The Hopf algebra of rooted trees

**Def. 1.** A rooted tree is a connected and simply connected set of edges and vertices such that each edge connects two vertices. We distinguish one vertex and orient all edges away from that vertex. Let \( \text{out}(v) \) be the number of edges outgoing from \( v \).

We consider the free commutative algebra of polynomials in rooted trees. The product is induced by the disjoint union, and we write \( e \) for the unit in this algebra (\( e \) from empty set).

For each rooted tree with root \( r \), we have \( \text{out}(r) \) rooted trees attached to the root.

\[
\text{out}(r) = 0 \quad \text{if } r = 1 \quad \text{if } r = 2 \quad \ldots
\]

\[
\begin{array}{c}
\vdots \\
\end{array}
\]

Let \( H \) be the algebra of rooted trees.

\( H \) is graded by the number of vertices:

\[
H = \bigoplus_{k \geq 0} H^{(k)}
\]

\[
H^{(0)} = \{ e \} \quad H^{(1)} = \{ \ldots \} \quad H^{(2)} = \{ \ldots \}
\]

\[
\vdots
\]
$H_L$ is the linear subspace.

$\emptyset$ is in $H_L$ and $H^{(2)}_L$, $\emptyset$ is in $H^{(2)}_L$ and in $H_L$.

Define a linear map $B : H \to H_L$ by

$$B_-(\emptyset) = 0, \quad B_-(\emptyset) = e, \quad B_-(T) = \cup_{i=1}^{\text{deg}(T)} T_i, \quad \text{deg}(T_i) = 1.$$

where $T_i$ are the trees attached to the root of $T$.

For $X \in H$, extend $B_-$ by a leibniz rule:

$$B_-(X_1 X_2) = X_1 B_-(X_2) + B_-(X_1) X_2.$$  

$$B_-(\emptyset) = e, \quad B_-(\emptyset) = 0, \quad B_-(\emptyset) = \emptyset, \quad B_-(\emptyset) = \ldots.$$

Let us now introduce a map $B_+ : H \to H$.

$$B_+(X) = \bigwedge_{i} X_i \quad \text{where} \quad X = \frac{1}{e_i}.$$  

$$B_+(\emptyset) = 0.$$  

For $T \in H_L$, $B_+(B_-(T)) = T = B_+(B_-(T)).$
But in general

\[ B_+(B_-(x)) \neq x. \]

Example: \[ B_+(B_-(\cdot \cdot)) = B_+(2 \cdot) = 2 B_+(\cdot) = 2 \cdot \neq \cdot \cdot \]

Let us now define a map (the co-product)

\[ \Delta: H \rightarrow H \otimes H \]

by

\[ \Delta(e) = e \otimes e \]

\[ \Delta([T_1 \ldots T_k]) = \Delta(T_1) \ldots \Delta(T_k) \]

\[ \Delta(B_+(x)) = B_+(x) \otimes e + \left( \text{id} \otimes B_+ \right) \circ \Delta(x). \]

Examples:

\[ \Delta(\cdot) = \Delta(B_+(e)) = B_+(e) \otimes e + \left( \text{id} \otimes B_+ \right) \circ \Delta(e) \]

\[ = \cdot \otimes e + e \otimes \cdot \]

\[ \Delta(\cdot \cdot) = \Delta(B_+(\cdot \cdot)) = \cdot \otimes e + \left( \text{id} \otimes B_+ \right) \circ \Delta(\cdot \cdot) \]

\[ = \cdot \otimes e + \cdot \otimes \cdot + e \otimes \cdot \]

\[ \Delta(\cdot \cdot \cdot) = \ldots, \Delta(\cdot \cdot \cdot \cdot) = \ldots \]
Let us introduce a map (the counit)

$$\overline{e} : H \rightarrow \mathbb{I}H \quad \text{by} \quad \overline{e}(e) = 1 \in \mathbb{I}H,$$

$$\overline{e}(x) = 0 \quad \text{else.}$$

Let $$E : \mathbb{I}H \rightarrow H$$ be the map $$\mathbb{I}H \ni q \rightarrow q \in H$$.

We set $$\gamma = E \circ \overline{e}$$, and $$\rho := \text{id} - \gamma$$.

To understand the significance of $$\Delta$$ (the co-product)

Let us remind ourselves what a product does in our algebra.

$$H \otimes H \xrightarrow{\text{id} \otimes H} H \otimes H$$

$$\otimes \text{not} \times \quad \text{in associativity}$$

$$H \otimes H \xrightarrow{\text{in}} H$$.

Reverse arrows

$$H \otimes H \otimes H \xleftarrow{\text{id} \otimes H \otimes H} H \otimes H$$

$$\otimes \text{id} \quad \times \quad \uparrow \Delta \quad \text{co-associativity.}$$

$$H \otimes H \xleftarrow{\Delta} H$$
In an algebra, we demand a unit
and an associative multiplication. In a coalgebra, we demand a counit and a coassociative multiplication.

\[(\otimes \text{id}) \cdot \Delta = (\text{id} \otimes \Delta) \cdot \Delta \quad (1)\]

\[(e \otimes \text{id}) \cdot \Delta = \text{id} = (\text{id} \otimes e) \cdot \Delta \quad (\text{if } \mathbb{H} \cong \mathbb{H}) \quad (2)\]

(2) is easy. (Do it yourself)
Do examples!
For (1).
Write \(\Delta (x) = x' \otimes x''\) (sum suppressed)

\[\Delta [B_+(X)] = B_+(x) \otimes e + (\text{id} \otimes B_+) \cdot \Delta (x)\]

\[(\text{id} \otimes \Delta) \cdot \Delta (x) = B_+(x) \otimes e \otimes e + (\text{id} \otimes \Delta \circ B_+) \cdot \Delta (x)\]

\[= B_+(x) \otimes e \otimes e + (\text{id} \otimes \Delta \circ B_+) (x' \otimes x'')\]

\[= B_+(x) \otimes e \otimes e + (\text{id} \otimes \Delta \circ B_+) \cdot \Delta (x)\]

\[= \text{id} \otimes e + \text{id} \otimes (\text{id} \otimes B_+) \cdot \Delta (x)\]

\[= \text{id} \otimes e + (\text{id} \otimes \Delta \circ B_+) \cdot \Delta (x)\]

\[= (\Delta \otimes \text{id}) \cdot \Delta (B_+(x))\]
Def. admissible cut:

Let \( T \) be the set of vertices of \( T \) and \( T^e \) be the set of edges. A subset \( C \subseteq T^e \) is called admissible if for any path from any \( v \in T^e \) towards the root, at most one edge which is traversed is in \( C \).

Examples: \( T^* \) admissible, \( T \) non-admissible.

Let \( P^* (T) \) be the product of rooted trees obtained by removing \( C \) while \( T \) is not connected to the root.

Let \( R^c (T) \) be the single component still connected to the root.

\[
\Delta (T) = T \otimes e + e \otimes T + \sum P^* (T) \otimes R^c (T).
\]

Examples.
The outcome: Coherence.

\( S : H \rightarrow H \)

\( W \circ (S \circ \text{id}) \circ \Delta = \text{moc} ((\text{id} \circ S) \circ \Delta = \mathbb{Z} \).

\( S(e) = e, \quad S(I) = -I - 2 \circ S \circ \text{id}(I) \circ R^{\circ}(I) \).

\( S(I) = -I - 2 \circ (-1) \circ \text{id}(I) \circ R^{\circ}(I) \).

Example: check both formulas:

\( S(\circ) = \circ \quad S(1) = -1 + \infty \)

\( S(1) = \ldots \quad S(\infty) = \ldots \)

If \( W \circ (S \circ \text{id}) \circ \Delta (1) = S(1) \circ S(\circ) \circ + 1 \)

\(-1 + \ldots + 1 = 0 = \gamma (1) \)

De \( W \circ (S \circ \text{id}) \circ \Delta \quad \text{and} \quad \text{moc} \circ \text{id} \circ S \circ \Delta \)

\( \text{check} \), \( \gamma \).

General note: Exercise. (Look the converse.)
Summary:

\((H, \Lambda, m, E, \mathcal{E}, S)\) is a Hopf algebra.

Putting Hopf algebras to use.

Ultimately, we want to use Hopf algebras to learn how to deal with singularities in QFT. As QFT is right now too hard for us, and its singularities are its hardest part, we train the use of Hopf algebras first in toy models.

Our first toy model is defined as follows.

Consider a map \(\phi : H \to \mathbb{V}\) (where \(\mathbb{V}\) can be an algebra, a ring or such) such that \(\phi(T_1, T_2) = \phi(T_1)\phi(T_2)\). \((*)\)

Note that this implies \(\phi(e) = 1_v\).

First example: \(\phi_a(e) = 1\).

\[
\phi_a(x) = \int_{x+\alpha}^{\infty} \frac{e^{-t}}{x+t} \, dt \quad \text{for } \alpha \in \mathbb{R}^+ \quad \text{for } \Re(z) > 0, \quad a > 0.
\]

\[
\phi_a(\mathcal{B}_+(X)) = \int_{x+\alpha}^{\infty} \phi_x(X) \quad \text{for } \alpha > 0.
\]

\(\text{All notations because of } (*)\)
Walk out

\[ \phi_a (0, \beta, \lambda, \alpha) \text{ using} \]

\[ \int_0^\infty \frac{x^{-k\xi}}{x + a} \, dx = B(k\xi, 1-k\xi) a^{-k\xi}. \]

Calculate

\[ \phi \circ S_r (\cdot, 1, \Lambda), \quad \phi \circ S_r (\cdot) = -\phi_a (T_1 - \Sigma \phi_a (SCP^2 (R))) \odot \phi_a (D^2 (R)) \]

let \( R : V \to \mathbb{V} \) be the map

\[ \phi_a \to <\phi_1>, \text{ (projection onto 1st part at unit scale)} \]

Define

\[ S_{R \phi_a} (T) = -R \left[ \phi_a (T) - \Sigma \phi_a (SCP^2 (R)) \phi_a (R^2 (R)) \right] \]

Do \[ \ldots \]
Thus, let \( R = \mathbb{R}^b \) be defined by
\[
R(\phi^a(T)) = \langle \phi^b(T) \rangle, \quad \text{where}
\]
\[
\langle \sum_{k=-r}^{0} c_k e^k \rangle = \sum_{k=-r}^{0} c_k e^k.
\]

Then,
\[
S^{\phi_a}_R(T) = -R(\phi^a(T)) - \mathbb{E} \left[ \prod_{n=1}^{\infty} \frac{\phi^a(\mathbb{R}^c(T)) \phi(\mathbb{R}^c(T))}{\mathbb{E}} \right]
\]

does not depend on \( \log b \) and \( (P = \text{id} - E \circ \epsilon) \)
\[
\mu (S^{\phi_a}_R \otimes \phi) \circ (\text{id} \otimes P) \circ \Delta(T) \text{ has}
\]
no poles \( \log a \) \( \log b \) and \( S^{\phi_a \otimes \phi}_R(T) \)
\[
= \mu (S^{\phi_a}_R \otimes \phi) \circ \Delta(T) \text{ is finite}, \quad \text{exists}
\]
Furthermore, \( S^{\phi_a}_R(T, T_2) = S^{\phi_a}_R(T_1) \otimes S^{\phi_a}_R(T_2) \)
is a homomorphism.

Before we consider the proof of this theorem, let us work out a few examples.
\[ T = 0. \]

\[ \Delta \left[ v \right] = \otimes e + e \otimes v_0. \]

\[ S_R^\phi (\cdot) = - R \left[ \phi (\cdot) \right] \]

\[ \phi_\cdot (\cdot) = \int \frac{x \pm \epsilon}{x + \epsilon} \, dx = B_\cdot (\epsilon) a^{-\epsilon} \]

\[ \left( B_\cdot (\epsilon) \right) = R^\epsilon \left( -j \epsilon \right) \]

\[ S_{Rx}^\phi (\cdot) = \left( B_\cdot (\epsilon) \right) \frac{a^{-\epsilon}}{1} = \frac{1}{\epsilon} \]

\[ R (1 + j \epsilon) = \epsilon + O (\epsilon) \]

\[ S_{Rx}^\phi (\cdot) = B_\cdot (\epsilon) a^{-\epsilon} \frac{1}{\epsilon} = \frac{1}{\epsilon} \ln a + O (\epsilon) \]

\[ T = 1 \]

\[ \Delta \left[ \nu \right] = \left( \otimes e + e \otimes \right) + \epsilon \otimes \]

\[ S_{Rz}^{\phi^2} (\cdot) = - R \left( \phi^2 (\cdot) \right) = R \left( S_R^\phi (\cdot) \phi_\cdot (\cdot) \right) \]

\[ \phi_\cdot (\cdot) = \int \int \frac{y \pm \epsilon}{y + \epsilon} \, dy \, dx \]

\[ = B_\cdot (\epsilon) B_\cdot (\epsilon) a^{-2\epsilon} = \frac{1}{2\epsilon^2 \left( 1 + O (\epsilon) \right)} \left( 1 - 2\epsilon \ln a + \ldots \right) \]

\[ = \frac{1}{2\epsilon^2} + \frac{\ln a}{\epsilon} + \ldots \]
\[ S_{\phi}^* (1) = - \langle B_1, B_2, e^{2x} \rangle \]
\[ + \langle B_1, e^{2x} \rangle B_2, e^{2x} \rangle \]
\[ = - \langle B_1, B_2, e^{2x} \rangle + \langle \frac{1}{2} B_1, e^{2x} \rangle e^{2x} \rangle \]

Expand in \( e^x \) no \( \frac{e^x}{x} \) terms survive.

\[ S_{\phi}^* \Phi (i) \]
\[ = \phi (i) - R [ \phi (i)] - R [ R [ \phi (i)] \phi (i) + R [ R [ \phi (i)] \phi (i)]] \]
\[ = [ i d - R ] ( m o ( S_{\phi}^* \otimes \phi ) \circ (i d \otimes P) \circ \Delta (i) ) \]

and in general,

\[ S_{\phi}^* \Phi (T) = (i d - R ) [ m o ( S_{\phi}^* \otimes \phi ) \circ (i d \otimes P) \circ \Delta (T) ] . \]

We implicitly assume that

\[ S_{\phi}^* (T_1, T_2) = S_{\phi}^* (T_1) S_{\phi}^* (T_2) \text{, which} \]

we will prove later in class.

As \( S_{\phi}^* \Phi \Phi \) is in the image of \( i d - R \), it is clear that \( S_{\phi}^* \Phi \Phi \)
exists at \( \varepsilon = 0 \) and from the results for \( \Gamma = 0 \% \\
we get \( S^\Phi \) does not depend on \( \log \beta \),

\[ S^\Phi \times \phi \text{ is finite and} \]

\[ m \cdot (S^\Phi \otimes \phi) \cdot (\text{id} \otimes P) \cdot A \equiv S^\Phi \times \phi - S^\Phi \]

has no \( \log \)-dependent poles.

The actually \( S^\Phi \times \phi \) \( (\varepsilon) \mid_{\varepsilon = 0} \) is a first order polynomial in \( \log a \).

We can prove similar properties for all \( \Gamma \) by induction over the number of vertices.

Use \( \Delta \circ B = B \circ \Delta + (\text{id} \otimes B) \circ \Delta \),

\[ S^\Phi \times \phi + (\Delta \circ B) (X) \]

\[ = m \cdot (S^\Phi \otimes \phi) \cdot \Delta \circ B (X) \]

\[ \Delta \circ B (X) \]
\[
\begin{align*}
\phi_\sigma (\phi_\sigma \otimes \phi_\sigma) \cdot \mathbb{I} \left[ (B_+ \otimes e) + (i \sigma \otimes B_+) \right] \Delta (X) \\
= S^\phi_\sigma (B_+ (X)) + (S^\phi_\sigma \otimes \phi_\sigma \, B_+) \left( \Sigma X' \otimes X' \right) \\
= S^\phi_\sigma (B_+ (X)) + \int_0^\infty \frac{e^{-x}}{x + \sigma} \, S^\phi_\sigma \ast \phi_\sigma (X)
\end{align*}
\]

In class, we called this \( B_+ \ast S^\phi_\sigma \ast \phi_\sigma (X) \).

It is easy to see that the integral has no log \( a \) dependent poles if

\[ S^\phi_\sigma \ast \phi_\sigma (X) \text{ is } \mathcal{C}_0 \text{ at } z = 0 \text{ at most a polynomial in } \log x, \text{ as } \]

\[ \int_0^\infty \left\{ \frac{x^{\sigma} \text{ Poly (log } x)}{x + \alpha} - \frac{x^{\sigma}}{x + \xi} \text{ Poly (log } x) \right\} \]

has no poles, and that after the subtraction of the poles \( \frac{x^{\sigma}}{x + \alpha} \) by \( S^\phi_\sigma \),
$S_k \ast \phi (T)$ is again a polynomial in $\log n$ of degree at most $n!$.

We will report this proof again in detail for Feynman graphs and in the exercises.
The sub Hopf algebra $H_{2n}$.

So far we introduced the Hopf algebra $H$ of undecorated rooted trees. We identified its coproduct $\Delta$, counit $\varepsilon$ and antipode $S$.

This Hopf algebra has a closed sub-Hopf algebra $H_{2n}$, i.e.

$$\Delta[x] = \sum x^{(1)} \otimes x^{(2)}$$

where $x, x^{(1)}, x^{(2)} \in H_{2n}$ and

$\Delta$ is the restriction of the coproduct $\Delta$ to $H_{2n}$.

To define them we need to operation of natural growth.

$$N: H \to H \quad N(x, x_2) = N(x_2) x_1 + x_1 N(x_2)$$

(r's a derivation) and

$$N(I) = \text{"sum over all ways of coloring our more edge and vertex"}$$

$\Rightarrow$ $N(e) = 0 \quad N(\cdot) = I \quad N(1) = I + 1$

\begin{align*}
N(\bar{1}) &= \bar{1} + \bar{1} \quad N(\bar{2}) = 2 \bar{1} + \bar{2}
\end{align*}
We define
\[ S_x := N^x(e). \]

We define \( S_x \) as the algebra of polynomials in generators \( S_x \), and make this commutative algebra into a Weyl algebra by using the previous \( e, d, S \).

The only non-trivial fact to be shown is that the composite maps \( S_x \) into Polynomials of six generators.

\[ \Delta (S_1) = S_1 \otimes e + e \otimes S_1 \]
\[ \Delta (S_2) = S_2 \otimes e + S_1 \otimes S_1 + e \otimes S_2 \]
\[ \Delta (S_3) = \Delta (S_1 + S_2) = S_3 \otimes e + e \otimes S_3 + \cdots + 2 \otimes 1 + \cdots \]
\[ + 1 \otimes \cdots \cdots = S_3 \otimes e + e \otimes S_3 + 3S_2 \otimes S_1 + 3S_2 \otimes S_1 + S_1 \otimes S_1 \]
\[ + S_1 \otimes S_1 \]

by inspection.
Let us prove this in general.

$S_i$ is by construction a sum of trees:

$S_i = \sum \delta_i \quad \text{say. (For example, } S_2 = t_1 + t_2) \quad \text{and so on.}$

Thus $\Delta(S_n) = e \otimes S_n + S_n \otimes e + \sum \sum P^{S}(t_i) \otimes R^{\nu}(t_j)$

We can write (using the $t_i$ of $S_n$)

$\Delta(S_{n+1}) = e \otimes S_{n+1} + S_{n+1} \otimes e$

$+ \sum \sum \left\{ N[P^{S}(t_i)] \otimes R^{\nu}(t_j) \right\}$

$+ P^{S}(t_i) \otimes N[R^{\nu}(t_j)]$

$+ n \delta_i \otimes S_n + \sum \sum \ell(R^{\nu}(t_i), P^{S}(t_j)) \otimes R^{\nu}(t_j)$,

where $\ell(t)$ gives the number of vertices at a tree $t$. 

(3)
What we did is we decomposed the cuts at $E$ into four classes:

1) The edge of a new group vertex is not part of the admissible cut; then we will have $\mathcal{N}$ acting on either the $P^g$ or the $R^g$ side. (Two cases)

2) On the new edge is part of the admissible cut. This gives us a factor $S_i$ always. If that edge is the only cut, we get

$$n_0 \otimes S_i$$

as a contribution.

Otherwise, that edge must have been from the $P^g$ part (otherwise the cut obtained by adding that edge would be the admissible cut. We get

$$n_0 \otimes \left[ R_i^g(t) \right]$$

part. 

What is the use of the Hopf algebras?
Consider formal diffeomorphism of the form
\[ x \mapsto x + \sum_{k=2}^{\infty} a_k x^k \quad (\text{for real } x \text{ and } a_k). \]

Let \( \phi(x) \), \( \psi(x) \) be two such diffeomorphisms.

\[ \phi(x) = x + \sum_{k=1}^{\infty} a_k x^k \quad \psi(x) = x + \sum_{k=2}^{\infty} b_k x^k \]

What are the Taylor coefficients of the composition
\[ \phi \circ \psi(x) ? \]

The answer is given by then in the following way.

Define \( \log^{(n)} [\phi'(x)] = \sum_{n=1}^{\infty} \log^{(n)} [\psi'(x)] = \sum_{n=1}^{\infty} \)

(These are the \( n \)-th derivatives of the log of the derivative of the diffeomorphism.)

Useful in understanding two different viewpoints on renormalization i.e. perturbative
It is easy to reconstruct ϕ, ψ from the knowledge of SN, S^N. (exponehtial and integrable).

So what we are after is the knowledge of S^N from SN, S^N.

Thus,

\[ S^N = m \cdot (\bar{\psi} \otimes \bar{\phi}) \cdot \Delta (S_N) \]

where \( S_N \in H^N \) and \( \bar{\psi}, \bar{\phi} \) are characters

\[ \bar{\phi} (S_{k_1}, \ldots, S_{k_p}) = S^N \quad \bar{\psi} (e) = 1 \]

and similarly for \( \bar{\phi} \).

That this is part of a larger theory given by Connes and Moscovici, and myself. We will not prove it here. It is useful in understanding two different viewpoints on renormalization i.e.,

\( C \)
renormalization using Hopf algebras as we do in these lectures and renormalizability is regarded as differentials of physical parameters driven by the renormalization group.

The equivalence of both viewpoints relies on theorems like the above.

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We are now ready to start considering the Lie algebra which appears in the dual of our Hopf algebra.
Then \( \mathbb{B} \) is unique up to isomorphism and universal: for any pair \((H_+, L)\), with \(H_+\) being a commutative Hopf algebra and \(L\) a Hochschild one cocycle (i.e. \(\Delta_+ 0_L = 1 \otimes 1 + (1 \otimes 1)_L 0_L\)), there exists a unique Hopf algebra morphism

\[
\varphi : H_+ \to H
\]

such that \(\log = \varphi \circ \mathbb{B}_+\).

\(\mathbb{B}_+\) is Hochschild closed by definition, but not exact: \(\mathbb{B}_+ (e) = 0\), while for \(T = \mathbb{B}_+, \ T (e) = 2 (e) e - (1 \otimes 2) \Delta (e) = 2 (e) e - e \varphi (e) = 0\).

Note that for any map \(\mathbb{B} : H \to \bigotimes_{n=0}^{\infty} H\), is given by

\[
\mathbb{B} (a) = (1 \otimes \varphi) \circ \Delta (a) + \sum_{n=1}^{\infty} (-1)^n \Delta (a) \otimes \varphi \big( \bigotimes_{i=0}^{n-1} a \big) \otimes e.
\]
So, if \( 2: H \rightarrow K \) (0 times),
\( n \) means \( \text{id} \otimes \text{id} \otimes \ldots \otimes \text{id} \) where \( \delta \) is in \( \mathcal{E} H \), then \( b^2(a) = (\text{id} \otimes \text{id}) \Delta(a) - \Delta^2(a) \in e \in H \)

\( h = 1: 2: H \rightarrow H \) \( b^2(a) = (\text{id} \otimes \text{id}) \Delta(a) - \Delta^2(a) + 2(\text{id}) \otimes e \in H \otimes e \)

and so on. Thus \( bB_+ = 0 \). Note \( b^2 = 0 \).

Now consider a pair \((H, L_1)\) where \( L_1 \) is a commutative bimodule algebra and \( L_1 \) is a 1-cocycle. \( L_1 \circ \varphi = \varphi \circ B_+ \) uniquely determines an algebra homomorphism \( \varphi: H \rightarrow K \).

\( \varphi(\overline{\mathcal{T}}) = \overline{\varphi(T)} \)

\( \varphi(\cdot) = L_1(1) \)

\( \varphi \circ B_+(\overline{\mathcal{T}}) = L_1 \varphi(\overline{\mathcal{T}}) \) by induction.

Need to check: \( (\varphi \otimes \varphi) \circ \Delta(a) = \Delta_2 \varphi(a) \).

Check on \( T = B_+(\overline{\mathcal{T}}) \).
\[ \Delta_1 L_1 (\varpi (\Pi T)) = L_1 (\varpi (\Pi T)) \otimes 1 \]
\[ + (\text{id} \otimes L_1) \Delta_1 \varpi (\Pi T). \]

le dextre

\[(\text{id} \otimes L_1) \circ \Delta_1 \varpi (\Pi T) = (\varpi \otimes \varpi) (\text{id} \otimes B_+ \otimes \varpi (\Pi T)) \]
\[ = (\text{id} \otimes L_1) \]
\[ \Rightarrow \Delta_1 \otimes L_1 \circ \varpi (\Pi T) = L_1 \circ \varpi (\Pi T) \otimes 1 \]
\[ + (\varpi \otimes \varpi) (\text{id} \otimes B_+ \otimes \varpi (\Pi T)). \]

\[ (\varpi \otimes \varpi) \circ \Delta_1 B_+ \otimes \varpi (\Pi T) = \varpi \circ B_+ \otimes \varpi (\Pi T) \otimes 1 \]
\[ + (\text{id} \otimes B_+) \Delta (\Pi T) \]
\[ = L_1 \circ \varpi (\Pi T) \otimes 1 + (\varpi \otimes \varpi) (\text{id} \otimes B_+) \Delta (\Pi T) \]
\[ \Rightarrow 1 \]
The Lie algebra \( \mathfrak{e} \)

Let \( \mathfrak{e}^{*} \) be the linear span of elements \( \mathfrak{e} \).

Define \( \mathfrak{e}_{1} + \mathfrak{e}_{2} = \sum \mathfrak{H}(T_{1}, T_{2}, T) \mathfrak{e}_{T} \)

\( \mathfrak{H}(T_{1}, T_{2}, T) = \# \) of simple admissible walks \( c \),
\( 1 < \ell = 1 \), and \( \Delta_{c} \mathfrak{P}(T) = T_{1}, \mathfrak{R}^{*}(T) = T_{2} \).

\( [\mathfrak{e}_{1} \mathfrak{e}_{2}] = \mathfrak{e}_{1} \mathfrak{e}_{2} - \mathfrak{e}_{2} \mathfrak{e}_{1} \)

is a Lie bracket.

\( \mathfrak{h}^{*} \) is the dual of the enveloping algebra \( \mathfrak{e} \).

Set \( \text{Ass}(T_{1}, T_{2}, T_{3}) = \mathfrak{e}_{1} * (\mathfrak{e}_{2} * \mathfrak{e}_{3}) - (\mathfrak{e}_{1} * \mathfrak{e}_{2}) * \mathfrak{e}_{3} \).

\( \text{Ass}(T_{1}, T_{2}, T_{3}) = \sum \mathfrak{H}(T_{1}, T_{2}, T_{3}, T) \mathfrak{e}_{T} \)

where \( \mathfrak{H} = \# \) of admissible walks \( n \), \( 1 < \ell = 2 \),
\( \mathfrak{P}(T) = T_{2}, T_{2}, T_{2}, T_{3} \).

Prop. \( \text{Ass}(T_{1}, T_{2}, T_{3}) = \text{Ass}(T_{2}, T_{2}, T_{3}) \).

(\( * \) is pre-Lie).
\[ \begin{align*}
    \text{Pre-lie: } & \quad 2_{t_1} \ast (2_{t_2} \ast 2_{t_3}) - (2_{t_1} \ast 2_{t_2}) \ast 2_{t_3} \\
    & \quad = 2_{t_1} \ast (2_{t_2} + 2_{t_3}) - (2_{t_1} + 2_{t_2}) \ast 2_{t_3}
\end{align*} \]

Then, if a map \( \ast : L \times L \to L \) is pre-lie, then \( \{x, y\} = x \ast y - y \ast x \) is a lie bracket, i.e., fulfills Jacobi.

**Proof.** \( L \times \{y, z\} = x \ast (y \ast z - z \ast y) - (y \ast z - z \ast y) \ast x \) 
+ cyclic terms.

Use pre-lie property \( \Box \)

\[ \begin{align*}
    & x \ast (y \ast z - z \ast y) - (y \ast z - z \ast y) \ast x \\
    + & y \ast (z \ast x - x \ast z) - (z \ast x - x \ast z) \ast y \\
    + & z \ast (x \ast y - y \ast x) - (x \ast y - y \ast x) \ast z \\
    & = 0
\end{align*} \]

So \( L \) becomes a Lie algebra.

Now \( \langle 2_{t_1}, P(S_{t_1}) \rangle = \frac{d}{dt} P(0) \) lin. form on \( H_{t_1} \in \pi^p_t \in \pi^t \)

\[ \langle 2_{t_1} + 5_{t_1}, 2_{t_1} \rangle = 1, \quad \text{Kronecker pairing} \]

\[ \Box \]
\( P \Rightarrow P(\theta) \) is the count.

\[
\langle \tilde{Z}_1, P \tilde{Q} \rangle = \langle \tilde{Z}_1, P \rangle \varepsilon(Q) + \varepsilon(P) \langle \tilde{Z}_1, \tilde{Q} \rangle
\]

\[
\Delta \tilde{Z}_1 = \tilde{Z}_1 \otimes 1 + 1 \otimes \tilde{Z}_1 \in \mathbb{C} \times \mathbb{C}
\]

\[
\langle \tilde{Z}_1 \tilde{Z}_2, P \rangle = \langle \tilde{Z}_1 \otimes \tilde{Z}_2, \Delta P \rangle
\]

Now \( \tilde{Z}_1 \tilde{Z}_2 = \tilde{Z}_2 \tilde{Z}_1 \) is a

action of (it fulfills \( \ast \)), hence we must show that \( \tilde{Z}_1 \tilde{Z}_2 = \tilde{Z}_2 \tilde{Z}_1 \)

\[
= \left[ \tilde{Z}_1 + \tilde{Z}_2 \right].
\]

Now, let \( H_{xy} \) be the kernel of \( \tilde{z} \).

Then \( \Delta \tilde{z} = \tilde{z} \otimes H_{xy} + H_{xy} \tilde{z} \)

\[ R_{xy} \in H_{xy} \otimes H_{xy} \]

Project onto \( H_{xy} \) and \( H_{xy} \)

\[
\begin{align*}
R_{\text{lin}} &= \frac{1}{2} \sum_{\tilde{z} \in (\mathbb{C}^2)^*} \tilde{z} \tilde{z} \otimes \tilde{z} \tilde{z} \\
&= \frac{1}{16} \sum_{i=1}^{16} \tilde{z} \tilde{z} \otimes \tilde{z} \tilde{z}
\end{align*}
\]
\[ \langle t_1, t_2, s_7 \rangle = \langle t_1, s_2, t_3 \rangle \]

counts the number of uses $T$ can be cut from $T$ as to obtain $T_2$. 

Furthermore, 

let us introduce some actions of $\mathbb{C}$ on $H$. 
(as derivations) 

i) $\mathbb{C} \times H \to \mathbb{C} \times H$ 

\[ \mathbb{C} \times \delta_{\tau_2} \to \mathbb{C} \times \delta_3 \]

\[ e_{\tau_2} \to e_{\tau_3} \]

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