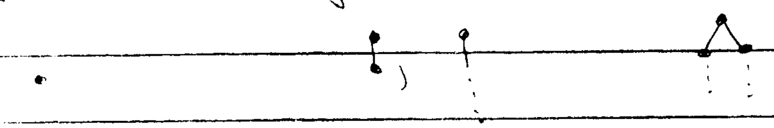
The Hopf algebra of rooted trees

Def. i) A rooted tree is a connected and simply connected set of edges and vertices such that each edge connects two vertices. We distinguish one vertex and orient all edges away from that vertex.
ii) fert (v) is the number of edges outgoing from v.

We consider the free commutative algebra of polynomials in rooted trees. The product is induced by the disjoint union, and we write e for the ~~un~~ unit in this algebra. (e from empty set)

For each rooted tree with root r, we have fert(r) rooted trees attached to the root.

$f(r) = 0$ $\qquad$ $f(r) = 1$ $\qquad$ $f(r) = 2$ $\quad \ldots$



Let H be the algebra of rooted trees.

It is graded by the number of vertices:

$$H = \bigoplus_{k \geq 0} H^{(k)}$$

$H^{(0)} \hat{=} \mathbb{Q} e \qquad H^{(1)} \hat{=} \mathbb{Q} \bullet \qquad H^{(2)} \hat{=} \{ \overset{\bullet}{\underset{\bullet}{|}}, \bullet\bullet \}, \ldots$

H_L is the linear subspace.

( is in H_L and $H^{(2)}$, $\circ\circ$ is in $H^{(2)}$, not in H_L)

Define a linear map $B_-: H \to H$ by

$$B_-(e) = 0. \quad B_-(\circ) = e \quad B_-(T) = \bigcup_{i=1\ldots \mathrm{fert}(r)} T_i, \quad T \in H_L$$

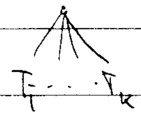
where T_i are the trees attached to the root of T.

For $X \in H$, extend B_- by a leibniz rule:

$$B_-(X_1 X_2) = X_1 B_-(X_2) + B_-(X_1) X_2.$$

$B_-(\circ) = e$, $B_-($  $) = \circ$, $B_-($  $) =$ , $B_-($  $) = \circ\circ$.

Let us now introduce a map $B_+: H \to H$.

$B_+(X) =$  where $X = \bigcup_{T_i}$.

$B_+(e) = \circ$.

For $T \in H_L$, $\quad B_-(B_+(T)) = T = B_+(B_-(T))$.

But in general

$B_+(B_-(X)) \neq X.$

Example: $B_+(B_-(\bullet\,\bullet)) = B_+(2\,\bullet) = 2\,B_+(\bullet) = 2\,\overset{\bullet}{\underset{\bullet}{|}} \neq \bullet\,\bullet$.

let us now define a map (the coproduct)

$\Delta : H \to H \otimes H$ by

$\Delta[e] = e \otimes e$

$\Delta[T_1 \dots T_k] = \Delta[T_1] \dots \Delta[T_k]$

$\Delta[B_+(X)] = B_+(X) \otimes e + (id \otimes B_+) \circ \Delta(X).$

Examples:

$\Delta[\bullet] = \Delta[B_+(e)] = B_+(e) \otimes e + (id \otimes B_+) \circ \Delta(e)$

$= \bullet \otimes e + e \otimes \bullet$

$\Delta[\overset{\bullet}{\underset{\bullet}{|}}] = \Delta[B_+(\bullet)] = \overset{\bullet}{\underset{\bullet}{|}} \otimes e + (id \otimes B_+)\,\Delta[\bullet]$

$= \overset{\bullet}{\underset{\bullet}{|}} \otimes e + \bullet \otimes \bullet + e \otimes \overset{\bullet}{\underset{\bullet}{|}}$

$\Delta[\overset{\bullet}{\underset{\bullet}{\overset{|}{\underset{|}{\bullet}}}}] = \dots, \quad \Delta[\overset{\bullet}{\bullet\!\wedge\!\bullet}] = \dots$

Let us further introduce a map (the counit)

$\bar{e}: H \to \mathbb{K}$ by $\bar{e}(e) = 1 \in \mathbb{K}$,

$\bar{e}(X) = 0$ else.

Let $E: \mathbb{K} \to H$ be the map $\mathbb{K} \ni q \to qe \in H$.

We set $\eta = E \circ \bar{e}$, and $P := id - \eta$.

To understand the significance of Δ (the coproduct)

let us remind ourselves what a product does for us in an algebra.

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "id \otimes m"] \arrow[d, "m \otimes id"'] & H \otimes H \arrow[d, "m"] \\
H \otimes H \arrow[r, "m"'] & H
\end{tikzcd}$$

associativity

Reverse arrows

$$\begin{tikzcd}
H \otimes H \otimes H & H \otimes H \arrow[l, "id \otimes \Delta"'] \\
H \otimes H \arrow[u, "\Delta \otimes id"] & H \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

coassociativity.

In an algebra, we demand a unit and an associative multiplication. In a coalgebra, we demand a counit and a coassociative multiplication.

$$(\Delta\otimes id)\circ\Delta = (id\otimes\Delta)\circ\Delta \qquad (1)$$

$$(\bar{e}\otimes id)\circ\Delta = id = (id\otimes\bar{e})\circ\Delta \qquad (\mathbb{K}\otimes H\simeq H) \qquad (2)$$

(2) is easy. (do it yourself)

Do examples!

For (1). Write $\Delta(X) = X'\otimes X''$ (sum suppressed)

$$\Delta[\underbrace{B_+(X)}_{T}] = B_+(X)\otimes e + (id\otimes B_+)\,\Delta(X).$$

$$(id\otimes\Delta)\circ\Delta[B_+(X)] = B_+(X)\otimes e\otimes e + (id\otimes\Delta\circ B_+)\circ\Delta(X)$$

$$= B_+(X)\otimes e\otimes e + (id\otimes\Delta\circ B_+)(X'\otimes X'')$$

$$= B_+(X)\otimes e\otimes e + X'\otimes(id\otimes B_+)\,\Delta(X'') + X'\otimes B_+(X'')\otimes e$$

$$= T\otimes e\otimes e + X'\otimes B_+(X'')\otimes e + (id\otimes id\otimes B_+)(id\otimes\Delta)\circ\Delta(X)$$

$$= T\otimes e\otimes e + X'\otimes B_+(X'')\otimes e + (id\otimes id\otimes B_+)(\Delta\otimes id)\circ\Delta(X)$$

$$= \overset{(\Delta\otimes id)}{\cancel{\Delta}}\circ B_+(X)\otimes e + (\Delta\otimes id)(id\otimes B_+)\circ\Delta(X)$$

$$= (\Delta\otimes id)\circ\Delta(B_+(X)). \qquad \square$$

Def. admissible cut.

We let $T^{[0]}$ be the set of vertices of T and $T^{[1]}$ be the set of edges. A subset $C \subset T^{[1]}$ is called admissible, if for any path from any $v \in T^{[0]}$ towards the root, at most one edge which is traversed is in C.

Examples!  ← admissible  ← non-admissible.

Let $P^C(T)$ be the product of rooted trees obtained by removing C which are not connected to the root.
Let $R^C(T)$ be the single component still connected to the root.

$$\Delta[T] = T \otimes e + e \otimes T + \sum_{\substack{\text{adm.}\\ \text{cuts } C\\ \text{of } T}} P^C(T) \otimes R^C(T).$$

Examples.

the antipode. Co-Inverse.

$S : H \to H$

$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = \varepsilon .$

$S(e) = e, \quad S(T) = -T - \sum_{\substack{\text{adm} \\ \text{cuts} \\ C}} S[P^C(T)]\, R^C(T).$

$S(T) = -T - \sum_{\substack{\text{all} \\ \text{cuts } C}} (-1)^{n_C} P^C(T)\, R^C(T).$

Example. check both formulas!

$S(\bullet) = -\bullet$ $S($  $) = -$  $+ \bullet\bullet$

$S($  $) = \dots$ $S($  $) = \dots$

Pf $m \circ (S \otimes id) \circ \Delta ($  $) = S($  $) + S(\bullet)\bullet +$ 

$= -$  $+ \bullet\bullet - \bullet\bullet +$  $= 0 = \varepsilon($  $)$

Do $m \circ (S \otimes id) \circ \Delta$ and $m \circ (id \otimes S) \circ \Delta$

for , .

General proof: exercise. (for the conversions)

Summary:

$(H, \Delta, m, E, \bar{e}, S)$ is a Hopf algebra.

Putting Hopf algebras to use.

Ultimately, we want to use Hopf algebras to learn how to deal with singularities in QFT. As QFT is right now too hard for us, and its singularities are its hardest part, we train the use of Hopf algebras first in toy models.

Our first toy model is defined as follows.

Consider a map $\phi: H \to V$ (where V can be an algebra, a ring or such)

such that $\phi(T_1 T_2) = \phi(T_1)\phi(T_2)$. $(*)$

Note that this implies $\phi(e) = 1_V$.

First example: $\phi_a(e) = 1$.

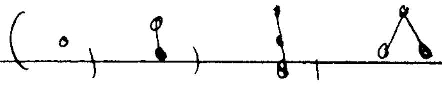
$$\phi_a(\bullet) = \int_0^\infty \frac{x^{-\varepsilon}}{x+a}\,dx \qquad 1 >> \mathrm{Re}(\varepsilon) > 0, \quad a > 0.$$

$\varepsilon \in \mathbb{C}$, $a \in \mathbb{R}_+$

$$\phi_a(B_+(X)) = \int_0^\infty \frac{x^{-\varepsilon}}{x+a}\,\phi_x(X).$$

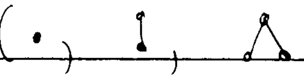
↑ well defined because of $(*)$

Walk out

$\phi_a($  $)$ using

$$\int_0^\infty \frac{x^{-k\varepsilon}}{x+a}\,dx = B(k\varepsilon, 1-k\varepsilon)\, a^{-k\varepsilon}.$$

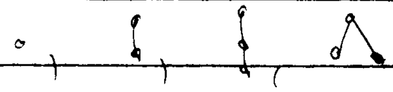
Calculate

$\phi \circ S($  $)$. $\phi_a \circ S(T) = -\phi_a(T) - \sum_{\substack{\text{ad-} \\ \text{cut } c_i}} \phi_a(S(P^{c}(T)))\cdot\phi_a(R^c(T))$

Let $R: V \to V$ be the map

$\phi_a \to \langle \phi_1 \rangle$ (projection onto pole part at unit scale)

Define $S_R^{\phi_a}(T) = -R[\phi_a(T)] - \sum_{\substack{\text{ad-} \\ \text{cuts} \\ C}} S_R^{\phi_a}(P^c(T))\,\phi_a(R^c(T))$.

Do 

⑦

Thm. Let $R \equiv R^b_{MS}$ be defined by

$R(\phi_a(T)) = \langle \phi_b(T) \rangle$, where

$\langle \dots \rangle$ means projection onto the pole part.

$$\left\langle \sum_{k=-r}^{+\infty} c_k \varepsilon^k \right\rangle = \sum_{k=-r}^{-1} c_k \varepsilon^k.$$

Then $$S_R^{\phi_a}(T) \equiv -R(\phi_a(T)) - \sum_{\substack{\text{adm}\\\text{cuts}\\C}} R\left[S_R^{\phi_a}(P^C(T))\, \phi_a(R^C(T))\right]$$

does not depend on $\log b$ and $(P = id - E\circ\bar{\varepsilon})$

$m \circ (S_R^{\phi_a} \otimes \phi) \circ (id \otimes P) \circ \Delta(T)$ has

<u>no poles</u> $\sim \log a$ ($\approx \log b$) and $S_R^{\phi_a} * \phi_a(T)$

$\equiv m \circ (S_R^{\phi_a} \otimes \phi) \circ \Delta(T)$ is finite. $\left(\begin{smallmatrix}\text{exists}\\ \text{at } \varepsilon = 0\end{smallmatrix}\right)$

Furthermore, $S_R^{\phi_a}(T_1 T_2) = S_R^{\phi_a}(T_1)\, S_R^{\phi_a}(T_2)$ is a character.

Before we consider the proof of this theorem, let us work out a few examples.

$T = \bullet$

$\Delta[\bullet] = \bullet \otimes e + e \otimes \bullet$

$S_R^{\phi}(\bullet) = -R[\phi(\bullet)]$

$\phi_a(\bullet) = \int_0^\infty \frac{x^{-\varepsilon}}{x+a}\,dx = B_1(\varepsilon)\, a^{-\varepsilon}$

where $B_j(\varepsilon) = B(j\varepsilon, 1-j\varepsilon) = \frac{\Gamma(1+j\varepsilon)\,\Gamma(1-j\varepsilon)}{j\varepsilon}$.

$S_R^{\phi_a}(\bullet) = -\langle B_1(\varepsilon)\, b^{-\varepsilon}\rangle = -\frac{1}{\varepsilon}$

$\Gamma(1+j\varepsilon) = 1 + O(\varepsilon)$

$S_R^{\phi_a} * \phi_a(\bullet) = B_1(\varepsilon)\, a^{-\varepsilon} - \frac{1}{\varepsilon} = \cancel{\varepsilon} - \ln a + O(\varepsilon)$.

$T = \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}$

$\Delta\left[\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right] = \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} \otimes e + e \otimes \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} + \bullet \otimes \bullet$

$S_R^{\phi_a}\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right) = -R\left(\phi_a\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right)\right) - R\left[S_R^{\phi_a}(\bullet)\, \phi_a(\bullet)\right]$.

$\phi_a\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right) = \int_0^\infty\int_0^\infty \frac{x^{-\varepsilon}}{x+a}\, \frac{y^{-\varepsilon}}{y+x}\, dy\, dx$

$= B_1(\varepsilon)\, B_2(\varepsilon)\, a^{-2\varepsilon} = \frac{1}{2\varepsilon^2}(1+O(\varepsilon))(1 - 2\varepsilon \ln a + \dots)$

$= \frac{1}{2\varepsilon^2} + \dots \frac{\ln a}{\varepsilon} + \dots$

③

$$S_R^{\phi_a}(T) = -\langle B_1(\varepsilon) B_2(\varepsilon) e^{-2\varepsilon}\rangle$$

$$+ \langle\langle B_1(\varepsilon) e^{-\varepsilon}\rangle B_1(\varepsilon) e^{-\varepsilon}\rangle$$

$$= -\langle B_1(\varepsilon) B_2(\varepsilon) e^{-2\varepsilon}\rangle + \langle \tfrac{1}{2} B_1(\varepsilon) e^{-\varepsilon}\rangle$$

Expand in ε $\Rightarrow$ no $\frac{\log b}{\varepsilon}$ terms survive.

$$S_R^{\phi_a} * \phi\,(T)$$

$$= \phi_a(T) - R[\phi_a(T)] - R[\phi_a(\cdot)]\,\phi_a(\cdot) + R[R[\phi_a(\cdot)]\,\phi_a(\cdot)]$$

$$= [id - R]\,(m \circ (S_R^{\phi_a} \otimes \phi_a)\cdot(id \otimes P)\circ\Delta(T))$$

and in general,

$$S_R^{\phi_a} * \phi\,(T) = (id - R)\,[m \circ (S_R^{\phi_a} \otimes \phi_a)\cdot(id \otimes P) \circ \Delta(T)].$$

We implicitly assume that

$S_R^{\phi_a}(T_1 T_2) = S_R^{\phi_a}(T_1)\, S_R^{\phi_a}(T_2)$, which we will prove later in class.

As $S_R^{\phi_a} * \phi_a$ is in the image of $id - R$, it is clear that $S_R^{\phi_a} * \phi_a$

exists at $\varepsilon = 0$ and from

the results for $T = \bullet$

we get $S_R^{\phi_a}$ does not depend on $\log b$,

$S_R^{\phi_a} * \phi_a$ ~~has~~ is finite and

$m \circ (S_R^{\phi_a} \otimes \phi_a) \circ (id \otimes P) \circ \Delta \equiv S_R^{\phi_a} * \phi_a - S_R^{\phi_a}$

has no log-dependent poles.

~~We~~ Actually, $S_R^{\phi_a} * \phi_a(\bullet)\big|_{\varepsilon=0}$ is a first

order polynomial in $\log a$.

We can prove similar properties for all

T by induction ~~of~~ over the number of

vertices.

Use $\Delta \circ B_+ = B_+ \otimes e + (id \otimes B_+) \circ \Delta$,

$S_R^\phi * \phi\, (B_+(X))$

$= m \circ (S_R^\phi \otimes \phi) \circ \Delta \circ B_+ (X)$

(5)

$$= m \circ (S_R^\phi \otimes \phi) \circ [(B_+ \otimes e) + (id \otimes B_+)] \Delta(X)$$

$$= S_R^\phi(B_+(X)) + (S_R^\phi \otimes \phi \circ B_+)(\Sigma X' \otimes X'')$$

$$= S_R^\phi(B_+(X)) + \underbrace{\int_0^\infty dx \frac{x^{-\varepsilon}}{x+a} S_R^{\phi_x} * \phi_x(X)}_{\text{in class, we called this } \bar{B}_+^{\phi_1 S_R^\phi * \phi}(X).}$$

Its easy to see that the integral has no log a dependent poles if $S_R^{\phi_x} * \phi_x(X)$ is at $\varepsilon = 0$ at most a polynomial in $\log x$, as

$$\int_0^\infty dx \left\{ \frac{x^{-\varepsilon}}{x+a} \operatorname{Poly}(\log x) - \frac{x^{-\varepsilon}}{x+\tilde{a}} \operatorname{Poly}(\log x) \right\}$$

has no poles, and that after the subtraction of the poles, ~~by~~ by S_R^ϕ, (14)

(6)

$S_R^{\phi} * \phi\,(T)$ is again a polynomial in $\log q$ of degree at most $n+1$.
We will repeat this proof again in detail for Feynman graphs, and in the exercises.

The sub Hopf algebra H_{CM}.

So far we introduced the Hopf algebra H of undecorated rooted trees. We identified its coproduct Δ, counit $\bar{e}$ and antipode S.

This Hopf algebra has a closed sub-Hopf algebra H_{CM}, ie

$\Delta[X] = \sum X' \otimes X''$ where $X, X', X'' \in H_{CM}$ and

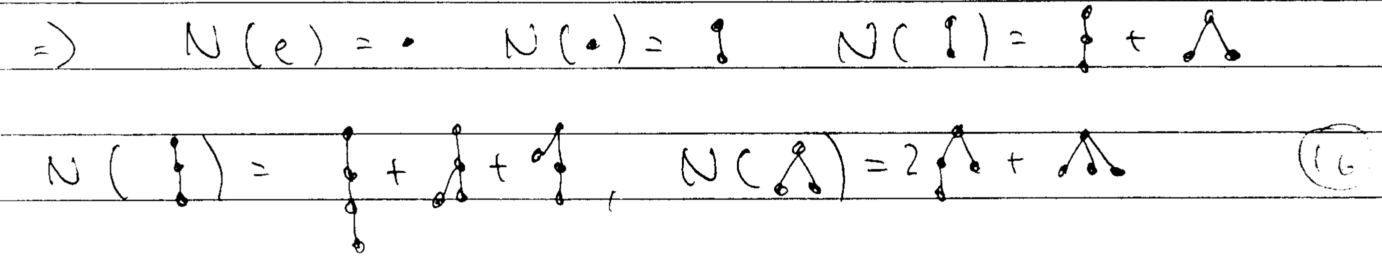
Δ is the restriction of our coproduct to H_{CM}.

To define H_{CM} we need the operation of natural growth.

$N : H \to H$ $\qquad N(X_1 X_2) = N(X_1) X_2 + X_1 N(X_2)$

(it's a derivation) and

$N(T)$ = "sum over all ways of adding one more edge and vertex"



We define

$$\delta_k := N^k(e).$$

We define H_{cn} as the algebra of polynomials in generators δ_k, and make this commutative algebra into a Hopf algebra by using the previous $\bar{e}, \Delta, S$.

The only non-trivial fact to be shown is that the coproduct maps δ_k into Polynomials of such generators.

$$\Delta(\delta_1) = \delta_1 \otimes e + e \otimes \delta_1$$

$$\Delta(\delta_2) = \delta_2 \otimes e + \delta_1 \otimes \delta_1 + e \otimes \delta_2$$

$$\Delta(\delta_3) = \Delta\Big(\overset{\bullet}{\underset{\underset{\bullet}{|}}{\overset{\bullet}{|}}} + \underset{\bullet\ \ \bullet}{\overset{\bullet}{\wedge}}\Big) = \delta_3 \otimes e + e \otimes \delta_3 + \bullet \otimes \overset{\bullet}{\underset{\bullet}{|}} + 2 \bullet \otimes \overset{\bullet}{\underset{\bullet}{|}}$$

$$+ \overset{\bullet}{\underset{\bullet}{|}} \otimes \bullet + \bullet\bullet \otimes \bullet = \delta_3 \otimes e + e \otimes \delta_3 + 3\,\delta_1 \otimes \delta_2 + \delta_2 \otimes \delta_1$$

$$+ \delta_1^2 \otimes \delta_1$$ by inspection.

Let us prove this in general.

δ_n is by construction a sum of trees:

$\delta_n = \sum_i t_i$ say. (for example, $\delta_3 =$  $+$ )

Thus
$$\Delta(\delta_n) = e \otimes \delta_n + \delta_n \otimes e + \sum_i \sum_{\substack{\text{all cuts} \\ C_i \text{ of } t_i}} P^{C_i}(t_i) \otimes R^{C_i}(t_i)$$

We can write (using the t_i of δ_n)

$$\Delta(\delta_{n+1}) = e \otimes \delta_{n+1} + \delta_{n+1} \otimes e$$
$$+ \sum_i \sum_{\substack{\text{all cuts} \\ C_i \text{ of } t_i}} \Big\{ N\big[P^{C_i}(t_i)\big] \otimes R^{C_i}(t_i)$$
$$+ P^{C_i}(t_i) \otimes N\big[R^{C_i}(t_i)\big] \Big\}$$
$$+ n\, \delta_1 \otimes \delta_n + \sum_i \sum_{\substack{\text{all cuts} \\ C_i \text{ of } t_i}} \ell\big(R^{C_i}(t_i)\big)\, P^{C_i}(t_i) \otimes$$
$$\otimes R^{C_i}(t_i),$$
where $\ell(t)$ gives the number of vertices of a tree t.

What we did is we decomposed the cuts of δ_{n+1} in four classes:

i) the edge of a new grown vertex is not part of the admissible cut: then we will have N acting on either the P^{G} or the R^{G} side. (two cases)

ii) or, the new edge is part of the admissible cut. This gives us a factor δ_1 always. If that edge is the only cut, we get $n\,\delta_1 \otimes \delta_n$ as a contribution.

Otherwise, that edge must have been grown from the R^c-part (otherwise the cut obtained by adding that edge would not be admissible)

then, we get the $\ell[R^{G_i}(t_i)]\cdots$ part. $\square$.

What is the use of the Hopf algebra H_{CK}?

different.

Consider formal diffeomorphism of the form

$$x \to x + \sum_{k=2}^{\infty} a_k x^k \quad \text{(for real } x \text{ say).}$$

let $\varphi(x)$, $\psi(x)$ be two such diff's.

$$\varphi(x) = x + \sum_{k=2}^{\infty} a_k x^k \qquad \psi(x) = x + \sum_{k=2}^{\infty} b_k x^k$$

What are the Taylor coefficients of the composition

$$\varphi \circ \psi(x) \; ?$$

The answer is given by H_{CM} in the following way.

Define $\log^{(n)}[\varphi'(x)] = \delta^n_\varphi$, $\log^{(n)}[\psi'(x)] = \delta^n_\psi$

(these are the n-th derivatives of the log of the derivative of the diffeomorphism)

useful in understanding two different viewpoints on renormalization: perturbative

It is easy to reconstruct φ, ψ from the knowledge of $\delta^n_\varphi, \delta^n_\psi$. (exponentiate and integrate).

So what we are after is the knowledge of $\delta^n_{\varphi\circ\psi}$ from $\delta^n_\varphi, \delta^n_\psi$.

Thm.

$$\delta^n_{\varphi\circ\psi} = m\circ(\bar\psi\otimes\bar\varphi)\circ\Delta(\delta_n),$$

where $\delta_n \in H_{CM}$ and $\bar\psi, \bar\varphi$ are characters

$$\bar\psi(\delta_{k_1}\ldots\delta_{k_r}) = \delta^{k_1}_\psi \ldots \delta^{k_r}_\psi, \quad \bar\psi(e) = 1,$$

and similarly for $\bar\varphi$.

That Thm is part of a larger theory given by Connes and Moscovici, and myself. We will not prove it here. It is useful in understanding two different view points on renormalization: perturbative renormalization using Hopf algebras as we do in these lectures and renormalization regarded as diffeomorphisms of physical parameters driven by the renormalization group.

The equivalence of both viewpoints relies on theorems like the above.

— o —

We are now ready to start considering the Lie algebra which appears in the dual of our Hopf algebra.

Thm 2. (H, B_+) is unique up to isomorphism and universal: for any pair (H_1, L), with H_1 being a commutative Hopf algebra and L a Hochschild one cocycle (ie $\Delta_1 \circ L = L \otimes 1 + (id \otimes L) \circ \Delta_1$) there exists a unique Hopf algebra morphism $\rho : H \to H_1$ s.t. $L \circ \rho = \rho \circ B_+$.

B_+ is Hochschild closed by definition, but not exact: $B_+(e) = \bullet$, while for $T = bZ$, $T(e) = Z(e)\, e - (id \otimes Z)\, \Delta(e)$

$$= Z(e)\, e - e\, Z(e) = 0.$$

Note that for any map $Z : H \to \underbrace{H \otimes \dots \otimes H}_{n\text{-times}}$

$bZ : H \to \underbrace{H \otimes \dots \otimes H}_{n+1 \text{ times}}$ is given by

$$bZ(a) = (id \otimes Z) \circ \Delta(a) - \Delta_{(1)} Z(a) + \Delta_{(2)} Z(a) + \dots + (-1)^j \Delta_{(j)} Z(a) + \dots + (-1)^n \Delta_{(n)} Z(a) + (-1)^{n+1} Z(a) \otimes e.$$

So, if $Z : H \to \mathbb{K}$ (σ times),
$\Delta_{(i)}$ means $id \otimes \dots \otimes \Delta \otimes \dots \otimes id$ where Δ is in i'th slot.
then $bZ(a) = (id \otimes Z)\Delta(a) - Z(a)e \in H$

$n=1$: $Z : H \to H$ $\quad bZ(a) = (id \otimes Z)\Delta(a) - \Delta \cdot Z(a) + Z(a) \otimes e \in H \otimes H$

and so on. Thus $bB_+ = 0$. Note $b^2 = 0$.

Now consider a pair (H_1, L_1) where L_1 is a commutative Hopf algebra and L_1 is a 1-cocycle. $L_1 \circ \rho = \rho \circ B_+$ uniquely determines an algebra homomorphism $\rho : H \to H_1$.

$$\rho(\Pi T) = \Pi \rho(T)$$

$$\rho(\bullet) = L_1(1)$$

$$\rho \circ B_+(\Pi T) = L_1 \rho(\Pi T) \text{ by induction.}$$

Need to check: $(\rho \otimes \rho) \circ \Delta(a) = \Delta_1 \rho(a)$.

Check on $T = B_+(\Pi T)$.

$$\Delta_1 L_1 (\rho(\overline{\pi} T)) = L_1 (\rho(\overline{\pi}(T))) \otimes 1$$

$$+ (id \otimes L_1)\, \Delta_1\, \rho(\overline{\pi}\, T).$$

Induction

$$(id \otimes L_1) \circ \Delta_1\, \rho(\overline{\pi}\, T) = (\rho \otimes \rho)(id \otimes B_+) \circ \Delta(\overline{\pi}\, T)$$

$\cancel{= (id \otimes L_1)}$

$$\Rightarrow \Delta_1 \circ L_1 \circ \rho(\overline{\pi}\, T) = L_1 \circ \rho(\overline{\pi}(T)) \otimes 1$$

$$+ (\rho \otimes \rho)(id \otimes B_+) \circ \Delta(\overline{\pi}(T)).$$

$$(\rho \otimes \rho) \circ \Delta \circ B_+ (\overline{\pi}\, T) = \rho \circ B_+ (\overline{\pi}\, T) \otimes \rho(e) \overset{\rho\otimes\rho}{+} (id \otimes B_+)\, \Delta(\overline{\pi}\, T)$$

$$= L_1 \circ \rho(\overline{\pi}\, T) \otimes 1 + (\rho \otimes \rho)(id \otimes B_+)\, \Delta(\overline{\pi}\, T) \quad \square.$$

The Lie algebra $\mathcal{L}$

Let $\mathcal{L}^{**}$ be the linear span of elements Z_T, T a rooted tree.

Define $Z_{T_1} * Z_{T_2} = \sum_T n(T_1, T_2; T)\, Z_T$

$n(T_1, T_2; T)$ = # of ~~simple~~ admissible cuts c, $|c| = 1$, such that $P^c(T) = T_1$, $R^c(T) = T_2$.

$\Rightarrow$ $[Z_{T_1}, Z_{T_2}] = Z_{T_1} * Z_{T_2} - Z_{T_2} * Z_{T_1}$

is a Lie bracket.

H_R is the dual of the enveloping algebra of $\mathcal{L}$.

Set $Ass(T_1, T_2, T_3) = Z_{T_1} * (Z_{T_2} * Z_{T_3}) - (Z_{T_1} * Z_{T_2}) * Z_{T_3}$.

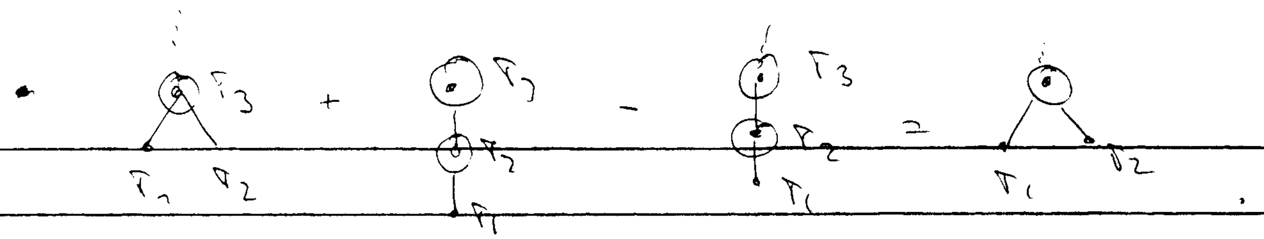
$Ass(T_1, T_2, T_3) = \sum_T n(T_1, T_2, T_3; T)\, Z_T$,

where n = # of admissible cuts C, $|C| = 2$,

$P^C(T) = T_1 T_2$, $R^C(T) = T_3$.

Prop. $Ass(T_1, T_2, T_3) = Ass(T_2, T_1, T_3)$.

($\hat{=}$ $*$ is pre-Lie).



Pre-lie: $Z_{\tau_1} * (Z_{\tau_2} * Z_{\tau_3}) - (Z_{\tau_1} * Z_{\tau_2}) * Z_{\tau_3}$

$$= Z_{\tau_2} * (Z_{\tau_1} * Z_{\tau_3}) - (Z_{\tau_2} * Z_{\tau_1}) * Z_{\tau_3}$$

Thm. If a map $* : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is pre-lie, then $[X, Y] = X * Y - Y * X$ is a Lie-bracket, ie fulfils Jacobi.

Proof. $[X[Y, Z]] = X * (Y * Z - Z * Y) - (Y * Z - Z * Y) * X$ + cyclic terms.

Use Pre-Lie property

$$\begin{aligned}
&\underline{X} * (\underline{Y * Z} - Z * Y) - (Y * Z - Z * Y) * X \\
+\ &\underline{Y} * (Z * X - \underline{X * Z}) - (Z * X - X * Z) * Y \\
+\ &Z * (X * Y - Y * X) - (\underline{X} * Y - \underline{Y * X}) * \underline{Z}
\end{aligned}$$

□

So $\mathcal{L}$ becomes a Lie algebra

Now $\langle Z_T, P(\delta_{T_1}) \rangle = \left(\frac{\delta}{\delta_T} P \right)(0)$ lin. fam on $\mathcal{H}$, ie $\in \mathcal{H}^*$

↑

$\langle Z_{T}, \delta_T \rangle = 1$. Kronecker pairing

$P \to P(0)$ is the counit

$$(*)\ \langle Z_{T}, PQ\rangle = \langle Z_T, P\rangle\, \varepsilon(Q) + \varepsilon(P)\,\langle Z_T, Q\rangle$$

$$\Rightarrow \quad \Delta Z_T = Z_T \otimes 1 + 1 \otimes Z_T \quad \in \mathcal{L} \times \mathcal{L}$$

$$\langle Z_1 Z_2, P\rangle = \langle Z_1 \otimes Z_2, \Delta P\rangle$$

~~$[Z_{T_1}, Z_{T_2}]$~~ Now $Z_{T_1} Z_{T_2} - Z_{T_2} Z_{T_1}$ is a derivation (it fulfils $(*)$), hence we must show that $Z_{T_1} Z_{T_2} - Z_{T_2} Z_{T_1}$ $= [Z_{T_1}, Z_{T_2}]$.

Now, let H_{Aug} be the kernel of $\bar{e}$.

Then $\Delta \delta_T = \delta_T \otimes e + e \otimes \delta_T + R_T$,

$$R_T \in \mathcal{H}_{Aug} \otimes \mathcal{H}_{Aug}.$$

projection onto $\mathcal{H}_L \cap \mathcal{H}_{Aug}$

$$R_T^{(lin)} = \sum_{\substack{C \\ |C|=1}} \delta_{T_C}\ \delta_{P^{C}(T)} \otimes \delta_{R^{C}(T)}.$$

Now $\langle Z_{T_1} Z_{T_2}, \delta_T \rangle = \langle Z_{T_1} \otimes Z_{T_2}, \Delta \delta_T \rangle$

counts the number of ways T_1 can be cut from T as to obtain T_2. $\square$

~~Furthermore,~~

Let us introduce some actions of $\mathcal{L}$ on H.
(as derivations)

i) $\mathcal{L} \times H \to H$ by $\quad Z_{T_1} \times \delta_{T_2} \to \sum_{\substack{\text{adm.}\\ \text{cuts}\\ c}} \langle Z_{T_1}, P^c(T_2) \rangle \, \delta_{R^c(T_2)}$

ii) $N_T : \mathcal{H}^* \otimes H \to H$ by

$$N_{T_1}(\delta_{T_2}) = \delta_{T_1 * T_2}.$$

for i) $\quad Z_{T_1} \times (Z_{T_2} \times \delta_{T_3}) - Z_{T_2} \times (Z_{T_1} \times \delta_{T_3})$

$$= \cancel{Z_{[T_1, T_2]}} \; [Z_{T_1}, Z_{T_2}] \times \delta_{T_3}$$

for ii) same action!

(more on that later)