Single Scale Quantities in Quantum Field Theory

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DESY

- LCE and Operator Expectation Values
- From floating point numbers to rational numbers
- From Moments to Functions
- Summation
- Difference Equations from Feynman Integrals
- From Sums to Complex Functions

0. A brief remark about MZV's

w = 27, d = 9 finished by J. Vermaseren this morning.

Computational details:

- run for 85 days at an 8 core machine with 96 Gbyte memory at DESY
- Data processed: $\sim$ 2 peta bytes = $31 \cdot 10^{12}$ terms
- J.V.: "The answer isn't quite what I expected (but is not 42)."

- Is everything like being expected ?

David, please stay calm, and wait for a few days. Jos is having a look with his 8 Gbyte editor into the 3.6 Gbyte result.

1. LCE and Operator Expectation Values

Integral cross sections are: no scale quantities

$\Longrightarrow$ Lectures by D. Broadhurst

Single differential distributions are: single scale quantities

- Anomalous dimensions
- Coefficient functions

- Nucleon Structure Functions $F_i(x), \quad x \in [0,1]$
- Parton Distributions $f_i(x), \quad x \in [0,1]$
- Wilson Coefficients $C_j^i(x), \; x \in [0,1]$
- Splitting Functions $P_i^j(x), \; x \in [0,1]$

$$F_i(x, Q^2) = \sum_j c_j C_j^i(x, Q^2/\mu^2) \otimes f_i(x, \mu^2)$$

$$\frac{\partial f_i(x, \mu^2)}{\partial \ln(\mu^2)} = P_i^j[x, a_s(\mu^2)] \otimes f_j(x, \mu^2)$$

$$A(x) \otimes B(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2)$$

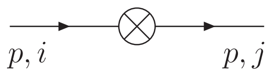
Mellin Convolution:

$$
\begin{aligned}
M[F(x)](N) &= \int_0^1 dx x^{N-1} \; F(x) \\
M[[A \otimes B](x)](N) &= M[A(x)](N) \cdot M[B(x)](N)
\end{aligned}
$$

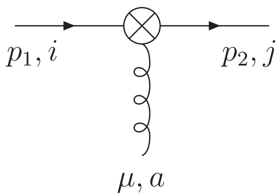
Work in Mellin space: much more simple structures are obtained.
2-point functions with local operator insertions:
$\Longrightarrow$ Splitting functions, massive operator matrix elements

$$
\begin{aligned}
O^{\mathsf{NS}}_{q,r;\mu_1,\ldots,\mu_N} &= i^{N-1}\mathbf{S}[\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi] - \text{trace terms} \;, \\
O^{\mathsf{S}}_{q;\mu_1,\ldots,\mu_N} &= i^{N-1}\mathbf{S}[\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi] - \text{trace terms} \\
O^{\mathsf{S}}_{g;\mu_1,\ldots,\mu_N} &= 2i^{N-2}\mathbf{SSp}[F^a_{\mu_1\alpha}D_{\mu_2}\ldots D_{\mu_{N-1}}F^{\alpha,a}_{\mu_N}] - \text{trace terms}
\end{aligned}
$$

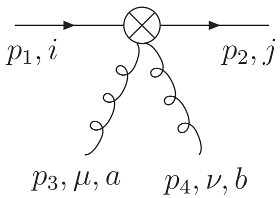
$$
A_{ij}^{\mathsf{S,NS}}\left(N, n_f+1, \frac{m^2}{\mu^2}\right) = \langle j|O_i^{\mathsf{S,NS}}|j\rangle = \delta_{ij} + \sum_{i=1}^{\infty} a_s^i A_{ij}^{(i),\mathsf{S,NS}}
$$



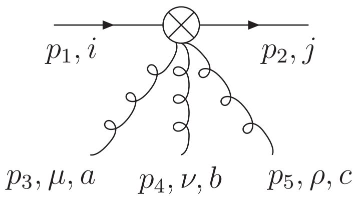
$$\delta^{ij} \not{\Delta} \gamma_\pm (\Delta \cdot p)^{N-1} , \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} , \quad N \geq 2$$



$$g^2 \Delta^\mu \Delta^\nu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right] ,$$
$$N \geq 3$$



$$g^3 \Delta^\mu \Delta^\nu \Delta^\rho \not{\Delta} \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta . p_2)^j (\Delta . p_1)^{N-m-2}$$
$$\Big[\; (t^a t^b t^c)_{ji} (\Delta . p_4 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_5 + \Delta . p_1)^{m-l-1}$$
$$+ (t^a t^c t^b)_{ji} (\Delta . p_4 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_4 + \Delta . p_1)^{m-l-1}$$
$$+ (t^b t^a t^c)_{ji} (\Delta . p_3 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_5 + \Delta . p_1)^{m-l-1}$$
$$+ (t^b t^c t^a)_{ji} (\Delta . p_3 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_3 + \Delta . p_1)^{m-l-1}$$
$$+ (t^c t^a t^b)_{ji} (\Delta . p_3 + \Delta . p_4 + \Delta . p_1)^{l-j-1} (\Delta . p_4 + \Delta . p_1)^{m-l-1}$$
$$+ (t^c t^b t^a)_{ji} (\Delta . p_3 + \Delta . p_4 + \Delta . p_1)^{l-j-1} (\Delta . p_3 + \Delta . p_1)^{m-l-1} \Big] ,$$
$$N \geq 4$$

$\gamma_+ = 1$, $\quad \gamma_- = \gamma_5$. For transversity, one has to replace: $\not{\Delta} \gamma_\pm \rightarrow \sigma^{\mu\nu} \Delta_\nu$.

The usual Feynman rules are extended by those of the composite operators.

Feynman Integrals are thus of the form : $F(N, \varepsilon)$
N $\in \mathbb{N}, \quad \varepsilon = D - 4$

Principle Mathematical Structure

J. Bümlein, Comput.Phys.Commun. 180 (2009) 2218, arXiv:0901.3106

- Feynman parameterization, performed repeatedly:

$$\frac{1}{A^\alpha \cdot B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{dx \;\; x^{\alpha-1} \; (1-x)^{\beta-1}}{[xA + (1-x)B]^{\alpha+\beta}}$$

- Numerator: Local Operator Insertion (all momentum integrals done):

$$(\Delta.k_i)^m \;,$$

- Carry out trivial Feynman parameter integrals:

$$\left.\frac{\Gamma(n_1 + r_1\varepsilon)\ldots\Gamma(n_k + r_k\varepsilon)}{\Gamma(m_1 + q_1\varepsilon)\ldots\Gamma(m_l + q_l\varepsilon)}\right|_{n_i,m_j \;\in\; \mathbf{Z}, r_i,q_j \;\in\; \mathbf{Q}}$$

- Solve all other Feynman parameter integrals via Mellin-Barnes:

$$\frac{1}{(A+B)^q} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\sigma\ A^\sigma\ B^{-q-\sigma}\frac{\Gamma(-\sigma)\Gamma(q+\sigma)}{\Gamma(q)}$$

- Final result after application of the residue theorem, Various infinite sums over:

$$\left.\frac{\Gamma(a_1 N + b_1(\sigma_a) + \bar{r}_1\varepsilon)\dots\Gamma(a_k N + b_k(\sigma_a) + \bar{r}_k\varepsilon)}{\Gamma(c_1 N + d_1(\sigma_a) + \bar{q}_1\varepsilon)\dots\Gamma(c_l N + d_l(\sigma_a) + \bar{q}_l\varepsilon)}\right|_{a_i\dots d_i\ \in\ \mathbf{Z}, \bar{r}_i, \bar{q}_j\ \in\ \mathbf{Q}}$$

- Expand in ε and carry out all sums.
- $\Longrightarrow$ nested but not necessarily harmonic sums in N.
- $F(N,\varepsilon)$ obeys

$$F(N,\varepsilon) = \sum_{k=-n_0}^{\infty} \varepsilon^k F_k(N), \quad n_0 \in \mathbb{N}$$

- For quantities without infrared problems: $n_0 =$ # loops

- The Functions $F_k(N+l)$

$$\sum_{l=0}^{m} c_l(N) \cdot F_k(N+l) = 0$$

  are recurrent, where $c_l(N)$ are polynomials.

- In practice, different, more economic ways are followed in the calculation.
- integration = antidifferentiation.
- It is a bit like vacation planning: can be done at different levels. Good to know, where to go; whether the area is crowded by wild animals, certain safety measures needed, rifles, ... extension of the desert to be crossed vs. water being carried along, etc.
- In short: one better knows and explores the target space before integrating.

2. From floating point numbers to rational numbers

Floating Point Numbers are easiest accessible in physics.
Also very many? Yes!
But at which precision??

Can one reconstruct functions, F(N), say anomalous dimensions, just in this way ?

- 1-dim integrals $\Longrightarrow$ $\zeta_{\vec{a}}$-values
- reconstruction of rational numbers with large numerators and denominators
- $\sim \#13.000/ \sim \#13.000$?

$\Longrightarrow$ Maple work sheet

High numerical precision needed!

3. From Moments to Functions

- Assume a large number of moments is known for a physical quantity of the above type.
- Assume we know that this function $F(N)$ is recurrent.
- $\Longrightarrow$ Algorithms exist to determine $F(N)$

J. Blümlein, M. Kauers, S. Klein, C. Schneider, Comput.Phys.Commun. 180 (2009) 2143.

- $\Longrightarrow$ 'Guessing' + Recurrency Check $\Longrightarrow$ minimal DEQ

Example:

$$F(N) = \frac{4}{(N+1)(N+3)(N+4)}\left[S_1(N) - \frac{(N^2+N-1)}{(N+1)(N+2)}\right]$$

- Need: 24 moments $\Longrightarrow$ DEQ of order 2 and degree 3

$$\begin{aligned}[N^3+7N^2+15N+9]F[N] + [2N^3+21N^2+69N+70]F[N+1] \\ +[N^3+14N^2+63N+90]F[N+2] &= 0\end{aligned}$$

F(1) = 1/12, F(2) = 13/270, F(3) = 11/360.

Problem :

- 1- and some smaller 2-loop problems may be solved in this way.
- Known methods allow to compute up to ~ 50 moments, MINCER, MATAD

S. Larin, F. Tkachov, J. Vermaseren, The FORM version of MINCER, NIKHEF-H-91-18
M. Steinhauser, Comput.Phys.Commun. 134 (2001) 335.

- 3-loop corrections : maximally 16 moments that far.

J. Blümlein and J. Vermaseren, Phys.Lett. B606 (2005) 130.

- What would be the resources to determine the 3-loop anomalous dimensions and Wilson coefficients this way?
- The corresponding DEQs would be the minimal ones.

J. Blümlein, M. Kauers, S. Klein, C. Schneider, Comput.Phys.Commun. 180 (2009) 2143.

- Can the corresponding DEQs be found practically?
- Can these recurrence be solved analytically ?

- Case study based on the known solution.

S. Moch, J. Vermaseren, A. Vogt, Nucl.Phys. B688 (2004) 101; B691 (2004) 129; B724 (2005) 3

```
C2qq3CF^3
N=3:
#11 digits / #10 digits

-98268084191 / 1166400000

N=500:
#1262 digits / #1256 digits

1641840770424196780953020619176376506284303544481262083057197600746507008493793994
4224110323441591630311482222058287688942209570859151121677307585313995100978363179
2518952817622034037186132846974627021672678012913675099511203807811938593043910803
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0115846094360239986233917872260722277322690450132376836253549152130116645670565045
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/
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496586232475091126597629976797375278827111116774593003520000000000000000000

N=5114:
#13388 digits / #13381 digits
```

Table 1: Run parameters for the unfolding of the non-singlet anomalous dimensions

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$P_{NS,0}$	14	2	3	0.05	0.087	1 [1]	0.55
$P^-_{NS,1,C_F^2}$	142	5	31	3.32	4.666	6 [10]	7.45
$P^-_{NS,1,C_AC_F}$	109	4	24	1.91	2.834	6 [7]	6.28
$P^-_{NS,1,C_FN_F}$	24	2	7	0.13	0.271	2 [2]	0.92
$P^+_{NS,1,C_F^2}$	142	5	31	3.35	4.707	6 [10]	7.45
$P^+_{NS,1,C_AC_F}$	109	4	23	1.88	2.703	6 [7]	6.23
$P^+_{NS,1,C_FN_F}$	24	2	7	0.09	0.271	2 [2]	0.89
$P^-_{NS,2,C_F^3}$	1079	16	192	3152.19	529.802	25 [68]	1194.41
$P^-_{NS,2,C_F^3\zeta_3}$	48	3	11	0.49	0.643	1 [1]	1.56
$P^-_{NS,2,C_AC_F^2}$	974	15	181	1736.08	450.919	25 [62]	1194.41
$P^-_{NS,2,C_AC_F^2\zeta_3}$	48	3	11	0.53	0.643	1 [1]	1.53
$P^-_{NS,2,C_A^2C_F}$	749	12	147	1004.12	242.892	25 [62]	1100.88
$P^-_{NS,2,C_A^2C_F\zeta_3}$	48	3	11	0.56	0.643	1 [1]	1.56
$P^-_{NS,2,C_FN_F^2}$	39	2	11	0.31	0.369	3 [3]	1.20
$P^-_{NS,2,C_F^2N_F}$	377	8	68	76.34	33.946	12 [24]	72.22
$P^-_{NS,2,C_F^2N_F\zeta_3}$	14	2	3	0.12	0.101	1 [1]	0.53
$P^-_{NS,2,C_AC_FN_F}$	356	7	62	65.25	23.830	12 [20]	52.67
$P^-_{NS,2,C_AC_FN_F\zeta_3}$	14	2	3	0.12	0.101	1 [1]	0.55
$P^+_{NS,2,C_F^3}$	1079	16	192	4713.27	527.094	25[68]	1165.22
$P^+_{NS,2,C_F^3\zeta_3}$	48	3	11	0.55	0.643	1[1]	1.562
$P^+_{NS,2,C_AC_F^2}$	974	15	178	1715.03	442.031	25[62]	889.047
$P^+_{NS,2,C_AC_F^2\zeta_3}$	48	3	11	0.61	0.643	1[1]	1.531
$P^+_{NS,2,C_A^2C_F}$	749	12	146	991.22	240.325	25[50]	516.812
$P^+_{NS,2,C_A^2C_F\zeta_3}$	48	3	11	0.61	0.643	1[1]	1.593
$P^+_{NS,2,C_F^2N_F}$	377	8	69	111.38	33.872	12[24]	71.235
$P^+_{NS,2,C_F^2N_F\zeta_3}$	14	2	3	0.15	0.101	1[1]	0.531
$P^+_{NS,2,C_AC_FN_F}$	307	7	61	48.62	23.808	12[24]	71.235
$P^+_{NS,2,C_AC_FN_F\zeta_3}$	14	2	3	0.15	0.101	1[1]	0.547
$P^+_{NS,2,C_FN_F^2}$	39	2	11	0.40	0.369	3[3]	1.172
$P^-_{NS,2,N_Fd_{abc}}$	39	2	11	0.55	0.369	3 [3]	1.19

Table 2: Run parameters for the unfolding of the unpolarized quarkonic Wilson Coefficients for the structure function $F_2(x,Q^2)$.

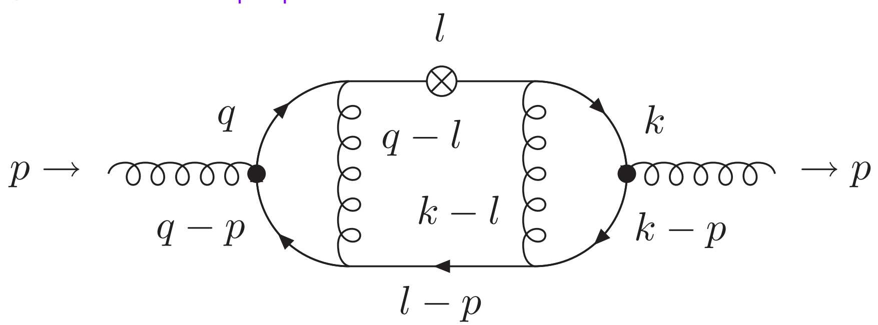
	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$C^{(1)}_{2,q,C_F}$	35	3	7	0.26	0.429	2[3]	1.13
$C^{(2)}_{2,q,C_F^2}$	689	11	137	1134.10	177.806	13[39]	258.24
$C^{(2)}_{2,q,C_AC_F}$	545	10	121	413.33	127.893	12[35]	178.73
$C^{(2)}_{2,q,C_F^2\zeta_3}$	15	2	3	0.27	0.100	1[1]	0.54
$C_{2,q,C_AC_F\zeta_3}$	15	2	3	0.27	0.112	1[1]	0.55
C_{2,q,N_FC_F}	71	4	16	2.68	1.655	4[10]	3.95
$C^{(3)}_{2,q,C_F^3}$	5114	35	938	1.78886×10^6	30394.173	58[289]	0.50924×10^6
$C^{(3)}_{2,q,C_F^3\zeta_3}$	284	8	64	31.02	32.363	7 [18]	27.60
$C^{(3)}_{2,q,C_F^3\zeta_4}$	19	2	5	0.08	0.163	1 [1]	0.47
$C^{(3)}_{2,q,C_F^3\zeta_5}$	19	2	5	0.08	0.163	1 [1]	0.47
$C^{(3)}_{2,q,C_F^2C_A}$	5059	35	930	1.69267×10^6	30122.380	60 [290]	0.47780×10^6
$C^{(3)}_{2,q,C_F^2C_A\zeta_3}$	284	8	64	34.00	33.400	7 [18]	28.53
$C^{(3)}_{2,q,C_F^2C_A\zeta_4}$	48	3	11	0.32	0.643	1[1]	1.01
$C^{(3)}_{2,q,C_F^2C_A\zeta_5}$	19	2	5	0.08	0.167	1 [1]	0.42
$C^{(3)}_{2,q,C_FC_A^2}$	4564	33	863	1.38918×10^6	24567.518	60 [258]	0.34941×10^6
$C^{(3)}_{2,q,C_FC_A^2\zeta_3}$	284	8	63	26.83	29.918	7 [17]	30.46
$C^{(3)}_{2,q,C_FC_A^2\zeta_4}$	48	3	11	0.32	0.643	1 [1]	1.01
$C^{(3)}_{2,q,C_FC_A^2\zeta_5}$	19	2	5	0.08	0.175	1 [1]	0.42
$C^{(3)}_{2,q,C_F^2N_F}$	1762	20	348	40237.45	2339.516	29 [107]	7548.56
$C^{(3)}_{2,q,C_F^2N_F\zeta_3}$	87	4	21	1.94	2.354	3 [5]	2.83
$C^{(3)}_{2,q,C_F^2N_F\zeta_4}$	15	2	3	0.07	0.101	1 [1]	0.34
$C^{(3)}_{2,q,C_FC_AN_F}$	1847	20	360	47661.64	2507.362	29 [111]	7525.89
$C^{(3)}_{2,q,C_FC_AN_F\zeta_3}$	89	4	24	2.47	2.935	3 [8]	3.19
$C^{(3)}_{2,q,C_FC_AN_F\zeta_4}$	15	2	3	0.06	0.101	1 [1]	0.34
$C^{(3)}_{2,q,C_FN_F^2}$	131	5	30	58.00	5.347	7 [22]	8.97
$C^{(3)}_{2,q,C_FN_F^2\zeta_3}$	15	2	3	0.06	0.101	1 [1]	0.38
$C^{(3)}_{2,q,dabc}$	1199	14	242	6583.27	738.498	14 [62]	841.24
$C^{(3)}_{2,q,dabc\zeta_3}$	109	4	25	2.33	3.164	2[7]	2.40
$C^{(3)}_{2,q,dabc\zeta_5}$	8	1	2	0.03	0.041	0[0]	0.10

4. Summation

3-Loop calculations of OMEs:

$\Longrightarrow$ reduction to multiply nested sums possible.

J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider in prep.



$$I_{1a} = S_\varepsilon^3 \frac{-\Gamma(2-3\varepsilon/2)}{(N+1)(N+2)(N+3)} \sum_{m,n=0}^{\infty} \Biggl\{ \sum_{t=1}^{N+2} \binom{3+N}{t} \frac{(t-\varepsilon/2)_m (2+N+\varepsilon/2)_{n+m} (3-t+N-\varepsilon/2)_n}{(4+N-\varepsilon)_{n+m}}$$
$$\times \Gamma\left[\begin{matrix} t, t-\varepsilon/2, 1+m+\varepsilon/2, 1+n+\varepsilon/2, 3-t+N, 3-t+N-\varepsilon/2 \\ 4+N-\varepsilon, 1+m, 1+n, 1+t+m+\varepsilon/2, 4-t+n+N+\varepsilon/2 \end{matrix}\right]$$
$$- \sum_{s=1}^{N+3} \sum_{r=1}^{s-1} \binom{s}{r} \binom{3+N}{s} (-1)^s \frac{(r-\varepsilon/2)_m (-1+s+\varepsilon/2)_{n+m} (s-r-\varepsilon/2)_n}{(1+s-\varepsilon)_{n+m}}$$
$$\times \Gamma\left[\begin{matrix} r, r-\varepsilon/2, s-r, 1+m+\varepsilon/2, 1+n+\varepsilon/2, s-r-\varepsilon/2 \\ 1+m, 1+n, 1+r+m+\varepsilon/2, 1+s-r+n+\varepsilon/2, 1+s-\varepsilon \end{matrix}\right] \Biggr\} .$$

Solution :

$$
\begin{aligned}
I_{1a} \quad = \quad & -\frac{4(N+1)S_1+4}{(N+1)^2(N+2)}\zeta_3 + \frac{2S_{2,1,1}}{(N+2)(N+3)} + \frac{1}{(N+1)(N+2)(N+3)}\Biggl\{ \\
& -2(3N+5)S_{3,1} - \frac{S_1^4}{4} + \frac{4(N+1)S_1-4N}{N+1}S_{2,1} + 2\left[(2N+3)S_1 + \frac{5N+6}{N+1}\right]S_3 \\
& +\frac{9+4N}{4}S_2^2 + \left[2\frac{7N+11}{(N+1)(N+2)} + \frac{5N}{N+1}S_1 - \frac{5}{2}S_1^2\right]S_2 + \frac{2(3N+5)S_1^2}{(N+1)(N+2)} \\
& +\frac{N}{N+1}S_1^3 + \frac{4(2N+3)S_1}{(N+1)^2(N+2)} - \frac{(2N+3)S_4}{2} + 8\frac{2N+3}{(N+1)^3(N+2)}\Biggr\} \\
& +O(\varepsilon)\ ,
\end{aligned}
$$

- Representation in terms of nested harmonic sums. (up to w =4)
- $\varepsilon \to 0$, non-ζ_3 term DEQ : 203 moments $\Longrightarrow$ order 8 and degree 26

Summation Methods :

- Telescoping
- Creative Telescoping
- Various Refinements of Creative Telescoping
- C. Schneider's packages Sigma, EvaluateInfiniteSums
- Summation in Product- and Difference fields

C. Schneider, Habilitation Thesis, JKU Linz, 2008

- J. Ablinger's package HarmonicSums

General form of Sums

$$\sum_{k_1=1}^{N_1(N)} \dots \sum_{k_m=1}^{N_m(k_{m-1},\dots,k_1,N)} R(k_1,\dots,k_m,N) \prod_{l=1}^{4} S_{\vec{a_l}}(s(k_i,N))$$
$$\times \Gamma \left[\begin{array}{c} s_1(k_i,N) \dots s_p(k_i,N) \\ s_1(k_i,N) \dots s_q(k_i,N) \end{array} \right] ,$$

with R a rational function, $s(k_i,N)$ a linear combination of the arguments with weight ± 1, $\vec{a_l}$ an index set, $p, q \in \mathbb{N}$, $N_i \in \mathbb{N} \cup \infty$

Some Exercises :

$\Longrightarrow$ Mathematica work sheet 1

An involved case:

$$\sum_{j_1=1}^{N-2}\sum_{n=1}^{\infty}(-1)^{j_1}B(n,N-j_1)\binom{N-2}{j_1}\frac{S_2(-j_1+n+N)}{n^2(j_1-N-2)}=$$

$$\Bigg\{(-1)^N\frac{6-23N+9N^2+2N^3}{2(-1+N)^2N^2(1+N)(2+N)}+\Bigg[\frac{1}{N+2}-\frac{27(-1)^N}{(N-1)N(N+1)(N+2)}\Bigg]S_1$$

$$-\frac{1}{N(N+2)}\Bigg\}S_2^2$$

$$+\Bigg[\frac{1}{N+2}-\frac{48(-1)^N}{(N-1)N(N+1)(N+2)}\Bigg]S_3S_2-\frac{2S_{-2}^2}{N(N+2)}$$

$$+\Bigg\{-(-1)^N\frac{7\big(12+6N-37N^2+6N^3+N^4\big)}{20(-1+N)^2N^2(1+N)(2+N)}+\Bigg[-(-1)^N\frac{21}{5(-1+N)N(1+N)(2+N)}$$

$$-\frac{7}{10(N+2)}\Bigg]S_1+\frac{7}{10N(N+2)}\Bigg\}\zeta_2^2+\Bigg\{(-1)^N\frac{6-23N+9N^2+2N^3}{2(-1+N)^2N^2(1+N)(2+N)}$$

$$+\Bigg[\frac{3(-1)^N}{(N-1)N(N+1)(N+2)}+\frac{3}{N+2}\Bigg]S_1-\frac{3}{N(N+2)}\Bigg\}S_4$$

$$+\Bigg[\frac{3}{N+2}-\frac{18(-1)^N}{(N-1)N(N+1)(N+2)}\Bigg]S_5+\Bigg[\frac{2S_{-2}^2}{N+2}+\frac{(-1)^N(3N-1)}{(N-1)^3N^3}\Bigg]S_1$$

$$+\frac{2}{2+N}S_{-2}S_{-3}+\Bigg\{-(-1)^N\frac{3\big(12-6N-14N^2+7N^3+12N^4+N^5\big)}{(-1+N)^3N^3(1+N)(2+N)}$$

$$+(-1)^N\frac{9S_1^2}{(-1+N)N(1+N)(2+N)}+(-1)^N\frac{3\big(6-23N+9N^2+2N^3\big)}{(-1+N)^2N^2(1+N)(2+N)}S_1$$

$$+\Bigg[\frac{3(-1)^N}{(N-1)N(N+1)(N+2)}-\frac{1}{N+2}\Bigg]S_2\Bigg\}S_{2,1}$$

$$+\Bigg[(-1)^N\frac{2\big(12-37N+9N^2+4N^3\big)}{(-1+N)^2N^2(1+N)(2+N)}+(-1)^N\frac{24}{(-1+N)N(1+N)(2+N)}S_1\Bigg]S_{3,1}$$

$$+\frac{2S_{3,2}}{N+2}+\Bigg[-\frac{12(-1)^N}{(N-1)N(N+1)(N+2)}-\frac{3}{N+2}\Bigg]S_{4,1}$$

$$-\frac{2S_{-2}S_{-2,1}}{2+N}+\frac{4S_{-3,-2}}{N+2}+\Bigg[-(-1)^N\frac{2\big(6-23N+9N^2+2N^3\big)}{(-1+N)^2N^2(1+N)(2+N)}$$

$$-(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}S_1\Bigg]S_{2,1,1}-\frac{2S_{-2,1,-2}}{N+2}$$

$$+(-1)^N\frac{1}{(-1+N)N(1+N)(2+N)}\Bigg[42S_{2,2,1}-24S_{3,1,1}+54S_{2,1,1,1}\Bigg]$$

$$-(-1)^N\frac{30}{(-1+N)N(1+N)(2+N)}S_3\tilde{S}_1\left(\frac{1}{2}\right)\tilde{S}_1(2)$$

$$+(-1)^N\frac{30}{(-1+N)N(1+N)(2+N)}S_1\tilde{S}_1\left(\frac{1}{2}\right)\tilde{S}_3(2)+\Bigg\{\frac{(-1)^N2^{N+2}}{(-1+N)^3N}$$

$$+\Bigg[-(-1)^N\frac{2\big(6-12N+7N^2+N^3\big)}{(-1+N)^3N^3}+(-1)^N\frac{6S_1^2}{(-1+N)N(1+N)(2+N)}$$

$$+(-1)^N\frac{2\big(6-23N+9N^2+2N^3\big)}{(-1+N)^2N^2(1+N)(2+N)}S_1\Bigg]\tilde{S}_1(2)$$

$$+\Bigg[(-1)^N\frac{2\big(6-23N+9N^2+2N^3\big)}{(-1+N)^2N^2(1+N)(2+N)}+(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}S_1\Bigg]\tilde{S}_2(2)$$

$$+(-1)^N\frac{6}{(-1+N)N(1+N)(2+N)}\tilde{S}_3(2)\Bigg\}\tilde{S}_{1,1}\left(\frac{1}{2},1\right)$$

$$+\Bigg\{\Bigg[(-1)^N\frac{12S_1^2}{(-1+N)N(1+N)(2+N)}+(-1)^N\frac{2\big(6-23N+9N^2+2N^3\big)}{(-1+N)^2N^2(1+N)(2+N)}S_1\Bigg]\tilde{S}_1\left(\frac{1}{2}\right)$$

$$+\Bigg[-(-1)^N\frac{2\big(6-23N+9N^2+2N^3\big)}{(-1+N)^2N^2(1+N)(2+N)}$$

$$-(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}S_1\Bigg]\tilde{S}_{1,1}\left(\frac{1}{2},1\right)\Bigg\}\tilde{S}_{1,1}(2,1)$$

$$+\Bigg[-(-1)^N\frac{2\big(6-12N+7N^2+N^3\big)}{(-1+N)^3N^3}+(-1)^N\frac{6S_1^2}{(-1+N)N(1+N)(2+N)}$$

$$
\begin{aligned}
&+(-1)^N\frac{2(6-23N+9N^2+2N^3)}{(-1+N)^2N^2(1+N)(2+N)}S_1\\
&+(-1)^N\frac{30}{(-1+N)N(1+N)(2+N)}S_2\Bigg]\tilde{S}_{1,2}\left(\frac{1}{2},2\right)\\
&-(-1)^N\frac{6}{(-1+N)N(1+N)(2+N)}\tilde{S}_{1,1}\left(\frac{1}{2},1\right)\tilde{S}_{1,2}(2,1)\\
&+\left[(-1)^N\frac{2(6-23N+9N^2+2N^3)}{(-1+N)^2N^2(1+N)(2+N)}+(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}S_1\right]\\
&\times\left[\tilde{S}_{1,3}\left(\frac{1}{2},2\right)-\tilde{S}_{1,3}\left(2,\frac{1}{2}\right)\right]\\
&+\frac{(-1)^N}{(-1+N)N(1+N)(2+N)}\Bigg\{30\tilde{S}_1\left(\frac{1}{2}\right)\tilde{S}_{1,3}(2,1)+36\tilde{S}_{1,4}\left(\frac{1}{2},2\right)\\
&+\left[30S_1\tilde{S}_1\left(\frac{1}{2}\right)-30\tilde{S}_2\left(\frac{1}{2}\right)+30\tilde{S}_{1,1}\left(\frac{1}{2},1\right)\right]\tilde{S}_{2,1}(1,2)\Bigg\}\\
&+\Bigg\{\Bigg[(-1)^N\frac{2(6-23N+9N^2+2N^3)}{(-1+N)^2N^2(1+N)(2+N)}\\
&+(-1)^N\frac{24}{(-1+N)N(1+N)(2+N)}S_1\Bigg]\tilde{S}_1\left(\frac{1}{2}\right)\\
&+\frac{(-1)^N}{(-1+N)N(1+N)(2+N)}\Bigg[-12\tilde{S}_{1,1}\left(\frac{1}{2},1\right)\tilde{S}_{2,1}(2,1)+30\tilde{S}_{2,3}\left(\frac{1}{2},2\right)\\
&-12\tilde{S}_{2,3}\left(2,\frac{1}{2}\right)-18\tilde{S}_1\left(\frac{1}{2}\right)\tilde{S}_{3,1}(2,1)+30\tilde{S}_{3,2}\left(\frac{1}{2},2\right)-30\tilde{S}_{4,1}\left(\frac{1}{2},2\right)\Bigg]\\
&+\Bigg[(-1)^N\frac{2(6-12N+7N^2+N^3)}{(-1+N)^3N^3}-(-1)^N\frac{6S_1^2}{(-1+N)N(1+N)(2+N)}\\
&-(-1)^N\frac{2(6-23N+9N^2+2N^3)}{(-1+N)^2N^2(1+N)(2+N)}S_1\\
&-(-1)^N\frac{30}{(-1+N)N(1+N)(2+N)}S_2\Bigg]\left[\tilde{S}_{1,1,1}\left(\frac{1}{2},1,2\right)\tilde{S}_{1,1,1}\left(\frac{1}{2},2,1\right)\right]\\
&+\Bigg[-(-1)^N\frac{2(6-23N+9N^2+2N^3)}{(-1+N)^2N^2(1+N)(2+N)}\\
&-(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}S_1\Bigg]\tilde{S}_1\left(\frac{1}{2}\right)\tilde{S}_{1,1,1}(1,2,1)\\
&+(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}\tilde{S}_{1,1}\left(\frac{1}{2},1\right)\tilde{S}_{1,1,1}(2,1,1)
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-(-1)^N\frac{2(6-23N+9N^2+2N^3)}{(-1+N)^2N^2(1+N)(2+N)}-(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}S_1\right]\\
&\times\left[\tilde{S}_{1,1,2}\left(\frac{1}{2},1,2\right)-\tilde{S}_{1,1,2}\left(2,\frac{1}{2},1\right)-2\tilde{S}_{1,1,2}\left(2,1,\frac{1}{2}\right)\right]\\
&-\frac{(-1)^N}{(-1+N)N(1+N)(2+N)}\Bigg[66\tilde{S}_{1,1,3}\left(\frac{1}{2},1,2\right)+36\tilde{S}_{1,1,3}\left(\frac{1}{2},2,1\right)\\
&+30\tilde{S}_{1,1,3}\left(1,\frac{1}{2},2\right)+30\tilde{S}_{1,1,3}\left(1,2,\frac{1}{2}\right)\Bigg]\\
&+\left[(-(-1)^N\frac{4(6-23N+9N^2+2N^3)}{(-1+N)^2N^2(1+N)(2+N)}-(-1)^N\frac{24}{(-1+N)N(1+N)(2+N)}S_1\right]\\
&\times\left[\tilde{S}_{1,2,1}\left(\frac{1}{2},2,1\right)-\tilde{S}_{1,2,1}\left(2,\frac{1}{2},1\right)-\frac{1}{2}\tilde{S}_{1,2,1}\left(2,1,\frac{1}{2}\right)\right]\\
&-(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}\tilde{S}_1\left(\frac{1}{2}\right)\tilde{S}_{1,2,1}(1,2,1)\\
&-(-1)^N\frac{1}{(-1+N)N(1+N)(2+N)}\Bigg[30\tilde{S}_{1,2,2}\left(\frac{1}{2},1,2\right)+36\tilde{S}_{1,2,2}\left(\frac{1}{2},2,1\right)\\
&+48\tilde{S}_{1,3,1}\left(\frac{1}{2},2,1\right)+30\tilde{S}_{1,3,1}\left(1,\frac{1}{2},2\right)+30\tilde{S}_{1,3,1}\left(1,2,\frac{1}{2}\right)+30\tilde{S}_{2,1,2}\left(\frac{1}{2},2,1\right)\\
&+30\tilde{S}_{2,1,2}\left(1,\frac{1}{2},2\right)-12\tilde{S}_{2,1,2}\left(2,\frac{1}{2},1\right)-24\tilde{S}_{2,1,2}\left(2,1,\frac{1}{2}\right)+30\tilde{S}_{2,2,1}\left(1,2,\frac{1}{2}\right)\\
&-24\tilde{S}_{2,2,1}\left(2,\frac{1}{2},1\right)-12\tilde{S}_{2,2,1}\left(2,1,\frac{1}{2}\right)+30\tilde{S}_{3,1,1}\left(\frac{1}{2},1,2\right)+30\tilde{S}_{3,1,1}\left(\frac{1}{2},2,1\right)\Bigg]\\
&+\Bigg[(-1)^N\frac{2(6-23N+9N^2+2N^3)}{(-1+N)^2N^2(1+N)(2+N)}\\
&+(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}S_1\Bigg]\Bigg[\tilde{S}_{1,1,1,1}\left(\frac{1}{2},1,2,1\right)\\
&-2\tilde{S}_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)-2\tilde{S}_{1,1,1,1}\left(2,1,\frac{1}{2},1\right)-2\tilde{S}_{1,1,1,1}\left(2,1,1,\frac{1}{2}\right)\Bigg]\\
&+\frac{(-1)^N}{(-1+N)N(1+N)(2+N)}\Bigg[66\tilde{S}_{1,1,1,2}\left(\frac{1}{2},1,2,1\right)+48\tilde{S}_{1,1,1,2}\left(\frac{1}{2},2,1,1\right)\\
&+30\tilde{S}_{1,1,1,2}\left(1,\frac{1}{2},2,1\right)+30\tilde{S}_{1,1,1,2}\left(1,2,\frac{1}{2},1\right)+30\tilde{S}_{1,1,2,1}\left(\frac{1}{2},1,1,2\right)\\
&+12\tilde{S}_{1,1,2,1}\left(\frac{1}{2},1,2,1\right)+48\tilde{S}_{1,1,2,1}\left(\frac{1}{2},2,1,1\right)+30\tilde{S}_{1,1,2,1}\left(1,\frac{1}{2},1,2\right)\\
&+30\tilde{S}_{1,1,2,1}\left(1,2,1,\frac{1}{2}\right)+30\tilde{S}_{1,2,1,1}\left(\frac{1}{2},1,1,2\right)+30\tilde{S}_{1,2,1,1}\left(\frac{1}{2},1,2,1\right)
\end{aligned}
$$

$$+12\tilde{S}_{1,2,1,1}\left(\frac{1}{2},2,1,1\right)+30\tilde{S}_{1,2,1,1}\left(1,1,\frac{1}{2},2\right)+30\tilde{S}_{1,2,1,1}\left(1,1,2,\frac{1}{2}\right)$$
$$+30\tilde{S}_{2,1,1,1}\left(1,\frac{1}{2},1,2\right)+30\tilde{S}_{2,1,1,1}\left(1,\frac{1}{2},2,1\right)+30\tilde{S}_{2,1,1,1}\left(1,1,\frac{1}{2},2\right)$$
$$+30\tilde{S}_{2,1,1,1}\left(1,1,2,\frac{1}{2}\right)+30\tilde{S}_{2,1,1,1}\left(1,2,\frac{1}{2},1\right)+30\tilde{S}_{2,1,1,1}\left(1,2,1,\frac{1}{2}\right)$$
$$-24\tilde{S}_{2,1,1,1}\left(2,\frac{1}{2},1,1\right)-24\tilde{S}_{2,1,1,1}\left(2,1,\frac{1}{2},1\right)-24\tilde{S}_{2,1,1,1}\left(2,1,1,\frac{1}{2}\right)$$
$$-12\tilde{S}_{1,1,1,1,1}\left(\frac{1}{2},1,2,1,1\right)-36\tilde{S}_{1,1,1,1,1}\left(\frac{1}{2},2,1,1,1\right)\Bigg]$$
$$+\Bigg\{(-1)^N\frac{3S_1^2}{(-1+N)N(1+N)(2+N)}+(-1)^N\frac{2+9N-5N^2}{(-1+N)^2N(1+N)(2+N)}$$
$$+(-1)^N\frac{6-11N+2N^2}{(-1+N)^2N^2(2+N)}S_1$$
$$+\left[\frac{3(-1)^N}{(N-1)N(N+1)(N+2)}+\frac{1}{N+2}\right]S_2\Bigg\}\zeta_3$$
$$+\zeta_2\Bigg\{\Big((-1)^N\frac{9S_1^2}{2(-1+N)N(1+N)(2+N)}$$
$$+(-1)^N\frac{2+3N-2^{2+N}N-2N^2-3 2^{1+N}N^2+6N^3-2^{1+N}N^3-3N^4}{(-1+N)^3N^2(1+N)(2+N)}$$
$$+\Bigg\{(-1)^N\frac{-12+N+27N^2-4N^3}{2(-1+N)^2N^2(1+N)(2+N)}$$
$$+\left[\frac{1}{-N-2}-\frac{6(-1)^N}{(N-1)N(N+1)(N+2)}\right]S_1$$
$$+\frac{1}{N(N+2)}\Bigg\}S_2+\left[-\frac{6(-1)^N}{(N-1)N(N+1)(N+2)}-\frac{2}{N+2}\right]S_3$$
$$-\frac{2S_{-2}}{N(N+2)}+\Bigg[(-1)^N\frac{-6+3N+18N^2-20N^3-3N^4+2N^5}{(-1+N)^3N^3(1+N)(2+N)}$$
$$+\frac{2S_{-2}}{N+2}\Bigg]S_1+\frac{3S_{-3}}{N+2}+(-1)^N\frac{12}{(-1+N)N(1+N)(2+N)}S_{2,1}$$
$$-\frac{2S_{-2,1}}{N+2}+\Big(-(-1)^N\frac{3S_1^2}{(-1+N)N(1+N)(2+N)}$$
$$+(-1)^N\frac{6-12N+7N^2+N^3}{(-1+N)^3N^3}+(-1)^N\frac{-6+23N-9N^2-2N^3}{(-1+N)^2N^2(1+N)(2+N)}$$
$$S_1\Big)\tilde{S}_1(2)$$
$$+\Big((-1)^N\frac{-6+23N-9N^2-2N^3}{(-1+N)^2N^2(1+N)(2+N)}$$
$$-(-1)^N\frac{6}{(-1+N)N(1+N)(2+N)}S_1\Big)\tilde{S}_2(2)$$
$$+\Big((-1)^N\frac{6-23N+9N^2+2N^3}{(-1+N)^2N^2(1+N)(2+N)}$$
$$+(-1)^N\frac{6}{(-1+N)N(1+N)(2+N)}S_1$$
$$-(-1)^N\frac{3}{(-1+N)N(1+N)(2+N)}\tilde{S}_3(2)\Bigg]\tilde{S}_{1,1}(2,1)$$
$$+\frac{(-1)^N}{(-1+N)N(1+N)(2+N)}\left[3\tilde{S}_{1,2}(2,1)-15\tilde{S}_{2,1}(1,2)+6\tilde{S}_{2,1}(2,1)-6\tilde{S}_{1,1,1}(2,1,1)\right]$$
$$+\left[\frac{3}{N+2}-\frac{18(-1)^N}{(N-1)N(N+1)(N+2)}\right]\zeta_3\Bigg\}$$
$$+\left[\frac{27(-1)^N}{(N-1)N(N+1)(N+2)}-\frac{9}{2(N+2)}\right]\zeta_5\,,$$

Generalized Harmonic Sums :

$$S_{b,\vec{a}}(x_1,\vec{y};N) = \sum_{k=1}^{N} \frac{{x_1}^k}{k^b} S_{\vec{a}}(\vec{y};k)$$

- quasi-shuffle algebra
- differential relations
- duplication relations
- associated generalized HPL
- analytic continuation to $N \in \mathbb{C}$

J. Ablinger, J. Blümlein, C. Schneider, 2010

5. Difference Equations from Feynman Integrals

For the type of Feynman integrals being considered only their dependence on N and ε is of relevance.

$$F(N,\varepsilon) = \int_0^1 dx_1 ... \int_0^1 dx_k \sum_{i=1}^{j} (P_i(x_1,...,x_k,\varepsilon))^N f_i(x_1,...,x_k,\varepsilon)\Theta(x_1,...,x_k)$$

- How can one find the associated minimal DEQ?
- One can find a DEQ, $\Longrightarrow$ Almquvist, Zeilberger 1990.
- Implementations: C. Koutschan: HolomonicFunctions. J. Ablinger

An Example :

$$F(N,\varepsilon) = \int_0^1 dz \int_0^1 dw z^{-\varepsilon/2} w^{-\varepsilon/2-1} \left[1 - w^{N+1} - (1-w)^{N+1}\right] \times (1-z)^{\varepsilon/2} (w + z - wz)^{\varepsilon - 1}$$

Derive the DEQ :

$\Longrightarrow$ Mathematica work sheet 2

$$
\begin{aligned}
&[(-3+e-n)(-2+e-n)(4+e+2n)(6+e+2n)]F[n+3]\\
&-[(-2+e-n)(4+e+2n)(-34+5e+e^2-28n+2en-6n^2)]F[n+2]\\
&+[(2+n)(72-28e-6e^2+e^3+116n\\
&-30en-3e^2n+64n^2-8en^2+12n^3)]F[n+1]\\
&+2[(-2+e-2n)(1+n)(2+n)^2]F[n] \qquad = 0
\end{aligned}
$$

Solve the DEQ :

$\Longrightarrow$ Mathematica work sheet 3

6. From Sums to Complex Functions

The basic functions need to be represented for $N \epsilon \mathbf{C}$.

General behaviour for complex N:

Meromorphic functions: with poles at the non–negative integers.

Representation: trough factorial series, $\psi^{(k)}$-functions and polynomials out of both.

$$\Omega(x) = \sum_{k=0}^{\infty} \frac{k! a_k}{x(x+1)\dots(x+k)}$$

The basic functions need to be represented for $N \epsilon \mathbf{C}$.

Three possibilities:

(i) highly accurate semi-analytic representations through `MINIMAX` polynomials in x

J. Blümlein, Comput.Phys.Commun. 133 (2000) 76.

J. Blümlein and S. Moch, Phys.Lett. B614 (2005) 53.

Example :

$$\mathbf{M}\left[\frac{\mathrm{Li}_2^2(-x) - \zeta_2^2/4}{1-x}\right](N) = \sum_{k=0}^{11} \frac{b_k^{(2)}}{N+k+1}$$

Inversion for $x \epsilon [10^{-6}, 0.98]$ more accurate than $5 \cdot 10^{-8}$.

$$\begin{array}{lcrlcr}
b_0^{(2)} & = & -0.6764520210934552\mathrm{D}{-0} & b_1^{(2)} & = & -0.6764520137562308\mathrm{D}{-0} \\
b_2^{(2)} & = & 0.3235476094265664\mathrm{D}{-0} & b_3^{(2)} & = & -0.1764446743143206\mathrm{D}{-0} \\
b_4^{(2)} & = & 0.1081940672246993\mathrm{D}{-0} & b_5^{(2)} & = & -0.7181309059958118\mathrm{D}{-1} \\
b_6^{(2)} & = & 0.4940999469881481\mathrm{D}{-1} & b_7^{(2)} & = & -0.3290941711692155\mathrm{D}{-1} \\
b_8^{(2)} & = & 0.1916664887064280\mathrm{D}{-1} & b_9^{(2)} & = & -0.8589741767655388\mathrm{D}{-2} \\
b_{10}^{(2)} & = & 0.2508898780543465\mathrm{D}{-2} & b_{11}^{(2)} & = & -0.3476710199486832\mathrm{D}{-3}
\end{array}$$

(ii) Known functional forms :

Analytic continuation of single harmonic sums :

$$S_k(N) = \frac{(-1)^k}{(k-1)!}\psi^{(k)}(N+1) + c_k; \quad c_1 = \gamma_E, c_k = \zeta_k, k \geq 2$$

$$S_{-k}(N) = \frac{(-1)^{(k+N)}}{(k-1)!}\beta^{(k)}(N+1) + d_k; \quad d_1 = -\ln(2); d_k = -(1-1/2^{k-1})\zeta_k, k \geq 2$$

$$\beta(N) = \frac{1}{2}\left[\psi\left(\frac{N+1}{2}\right) - \left(\frac{N}{2}\right)\right]$$

Recursion relation :

$$\psi(N+1) = \psi(N) + \frac{1}{N}$$

Asymptotic representation :

$$\psi(N) = \ln(N) - \frac{1}{2N} - \frac{1}{12N^2} + \frac{1}{120N^4} - \frac{1}{256N^6} + \frac{1}{240N^8} + O\left(\frac{1}{N^{10}}\right)$$

analytic representations :

J. Blümlein, Comput.Phys.Commun. 180 (2009) 2218; Clay Inst. Proc: arXiv:0901.0837
J. Ablinger, J. Blümlein, C. Schneider, 2010

(iii) New functions :

Example :

$$F_5(N) := \mathbf{M}\left[\left(\frac{S_{1,2}(x)}{1-x}\right)_+\right]; \qquad S_{1,2}(x) \leftrightarrow \mathrm{Li}_3(1-x)$$

Recursion relation :

$$F_5(N+1) = -F_5(N) + \frac{\zeta_3}{N+1} - \frac{S_1^2(N+1) + S_2(N+1)}{2(N+1)^2}$$

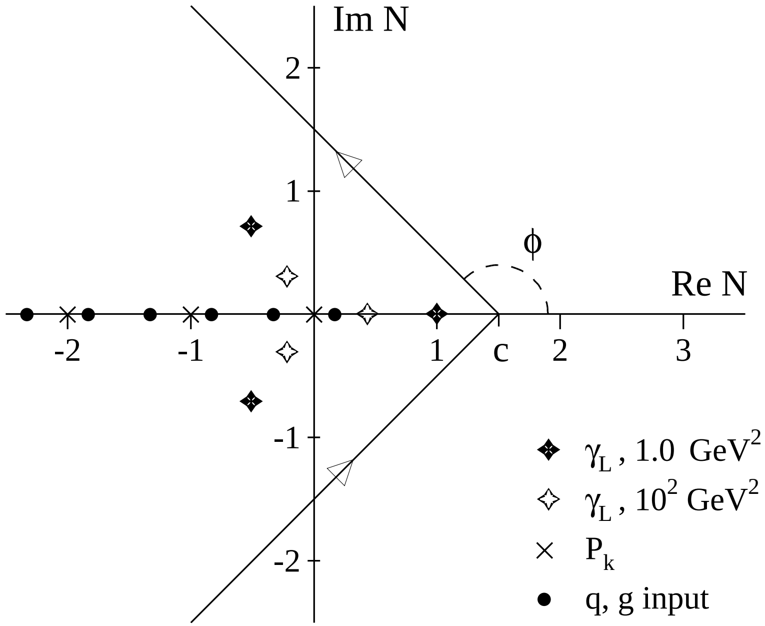
Asymptotic representation :

$$\mathbf{M}\left[\frac{\mathrm{Li}_3(1-x)}{1-x}\right](N) = \frac{1}{N} + \frac{1}{8N^2} - \frac{11}{216N^3} - \frac{1}{288N^4} + \frac{1243}{54000N^5} - \frac{49}{7200N^6} + O\left(\frac{1}{N^7}\right)$$

Apply :

$$F_5(N) = -\mathbf{M}\left[\frac{\mathrm{Li}_3(1-x)}{1-x}\right](N) + \frac{\zeta_2}{2}\left[S_1^2 + S_2\right] - 2\zeta_3 S_1 + \frac{2}{5}\zeta_2^2 + \frac{1}{2}\left[S_1 S_3 - S_1^2 S_2\right]$$
$$+S_4 - S_1 S_{2,1} - 2S_{3,1}, \qquad S_{k_1,\ldots,k_m} \equiv S_{k_1,\ldots,k_m}(N) .$$

Mellin inversion:



$$f(x) = \frac{1}{\pi} \int_0^\infty dz Im \left[e^{i\phi}\, x^{-C} \mathbf{M}[f](N = C) \right], \quad C = c + z e^{i\phi}.$$

7. Summary

- The single scale Feynman parameter integrals $F(N, \varepsilon)$ are recurrent quantities.
- Starting from the Feynman rules - including those for the local operators - one may easily perform all momentum integrals.
- The integration of the Feynman parameter integrals is understood in case of simple topologies in the general N case.
- Most likely all the Feynman parameter integrals imply hypergeometric structures in general.
- Current research targets at finding the associate low(est) order difference equation for the respective integral.
- The efficient computation of the initial values is an issue in its own right.
- Generalizations of harmonic sums do definitely appear in intermediary results.
- Analytic continuation to complex values of N is a solved issue.
- May one compute all Feynman integrals just referring to the functions appearing in the result ?