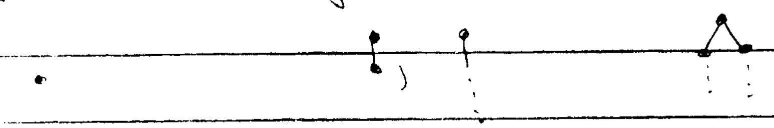
The Hopf algebra of rooted trees

Def. i) A rooted tree is a connected and simply connected set of edges and vertices such that each edge connects two vertices. We distinguish one vertex and orient all edges away from that vertex.
ii) fert (v) is the number of edges outgoing from v.

We consider the free commutative algebra of polynomials in rooted trees. The product is induced by the disjoint union, and we write e for the ~~un~~ unit in this algebra. (e from empty set)

For each rooted tree with root r, we have fert(r) rooted trees attached to the root.

$f(r) = 0$ $\quad$ $f(r) = 1$ $\quad$ $f(r) = 2$ $\quad \ldots$



Let H be the algebra of rooted trees.

H is graded by the number of vertices:

$$H = \bigoplus_{k \geq 0} H^{(k)}$$

$$H^{(0)} \hat{=} q e \qquad H^{(1)} \hat{=} q \bullet \qquad H^{(2)} \hat{=} \{ \;, \bullet\bullet \}, \ldots$$

H_L is the linear subspace.

( is in H_L and $H^{(2)}$, $\bullet\bullet$ is in $H^{(2)}$, not in H_L)

Define a linear map $B_-: H \to H$ by

$$B_-(e) = 0. \quad B_-(\bullet) = e \quad B_-(T) = \bigcup_{i=1\ldots \mathrm{fert}(r)} T_i, \quad T \in H_L$$

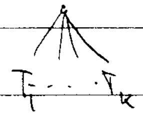
where T_i are the trees attached to the root of T.

For $X \in H$, extend B_- by a Leibniz rule:

$$B_-(X_1 X_2) = X_1 B_-(X_2) + B_-(X_1) X_2.$$

* $B_-(\bullet) = e$, $B_-($  $) = \bullet$, $B_-($  $) =$ , $B_-($  $) = \bullet\bullet$.

Let us now introduce a map $B_+: H \to H$.

$B_+(X) =$  where $X = \bigcup_{T_i}$.

$B_+(e) = \bullet$.

For $T \in H_L$, $B_-(B_+(T)) = T = B_+(B_-(T))$.

But in general

$B_+(B_-(X)) \neq X$.

Example: $B_+(B_-(\bullet\bullet)) = B_+(2\bullet) = 2\,B_+(\bullet) = 2$  $\neq \bullet\bullet$.

let us now define a map (the coproduct)

$\Delta : H \to H \otimes H$ by

$\Delta[e] = e \otimes e$

$\Delta[T_1 \dots T_k] = \Delta[T_1] \dots \Delta[T_k]$

$\Delta[B_+(X)] = B_+(X) \otimes e + (id \otimes B_+) \circ \Delta(X)$.

Examples:

$\Delta[\bullet] = \Delta[B_+(e)] = B_+(e) \otimes e + (id \otimes B_+) \circ \Delta(e)$

$= \bullet \otimes e + e \otimes \bullet$

$\Delta[$  $] = \Delta[B_+(\bullet)] =$  $\otimes e + (id \otimes B_+)\,\Delta[\bullet]$

$=$  $\otimes e + \bullet \otimes \bullet + e \otimes$ 

$\Delta[$  $] = \dots$, $\Delta[$  $] = \dots$

Let us further introduce a map (the counit)

$\bar{e}: H \to \mathbb{K}$ by $\bar{e}(e) = 1 \in \mathbb{K}$,

$\bar{e}(X) = 0$ else.

Let $E: \mathbb{K} \to H$ be the map $\mathbb{K} \ni q \to qe \in H$.

We set $\eta = E \circ \bar{e}$, and $P := id - \eta$.

To understand the significance of Δ (the coproduct)

let us remind ourselves what a product does for us in an algebra.

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "id \otimes m"] \arrow[d, "m \otimes id"'] \arrow[dr, phantom, "\times"] & H \otimes H \arrow[d, "m"] \\ H \otimes H \arrow[r, "m"'] & H \end{tikzcd}$$

associativity.

Reverse arrows

$$\begin{tikzcd} H \otimes H \otimes H & H \otimes H \arrow[l, "id \otimes \Delta"'] \\ H \otimes H \arrow[u, "\Delta \otimes id"] \arrow[ur, phantom, "\times"] & H \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

coassociativity.

In an algebra, we demand a unit and an associative multiplication. In a coalgebra, we demand a counit and a coassociative multiplication.

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \qquad (1)$$

$$(\bar{e} \otimes id) \circ \Delta = id = (id \otimes \bar{e}) \circ \Delta \qquad (\mathbb{K} \otimes H \simeq H) \qquad (2)$$

(2) is easy. (do it yourself)
Do examples!

For (1). Write $\Delta(X) = X' \otimes X''$ (sum suppressed)

$$\Delta[\underbrace{B_+(X)}_{T}] = B_+(X) \otimes e + (id \otimes B_+)\,\Delta(X).$$

$$(id \otimes \Delta) \circ \Delta[B_+(X)] = B_+(X) \otimes e \otimes e + (id \otimes \Delta \circ B_+) \circ \Delta(X)$$

$$= B_+(X) \otimes e \otimes e + (id \otimes \Delta \circ B_+)(X' \otimes X'')$$

$$= B_+(X) \otimes e \otimes e + X' \otimes (id \otimes B_+)\,\Delta(X'') + X' \otimes B_+(X'') \otimes e$$

$$= T \otimes e \otimes e + X' \otimes B_+(X'') \otimes e + (id \otimes id \otimes B_+)(id \otimes \Delta) \circ \Delta(X)$$

$$= T \otimes e \otimes e + X' \otimes B_+(X'') \otimes e + (id \otimes id \otimes B_+)(\Delta \otimes id) \circ \Delta(X)$$

$$= \overset{(\Delta \otimes id)}{\Delta} \circ B_+(X) \otimes e + (\Delta \otimes id)(id \otimes B_+) \circ \Delta(X)$$

$$= (\Delta \otimes id) \circ \Delta(B_+(X)). \qquad \square$$

Def. admissible cut.

We let $T^{[0]}$ be the set of vertices of T and $T^{[1]}$ be the set of edges. A subset $C \subset T^{[1]}$ is called admissible, if for any path from any $v \in T^{[0]}$ towards the root, at most one edge which is traversed is in C.

Examples!  ← admissible  ← non-admissible.

Let $P^C(T)$ be the product of rooted trees obtained by removing C which are not connected to the root.
Let $R^C(T)$ be the single component still connected to the root.

$$\Delta[T] = T \otimes e + e \otimes T + \sum_{\substack{\text{adm.} \\ \text{cuts } C \\ \text{of } T}} P^C(T) \otimes R^C(T).$$

Examples.

the antipode. Co-Inverse.

$S: H \to H$

$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = \varepsilon.$

$$S(e) = e, \quad S(T) = -T - \sum_{\substack{\text{adm} \\ \text{cuts} \\ C}} S[P^C(T)]\, R^C(T).$$

$$S(T) = -T - \sum_{\substack{\text{all} \\ \text{cuts } C}} (-1)^{n_C} P^C(T)\, R^C(T).$$

Example. check both formulas!

$S(\bullet) = -\bullet$ $\quad$ $S($$) = -$ $+ \bullet\bullet$

$S($$) = \ldots$ $\quad$ $S($$) = \ldots$

Pf $\quad m \circ (S \otimes id) \circ \Delta ($$) = S($$) + S(\bullet)\bullet +$ 

$= -$ $+ \bullet\bullet - \bullet\bullet +$  $= 0 = \varepsilon($$)$

Do $m \circ (S \otimes id) \circ \Delta$ and $m \circ (id \otimes S) \circ \Delta$

for , .

General proof: exercise. (for the curious)

Summary:

$(H, \Delta, m, E, \bar{e}, S)$ is a Hopf algebra.

Putting Hopf algebras to use.

Ultimately, we want to use Hopf algebras to learn how to deal with singularities in QFT. As QFT is right now too hard for us, and its singularities are its hardest part, we train the use of Hopf algebras first in toy models.

Our first toy model is defined as follows.

Consider a map $\phi : H \to V$ (where V can be an algebra, a ring or such)

such that $\phi(T_1 T_2) = \phi(T_1)\phi(T_2)$. $(*)$

Note that this implies $\phi(e) = 1_V$.

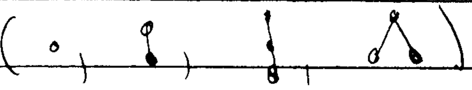
First example: $\phi_a(e) = 1$.

$$\phi_a(\bullet) = \int_0^\infty \frac{x^{-\varepsilon}}{x+a}\,dx \qquad 1 >> \mathrm{Re}(\varepsilon) > 0, \quad \varepsilon \in \mathbb{C}, \quad a > 0, \ a \in \mathbb{R}_+.$$

$$\phi_a(B_+(X)) = \int_0^\infty \frac{x^{-\varepsilon}}{x+a}\,\phi_x(X).$$

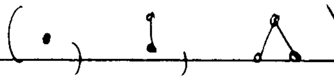
well defined because of $(*)$

Walk out

$\phi_a($  $)$ using

$$\int_0^\infty \frac{x^{-k\varepsilon}}{x+a}\,dx = B(k\varepsilon, 1-k\varepsilon)\, a^{-k\varepsilon}.$$

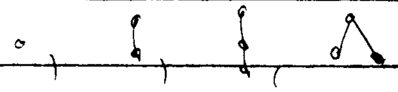
Calculate

$\phi \circ S($  $)$. $\phi_a \circ S(T) = -\phi_a(T) - \sum_{\substack{\text{ad-}\\ \text{cut } C_i}} \phi_a(S(P^{c}(T))) \cdot \phi_a(R^c(T))$

Let $R : V \to V$ be the map

$\phi_a \to \langle \phi_1 \rangle$. (projection onto pole part at unit scale)

Define $S_R^{\phi_a}(T) = -R[\phi_a(T)] - \sum_{\substack{\text{ad-}\\ \text{cuts}\\ C}} S_R^{\phi_a}(P^c(T))\, \phi_a(R^c(T))$.

Do 

Thm. Let $R \equiv R^b_{MS}$ be defined by

$R(\phi_a(T)) = \langle \phi_b(T) \rangle$, where

$\langle \ldots \rangle$ means projection onto the pole part.

$$\left\langle \sum_{k=-r}^{+\infty} c_k \varepsilon^k \right\rangle = \sum_{k=-r}^{-1} c_k \varepsilon^k.$$

Then $S_R^{\phi_a}(T) \equiv -R(\phi_a(T)) - \sum_{\substack{\text{adm}\\ \text{cuts}\\ C}} R\left[S_R^{\phi_a}(P^C(T))\, \phi_a(R^C(T))\right]$

does not depend on $\log b$ and $(P = id - E \circ \bar{e})$

$m \circ (S_R^{\phi_a} \otimes \phi) \circ (id \otimes P) \circ \Delta(T)$ has

no poles $\sim \log a$ (or $\log b$) and $S_R^{\phi_a} * \phi_a(T)$

$\equiv m \circ (S_R^{\phi_a} \otimes \phi) \circ \Delta(T)$ is finite. (exists at $\varepsilon = 0$)

Furthermore, $S_R^{\phi_a}(T_1 T_2) = S_R^{\phi_a}(T_1)\, S_R^{\phi_a}(T_2)$ is a character.

Before we consider the proof of this theorem, let us work out a few examples.

$T = \bullet$

$\Delta[\bullet] = \bullet \otimes e + e \otimes \bullet$

$S_R^{\phi}(\bullet) = -R[\phi(\bullet)]$

$\phi_a(\bullet) = \int_0^\infty \frac{x^{-\varepsilon}}{x+a}\, dx = B_1(\varepsilon)\, a^{-\varepsilon}$

where $B_j(\varepsilon) = B(j\varepsilon, 1-j\varepsilon) = \frac{\Gamma(1+j\varepsilon)\,\Gamma(1-j\varepsilon)}{j\varepsilon}$.

$S_R^{\phi_a}(\bullet) = -\langle B_1(\varepsilon)\, b^{-\varepsilon}\rangle = -\frac{1}{\varepsilon}$

$\Gamma(1+j\varepsilon) = 1 + O(\varepsilon)$

$S_R^{\phi_a} * \phi_a(\bullet) = B_1(\varepsilon)\, a^{-\varepsilon} - \frac{1}{\varepsilon} = -\ln a + O(\varepsilon)$.

$T = \overset{\bullet}{\underset{\bullet}{|}}$

$\Delta\left[\overset{\bullet}{\underset{\bullet}{|}}\right] = \overset{\bullet}{\underset{\bullet}{|}} \otimes e + e \otimes \overset{\bullet}{\underset{\bullet}{|}} + \bullet \otimes \bullet$

$S_R^{\phi_a}\left(\overset{\bullet}{\underset{\bullet}{|}}\right) = -R\left(\phi_a\left(\overset{\bullet}{\underset{\bullet}{|}}\right)\right) - R\left[S_R^{\phi_a}(\bullet)\, \phi_a(\bullet)\right]$.

$\phi_a\left(\overset{\bullet}{\underset{\bullet}{|}}\right) = \int_0^\infty\!\!\int_0^\infty \frac{x^{-\varepsilon}}{x+a}\, \frac{y^{-\varepsilon}}{y+x}\, dy\, dx$

$= B_1(\varepsilon)\, B_2(\varepsilon)\, a^{-2\varepsilon} = \frac{1}{2\varepsilon^2}(1 + O(\varepsilon))(1 - 2\varepsilon \ln a + \dots)$

$= \frac{1}{2\varepsilon^2} + \dots \frac{\ln a}{\varepsilon} + \dots$

③

$$S_R^{\phi_a}(\,\text{I}\,) = -\left\langle B_1(\varepsilon)\, B_2(\varepsilon)\, b^{-2\varepsilon} \right\rangle + \left\langle \left\langle B_1(\varepsilon)\, b^{-\varepsilon} \right\rangle B_1(\varepsilon)\, b^{-\varepsilon} \right\rangle$$

$$= -\left\langle B_1(\varepsilon)\, B_2(\varepsilon)\, b^{-2\varepsilon} \right\rangle + \left\langle \tfrac{1}{2} B_1(\varepsilon)\, b^{-\varepsilon} \right\rangle$$

Expand in ε $\Rightarrow$ no $\frac{\log b}{\varepsilon}$ terms survive.

$$S_R^{\phi_a} * \phi\,(\,\text{I}\,)$$

$$= \phi_a(\,\text{I}\,) - R[\phi_a(\text{I})] - R[\phi_a(\bullet)]\,\phi_a(\bullet) + R[R[\phi_a(\bullet)]\,\phi_a(\bullet)]$$

$$= [id - R]\left(m \circ (S_R^{\phi_a} \otimes \phi_a) \cdot (id \otimes P) \circ \Delta(\,\text{I}\,) \right)$$

and in general,

$$S_R^{\phi_a} * \phi\,(T) = (id - R)\left[m \circ (S_R^{\phi_a} \otimes \phi_a) \cdot (id \otimes P) \circ \Delta\,(T) \right].$$

We implicitly assume that

$S_R^{\phi_a}(T_1 T_2) = S_R^{\phi_a}(T_1)\, S_R^{\phi_a}(T_2)$, which we will prove later in class.

As $S_R^{\phi_a} * \phi_a$ is in the image of $id - R$, it is clear that $S_R^{\phi_a} * \phi_a$

exists at $\varepsilon = 0$ and from

the results for $T = \bullet$

we get $S_R^{\phi_a}$ does not depend on $\log b$,

$S_R^{\phi_a} * \phi_a$ ~~has~~ is finite and

$m \circ (S_R^{\phi_a} \otimes \phi_a) \circ (id \otimes P) \circ \Delta \equiv S_R^{\phi_a} * \phi_a - S_R^{\phi_a}$

has no log-dependent poles.

~~We~~ Actually, $S_R^{\phi_a} * \phi_a(\bullet)\Big|_{\varepsilon=0}$ is a first

order polynomial in $\log a$.

We can prove similar properties for all

T by induction ~~of~~ over the number of

vertices.

Use $\Delta \circ B_+ = B_+ \otimes e + (id \otimes B_+) \circ \Delta$,

$S_R^{\phi} * \phi \, (B_+(X))$

$= m \circ (S_R^{\phi} \otimes \phi) \circ \Delta \circ B_+(X)$

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$$= m \circ (S_R^\phi \otimes \phi) \circ [(B_+ \otimes e) + (id \otimes B_+)] \Delta(X)$$

$$= S_R^\phi(B_+(X)) + (S_R^\phi \otimes \phi \circ B_+)(\Sigma X' \otimes X'')$$

$$= S_R^\phi(B_+(X)) + \underbrace{\int_0^\infty dx \frac{x^{-\varepsilon}}{x+a} S_R^{\phi_x} * \phi_x(X)}_{\text{in class, we called this } \bar{B}_+^{\phi_1 \, S_R^\phi * \phi}(X).}$$

Its easy to see that the integral has no log a dependent poles if $S_R^{\phi_x} * \phi_x(X)$ is at $\varepsilon = 0$ at most a polynomial in $\log x$, as

$$\int_0^\infty dx \left\{ \frac{x^{-\varepsilon}}{x+a} \mathrm{Poly}(\log x) - \frac{x^{-\varepsilon}}{x+\tilde{a}} \mathrm{Poly}(\log x) \right\}$$

has no poles, and that after the subtraction of the poles by S_R^ϕ,

$S_R^\phi * \phi\,(T)$ is again a polynomial in $\log q$ of degree at most $n+1$. We will repeat this proof again in detail for Feynman graphs, and in the exercises.