The Lie- and Hopf-algebras of quantum fields: numbers, polylogs, Dyson-Schwinger eqs

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- "mixed Tate Hodge structure for polylog"
  (S. Bloch "Function theory of polylogs")
  " H. Gangl thanks!
- DSE for polylogs
- ambiguities and ambiguities
- similarities with QFT
- QFT example

Bergbauer, Broadhurst, Connes, Ebrahimi-Fard, Foissy, Marcolli, ...

Bloch, Goncharov, Hoang, ... Zagier

Violations of scaling and MZV's in QFT

Dirk Kreimer

- some remarks on polylog vs DSE
- DSE, Renormalization, Hopf algebra, Hochschild, ...
- what shows up solving DSE's?
- MZVs from non-vanishing β-fct ⟺ perturbative progression

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

M_{ij}

$$\begin{array}{cccc|c}
1 & 0 & 0 & 0 & \alpha^0 \\
-\mathrm{Li}_1(z) & 2\pi i & 0 & 0 & \alpha^1 \\
-\mathrm{Li}_2(z) & 2\pi i \cdot \log z & (2\pi i)^2 & 0 & \alpha^2 \\
-\mathrm{Li}_3(z) & 2\pi i \,\frac{(\log z)^2}{2!} & (2\pi i)^2 \log z & (2\pi i)^3 & \alpha^3 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\rightarrow \ln^k \;.. & \ln^{k-1} \;\ldots & \ldots \;\ln^1 & \ldots \;\ln^0 &
\end{array}$$

DSE for Li_n:

$$F(\alpha, a) = \frac{a}{1-a} - \alpha \int_0^a dz \, \frac{F(\alpha, z)}{z}$$

Setting $F(\alpha, a) = \sum_{k=0}^{\infty} \alpha^k f_k(a)$

$bB_+ = 0$

$\Delta \circ B_+ = B_+ \otimes 1 + (\mathrm{id} \otimes B_+)\Delta$

$$\Rightarrow \quad f_0 = \frac{a}{1-a}$$

$$f_1 = -\int_0^a dz \, \frac{1}{1-z} = + \log(1-a)$$

$\vdots$

$$\phi(t_n) \equiv \phi(B_+(t_{n-1}))(z)$$
$$= \int \frac{\phi(t_{n-1})(x)}{x} \, dx$$

$$X = c_1 \mathbb{1}_H + \alpha\, B_+\left(\frac{1}{c_1}\bar{e}(X) + P(X)\right)$$

$$\Leftrightarrow \quad = c_1 \mathbb{1}_H + \alpha\, B_+(\mathbb{1}_H) + \alpha^2\, B_+(B_+(\mathbb{1}_H)) + \dots$$

$$h = h_{aug} + c\,\mathbb{1}_H$$

$\phi(X)$:

$$\phi(B_+(h))(z) = \int_0^z \frac{\phi(h_{aug})(x)}{x}\,dx + c\int_0^z \frac{\phi(\mathbb{1}_H)}{x-1}\,dx$$

$\phi(B_+)$ in tegral operator

Define characters

- $\phi(t_n) = Li_n(z) \qquad L(t_n) = \dfrac{\ln^n(t)}{n!}$

$$\tilde{a}_p(z) := \left[Li_p(z) - \dots + (-1)^j\, Li_{p-j}(t)\, \frac{\ln^j(t)}{j!} + \dots + (-1)^{p-1}\, Li_1(z)\, \frac{\ln^{p-1}(z)}{(p-1)!}\right.$$

$$\tilde{a}_p(z) = m \circ (L^{-1} \otimes Li) \circ (id \otimes P) \circ \Delta(t_p)$$

$L^{-1} \equiv L \circ S$, S the antipode

P projection into H_{aug}

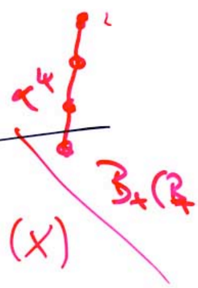
another ambiguity : removal of divergences

$$F(\alpha, a; z) = 1 - \alpha \int_a^\infty dx \; x^{-z-1} \; F(\alpha, x; z)$$

1	0	0	α^0
$-\ln a$	$\frac{1}{z}$	0	α^1
$+\ln^2 a$	$-\frac{1}{z} \ln a$	$\frac{1}{2z^2}$ ...	α^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\left(= \left\{ \exp \frac{\alpha}{z} \right\} \left\{ \exp -\alpha \ln a \right\} \left\{ \exp \alpha z \ln^2 a \right\} \{ \ldots \} \ldots \right)$$

$$F_R = Z(\alpha) - \alpha \int_a^\infty dx \; x^{-z-1} \; F_R(\alpha, x; z)$$



$$bB_+ = 0$$

$$X = 1 + \alpha B_+(X)$$

$$Z(\alpha) = S_R^\phi(X)$$

$$F_R = S_R^\phi * \phi$$

$$Z(\alpha) = \exp -\alpha z$$

$$F_R(\alpha, a; 0) = a^{-\alpha}$$

$$F_R = 1 - \alpha \int_1^a \frac{F(\alpha, x)}{x} dx$$

ambiguity: $a \to \tilde{a} = a \, e^{\gamma(\alpha)}$

$$\gamma(\alpha) = \sum_{k=0}^\infty \gamma_k \alpha^k$$

$$\Delta \circ B_+ = B_+ \otimes 1 + (\mathrm{id} \otimes B_+) \Delta$$

$$L \circ S = S_R^{Li} = -R\left[m \circ (S_R^{Li} \otimes Li)(\mathrm{id} \otimes P) \circ \Delta \right]$$

$$\equiv -R[\overline{Li}] \qquad \overline{Li} \leftrightarrow \overline{R}$$

$$R[Li] \equiv L$$

$$\Delta[t_2] = t_2 \otimes 1_H + 1_H \otimes t_2 + t_1 \otimes t_1$$

$$\overline{Li}(t_2) = Li(t_2) - L(t_1)\, Li(t_1)$$

$$= Li_2(z) - \log z \; Li_1(z)$$

roles of B_+:
- eq. of motion
- locality <-> ambiguities
- algebraic structures like shuffle algebras

Hopf algebra

$\downarrow$

DSE

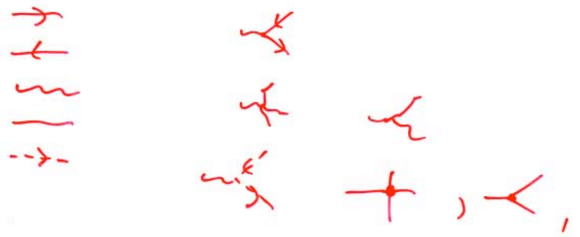
$\downarrow$

control of ambiguities thanks to B_+ operators (Hochschild cohomology) (gradings by α, $\ln^k$)

$\downarrow$

solution to DSE

Determine edges and vertices



free QFT

local renormalizable interactions

set of 1PI graphs

Hopf algebras, Lie algebras

quantum equations of motion?

(usually, functional integral or Gell-Mann-Low formula)

Hochschild cohomology

↓

non-perturbative DSE

-> identify primitives

-> Hochschild one-cocycles

-> DSE, locality, Slavnov-Taylor

-> no short-distance singularities

↳ self-regulated

$\mathcal{R} = \{$  $\}$

free propagators and local interactions given

$\Downarrow$

set of 1PI graphs and power counting given

$\forall\, r \in \mathcal{R}$

$$\Gamma^{r} = 1 + \sum_{\Gamma} (g^2)^{|\Gamma|} \frac{\Gamma}{\mathrm{Sym}(\Gamma)}$$

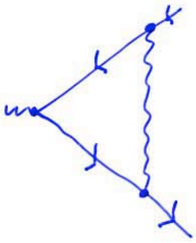
formal series with coeffs in H

where to get it from?

$$G^{r} = 1 + \sum_{\Gamma} (g^2)^{|\Gamma|} \frac{\phi(\Gamma)}{\mathrm{Sym}(\Gamma)}$$

Green-function in perturbation theory

Can we get the DSE from $\mathcal{R}$?

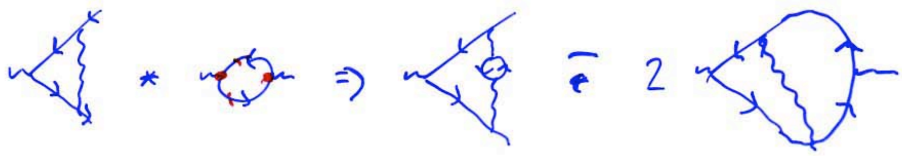


$\Gamma^{[1]}_{int}$ int. edges

$\Gamma^{[1]}_{ext}$ ext. edges

$\Gamma^{[0]}$ vertices

$\Rightarrow$ pre-Lie bracket, Lie bracket, Hopf algebra



$$[a,b] = a*b - b*a$$

$$(a*b)*c - a*(b*c) = (a*c)*b - a*(c*b)$$

Hopf algebra

$$\Delta[\Gamma] = \Gamma \otimes e + e \otimes \Gamma + \sum_\gamma \gamma \otimes \Gamma/\gamma$$

$$S[\Gamma] = -\Gamma - \sum_\gamma S(\gamma)\,\Gamma/\gamma$$

$$\phi: H \to V_{\underline{R}} \qquad \underbrace{R[xy]}_{\phi \circ S} + R[x]R[y] = R[R[x]y] + R[xR[y]]$$

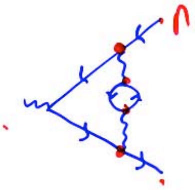
twist to ϕ_-

$$\underbrace{S^\phi_R(\Gamma) = -R\left[\phi(\Gamma) + \sum_\gamma S^\phi_R(\gamma)\,\phi(\Gamma/\gamma)\right]}_{\phi_-}$$

character group

$$S^\phi_R * \phi = \underbrace{(id - R)\,[\qquad " \qquad]}_{\phi_+}$$

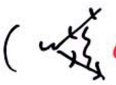
$\int_a^{\infty} \frac{dx}{x}$

 $\equiv \Gamma$

$\Delta[\Gamma] = \Gamma \otimes e + e \otimes \Gamma +$  $\otimes$ 

$$\phi(\Gamma) = \int \underbrace{\prod_e \mathrm{Prop}(k_e) \prod_v \delta\Big(\sum_{j \cup \partial j} k_j\Big)}_{\mathcal{D}(\Gamma)}$$

$S_R^\phi(\Gamma) = -R\big[\phi(\Gamma)\big] + S_R^\phi($  $)\ \phi($  $)\big]$

$= -R\Big[\int \mathcal{D}($  $)\ \underbrace{S_R^\phi * \phi(}_{}$  $)\Big]$

$[\mathrm{id} - R]\int \mathcal{D}($  $)$

$S_R^\phi * \phi(\Gamma) = \phi(\Gamma) - R[\phi(\Gamma)] - \phi($  $)\ R[\phi($  $)]$

$+ R[\phi($  $)\ R[\phi($  $)]]$

$= [\mathrm{id} - R]\Big\{\phi(\Gamma) - R[\phi($  $)]\ \phi($  $)\Big\}$

$$R[ab] + R[a]R[b] = R[R[a]\,b] + R[a\,R[b]]$$

Locality

$$b\, B_+^{k,\Gamma} = 0$$

$$\Rightarrow \quad \Delta B_+^{k,\Gamma} = \underline{B_+^{k,\Gamma} \otimes e} + (\mathrm{id} \otimes B_+^{k,\Gamma}) \cdot \Delta$$

counterterm

augm. ideal

$$S_R^\phi = -R\, \underbrace{\left[m_v \cdot (S_R^\phi \otimes \phi)(\mathrm{id} \otimes P)\, \Delta \right]}_{\bar{R}_\phi} \qquad P(e) = 0$$

$$\bar{R}_\phi [B_+^{\cdots}(X)]$$

$$= m_v (S_R^\phi \otimes \phi)(\mathrm{id} \otimes B_+^{\cdots})\, \Delta(X)$$

$$\Downarrow$$

$$\bar{R}_\phi (B_+^{\cdots}(X)) = \int D(\ldots)\; S_R^\phi * \phi\, (X)$$

induction proof of locality by induction over the augmentation degree "# of (sub-) divs"

start of induction: primitive graphs can be renormalized by local c.t.

$\exists$ $H_{\mathcal{R}}$ Hopf algebra for $\mathcal{R}$

and a pre-Lie structure of graph insertions

$$\Delta[\gamma] = \gamma \otimes e + e \otimes \gamma$$

$$B_+^{\gamma} : H \to H$$

section coefficients of $H_{\mathcal{R}}$

$$\frac{1}{\mathrm{sym}(\gamma)} B_+^{\gamma}(\underline{\underline{X}}) = \sum_{\Gamma} \frac{n(X, \gamma, \Gamma)}{\#\text{ insertions} \cdot \text{max forests }(\Gamma)} \Gamma$$

there are distinguished graphs:

$$\Delta(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$$

for each γ in $H_{\mathcal{R}}$, $\exists$ $X_\gamma = \Gamma^{\mathrm{res}(\gamma)} \underbrace{\prod_{m} (\Gamma^{m})^{\frac{n_m}{\gamma}}}_{(X_{\mathrm{coupl}})^{|\gamma|}}$

let $$B^{k,r} = \sum_{\substack{|\gamma| = k \\ \mathrm{res}(\gamma) = r}} \frac{1}{\mathrm{sym}(\gamma)} B_+^{\gamma}$$

$$\Gamma^{r} = 1 + \sum_{\gamma} \frac{|g^2|^{|\gamma|}}{\mathrm{sym}(\gamma)} B_+^{\gamma}(X_\gamma) = 1 + \sum_{k} B_+^{k,r}(X_\gamma)$$

is the Dyson-Schwinger equation.

$$X_\gamma = \Gamma^{r} (X_{\mathrm{coupl}})^{|\gamma|} \leftarrow \text{loop number}$$

$$X_{\mathrm{coupl}} = \frac{X^{\mathrm{vertex}}}{\prod (X^{\mathrm{prop}})^{1/2}}$$

sub-Hopf algebras

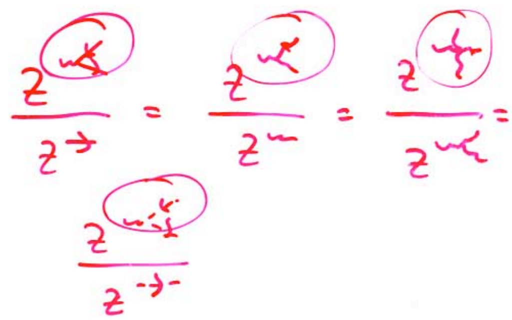
let $\Gamma^{\underline{r}} = 1 \pm \sum_k g^{2k} c_k^{\underline{r}}$ $\qquad S_R^{\phi}(\Gamma^{\underline{r}})$

$\Delta c_k^{\underline{r}} = \mathrm{Pol}_j(\{c_m^{\underline{\ell}}\}) \otimes c_{k-j}^{\underline{r}}$

this is highly non-trivial in theories with more than one vertex.

Connection with gauge symmetry

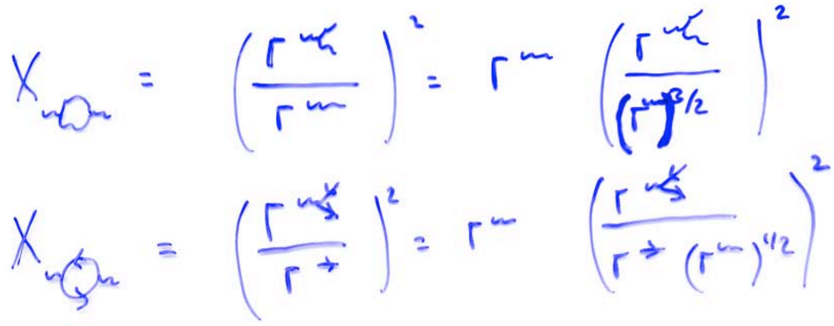
$X_v = X_{v'}$ $\qquad \boxed{\ker \phi_-}$



where $X_v = \dfrac{\Gamma^{v}}{\prod_{\underline{m} \in v^{[1]}_{ext}} (\Gamma^{\underline{m}})^{1/2}}$

$\Downarrow$

sub Hopf algebra exists and is well-defined



$\vdots$

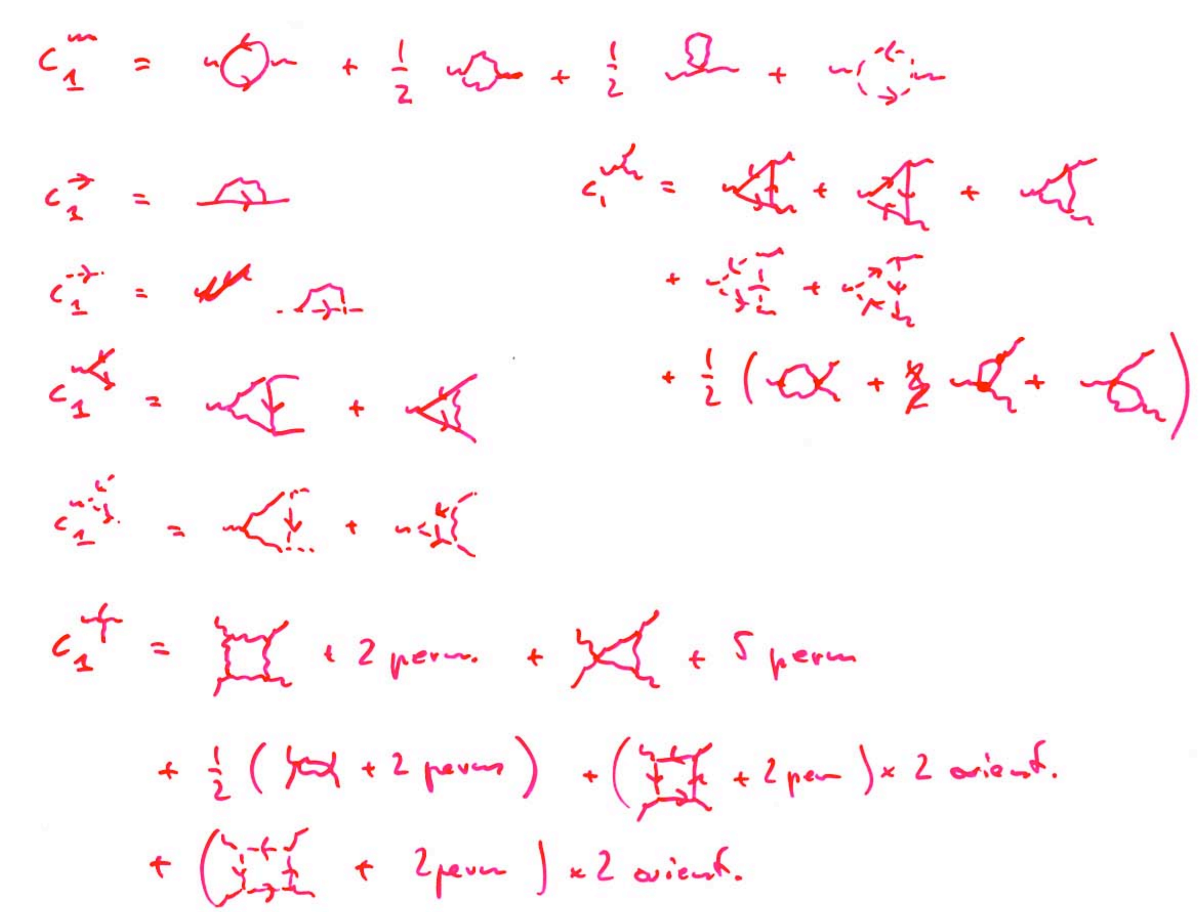
$$\Gamma^{\underline{r}} = 1 + \sum_{\substack{\gamma \text{ primitive} \\ \mathrm{res}(\gamma) = \underline{r}}} \frac{(g^2)^{|\gamma|} \; B_+^{\gamma}(X_\gamma)}{\mathrm{Sym}(\gamma)} = 1 + \sum_{k \geq 1} g^{2k} c_k^{\underline{r}}$$

B_+^{γ} : normalized insertions into γ

s.t. $\quad b B_+^{\gamma} = 0$

X_γ products of $\Gamma^{\underline{r}}$

QCD



so

$$\Gamma^{\Sigma} \quad (\Gamma)^2 \, (\Gamma)^2$$

$$c_2 = \frac{1}{2} B_+ \left(X \right) + \frac{1}{2} B_+ \left(X \right)$$

$$+ B_+ \left(X \right) + B_+ \left(X \right)$$

expand $X_{\dots}$ to order g^2

$$c_2 = \frac{1}{2} B_+ \left(2 c_1 + 2 c_1 \right)$$

$$+ \frac{1}{2} B_+ \left(c_1 + c_1 \right)$$

$$+ B_+ \left(2 c_1 + 2 c_1 \right)$$

$$+ B_+ \left(2 c_1 + 2 c_1 \right)$$

where

$$X = (\Gamma)^2 \, (\Gamma)^{-2}$$

$$X = \Gamma \, (\Gamma)^{-1}$$

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$$X = (\Gamma)^2 \, (\Gamma)^{-2}$$

you get

$$\Delta \dot{c}_2^{\,m} = c_2^{\,m} \otimes e + e \otimes c_2^{\,m}$$

$$+ \tfrac{1}{2}\left(2\, c_1 + 2\, c_1\right) \otimes$$

$$+ \tfrac{1}{2}\left(c_1 + c_1\right) \otimes$$

$$+ \left(2\, c_1 + 2\, c_1\right) \otimes$$

$$+ \left(2\, c_1 + 2\, c_1\right) \otimes$$

$$S_R^{\phi}\left(c_2^{\,m}\right) =$$

$$- R\left[\phi\left(c_2^{\,m}\right)\right] - \tfrac{1}{2} R\left[S_R^{\phi}\left(2\, c_1 + 2\, c_1\right)\phi\left(\right)\right]$$

Using $$\frac{S_R^{\phi}\left(\Gamma\right)}{S_R^{\phi}\left(\Gamma\right)} = \frac{S_R^{\phi}\left(\Gamma\right)}{S_R^{\phi}\left(\Gamma\right)} = \ldots$$

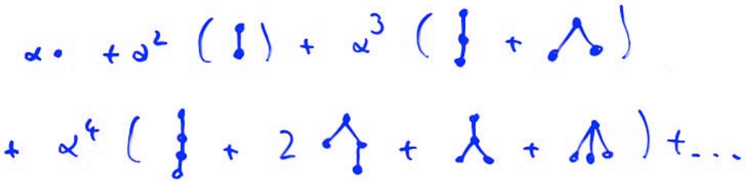
$$= S_R^{\phi}\left(X_{gauge}\right)$$

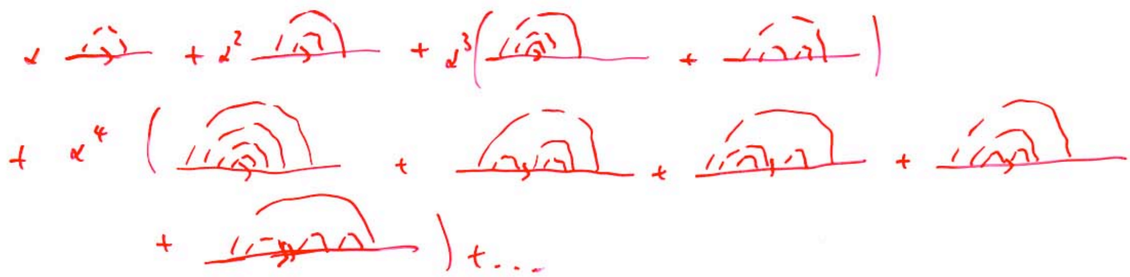
$$\hookrightarrow \quad \Delta c_2^{\,m} = c_2^{\,m} \otimes e + e \otimes c_2^{\,m} + c_1^{\,gauge} \otimes c_1^{\,m}$$

$H \triangleq$ rooted trees, $\quad B_+ : H \to H \quad bB_+ = 0$

$$X = \alpha B_+\left(\frac{1}{1-X}\right)$$

$\Rightarrow X =$





$$G\left(\alpha, \frac{q^2}{\mu^2}\right) = 1 - \Sigma\left(\alpha, \frac{q^2}{\mu^2}\right) \qquad L = \log \frac{q^2}{\mu^2}$$

$$\frac{\partial \log G}{\partial L} = \gamma(\rho)\Big|_{\rho = \frac{\alpha}{(1-\Sigma)^2}} \quad [\alpha, L] \quad \leftarrow X^2_{\text{couple}}$$

where $\gamma = \left.\frac{\partial \log G}{\partial L}\right|_{L=0} \equiv \gamma(\alpha)$

$$\alpha = \left(\frac{g}{4\pi}\right)^2$$

DSE + RG

$$\sqrt{\frac{\alpha}{2\pi}} = \exp(\rho_0^2)\,\mathrm{erfc}(\rho_0)$$

$$\rho_0 = \frac{\gamma + 2}{\sqrt{2\alpha}} \qquad \mathrm{erfc}(\rho_0) = \frac{2}{\sqrt{\pi}} \int_{\rho_0}^{\infty} dx\, e^{-x^2}$$

For a primitive graph γ

set

$$\mathrm{res}(\gamma) := \lim_{\rho \to 0} \rho \int d^D k \left(\frac{k^2}{\mu^2}\right)^{-\rho} \mathrm{Ker}[k, q]\Big|_{q^2=\mu^2}$$

and $$\mathcal{L}_k(\gamma) := \lim_{\rho \to 0} \rho^k \int d^D k \log^{k-1}\left(\frac{k^2}{\mu^2}\right) \left(\frac{k^2}{\mu^2}\right)^{\rho} \mathrm{Ker}[k, q]\Big|_{q^2=\mu^2}$$

$$L = \log \frac{q^2}{\mu^2} \qquad |\gamma| = \alpha^{\#(\gamma)}$$

$$\zeta(\alpha; L) = \prod_\gamma \frac{1}{1 - e^{|\gamma| \mathrm{res}(\gamma) L}}$$

$$L_k(\alpha) = \prod_\gamma \frac{1}{1 - \mathcal{L}_k(\gamma)|\gamma|}$$

$$G(\alpha, L) \Leftrightarrow \zeta(\alpha, L) \prod_{k=1}^{\infty} L_k(\alpha)$$

missing: action of permutations of insertion places.

let $G_1(\alpha; z) = 1 + \alpha \int_0^\infty\int_0^\infty G_1(\alpha; x) \frac{1}{(y+x+z)^2} \frac{dx\,dy}{\pi}$ - subtr. at $z=1$

$$G_2(\alpha; z) = 1 + \alpha \iint G_2(\alpha; x)^{1/2} G_2(\alpha; y)^{1/2} \frac{1}{(y+x+z)^2} \frac{dx\,dy}{\pi} - \ldots$$

$$G_i(\alpha; z) = z^{-2\pi \gamma_i(\alpha)}$$

$$\hookrightarrow \quad 1 = \frac{\alpha}{\sin(2\pi\gamma_1(\alpha))}$$

$$\Rightarrow \quad \gamma_1 = \sum_{k\geq 1} c_k \alpha^k \qquad c_k \in \mathbb{Q}$$

$$\gamma_2 = \sum_{k\geq 1} d_k \alpha^k \qquad d_k \in \mathbb{Q}[\zeta(k)]$$

log-insertion along blobs

$\Leftrightarrow$ permutation of blobs $\to$

transc. extensions

$$\log^{r_1}(k_1^2)\, P(k_1^2)\, \log^{r_2}(k_2^2)\, P(k_2)$$

$$\updownarrow$$

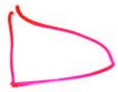
$$\log^{r_2}(k_1^2)\, P(k_1^2)\, \log^{r_1}(k_2^2)\, P(k_2)$$

$$\sum_{\Gamma \in \mathcal{R}} r^{\Gamma} \to \prod_{\gamma}^{\cdot} \frac{1}{1-\gamma}$$

each Euler factor
is a sub-Hopf algebra

$$G^{\Gamma}_{\gamma} = 1 + B_{+}^{\gamma}(G^{\Gamma}_{\gamma})$$

absorb short distance singularities in
2pt vertex functions



$$\frac{G^{\Gamma\to 2}}{\prod_{m \in \Gamma^{[1]}_{ext}} (G^{m})^{1/2}} \to \text{const for fixpt}$$

$$G^{\Gamma\to 2} = 1 + \sum_{\gamma} \frac{(g^2)^{|\gamma|}}{\mathrm{Sym}(\gamma)} \int D(\gamma)\, G^{\Gamma\to 2} \underset{\substack{\downarrow \\ \text{const}}}{(G_{\text{coupl}})^{|\gamma|}}$$

$\downarrow$

cocommutative Hopf algebra, at fixpt,
self-regulation by anom dim.

but even away from fixpoint you can
solve DSE once you know $\int D(\gamma)$.