ of all the algebraic relations that follow from the shuffle and stuffle algebras algebras of MZVs, combined with duality and the reduction of even zeta values to powers of π^2? Note that $D_{w,d}$ is an upper limit for the number of irreducible MZVs at this weight and depth. There may conceivably (but rather improbably) be fewer, since we cannot rule out the possibility of additional integer relations. [We cannot even prove that $\zeta(3)/\pi^3$ is irrational.]

In 1996, Dirk Kreimer and I conjectured that the answer to this question is given by the generating function

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{D_{w,d}} \stackrel{?}{=} 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)} \tag{8}$$

which produces the following table of values, with underlined values verified by Jos Vermaseren.

To explain how I guessed the final term in the generating function (8), I shall need to discuss alternating Euler sums.

w /d	1	2	3	4	5	6	7	8	9	10
3	$\underline{1}$									
4										
5	$\underline{1}$									
6										
7	$\underline{1}$									
8		$\underline{1}$								
9	$\underline{1}$									
10		$\underline{1}$								
11	$\underline{1}$		$\underline{1}$							
12		$\underline{1}$		$\underline{1}$						
13	$\underline{1}$		$\underline{2}$							
14		$\underline{2}$		$\underline{1}$						
15	$\underline{1}$		$\underline{2}$		$\underline{1}$					
16		$\underline{2}$		$\underline{3}$						
17	$\underline{1}$		$\underline{4}$		$\underline{2}$					
18		$\underline{2}$		$\underline{5}$		$\underline{1}$				
19	$\underline{1}$		$\underline{5}$		$\underline{5}$					
20		$\underline{3}$		$\underline{7}$		$\underline{3}$				
21	$\underline{1}$		$\underline{6}$		$\underline{9}$		$\underline{1}$			
22		$\underline{3}$		$\underline{11}$		$\underline{7}$				
23	$\underline{1}$		$\underline{8}$		$\underline{15}$		$\underline{4}$			
24		$\underline{3}$		$\underline{16}$		$\underline{14}$		$\underline{1}$		
25	$\underline{1}$		$\underline{10}$		$\underline{23}$		$\underline{11}$			
26		$\underline{4}$		$\underline{20}$		$\underline{27}$		$\underline{5}$		
27	$\underline{1}$		$\underline{11}$		$\underline{36}$		$\underline{23}$		2	
28		$\underline{4}$		$\underline{27}$		$\underline{45}$		16		
29	$\underline{1}$		$\underline{14}$		$\underline{50}$		$\underline{48}$		7	
30		$\underline{4}$		$\underline{35}$		$\underline{73}$		37		2

Table 1: Number of basis elements for MZVs as a function of weight and depth in a minimal depth representation. Underlined are the values we have verified with our programs.

1.6 MZVs in QFT

The counterterms in the renormalization of the coupling in ϕ^4 theory, at L loops, may involve MZVs with weights up to $2L-3$. Those associated with subdivergence-free diagrams may be obtained from finite massless 2-point diagrams with one less loop.

The first irreducible MZV of depth 2, namely $\zeta(5,3)$, occurs in a counterterm coming from the most symmetric 6-loop diagram for the ϕ^4 coupling, in which each of the 4 vertices connected to an external line is connected to each of the 3 other vertices, giving 12 internal propagators (or edges, as mathematicians prefer to call them). It hence diverges, at large loop momenta, in the manner of $\int \mathrm{d}^{24}k/k^{24}$. Its contribution to the β-function of ϕ^4-theory is scheme-independent and may be computed to high accuracy by using Gegenbauer polynomial expansions in x-space, which give the counterterm as a 4-fold sum that is far from obviously a MZV. Accelerated convergence of truncations of this sum gave an empirical $\mathbf{Q}$-linear of combination of $\zeta(5)\zeta(3)$ with

$$\zeta(5,3) - \frac{29}{12}\zeta(8)$$

and the latter combination was found to occur in another 6-loop counterterm. I shall attempt to demystify the multiple of $\zeta(8)$ after discussing alternating Euler sums.

At 7 loops, Dirk Kreimer and I found the combination

$$\zeta(3,5,3) - \zeta(3)\zeta(5,3)$$

in 3 different counterterms, where it occurs in combination with rational multiples of $\zeta(11)$ and $\zeta^2(3)\zeta(5)$.

2 Alternating Euler sums

My second topic is closely related to the first, namely alternating sums of the form

$$\sum_{n_1>n_2>\ldots>n_k>0}^{\infty} \frac{\varepsilon_1^{n_1}\ldots\varepsilon_k^{n_k}}{n_1^{s_1}\ldots n_k^{s_k}}$$

with positive integers s_j and signs $\varepsilon_j = \pm 1$. We may compactly indicate the presence of an alternating sign, when $\varepsilon_j = -1$, by placing a bar over the corresponding integer exponent s_j. Thus we write

$$\begin{aligned} \zeta(\overline{3},\overline{1}) &= \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} \\ \zeta(3,\overline{6},3,\overline{6},3) &= \sum_{j>k>l>m>n>0} \frac{(-1)^{k+m}}{j^3 k^6 l^3 m^6 n^3} \end{aligned}$$

using the same symbol ζ as we did for the MZVs. Such sums may be studied using `EZFace` and the `DataMine`.

2.1 Three-letter alphabet

Alternating sums have a stuffle algebra, from their representation as nested sums, and a shuffle algebra, from their representation as iterated integrals. In the integral representation we need a third letter, C, in our alphabet, corresponding to the differential form $\mathrm{d}x/(1+x)$. Consider

$$Z(ABC) = \int_0^1 \frac{\mathrm{d}x}{x} \int_0^x \frac{\mathrm{d}y}{1-y} \int_0^y \frac{\mathrm{d}z}{1+z}.$$

The z-integral gives $\log(1+y) = -\sum_{j>0}(-y)^j/j$ and hence

$$Z(ABC) = -\sum_{j>0} \int_0^1 \frac{\mathrm{d}x}{x} \int_0^x \frac{\mathrm{d}y}{1-y} \frac{(-y)^j}{j}.$$

Expanding $1/(1-y) = \sum_{k>0} y^{k-1}$ and integrating over y we obtain

$$Z(ABC) = -\sum_{k>0}\sum_{j>0}\int_0^1 \frac{\mathrm{d}x}{x}\frac{x^{j+k}}{j+k}\frac{(-1)^j}{j}$$

and the final integration gives

$$Z(ABC) = -\sum_{k>0}\sum_{j>0}\frac{1}{(j+k)^2}\frac{(-1)^j}{j}.$$

Finally, the substitution $k = m - j$ gives

$$Z(ABC) = -\sum_{m>j>0}\frac{(-1)^j}{m^2 j} = -\zeta(2,\overline{1}).$$

It takes a bit of practice to translate between words and sums. Here's another example:

$$Z(ACCAC) = (-1)^3\sum_{l>0}\sum_{k>0}\sum_{j>0}\frac{(-1)^l}{(j+k+l)^2}\frac{(-1)^k}{j+k}\frac{(-1)^j}{j^2}$$

gives

$$Z(ACCAC) = -\sum_{m>n>j>0}\frac{(-1)^m}{m^2 n j^2} = -\zeta(\overline{2},1,2)$$

after the substitutions $l = m - n$ and $k = n - j$.

Going from sums to words is quite tricky. For example, try to find the word W and the sign $\varepsilon(W)$ such that

$$\zeta(3,\overline{6},3,\overline{6},3) = \varepsilon(W)Z(W).$$

Note that $\varepsilon(W)$ is $+1$ or -1 according as whether there is an odd or even number of letters C in the word W. The word W begins $AABAAAAACAA\ldots$. The next letter is either B or C, but which is it?

2.2 Shuffles and stuffles

The 6 shuffles in

$$\mathcal{S}(ab,xy) = \{abxy, axby, xaby, axyb, xayb, xyab\}$$

give 6 different words, with $a = A$, $b = B$, $x = y = C$:

$$\begin{aligned} Z(AB)Z(CC) &= Z(ABCC) + Z(ACBC) + Z(CABC) \\ &\quad + Z(ACCB) + Z(CACB) + Z(CCAB) \end{aligned}$$

which translates to

$$\zeta(2)\zeta(\overline{1},1) = \zeta(2,\overline{1},1) + \zeta(\overline{2},\overline{1},\overline{1}) + \zeta(\overline{1},\overline{2},\overline{1}) + \zeta(\overline{2},1,\overline{1}) + \zeta(\overline{1},2,\overline{1}) + \zeta(\overline{1},1,\overline{2}).$$

The stuffles for this product are

$$\zeta(2)\zeta(\overline{1},1) = \zeta(2,\overline{1},1) + \zeta(\overline{3},1) + \zeta(\overline{1},2,1) + \zeta(\overline{1},3) + \zeta(\overline{1},1,2).$$

2.3 Transforming words

The transformation $x = (1-y)/(1+y)$ gives

$$\begin{aligned} \mathrm{d}\log(x) &= \mathrm{d}\log(1-y) - \mathrm{d}\log(1+y) \\ \mathrm{d}\log(1-x) &= \mathrm{d}\log(y) - \mathrm{d}\log(1+y) \\ \mathrm{d}\log(1+x) &= -\mathrm{d}\log(1+y) \end{aligned}$$

and maps $x = 0$ and $x = 1$ to $y = 1$ and $y = 0$. Thus, if we take a word W, write it backwards, and make the transformations

$$\begin{aligned} A &\to (B + C) \\ B &\to (A - C) \end{aligned}$$

we may obtain an expression for $Z(W)$ by expanding the brackets.

For example the transformation

$$AB \to (A - C)(B + C) = AB + AC - CB - CC$$

gives

$$Z(AB) = Z(AB) + Z(AC) - Z(CB) - Z(CC).$$

Combining this with the shuffle

$$Z(C)Z(C) = Z(CC) + Z(CC)$$

we obtain

$$0 = Z(AC) - Z(CB) - \frac{1}{2}Z(C)Z(C) = -\zeta(\overline{2}) + \zeta(\overline{1},\overline{1}) - \frac{1}{2}\zeta(\overline{1})\zeta(\overline{1}).$$

Combining this with the stuffle

$$\zeta(\overline{1})\zeta(\overline{1}) = \zeta(\overline{1},\overline{1}) + \zeta(2) + \zeta(\overline{1},\overline{1})$$

we obtain

$$\zeta(\overline{2}) = -\frac{1}{2}\zeta(2)$$

which is also obtainable as follows.

2.4 Doubling relations

For $a > 1$ we have

$$\zeta(a) + \zeta(\overline{a}) = \sum_{n>0} \frac{1 + (-1)^n}{n^a} = \sum_{k>0} \frac{2}{(2k)^a} = 2^{1-a}\zeta(a)$$

by the substitution $n = 2k$. Hence

$$\zeta(\overline{a}) = (2^{1-a} - 1)\zeta(a).$$

At $a=2$, we obtain $\zeta(\overline{2})=-\zeta(2)/2$, as above. Note also that $\zeta(\overline{1})=-\log(2)$.

We may take any MZV and convert it into a combination of MZVs and alternating sums, by doubling the summation variables. For example, we obtain

$$\begin{aligned} 2^{2-a-b}\zeta(a,b) &= \sum_{m>n>0}\frac{2}{(2m)^a}\frac{2}{(2n)^b} \\ &= \sum_{j>k>0}\frac{1+(-1)^j}{j^a}\frac{1+(-1)^k}{k^a} \\ &= \zeta(a,b)+\zeta(\overline{a},b)+\zeta(a,\overline{b})+\zeta(\overline{a},\overline{b}) \end{aligned}$$

by the transformations $j=2m$ and $k=2n$.

More complicated doubling relations were used in constructing the `DataMine`. With these, it was possible to avoid using the time-consuming transformations $A\to(B+C)$ and $B\to(A-C)$ as algebraic input. It was verified that the output, obtained by shuffling, stuffling and doubling, satisfied the relations that follow from word transformation.

2.5 Conjectured enumeration of irreducibles

Before considering the enumeration of irreducible MZVs, in the (A,B) alphabet, I already had a rather simple conjecture for the generator of the number, $E_{w,d}$, of irreducible sums of weight w and depth d in the (A,B,C) alphabet, namely

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{E_{w,d}} \stackrel{?}{=} 1-\frac{x^3y}{(1-xy)(1-x^2)}. \tag{9}$$

If this be true, it is easy to obtain $E_{w,d}$ by Möbius transformation of the binomial coefficients in Pascal's triangle. Let

$$T(a,b)=\frac{1}{a+b}\sum_{c|a,b}\mu(c)\,\frac{(a/c+b/c)!}{(a/c)!(b/c)!} \tag{10}$$

where the sum is over all positive integers c that divide both a and b and the Möbius function is defined by

$$\mu(c)=\begin{cases}1 & \text{when } c=1\\ 0 & \text{when } c \text{ is divisible by the square of a prime}\\ (-1)^k & \text{when } c \text{ is the product of } k \text{ distinct primes.}\end{cases} \tag{11}$$

When w and d have the same parity, and $w>d$, one obtains from (9)

$$E_{w,d}=T\left(\frac{w-d}{2},d\right). \tag{12}$$

The `DataMine` now provides extensive evidence to support this conjecture. It was verified at depth 6 up to weight 12, solving the algebraic input in rational arithmetic, and then up to weight 18, using arithmetic modulo a 31-bit prime. At depth 5, the corresponding weights are 17 and 21. At depth 4, they are 22 and 30.

2.6 Pushdown

Now consider the integers $M_{w,d}$ generated by an even simpler process:

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{M_{w,d}} = 1 - \frac{x^3 y}{1-x^2}. \qquad (13)$$

But what is the question, to which this is the answer?

I conjectured that $M_{w,d}$ is the number of irreducible sums of weight w and depth d in the (A,B,C) alphabet that suffice for the evaluation of MZVs in the (A,B) alphabet.

As already hinted, the first place that this conjecture becomes non-trivial is at weight 12, where the enumerations $M_{12,4}=0$ and $M_{12,2}=2$ are to be contrasted with the enumerations $D_{12,4}=1$ and $D_{12,2}=1$ of irreducible MZVs. The conjecture requires that

$$\zeta(6,4,1,1) = \sum_{k>l>m>n>0} \frac{1}{k^6 l^4 m n}$$

be reducible to sums of lesser depth, if we include an alternating double sum in the basis.

In 1996, I found such a “pushdown” empirically, using the integer-relation search routine `PSLQ`. It took another decade to prove such an integer relation, by the laborious process of solving all the known algebraic relations in the (A,B,C) alphabet at weight 12 and depths up to 4. Jos Vermaseren derived this proven identity from the `DataMine`:

$$\begin{aligned}\zeta(6,4,1,1) \;=\; & -\frac{64}{27}A(7,5) - \frac{7967}{1944}\zeta(9,3) + \frac{1}{12}\zeta^4(3) + \frac{11431}{1296}\zeta(7)\zeta(5)\\ & -\frac{799}{72}\zeta(9)\zeta(3) + 3\zeta(2)\zeta(7,3) + \frac{7}{2}\zeta(2)\zeta^2(5) + 10\zeta(2)\zeta(7)\zeta(3)\\ & +\frac{3}{5}\zeta^2(2)\zeta(5,3) - \frac{1}{5}\zeta^2(2)\zeta(5)\zeta(3) - \frac{18}{35}\zeta^3(2)\zeta^2(3) - \frac{5607853}{6081075}\zeta^6(2)\end{aligned}$$

where

$$A(7,5) = Z(AAAAAA(B-C)AAAAB) = \zeta(7,5) + \zeta(\overline{7},\overline{5}).$$

It is now proven that all MZVs of weight up to 12 are reducible to $\mathbf{Q}$-linear combinations of $\zeta(5,3)$, $\zeta(7,3)$, $\zeta(3,5,3)$, $\zeta(9,3)$, $\zeta(\overline{7},\overline{5})$, single zeta values, and products of these terms.

I can now explain the rather simple-minded procedure that Dirk Kreimer and I used in 1996 to arrive at the conjecture

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{D_{w,d}} \stackrel{?}{=} 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)}$$

for the number $D_{w,d}$ of irreducible sums in the (A,B) alphabet of pure MZVs. We added the third term to the much simpler conjectured generator for the much complicated question answered by $M_{w,d}$, namely the number of irreducibles in the (A,B,C) alphabet that suffice for reductions of MZVs. The numerator, $x^{12}y^2(1-y^2)$, of this term was determined by the single pushdown observed at weight 12, from an MZV of depth 4 to an alternating sum of depth 2. The denominator, $(1-x^x)(1-x^6)$, was chosen to agree with the empirical number $D_{2n,2} = \lceil n/3 \rceil - 1$ of double non-alternating irreducible sums of weight $2n$. Then we assumed that the enumeration of all other pushdowns would be generated by filtration. It was possible to check this, in a few cases, using PSLQ in 1996.

The list of explicit pushdowns that have now been obtained, in accord with the conjecture, has grown since then.

At weights 15, 16, 17, we have found pushdowns from MZVs to these alternating sums: $\zeta(\overline{6}, 3, \overline{6})$, $\zeta(\overline{13}, \overline{3})$, $\zeta(\overline{6}, 5, \overline{6})$.
At weight 18, there were pushdowns to $\zeta(\overline{15}, \overline{3})$ and $\zeta(6, \overline{5}, \overline{4}, 3)$.
At weight 19, to $\zeta(\overline{8}, 3, \overline{8})$ and $\zeta(\overline{6}, 7, \overline{6})$.
At weight 20, to $\zeta(\overline{17}, \overline{3})$, $\zeta(8, \overline{5}, \overline{4}, 3)$ and $\zeta(6, \overline{5}, \overline{6}, 3)$.

Our most ambitious efforts were at weight 21, where 3 MZVs of depth 5 are pushed down to the alternating sums $\zeta(\overline{8}, 5, \overline{8})$, $\zeta(\overline{6}, 9, \overline{6})$and $\zeta(\overline{8}, 3, \overline{10})$. Moreover the first pushdown from an MZV of depth 7 to an alternating sum of depth 5 is predicted at weight 21. A demanding `PSLQ` computation gave a relation of the form

$$\zeta(6,2,3,3,5,1,1) = -\frac{326}{81}\zeta(3,\overline{6},3,\overline{6},3) + \ldots \tag{14}$$

where the remaining 150 terms are formed by MZVs with depth no greater than 5, and their products. At such weight and depth, it becomes rather non-trivial to decide on a single alternating sum that might replace a MZV of greater depth. It took several attempts to discover that the alternating sum

$$\zeta(3,\overline{6},3,\overline{6},3) = \sum_{j>k>l>m>n>0} \frac{(-1)^{k+m}}{(jk^2lm^2n)^3}$$

is an "honorary MZV" that performs this task.

2.7 Suppression of π in massless diagrams

Now I can demystify, somewhat, the combination

$$\zeta(5,3) - \frac{29}{12}\zeta(8)$$

that occurs in scheme-independent counterterms of ϕ^4 theory at 6 loops. Dirk Kreimer and I discovered that the combinations

$$N(a,b) = \zeta(\overline{a}, b) - \zeta(\overline{b}, a),$$

with distinct odd integers a and b, simplify the results for counterterms. In particular, the use of

$$N(3,5) = \frac{27}{80}\left(\zeta(5,3) - \frac{29}{12}\zeta(8)\right) + \frac{45}{64}\zeta(3)\zeta(5)$$

removes all powers of π from both subdivergence-free diagrams that contribute to the 6-loop β-function. In each case, the contribution is a $\mathbf{Z}$-linear combination of $N(3,5)$ and $\zeta(3)\zeta(5)$.

At higher loop numbers, Oliver Schnetz has found that $N(3,7)$ suppresses the appearance π^{10}. However, at 8 loops he found that $N(3,9)$ and $N(5,7)$ are not sufficient to remove π^{12}. Like the maths, the physics becomes different at weight 12.

2.8 Magnetic moment of the electron

The magnetic moment of an electron, with charge $-e$ and mass m, is slightly greater than the Bohr magneton

$$\frac{e\hbar}{2m} = 9.274 \times 10^{-24}\ \mathrm{J\ T}^{-1}$$

which was the value predicted by Dirac. Here I included $\hbar = h/(2\pi)$, which we usually set to unity in QFT.

Using perturbation theory, we may expand in powers of the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.035999\ldots}.$$

In QFT, we usually set $\varepsilon_0 = 1$ and $c = 1$ and expand in powers of $\alpha/\pi = e^2/(4\pi^2)$, obtaining a perturbation expansion

$$\frac{\text{magnetic moment}}{\text{Bohr magneton}} = 1 + A_1\frac{\alpha}{\pi} + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + \ldots$$

which is known up to 3 loops.

In 1947, Schwinger found the first correction term $A_1 = \frac{1}{2}$. In 1950, Karplus and Kroll claimed the value

$$28\zeta(3) - 54\zeta(2)\log(2) + \frac{125}{6}\zeta(2) - \frac{2687}{288} \;=\; -2.972604271\ldots$$

for the coefficient of the next correction. It turned out that they had made a mistake in this rather difficult calculation. The correct result

$$A_2 \;=\; \frac{3}{4}\zeta(3) - 3\zeta(2)\log(2) + \frac{1}{2}\zeta(2) + \frac{197}{144} \;=\; -0.3284789655\ldots$$

was not obtained until 1957. Not until 1996 was the next coefficient

$$\begin{aligned} A_3 \;&=\; -\frac{215}{24}\zeta(5) + \frac{83}{12}\zeta(3)\zeta(2) - \frac{13}{8}\zeta(4) - \frac{50}{3}\zeta(\overline{3},\overline{1}) \\ &\quad + \frac{139}{18}\zeta(3) - \frac{596}{3}\zeta(2)\log(2) + \frac{17101}{135}\zeta(2) + \frac{28259}{5184} \\ &=\; 1.181241456\ldots \end{aligned} \tag{15}$$

found, by Stefano Laporta and Ettore Remiddi. The irrational numbers appearing on the second line are those already seen in A_2. On the first line we see zeta values and a new number, namely the alternating double sum

$$\zeta(\overline{3},\overline{1}) = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} \approx -0.11787599965050932684101395083413761871522\ldots$$

I visited Stefano and Ettore in Bologna when they were working on this formidable calculation and recommended to them a method of integration by parts, in D dimensions, that I had found useful for related calculations in the quantum field theory of electrons and photons. Here $D = 4 - 2\varepsilon$ is eventually set to 4, the number of dimensions of space-time. But it turns out to be easier if we keep it as a variable until the final stage of the calculation. Then if we find parts of the result that are singular at $\varepsilon = 0$ we need not worry: all that matters is that the complete result is finite. Based on my D-dimensional experience, I expected their final result to look simplest when written in terms of $\zeta(\overline{3}, \overline{1})$.

The D-dimensional calculation that informed this intuition involved three-loop massive diagrams contributing to charge renormalization in QED. These yielded Saalschützian F_{32} hypergeometric series, with parameters differing from $\frac{1}{2}$ by multiples of ε, namely

$$W(a_1, a_2; a_3, a_4) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - a_1\varepsilon)_n(\frac{1}{2} - a_2\varepsilon)_n}{(\frac{1}{2} + a_3\varepsilon)_{n+1}(\frac{1}{2} + a_4\varepsilon)_{n+1}}$$

with $(\alpha)_n \equiv \Gamma(\alpha + n)/\Gamma(\alpha)$. In particular, I needed the expansions of $W(1, 1; 1, 0)$ and $W(1, 0; 1, 1)$ in ε. The result for the most difficult three-loop diagram had the value $\pi^2 \log(2) - \frac{3}{2}\zeta(3)$ at $\varepsilon = 0$. Noting that this also occurs in the two-loop contribution to the magnetic moment, I expanded the charge-renormalization result to $O(\varepsilon)$, where I found only $\zeta(\overline{3}, \overline{1})$ and $\zeta(4)$. I thus hazarded the guess that these two sums would exhaust the weight-4 contributions to the magnetic moment at 3 loops, which happily is the case.

One may also write (15) in terms of a polylog that is not evaluated on the unit circle, such as

$$\mathrm{Li}_4(1/2) = \sum_{n=1}^{\infty} \frac{1}{n^4}\left(\frac{1}{2}\right)^n = -\frac{1}{24}\log^4(2) + \frac{1}{4}\zeta(2)\log^2(2) + \frac{1}{4}\zeta(4) - \frac{1}{2}\zeta(\overline{3}, \overline{1}),$$

but then the result for A_3 will acquire extra terms, involving powers of $\log^2(2)$.

3 Polylogs of the sixth root of unity

Here I shall review results obtained for massive diagrams that are needed in other parts of the standard model.

How many ways of colouring a tetrahedron with mass?

Fig 1: Symmetries of the tetrahedron

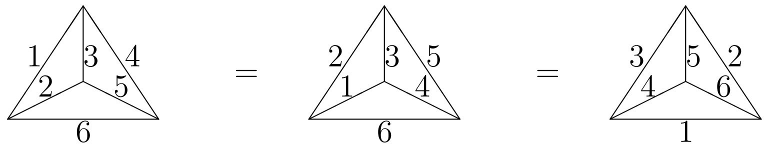
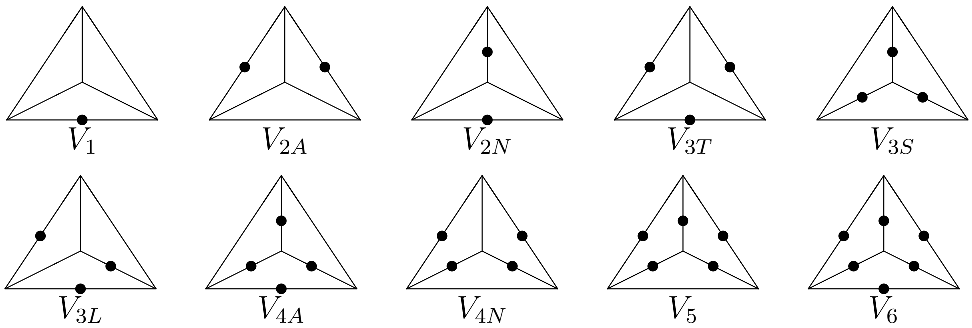


Fig 2: Colourings of the tetrahedron by mass (denoted by a blob)



In $D = 4 - 2\varepsilon$ dimensions, we have

$$V_j = \left(\frac{1}{3\varepsilon} + 1\right) 6\zeta(3) + 3\zeta(4) - F_j + O(\varepsilon)$$

where

$$F_j = \lim_{\varepsilon \to 0}(V_1 - V_j)$$

may be calculated as a finite integral in 4-dimensions of the difference of two finite 2-loop 2-point functions.

What was known before 1998?

$$\begin{aligned} F_1 &= 0 \\ F_{2A} &= 8\zeta(4) \\ F_{3T} &= 12\zeta(4) \end{aligned}$$

from expansion of Γ functions:

$$\begin{aligned} 3\varepsilon^4(1-\varepsilon)(1-2\varepsilon)V_1 &= \frac{G_{3,-3}G_{-1,-1}}{G_{1,-2}} - \frac{G_{3,-1}G_{2,-1}}{G_{1,1}} \\ 6\varepsilon^4(1-\varepsilon)(1-2\varepsilon)V_{2A} &= \frac{G_{3,-1}(G_{2,-1} - 3G_{2,1} + 2G_{2,2})}{G_{1,1}} \\ 4\varepsilon^4(1-\varepsilon)(1-2\varepsilon)V_{3T} &= \frac{G_{3,-1}G_{2,2}}{G_{1,1}} - 1 \end{aligned}$$

with

$$\begin{aligned} G_{\mu,\nu} &= \frac{\Gamma(1+\mu\varepsilon)\Gamma(1+\nu\varepsilon)}{\Gamma(1+\mu\varepsilon+\nu\varepsilon)} \\ &= 1 - \mu\nu\zeta(2)\varepsilon^2 + \mu\nu(\mu+\nu)\zeta(3)\varepsilon^3 - \mu\nu(\mu^2 + \tfrac{1}{4}\mu\nu + \nu^2)\zeta(4)\varepsilon^4 + O(\varepsilon^5) \end{aligned}$$

and my result

$$F_{4N} = 17\zeta(4) + 16\zeta(\overline{3}, \overline{1})$$

obtained in the course of charge renormalization in QED.

3.1 Numerics

Table 2: $\overline{\text{MS}}$ finite parts

V_j	z_j	u_j	s_j	v_j	$\overline{V}_j$
V_1	3				10.4593111200909802869464400586922036529141
V_{2A}	-5				1.8007252504018747548184104863628604307161
V_{2N}	$-\frac{13}{2}$	-8			1.1202483970392420822725165482242095262757
V_{3T}	-9				−2.5285676844426780112456042998018111803828
V_{3S}	$-\frac{11}{2}$		-4		−2.8608622241393273502727845677732419175614
V_{3L}	$-\frac{15}{4}$		-6		−3.0270094939876520197863747017589572861507
V_{4A}	$-\frac{77}{12}$		-6		−5.9132047838840205304957178925354050268834
V_{4N}	-14	-16			−6.0541678585902197393693995691614487948131
V_5	$-\frac{469}{27}$		$\frac{8}{3}$	-16	−8.2168598175087380629133983386010858249695
V_6	-13	-8	-4		−10.0352784797687891719147006851589002386503

3.2 Number-theory content

$$\begin{aligned}\overline{V}_j &= \lim_{\varepsilon \to 0} \left(V_j - \frac{6\zeta(3)}{3\varepsilon} \right) = 6\zeta(3) + 3\zeta(4) - F_j \\ &= 6\zeta(3) + z_j \, \zeta(4) + u_j \, \zeta(\overline{3}, \overline{1}) + s_j \, \text{Cl}_2^2(\pi/3) + v_j \, V_{3,1}\end{aligned}$$

where

$$\text{Cl}_2(\theta) = \sum_{n>0} \frac{\sin(n\theta)}{n^2} = \Im \text{Li}_2(\exp(\text{i}\theta))$$

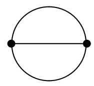
and

$$V_{3,1} = \sum_{m>n>0} \frac{(-1)^m \cos(2\pi n/3)}{m^3 n}.$$

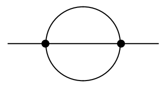
4 Singular values of elliptic integrals in QFT

Here I shall review result for series of diagrams that begin with polylogs and then produce elliptic integrals that, rather amazingly, reduce to products of Γ values.

In 2 space-time dimensions, these massive diagrams



V_3



S_4

are finite. V_3 evaluates to a multiple of

$$L_{-3}(2) = \sum_{n\geq 0} \frac{1}{(3n+1)^2} - \sum_{n\geq 0} \frac{1}{(3n+2)^2}$$

which can also be written in terms of a dilog of the sixth root of unity. When the external particle in S_4 is on its mass-shell, S_4 evaluates to a multiple of $\zeta(2)$. V_4, the "banana diagram" with 4 lines, evaluates to a multiple of $\zeta(3)$. The finite parts of the corresponding diagrams in 4 space-time dimensions involve the same constants.

What types of number come at higher loops?

4.1 Bessel moments

David Bailey, Jon Borwein, Larry Glasser and I became engrossed by this question in a study entitled *Elliptic integral evaluations of Bessel moments.*

In even space-time dimensions, the Fourier transform of the propagator $1/(p^2 - m^2)$ is a Bessel function and the evaluation of such Feynman diagrams amounts to evaluating integrals like

$$\begin{aligned} c_{n,k} &= \int_0^\infty t^k K_0^n(t)\,\mathrm{d}t \\ s_{n,k} &= \int_0^\infty t^k I_0(t) K_0^{n-1}(t)\,\mathrm{d}t \\ t_{n,k} &= \int_0^\infty t^k I_0^2(t) K_0^{n-2}(t)\,\mathrm{d}t \end{aligned}$$

with $V_n = c_{n,1}$, $S_n = s_{n,1}$, in 2 dimensions. The integral $t_{n,1}$ comes from cutting a line in S_n, just as S_n came from cutting a line in V_n.

We did not succeed in determining the number-theory content of V_5. We found a beautiful conjecture for S_5, with a wonderful relation to our (eventually) proven result for $t_{5,1}$.

4.2 Diamond mining

They key that unlocked our proofs and conjectures was, rather unexpectedly, the diamond lattice of the element carbon.

It is not hard to show that $s_{4,k}$ evaluates to a multiple of π^2 whenever k is odd. If we write

$$s_{4,2k+1} = \frac{\pi^2}{16}\left(\frac{k!}{4^k}\right)^2 b_k$$

then the integers b_k, for $k = 0, 1, 2, \ldots$, form the sequence

1, 4, 28, 256, 2716, 31504, 387136, 4951552, 65218204, 878536624, 12046924528 ...

which is entry A2895 in Neil Slonae's online encyclopedia of integer sequences. This sequence counts the number of self-returning walks of length $2k$ on a diamond lattice and is generated by

$$I_0^4(2t) = \sum_{k=0}^{\infty} b_k \left(\frac{t^k}{k!}\right)^2.$$

The closed form is

$$b_k = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2k-2j}{k-j} \binom{2j}{j}.$$

4.3 The journey to Birmingham

Condensed-matter theorists study triple integrals related to Green functions on crystal lattices. When these first arose, in the late 1930's, such integrals could not be evaluated by the best analysts available in Oxford and Cambridge.

But they soon "made the journey to Birmingham", where they were evaluated as squares of elliptic integrals.

Quarterly Journal of Mathematics (Oxford), **10** (1939), 266–276, *Three Triple Integrals*, by G.N. Watson (Birmingham), received 3 June 1939.

"The desirability of investigating the triple integrals has arisen as a consequence of their having appeared in a recent paper in ferromagnetic anisotropy by F. van Peype, a student of H.A. Kramers. The problem of evaluating them was proposed by Kramers to R.H. Fowler who communicated it to G.H. Hardy. The problem then became common knowledge first in Cambridge and subsequently in Oxford, whence it made the journey to Birmingham without difficulty.

It is possible to express all three of the integrals in terms of surds, the number π and certain complete elliptic integrals. The elliptic integrals that occur in [the face-centred and body-centred cases] are easily expressible in terms of gamma functions whose arguments are simple fractions. That is not the case, so far as I know, with the elliptic integral occurring in [the simple cubic case] ..."

In fact it was later shown that Watson's most difficult integral is an algebraic multiple of $\Gamma^2(\frac{1}{24})\Gamma^2(\frac{11}{24})/\pi^3$. I recently communicated its evaluation to York Schröder, who needed this result in his study of quark-gluon plasma.

4.4 Elliptic integrals at the 15th singular value

We say that the complete elliptic integral

$$K(\theta) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1 - \sin^2\theta \sin^2\phi}}$$

is evaluated at the N-th singular value if

$$K(\pi/2-\theta) = \sqrt{N}K(\theta).$$

In this very special case, it may be evaluated in terms of the Γ function at rational arguments.

Watson's most difficult integral was at the 6th singular value. Following leads from Stefano Laporta, we discovered that QFT chooses the 15th singular value. Here we encounter the constant

$$C = \frac{\pi}{16}\left(1-\frac{1}{\sqrt{5}}\right)\left(1+2\sum_{n=1}^{\infty}\exp\left(-n^2\pi\sqrt{15}\right)\right)^4$$

which may be reduced to Γ values at integer multiples of $\frac{1}{15}$.

After much work, we proved that

$$\begin{aligned}
\frac{2t_{5,1}}{\sqrt{15}\pi} &= C\\
\frac{2t_{5,3}}{\sqrt{15}\pi} &= \left(\frac{2}{15}\right)^2\left(13C+\frac{1}{10C}\right)\\
\frac{2t_{5,5}}{\sqrt{15}\pi} &= \left(\frac{4}{15}\right)^3\left(43C+\frac{19}{40C}\right)
\end{aligned}$$

and checked our rather appealing conjectures

$$\begin{aligned}
\frac{s_{5,1}}{\pi^2} &\stackrel{?}{=} C\\
\frac{s_{5,3}}{\pi^2} &\stackrel{?}{=} \left(\frac{2}{15}\right)^2\left(13C-\frac{1}{10C}\right)\\
\frac{s_{5,5}}{\pi^2} &\stackrel{?}{=} \left(\frac{4}{15}\right)^3\left(43C-\frac{19}{40C}\right)
\end{aligned}$$

at 1200-digit accuracy.

The evaluation of C in terms of

$$\sum_{n=1}^{\infty}\exp\left(-n^2\pi\sqrt{15}\right)$$

took a fraction of a second. Evaluation of 1200 good digits of the corresponding Feynman amplitude $S_5 = s_{5,1}$ takes considerably longer.

Finally, I give a conjectured relation to Γ values

$$\frac{\sqrt{5}}{2}\int_0^\infty t^3 I_0(t)K_0^4(t)\,\mathrm{d}t \stackrel{?}{=} \frac{13\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{30^3} - \frac{\Gamma\left(\frac{7}{15}\right)\Gamma\left(\frac{11}{15}\right)\Gamma\left(\frac{13}{15}\right)\Gamma\left(\frac{14}{15}\right)}{15} \tag{16}$$

whose proven counterpart, from diamond mining, is

$$\frac{\pi^3}{4\sqrt{3}}\sum_{n=1}^{\infty}\frac{n^2 b_n}{64^n} = \frac{13\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{30^3} + \frac{\Gamma\left(\frac{7}{15}\right)\Gamma\left(\frac{11}{15}\right)\Gamma\left(\frac{13}{15}\right)\Gamma\left(\frac{14}{15}\right)}{15}. \tag{17}$$

When QFT first moves on from polylogs to elliptic integrals, it does so in the most graceful way imaginable.

5 Summation of polylogs

Here I shall report results from summing infinite series of polylogarithms in QFT.

5.1 Sum of zeta-valued diagrams

The sum of ladder diagrams for the derivative of the self energy in massless ϕ^3-theory, in 4 space-time dimensions, has a perturbative expansion

$$\Sigma'_{\rm lad}(w^2) = \frac{\alpha}{2} + \sum_{L=2}^{\infty} \frac{1-L}{2} \binom{2L}{L} \zeta(2L-1)(-\alpha)^L$$

which diverges for $\alpha \equiv (\lambda/4\pi w)^2 > \frac{1}{4}$.

Q: What is the value for $\lambda \to \infty$?

A: The answer is $\frac{1}{24}$.

The function approximated by perturbation theory is

$$\Sigma'_{\rm lad}(w^2) = \frac{1}{24} - \frac{1}{4\pi^2} \int_{\lambda/2w}^{\infty} \frac{{\rm d}t}{\sqrt{1-(\lambda/2wt)^2}} \frac{t^2}{\sinh^2 t}\,.$$

Noting that $\frac{1}{24} = -\frac{1}{2}\zeta(-1)$, I found that

$$\sum_{L=0}^{\infty} \frac{1-L}{2} \binom{2L}{L} \zeta(2L-1)(-1)^L \left(\frac{x}{4\pi}\right)^{2L} = x^{5/2} \exp(-x) F(x)$$

where $F(\infty)$ is finite. Moreover, with a bit more work, I derived the asymptotic expansion

$$F(x) = \sum_{n=0}^{N-1} \frac{4n^2-8n+15}{2^{n+11/2}\,\pi^{3/2}\,n!} \frac{\Gamma(\frac{3}{2}+n)}{\Gamma(\frac{7}{2}-n)} \frac{1}{x^n} + O(1/x^N)$$

at strong coupling.

Perturbation theory overlaps nicely with the strong-coupling result.

Number of terms giving 3% accuracy for Σ' for various couplings, using perturbation theory (P), or the strong-coupling expansion (S):

λ/w :	0.5	1.0	1.5	2.0	2.5	3.0	3.35	3.5	4.0	4.5	5.0	5.5	6.0
P :	1	2	2	2	3	4	4						
S :							4	4	3	2	2	2	1

5.2 Sum of polylogarithmic diagrams

Now consider the 4-point ladder diagram with L loops. In principle, it depends on 6 kinematic variables, k_1^2, k_2^2, k_3^2, k_4^2, $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$, here assumed to be positive (i.e. time-like). In fact it turns out to depend only on the two ratios

$$X \equiv \frac{k_1^2 k_3^2}{st}, \quad Y \equiv \frac{k_2^2 k_4^2}{st}.$$

Spencer Bloch will tell you, on the basis of pure thought, why we expect a dilogarithm at $L = 1$. It might require a bit more thought to explain why the L-loop term is

$$\Phi^{(L)}(X,Y) = \frac{2}{\mu L!} \sum_{j=L}^{2L} \frac{j!}{(j-L)!\,(2L-j)!} \left(\log \frac{X}{Y}\right)^{2L-j} \Im \mathrm{Li}_j \left(\sqrt{\frac{Y}{X}}\ \exp(\mathrm{i}\phi)\right)$$

when

$$\mu = \sqrt{4XY - (X + Y - 1)^2}$$

is real and positive and hence

$$\phi = \arccos\left(\frac{X + Y - 1}{\sqrt{4XY}}\right)$$

is an angle between 0 and π.

Andrei Davydychev and I tried, on and off, for 17 years to sum the series

$$\mathcal{D} = \frac{1}{t}\left\{1 + \sum_{L=1}^{\infty} \left(-\frac{\kappa^2}{4}\right)^L \Phi^{(L)}(X,Y)\right\}$$

for very large coupling $\kappa = \lambda/(2\pi\sqrt{s})$.

Q: What is the value as $\kappa \to \infty$?

A: The answer is 0.

This is apparent from our exact result

$$\mathcal{D} = \frac{2}{t\mu} \int_{\kappa}^{\infty} \frac{z\ \mathrm{d}z}{\sqrt{z^2 - \kappa^2}} \frac{\sinh\left[(\pi - \phi)z\right]}{\sinh(\pi z)} \cos\left(\tfrac{1}{2}\log(X/Y)\sqrt{z^2 - \kappa^2}\right)$$

which is valid for all couplings. I guessed the answer and Andrei proved it, using an identity that we eventually traced back to a paper by Cauchy in 1826, entitled *Analyse transcendante*, with the ambitious subtitle *Recherche d'une formule générale qui fournit la valeur de la plupart des intégrales définies connues et celle d'un grand nombre d'autres.*

The sum is even more beautiful than its polylogarithmic parts.