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Chapter 1

Introduction

The goal of this dissertation is to begin the analysis of a class of combinatorial Lie algebras, which has been introduced by Alain Connes and Dirk Kreimer in [7], in their approach to the renormalization of perturbative quantum field theories [5, 6, 2], and [9] for the general framework. In their approach, a main role is played by the Hopf algebra structure defined over the set of Feynman diagrams underlying the theory. The main features of such a Hopf algebra are captured by the Hopf algebra of rooted trees $\mathcal{H}_{rt}$ in its bare and dressed version. The properties of such a combinatorial Hopf algebra, becomes then crucial for the understanding of the combinatorics which is behind renormalization [12, 13, 14].

$\mathcal{H}_{rt}$ is a commutative, $\mathbb{Z}_{\geq 0}$-graded and connected ($\ker\epsilon = \bigoplus_{i>0} H_i$) Hopf algebra. By the Milnor-Moore theorem [21], its dual $\mathcal{H}^\star_{rt}$ is isomorphic to the universal enveloping algebra of a Lie algebra $P(\mathcal{H}^\star_{rt})$, which can be faithfully represented into the Lie algebra of the infinitesimal characters of $\mathcal{H}_{rt}$. From a more detailed analysis [7], it follows that the Lie algebra $P(\mathcal{H}^\star_{rt})$ has two other distinguished representations, $\mathcal{D}^+$ and $\mathcal{D}^-$, where the former is the Lie algebra of the insertion operators and the latter the Lie algebra of the elimination operators. Since both $\mathcal{D}^+$ and $\mathcal{D}^-$ are Lie algebras of derivations for $\mathcal{H}_{rt}$, it is natural to seek for a larger Lie algebra which contains both $\mathcal{D}^+$ and $\mathcal{D}^-$ as sub Lie algebras. Such a Lie algebra is the insertion elimination Lie algebra $\mathcal{L}_L$ introduced in [7].

Since the full insertion elimination Lie algebra is a quite complicated object, it is natural to seek for some distinguished sub Hopf algebra of $\mathcal{H}_{rt}$, and then begin the analysis of the insertion elimination Lie algebra naturally attached to it. The choice of the ladder Hopf algebra $\mathcal{H}_L$ is then quite a natural one.

On the one hand, it is a fairly simple Hopf algebra, on the other hand it has a non trivial physical content, [3], [16].
The main achievement of the present work is the description of the structure of the insertion elimination Lie algebra, which is naturally associated to the ladder Hopf algebra of rooted trees.

The outline of the present work is the following:

In the first chapter, we introduce all the algebraic structures which are used in the following chapters.
The second chapter is devoted to the detailed analysis of the Hopf algebra of rooted trees. In particular we give a summary of the results contained in [21], suitable to the present purposes.
The third and the fourth chapters are the core of the present work. In the third chapter, we introduce and motivate the class of the insertion-elimination Lie algebras. The fourth chapter contains the analysis of the structure of the insertion-elimination Lie algebra which is naturally associated to the Hopf algebra of the ladder rooted trees. There, we describe the structure of such a Lie algebra, its relations with some other well known infinite dimensional Lie algebra, and finally, we describe its cohomology in some details .
In the fifth chapter, we give a survey of the theory of the extensions of Lie algebras. There we carefully describe this theory, which is particularly relevant for what is discussed in chapter four.
We conclude the exposition with two appendices, where the main results about the cohomology of Lie algebras and the cohomology of the general linear group are stated.

Chapter 2

Hopf algebras

In this chapter all the algebraic structures relevant for the present work will be introduced. A particular care will be taken of the class of connected graded Hopf algebras, for which a structure theorem will be proved. The last section contains a detailed account of the Milnor-Moore theorem, which is a key result for the topic of the present work. The references for the present chapter are: [11] and [21].

2.1 Algebras

We will introduce in this section the main notions from the theory of Hopf algebras. In what follows, we will assume that the base field k is the field of complex numbers $\mathbb{C}$ of the field of real numbers $\mathbb{R}$ and all the tensor products will be assumed over the field k.

Definition 1 *A k-algebra with unit is a k-vector space A together with two linear maps, multiplication $m : A \otimes A \longrightarrow A$ and unit $u : k \longrightarrow A$ such that the following two diagrams are commutative:*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id}\otimes m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

(Associativity)

$$\begin{tikzcd}
k \otimes A \arrow[r, "u\otimes \mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "m"] & A \otimes k \arrow[l, "\mathrm{id}\otimes u"'] \arrow[dl, "\cong"] \\
 & A &
\end{tikzcd}$$

(Unit)

Definition 2 *Let V and W be two k-vector spaces. Define:*

$$\tau : V \otimes W \longrightarrow W \otimes V \tag{2.1}$$

saying that $\tau(v \otimes w) = w \otimes v$ for each $v \otimes w \in V \otimes W$. The map τ is called twist map.

Definition 3 *(A, u) is said to be commutative if $\tau \circ m = m$, i.e if the following diagram is commutative:*

$$\begin{tikzcd}
A \otimes A \arrow[rr, "m"] \arrow[dr, "\tau"'] & & A \\
 & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

(Commutativity)

Algebras over a given field k form a category, whose morphisms are defined as follow:

Definition 4 *A morphism ϕ between two algebras (A_1, m_1, u_1) and (A_2, m_2, u_2) is a linear map $\phi : A_1 \longrightarrow A_2$ such that $m_2 \circ (\phi \otimes \phi) = \phi \circ m_1$ and $\phi \circ u_1 = u_2$.*

To show that algebras and their morphisms form a category we need only to check that:

Proposition 1 *Given $\phi_1 : A_1 \longrightarrow A_2$ and $\phi_2 : A_2 \longrightarrow A_3$, morphisms of algebras $\phi_2 \circ \phi_1 : A_1 \longrightarrow A_3$ is a morphism of algebras;*

Proof We need to check that:

$$m_3 \circ \big((\phi_2 \circ \phi_1) \otimes (\phi_2 \circ \phi_1)\big) = (\phi_2 \circ \phi_1) \circ m_1.$$

$m_3 \circ \big((\phi_2 \circ \phi_1) \otimes (\phi_2 \circ \phi_1)\big) = m_3 \circ \big(\phi_2(\phi_1) \otimes \phi_2(\phi_1)\big)$. Since ϕ_2 is an algebra morphism the last term of the previous equality can be written as: $\phi_2\big(m_2(\phi_1 \otimes \phi_1)\big)$. Since also ϕ_1 is an algebra morphism we can rewrite the last formula as: $\phi_2(\phi_1 \circ m_1)$, that is what we wanted to show. ♠

From now on, by algebra will be meant associative algebra unless specified differently.

Example 1 *The ground field k with multiplication $m_k : k \otimes k \longrightarrow k$ (which corresponds to the natural multiplication) and unit $u_k : k \longrightarrow k$, defined by $u_k(1) = 1$, is a commutative k-algebra. For any given algebra (A, m, u), the unit $u : k \longrightarrow A$ is a morphism between the algebra (k, m_k, u_k) and the algebra (A, m, u).*

Example 2 *Let (A, m_A, u_A) and (B, m_B, u_B) be two algebras over the field k. We can define an algebra structure on the tensor product $A \otimes B$ via the following: $m_\otimes\big((a \otimes b) \otimes (a' \otimes b')\big) = m_A(a \otimes a') \otimes m_B(b \otimes b')$, for $a, a' \in A$ and $b, b' \in B$. The unit is given by $u_A \otimes u_B$.*

For any given algebra A, we can define the following notions:

Definition 5 *A given subvector space $B \subset A$ is called an A sub-algebra if for each $x, y \in B$, $m(x, y) \in B$, i.e if and only if the restriction of the multiplication map to B takes values in B.*

Moreover we also have the following:

Definition 6 *A subalgebra $I \subset A$ is called right (left) ideal if $m(A, I) \subset I$ $(m(I, A) \subset I)$. If $I \subset A$ is called bilateral if it is left and right ideal.*

Proposition 2 *If I is a bilateral ideal then the quotient space $\overline{A} = A/I$ has a natural structure of algebra:* $\overline{m} : \overline{A} \otimes \overline{A} \longrightarrow \overline{A}$, $\overline{m}(\overline{x}, \overline{y}) = \overline{m(x, y)}$.

Proof The only thing we need to check is that the multiplication is well defined. This follows from the hypothesis that I is bilateral. ♠

Let us now introduce one more notion:

Definition 7 *An augmentation for A is an algebra morphism $\varepsilon : A \longrightarrow k$ (where k is endowed with the algebra structure defined in example 1). An algebra (A, m, u) with an augmentation map will be called augmented algebra.*

Proposition 3 *Let A be an augmented algebra, with augmentation map ε. Then $\ker \varepsilon \subseteq A$ is an ideal and it is called augmentation ideal.*

Proof Since ε is an algebra morphism, we have that for $x \in \ker \varepsilon$, and $y \in A$, $\varepsilon(m(x, y)) = m_k(\varepsilon(x), \varepsilon(y)) = 0$. ♠

2.2 Coalgebras

The dual notion of a k-algebra is the one of a k-coalgebra. In this section we will introduce and discuss some of the most elementary properties of such an algebraic structure.

Definition 8 *A k-coalgebra is a k-vector space C together with two k-linear maps: the comultiplication $\Delta : C \longrightarrow C \otimes C$ and the counit $\varepsilon : C \longrightarrow k$, such that the following two diagrams are commutative:*

$$\begin{array}{ccc} C \otimes C & \xrightarrow{\mathrm{id}\otimes\Delta} & C \otimes C \otimes C \\ {\scriptstyle\Delta}\Big\uparrow & & \Big\uparrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ C & \xrightarrow{\Delta} & C \otimes C \end{array}$$

(Coassociativity)

$$\begin{tikzcd} k \otimes C & C \otimes C \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\varepsilon"] & C \otimes k \\ & C \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

(Counit)

Example 3 *The ground field k endowed with the maps: comultiplication $\Delta_k : k \longrightarrow k \otimes k$, $1 \rightsquigarrow 1 \otimes 1$ and counit $\varepsilon_k : k \longrightarrow k$, $1 \rightsquigarrow 1$ is easily checked to be a coalgebra.*

The notion of morphism between coalgebras is given in the following definition:

Definition 9 *A morphism between two coalgebras $(C_1, \Delta_1, \epsilon_1)$ and $(C_2, \Delta_2, \epsilon_2)$ is a linear map $\psi : C_1 \longrightarrow C_2$ such that: $(\psi \otimes \psi) \circ \Delta_1 = \Delta_2 \circ \psi$ and $\varepsilon_1 = \varepsilon_2 \circ \psi$.*

In particular, coalgebras and their morphisms form a category. As in the algebra case, we only need to check that:

Proposition 4 *Given $\psi_1 : C_1 \longrightarrow C_2$ and $\psi_2 : C_2 \longrightarrow C_3$ morphisms of algebras $\psi_2 \circ \psi_1 : C_1 \longrightarrow C_3$ is a morphism of algebras;*

Proof The proof of this statement is completely analogous to the one given for the algebras' case. ♠

Definition 10 *An augmentation for (C, Δ, ε) is a coalgebra morphism $u : k \longrightarrow C$, where we think of k as the coalgebra $(k, \Delta_k, \varepsilon_k)$, defined in example 3.*

The notion of commutativity is given in the following definition:

Definition 11 *C is cocommutative if $\tau \circ \Delta = \Delta$. Equivalently, C is cocommutative if the following diagram is commutative:*

$$\begin{tikzcd} C \otimes C & & C \arrow[ll, "\Delta"'] \arrow[dl, "\Delta"] \\ & C \otimes C \arrow[ul, "\tau"] & \end{tikzcd}$$

(Cocommutativity)

In any given coalgebra C, we can individuate a particular subset of elements, which are called primitive. These are defined as follows:

Definition 12 *We say that $x \in C$ is primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$. We will denote the set of primitive elements in C as $P(C)$.*

In particular we have:

Proposition 5 *$P(C) \subseteq \ker \varepsilon$.*

Proof This follows from the property of the counit map: $x = (\varepsilon \otimes \mathrm{id}_C) \circ \Delta(x) = x + \varepsilon(x)$. ♠

Example 4 *The field k has a natural coalgebra structure: $\Delta_k(1) = 1 \otimes 1$, $\varepsilon_k(1) = 1$ (and use now the linearity of the maps Δ and ε). Moreover, for any coalgebra (C, Δ, ε), $\varepsilon : C \longrightarrow k$ is a map of coalgebras.*

Example 5 *For every coalgebra (C, Δ, ε), we can define a new coalgebra $(C, \Delta^{\mathrm{op}}, \varepsilon)$ called opposite coalgebra, where $\Delta^{\mathrm{op}} = \tau \circ \Delta$.*

Moreover:

Example 6 *1) The dual vector space C^* of a coalgebra (C, Δ, ε) is endowed with a natural algebra structure. In fact, define:*

$$m_* = \Delta^t|_{C^* \otimes C^*}(C \otimes C)^* \longrightarrow C^*,$$

$$\langle \Delta^t(\phi \otimes \psi), x \rangle = \langle \phi \otimes \psi, \Delta(x) \rangle.$$

The associativity of m_* follows from the coassociativity of Δ. The unit is defined by taking the transpose of the counit map:

$$u_* = \varepsilon^t : k^* \simeq k \longrightarrow C^*.$$

2) The dual vector space A^* of a finite dimensional algebra (A, m, u) has a natural coalgebra structure. In fact, we can define comultiplication and counit by taking the transpose of the multiplication map and the transpose of the unit map, i.e:

$$\Delta_* = m^t : A^* \longrightarrow (A \otimes A)^* \simeq A^* \otimes A^* \text{ and } \varepsilon_* = u^t : A^* \longrightarrow k^* \simeq k.$$

Let us conclude this section with one more example.

Example 7 *Let $(C_1, \Delta_1, \varepsilon_1)$ and $(C_2, \Delta_2, \varepsilon_2)$ be two coalgebras. The tensor product $C_1 \otimes C_2$ has a coalgebra with co-multiplication $\Delta_\otimes = (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta_1 \otimes \Delta_2)$ and co-unit $\varepsilon_1 \otimes \varepsilon_2$.*

2.3 Lie algebras

In this subsection we introduce another algebraic structure which will play the most fundamental role in the following exposition.
Let $\mathfrak{g}$ a k-vector space.

Definition 13 *$\mathfrak{g}$ is called Lie algebra if it is endowed with a bilinear map:*

$$B : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

such that:
1) B is antisymmetric: $B \circ \tau = -B$;
2) B fulfills the following identity:

$$B(x, B(y, z)) + B(z, B(x, y)) + B(y, B(z, x)) = 0$$

for each $x, y, z \in \mathfrak{g}$.

We will denote with the bracket [,] the bilinear form B.

Example 8 *Let (A, m, u) be any algebra. We can define a Lie algebra structure on A, defining the bracket between two elements $x, y \in A$ as $[x, y] = m(x, y) - m(y, x)$. The antisymmetry follows from the very definition, while the Jacobi identity is an easy consequence of the associativity of the product m. We will indicate with $L(A)$ the Lie algebra defined on A by this bracket. In particular given a vector space V, the vector space $\mathrm{End}(V)$ is an associative algebra with product defined by the composition: ϕ, $\psi \in \mathrm{End}(V)$, $m(\phi, \psi) = \psi \circ \psi \in \mathrm{End}(V)$. The Lie algebra $L(\mathrm{End}(V))$ is defined by the bracket $[\phi, \psi] = \phi \circ \psi - \psi \circ \phi$ for each ϕ,ψ in $\mathrm{End}(V)$.*

Example 9 *Let us consider a 3-dimensional k-vector space generated by x, y, h. Let us define a Lie algebra structure on V, via the following: $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$ (and $[y, x] = -h$, $[x, h] = -2x$, $[y, h] = 2y$). Such a bracket fulfills the Jacobi identity, so that it defines a Lie algebra structure on such a vector space. This Lie algebra is usually called $\mathfrak{sl}_2$*

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras over the field k, and $\phi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ a morphism of k-modules.

Definition 14 *We say that ϕ is a Lie algebra morphism if:*

$$\phi[x, y]_{\mathfrak{g}_1} = [\phi(x), \phi(y)]_{\mathfrak{g}_2}, \ \forall x, y \in \mathfrak{g}_1.$$

Remark 1 *Lie algebras and their morphisms form a category.*

Definition 15 *A Lie algebra $\mathfrak{g}$ is called commutative or abelian if $[x, y] = 0$ for each $x, y \in \mathfrak{g}$.*

Example 10 *Every vector space V can be endowed with the structure of abelian Lie algebra by saying that: for each $x, y \in V$, $[x, y] = 0$.*

Lie subalgebras and Lie ideals are defined as follows:

Definition 16 *A subvector space $\mathfrak{a} \subset \mathfrak{g}$ is a sub Lie algebra if for each $x, y \in \mathfrak{a}$, $[x, y] \in \mathfrak{a}$.*

Definition 17 *A given subvector space $\mathfrak{a} \subset \mathfrak{g}$ is called an ideal of $\mathfrak{g}$, if $[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{g}$.*

Proposition 6 *If* $\mathfrak{a} \subset \mathfrak{g}$ *is an ideal, the quotient* $\mathfrak{g}/\mathfrak{a}$ *has a structure of a Lie algebra.*

Proof The proof of the statement is completely analogous to the one given for the algebra case. ♠

The previous proposition gives us a way to define a new Lie algebra starting from a given Lie algebra $\mathfrak{g}$ and a given ideal $\mathfrak{a}$. However, this is not the only way to define new Lie algebras from old ones. In fact:

Proposition 7 *If* $\mathfrak{g}$ *and* $\mathfrak{t}$ *are two Lie algebras, the cartesian product* $\mathfrak{A} = \mathfrak{g} \times \mathfrak{t}$ *has a natural structure of Lie algebra given by:*

$$\big[(\xi_1, x_1), (\xi_2, x_2)\big]_{\mathfrak{A}} = \big([\xi_1, \xi_2]_{\mathfrak{g}}, [x_1, x_2]_{\mathfrak{t}}\big).$$

Proof Antisymmetry and Jacobi identity follow immediately from the definition of the bracket $[\ ,\ ]_{\mathfrak{A}}$ and from antisymmetry and Jacobi identity of the brackets $[\ ,\ ]_{\mathfrak{g}}$ and $[\ ,\ ]_{\mathfrak{t}}$. ♠

One more definition:

Definition 18 *A Lie algebra* $\mathfrak{g}$ *is called simple if it has no non trivial ideal. It is said semi-simple if it has no non trivial abelian ideal.*

The center of the Lie algebra $\mathfrak{g}$ is defined as follows:

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \text{ such that } [x, y] = 0 \text{ for each } y \in \mathfrak{g}\}.$$

In particular $Z(\mathfrak{g})$ is an abelian ideal.

Example 11 *The Lie algebra* $\mathfrak{sl}_2$ *is simple. In particular its center is trivial.*

2.3.1 Universal enveloping algebra

An universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is an associative algebra which is constructed starting from any given Lie algebra $\mathfrak{g}$. In what follows we will describe its construction and its main properties.
Let us consider a Lie algebra $\mathfrak{g}$, and let us think of it as a k-vector space:

Definition 19 *The tensor algebra $T(\mathfrak{g})$ is the following graded associative algebra generated by $\mathfrak{g}$:*

$$T(\mathfrak{g}) = k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots \oplus \mathfrak{g}^{\otimes n} \oplus \cdots = \bigoplus_{k \geq 0} \mathfrak{g}^{\otimes k}.$$

Here $\mathfrak{g}^{\otimes n}$ denotes the tensor product (over the field k) of n copies of $\mathfrak{g}$, whose elements are (finite) linear combinations of terms $\xi_1 \otimes \cdots \otimes \xi_n$, with $\xi_k \in \mathfrak{g}$. The algebra structure is given by concatenation: $(\eta_1, \eta_2) \longrightarrow \eta_1 \otimes \eta_2 \; \forall \, \eta_1, \eta_2 \in T(\mathfrak{g})$. A given element η is (homogeneous) of degree n if and only if $\eta \in \mathfrak{g}^{\otimes n}$.

Let us write: $j : \mathfrak{g} \longrightarrow T(\mathfrak{g})$ for the inclusion map. Then $T(\mathfrak{g})$ is generated, as k-algebra, by $j(\mathfrak{g})$. From this, we conclude that T is a functor from the category of k-modules to the category of associative (unital) k-algebras.
We also have a presentation of $T(\mathfrak{g})$ as an algebra: $T(\mathfrak{g})$ is a free algebra with generators $j(\xi)$ $\xi \in \mathfrak{g}$, which are subject to the (k-module) relations in $j(\mathfrak{g})$:

$$a j(\xi) = j(a\xi) \text{ and } j(\xi_1) + j(\xi_2) = j(\xi_1 + \xi_2), \tag{2.2}$$

for any $a \in k$, ξ, ξ_1 and $\xi_2 \in \mathfrak{g}$.

We can now define:

Definition 20 *The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ by the 2-sided ideal generated by the relations:*

$$j([\xi_1, \xi_2]) = j(\xi_1) \otimes j(\xi_2) - j(\xi_2) \otimes j(\xi_1), \;\; \xi_1, \xi_2 \in \mathfrak{g}. \tag{2.3}$$

Equivalently, $\mathcal{U}(\mathfrak{g})$ is the free algebra generated by $j(\xi)$, $\xi \in \mathfrak{g}$, subjected to the relations (2.2) as well as the relation (2.3).

Since the ideal generated by the relation (2.3) is not homogeneous, the grading in $T(\mathfrak{g})$ does not induce a grading in $\mathcal{U}(\mathfrak{g})$. Nevertheless, such an ideal preserve the natural filtration in $T(\mathfrak{g})$, $T(\mathfrak{g}) : \mathbb{C} = T_0 \subset T_1 \subset \cdots \subset T_n \subset \cdots$, where $T_n = \bigoplus_{i=0}^{n} \mathfrak{g}^i$, so that $\mathcal{U}(\mathfrak{g})$ is naturally a filtered algebra:

$$\mathcal{U}(\mathfrak{g}) : \mathbb{C} = \mathcal{U}(\mathfrak{g})_0 \subset \mathcal{U}(\mathfrak{g})_1 \subset \cdots \subset \mathcal{U}(\mathfrak{g})_n \subset \cdots .$$

Such a filtration will induce a grading:

$$\bigoplus_{k \geq 0} \mathcal{U}(\mathfrak{g})^k \, , \;\; \mathcal{U}(\mathfrak{g})^k = \mathcal{U}(\mathfrak{g})_k / \mathcal{U}(\mathfrak{g})_{k-1} \text{ and } \mathcal{U}(\mathfrak{g})_{-1} = 0,$$

and with respect to this grading we will think of $\mathcal{U}(\mathfrak{g})$ as a graded algebra. Moreover, a natural inclusion $i_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is defined.

The universal enveloping is characterized by the following universal property:

Theorem 1 *(Universal property)*
Let $\mathfrak{g}$ be a Lie algebra and A an algebra. For any morphism of Lie algebras: $f : \mathfrak{g} \longrightarrow L(A)$, there exists a unique morphism of associative algebras: $\phi : \mathcal{U}(\mathfrak{g}) \longrightarrow A$, such that:

$$\phi \circ i_{\mathfrak{g}} = f,$$

i.e ϕ is the only morphism of associative algebras which makes the following triangle commutative:

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{g}) \arrow[dr, "\phi"] & \\
\mathfrak{g} \arrow[u, "i_{\mathfrak{g}}"] \arrow[r, "f"'] & L(A)
\end{tikzcd}$$

(Universal)

Theorem 1 can be rephrased by saying that, for every Lie algebra $\mathfrak{g}$ and any associative algebra A, we have a natural bijection:

$$\mathrm{Hom}_{\mathrm{Lie}}(L, L(A)) \cong \mathrm{Hom}_{\mathrm{Ass}}(\mathcal{U}(\mathfrak{g}), \mathrm{A}).$$

The universal property described above and the explicit construction of $\mathcal{U}(\mathfrak{g})$ described before are equivalent to say that the universal enveloping algebra is unique.

2.4 Bialgebras and Hopf algebras

Let B be a vector space having an algebra (B, m, u) and a coalgebra (B, Δ, ε) structure. The tensor product $B \otimes B$ is endowed with the structure of algebra and the one of coalgebra as described in the examples (2), (7). We can now investigate the compatibility of these two algebraic structures. We start with the following definition:

Definition 21 *A bialgebra is a k-vector space endowed with an algebra structure (m, u) and a coalgebra structure (Δ, ε), such that (m, u) and (Δ, ε) are respectively coalgebra and algebra morphisms.*

The following theorem tells us that we need to check only one of the two compatibility conditions:

Theorem 2 *The maps (m, u) are coalgebra morphisms if and only if the maps (Δ, ε) are algebra morphisms.*

Proof If m is a coalgebra morphism the following diagrams are commutative:

$$\begin{tikzcd} H \otimes H \arrow[r, "m"] \arrow[d, "(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\Delta\otimes\Delta)"] & H \arrow[d, "\Delta"] \\ (H \otimes H) \otimes (H \otimes H) \arrow[r] & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"] & H \arrow[d, "\mathrm{id}"] \\ H \arrow[r, "\varepsilon"] & H \end{tikzcd}$$

and similarly:

$$\begin{tikzcd} k \arrow[rr, "u"] \arrow[dr, "\mathrm{id}"'] & & H \arrow[dl, "\varepsilon"] \\ & k & \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "u"] \arrow[d, "\mathrm{id}"] & H \arrow[d, "\Delta"] \\ k \otimes k \arrow[r, "u\otimes u"] & H \otimes H \end{tikzcd}$$

These four diagrams are equivalent to the commutativity of the following four diagrams:

$$\begin{tikzcd} k \arrow[r, "u"] \arrow[d, "\mathrm{id}"] & H \arrow[d, "\Delta"] \\ k \otimes k \arrow[r, "u\otimes u"] & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \arrow[r] \arrow[d, "m"] & (H \otimes H) \otimes (H \otimes H) \arrow[d, "(m\otimes m)(\mathrm{id}\otimes\tau\otimes\mathrm{id})"] \\ H \arrow[r, "\Delta"] & H \otimes H \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[rr, "u"] \arrow[dr, "\mathrm{id}"'] & & H \arrow[dl, "\varepsilon"] \\ & k & \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"] & k \otimes k \arrow[d, "\mathrm{id}"] \\ H \arrow[r, "\varepsilon"] & k \end{tikzcd}$$

which express the fact that Δ and ε are morphisms of algebras. ♠

The set of primitive elements of a bialgebra B has the following remarkable property:

Proposition 8 *If B is a bialgebra, then $P(B)$ is a sub Lie algebra of the Lie algebra $L(B)$.*

Proof Let $x, y \in P(B)$. Then $\Delta([x,y]) = [\Delta(x), \Delta(y)] = [x \otimes 1 + 1 \otimes x, y \otimes 1 + 1 \otimes y] = [x,y] \otimes 1 + 1 \otimes [x,y]$. ♠

Let us now consider the bialgebra B and the vector space of the endomorphisms of B, $\mathrm{Hom}_k(B,B)$. For any $\phi, \psi \in \mathrm{Hom}_k(B,B)$, we can define the following linear map:

$$\phi \star \psi = m \circ (\phi \otimes \psi) \circ \Delta, \tag{2.4}$$

Definition 22 *The operation $\star$ defined in (2.4), which is clearly bilinear, is called convolution product.*

Now it follows that:

Proposition 9 $(\mathrm{Hom}_k(B,B), \star, u \circ \varepsilon)$ *is a k-algebra over the field k.*

Proof The associativity of the product $\star$ follows from the coassociativity of Δ and the associativity of m.♠

Definition 23 *An element $S \in \mathrm{Hom}_k(B,B)$, which has the property $S \star \mathrm{id_B} = \mathrm{id_B} \star S = u \circ \varepsilon$ is called antipode*

Definition 24 *A bialgebra endowed with an antipode is called Hopf algebra.*

If a given bialgebra has an antipode, this is unique:

Proposition 10 *If S_1 and S_2 are two antipodes then $S_1 = S_2$*

Proof This follows from the associativity of the convolution product defined in (2.4) and from the definition of antipode (see definition 23); in fact:

$$S_1 = S_1 \star (u \circ \varepsilon) = S \star (\mathrm{id_H} \star S_2) = (S_1 \star \mathrm{id}_H) \star S_2 = (u \circ \varepsilon) \star S_2 = S_2.$$

♠

We can summarize what has been discussed in this section by saying that: a *Hopf algebra H* is a bialgebra endowed with an antipode S.
Let us now go back to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ introduced in the previous section, and let us show that it has a natural Hopf algebra structure:

Theorem 3 *The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ has a structure of Hopf algebra.*

Proof $\mathcal{U}(\mathfrak{g})$ is an algebra by its very definition. Let us now define on $\mathcal{U}(\mathfrak{g})$ a compatible coalgebra structure. As a consequence of the universal property, we first observe, that the universal enveloping algebra $\mathcal{U}(\mathfrak{g} \times \mathfrak{g})$ is isomorphic as a k-algebra to $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$. Then, since $\mathcal{U}(\mathfrak{g})$ is generated by $\mathfrak{g}$, the diagonal morphism

$$\mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}, \ \ \xi \rightsquigarrow (\xi, \xi), \ \xi \in \mathfrak{g}$$

induces an algebra morphism:

$$\Delta : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}),$$

whose restriction to $\mathfrak{g} \simeq j(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ is:

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi.$$

Such a map defines a coproduct on $\mathcal{U}(\mathfrak{g})$. The counit is defined extending the linear map:

$$\varepsilon : \mathcal{U}(\mathfrak{g}) \longrightarrow k, \ \ \varepsilon(\xi) = 0 \, \forall \xi \in \mathfrak{g},$$

to an algebra morphism. Finally the antipode is defined on the generators by $S(\xi) = -\xi$ and it is then extended to an algebra morphism to $\mathcal{U}(\mathfrak{g})$. It is easy to check the compatibility of the maps just defined, so that we can summarize what has been done saying that $(\mathcal{U}(\mathfrak{g}), m, \Delta, u, \varepsilon, S)$ is a Hopf algebra. ♠

Let us give two more examples of Hopf algebras, both associated with a finite group G. We will then end the present section, briefly discussing the duality between finite dimensional Hopf algebras.

Example 12 *(group algebra) Let G be any finite group. The group algebra kG is the k-vector space, freely generated by the elements g of the group G. The algebra structure over kG is induced by the group structure defined in G; if $a = \sum_i \alpha_i g_i$ and $b = \sum_j \beta_j g_j$ (both sums are finite), define the multiplication via:*

$$m(a, b) = \sum_{i,j} \alpha_i \beta_j (g_i \cdot g_j)$$

while the unit is defined via the following $u(1) = e$*, where e is the unit element in* G*. The coalgebra structure is defined as follows: given* $c = \sum_k \gamma_k g_k$*, the coproduct is*

$$\Delta(c) = \sum_k \gamma_k g_k \otimes g_k,$$

while the counit map is given by

$$\varepsilon(c) = \sum_k \gamma_k.$$

It is easy to show that the algebra and the coalgebra structure so defined are compatible, so that kG *is a bialgebra. Finally the antipode is defined as follows:* $S(g) = g^{-1}$*, for* $g \in G$ *and then extended to a morphism of* kG *by linearity. Also, the compatibility of the antipode so defined with the bialgebra structure is easily checked. Summarizing:*

Theorem 4 *The group algebra* kG *endowed with algebra, coalgebra and antipode as defined above, is a Hopf algebra.*

Example 13 *(functions on a finite group* G*)*
Let G *be a finite group, with multiplication map:*

$$\cdot : G \times G \longrightarrow G, \ \ (g_1, g_2) \rightsquigarrow g_1 \cdot g_2.$$

The set of k*-valued functions on* G*,* $\mathcal{F}(G)$*, is a vector space generated by delta type functions, i.e by the functions* δ_g*,* $g \in G$*, defined as follows:*

$$\begin{cases} \delta_{g_1}(g_2) & = 1 \ \ \text{if } g_1 = g_2, \\ \delta_{g_1}(g_2) & = 0 \ \ \text{if } g_1 \neq g_2. \end{cases} \tag{2.5}$$

Let us endow $\mathcal{F}(G)$ *with a structure of Hopf algebra. The multiplication map* $m : \mathcal{F}(G) \otimes \mathcal{F}(G) \longrightarrow \mathcal{F}(G)$*, is defined by:*

$$m(\delta_{g_1}, \delta_{g_2}) = \delta_{g_1}(g_2)\delta_{g_1},$$

and then extended by linearity. The identity function in $\mathcal{F}(G)$ *will be used to define the unit map:*

$$u : k \longrightarrow \mathcal{F}(G)$$
$$1 \rightsquigarrow \sum_{g \in G} \delta_g.$$

Moreover, since G is a finite group, we have an isomorphism of vector spaces:

$$\mathcal{F}(G \times G) \longrightarrow \mathcal{F}(G) \otimes \mathcal{F}(G),$$

$$\delta_{(g_1,g_2)} \rightsquigarrow \delta_{g_1} \otimes \delta_{g_2},$$

where $G \times G$ is endowed with the product group structure. Using this isomorphism and the map:

$$\Delta : \mathcal{F}(G) \longrightarrow \mathcal{F}(G \times G),$$

$$\delta_g \rightsquigarrow \sum_{(g_1,g_2),\ g_1 \cdot g_2 = g} \delta_{(g_1,g_2)},$$

we can define a linear map, still denoted by Δ,

$$\Delta : \mathcal{F}(G) \longrightarrow \mathcal{F}(G) \otimes \mathcal{F}(G),$$

$$\delta_g \rightsquigarrow \sum_{(g_1,g_2),\ g_1 \cdot g_2 = g} \delta_{g_1} \otimes \delta_{g_2}.$$

It is easy to show that such a map defines a coproduct. The coalgebra structure on $\mathcal{F}(G)$ is completed by defining the counit as the linear map:

$$\varepsilon : \mathcal{F}(G) \longrightarrow k, \ \ \varepsilon(\delta_g) = \delta_g(e).$$

It is clear by their very definition that such maps are compatible, so that they define a bialgebra structure $(\mathcal{F}(G), m, \Delta, u, \varepsilon)$ on the algebra of functions on G. Finally, the antipode is given by:

$$S : \mathcal{F}(G) \longrightarrow \mathcal{F}(G).$$

$$\delta_g \rightsquigarrow \delta_{g^{-1}}.$$

We have already remarked that, at least in the finite dimensional case, the notion of algebra and the one of coalgebra are dual to each other. Let us briefly discuss the case of a finite dimensional Hopf algebra $(H, m, \Delta, u, \varepsilon, S)$. In this case we have an isomorphism of k-vector spaces $H^* \otimes H^* \simeq (H \otimes H)^*$. Let us define the transposed maps: $m_* = \Delta^t : H^* \otimes H^* \longrightarrow H^*$, $\Delta_* = m^t : H^* \longrightarrow H^* \otimes H^*$, $u_* = \varepsilon^t : k \longrightarrow H^*$, $\varepsilon_* = u^t : H^* \longrightarrow k$ and $S_* = S^t : H^* \longrightarrow H^*$. We can now state the following theorem:

Theorem 5 *$(H^*, m_*, \Delta_*, u_*, \varepsilon_*, S_*)$ is a Hopf algebra. In particular the category of finite dimensional Hopf algebras is involutive, with involution functor given by the adjunction.*

Proof Since H is a finite dimensional vector space, $(H \otimes H)^* \simeq H^* \otimes H^*$. To prove the theorem we need to show that the maps m_*, Δ_* etc, define a product, coproduct, etc, for H^*. For example, the associativity of $m_* = \Delta^t$ is a consequence of the coassociativity of Δ and similarly, the coassociativity of $\Delta_* = m^t$ will follow from the associativity of m. Moreover it is necessary to show that (m_*, u_*) is a morphism of coalgebra, i.e that they are compatible with $(\Delta_*, \varepsilon_*)$ (see theorem 2). All these are simple proofs which follow directly by the definitions of the maps involved. ♠

Example 14 *For a given finite group G, the Hopf algebra $\mathcal{F}(G)$ is the dual Hopf algebra of kG.*

In the next section, we discuss more in depth the duality for Hopf algebras dropping the hypothesis of finite dimensionality.

2.5 The Milnor-Moore theorem

In this section, we discuss a structure theorem for cocommutative Hopf algebras. We mainly work with infinite dimensional graded Hopf algebras. Let us start by introducing their main definitions and properties. All the results contained in the present sections are taken from [21] (warning for the reader: we will work with Hopf algebras over a field, which will be always $\mathbb{C}$ or $\mathbb{R}$. In the reference [21] the authors work with Hopf algebras over a commutative ring).

Definition 25 *A $\mathbb{Z}$-graded Hopf algebra is a Hopf algebra H whose underlying vector space is $\mathbb{Z}$-graded, i.e $H = \bigoplus_{i\in\mathbb{Z}} H_i$ and such that product, coproduct and antipode respect the grading: $m : H_n \otimes H_k \longrightarrow H_{n+k}$, $\Delta(H_n) \longrightarrow \bigoplus_{p+q=n} H_p \otimes H_q$ and $S : H_n \longrightarrow H_n$ for each $n, m \in \mathbb{Z}$.*

Example 15 *The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, is a graded Hopf algebra.*

Remark 2 *In what follows we will consider Hopf algebras graded over $\mathbb{Z}_{\geq 0}$.*

Definition 26 *A graded Hopf algebra $H = \bigoplus_{i\in\mathbb{Z}_+} H_i$ is said to be of finite type if each of its homogeneous components H_i are vector spaces of finite dimension.*

We want now to discuss some of the consequences of the property of being a graded Hopf algebra of finite type.

Definition 27 *Let H be a graded Hopf algebra of finite type. Then $H^\star = \bigoplus_{i\in\mathbb{Z}_+} H_i^*$ is called the restricted dual of H.*

Remark 3 *If H is an infinite dimensional vector space, then the restricted dual $H^\star$ is strictly contained in the space of linear functional $H^* = \mathrm{Hom}_k(H,k)$ on H. In particular, a given linear map $f \in H^*$ belongs to $H^\star$, if and only if $f|\, H_l = 0$ for all $l \in \mathbb{Z}_+$, but for a finite number. If H is a finite dimensional vector space, then $H^\star \simeq \mathrm{Hom}_k(H,k)$.*

Let $H = \bigoplus_{i\in\mathbb{Z}_+} H_i$ be an Hopf algebra of finite type. Let us indicate with $\varepsilon_* \doteq u^t : H^\star \longrightarrow k$, $m_* \doteq \Delta^t : H^\star \otimes H^\star \longrightarrow H^\star$, $u_* \doteq \varepsilon^t : k \longrightarrow H^\star$, $\Delta_* \doteq m^t : H^\star \longrightarrow H^\star \otimes H^\star$ and $S_* = S^t : H^\star \longrightarrow H^\star$ the adjoint maps (with respect to the pairing between H and $H^\star$) of the unit, coproduct, counit, multiplication and antipode maps.

Theorem 6 *Under the previous assumptions, we have that $(H^\star, m_*, \Delta_*, u_*, \varepsilon_*, S_*)$ is a Hopf algebra with multiplication given by m_*, unit given by u_*, coproduct given by Δ_*, counit by ε_* and antipode by S_*.*

Proof Let us first note the following isomorphism:

$$(H \otimes H)^\star \simeq \bigoplus_{n\geq 0} \bigoplus_{i+j=n} \left(H_i^* \otimes H_j^*\right) \simeq H^\star \otimes H^\star,$$

which is a direct consequence of the property of the restricted dual. The proof of the statement consists in proving that the maps Δ_*, m_* etc, define a coproduct, a product etc, for $H^\star$. All of this follows from the definitions of the maps with the lower stars and from the properties of product, coproduct, unit counit and antipode. The discussion follows verbatim the final dimensional case discussed in theorems 2 and 5.♠

Remark 4 *The previous theorem is the infinite dimensional generalization of the theorem 5. In particular, we can say that the category of graded and finite type Hopf algebras has an involution which is given by taken the restricted dual.*

In what follows, we consider a particular class of graded Hopf algebras, which we introduce with the following definition:

Definition 28 *A graded Hopf algebra is called connected if* $H_0 \simeq k$.

Proposition 11 *If H is a graded Hopf algebra of finite type, then* $\ker \varepsilon = \{x \in H |\, x \neq \alpha 1_H,\ \alpha \in k\}$. *In particular, if H is connected, graded, and of finite type then* $\ker \varepsilon \simeq \bigoplus_{i>0} H_i$.

Proof Since H is graded and of finite type, also $H^\star$ is a graded and of finite type Hopf algebra. In particular $1_{H^\star} = u_*(1)$ is the only element in $H^\star$ such that: $0 = \langle 1_{H^*}, x\rangle = \langle u_*(1), x\rangle = \langle 1, \varepsilon(x)\rangle = \varepsilon(x)$, for each $x \neq 1_H$. The second statement it is now clear. ♠

In the case of connected Hopf algebras, the coproduct is characterized by the following proposition:

Proposition 12 *For any given element $x \in H$, $\Delta(x) = x \otimes 1 + 1 \otimes x + \Sigma \otimes \Sigma'$, where $\Sigma \otimes \Sigma' \in \ker \varepsilon \otimes \ker \varepsilon$.*

Proof Suppose that $x \in H_i$. Since H is graded, we have: $\Delta(x) \in \bigoplus_{p+q=i} H_p \otimes H_q$. Since H is connected, we have that $\Delta(x) = \alpha(y \otimes 1) + \beta(1 \otimes z) + \sum_{k,l} z_k \otimes z_l$, (in fact H connected means $H_0 \simeq k$). From the co-unit property: $(\varepsilon \otimes \mathrm{Id}) \circ \Delta(x) = x = (\mathrm{Id} \otimes \varepsilon) \circ \Delta(x)$, it follows that $x \otimes 1 = \alpha(y \otimes 1)$ and $1 \otimes x = \beta(1 \otimes z)$, so that $\alpha = \beta = 1$ and $z = y = x$. ♠

Definition 29 *The kernel of the counit ε is an ideal in H, which is called augmentation ideal.*

Given an Hopf algebra H, let us restrict the multiplication map to the kernel of the counit map:

$$m : \ker \varepsilon \otimes \ker \varepsilon \longrightarrow \ker \varepsilon.$$

In particular, we can consider the cokernel of such a map, which will be denoted with: $i(H) = \ker \varepsilon / m(\ker \varepsilon \otimes \ker \varepsilon)$.

We can now define the following set of elements in H:

Definition 30 *An element $x \in \ker \varepsilon \subset H$ is called indecomposable if and only if it has non trivial class in $i(H)$, i.e if and only if it cannot be written as a linear combination of products of elements in* $\ker \varepsilon$.

The set of indecomposable elements in H will be denoted with $I(H)$.
The following result will be important for what follows:

Theorem 7 *Let H be a connected, graded Hopf algebra of finite type and $H^\star$ its restricted dual. Then, the space of primitive elements $P(H)$ is in one to one correspondence with the space $I(H^\star)$ of indecomposable elements in the dual Hopf algebra $H^\star$.*

Proof Let x be an homogeneous element in $P(H)$ (say $\deg \; x = i$). Suppose that Z_x is the dual form of x, $\langle Z_x, x\rangle = 1$ and zero otherwise (in particular $Z_x \in H_i^*$). Let now suppose that $Z_x = m_*(\sum_{k,l} \alpha_{l,k} Z_l \otimes Z_k)$, where $Z_k, Z_l \neq 1_H^*$, for each k, l and $Z_k \in H_k^*$ and $Z_l \in H_l^*$ with $k + l = i$, for each k, l. Then we can write: $1 = \langle Z_x, x\rangle = \langle m_*(\sum_{k,l} \alpha_{k,l} Z_k \otimes Z_l), x\rangle = \sum_{k,l} \alpha_{k,l} \langle \Delta^t(Z_k \otimes Z_l), x\rangle = \sum_{k,l} \alpha_{k,l} \langle Z_k \otimes Z_l, \Delta(x)\rangle = \sum_{k,l} \alpha_{k,l} \langle Z_k \otimes Z_l, x \otimes 1_H + 1_H \otimes x\rangle$. From the hypothesis $1 = \langle Z_x, x \otimes 1_H + 1_H \otimes x\rangle$ we get now a contradiction. So we get a (linear) map $\Gamma : P(H) \longrightarrow I(H^\star)$, defined as follows: $x \rightsquigarrow Z_x$, where $\langle Z_x, x\rangle = 1$. Such a map is clearly injective. The fact that the map just defined is surjective follows easily by similar argument. ♠
The following corollaries are almost self evident:

Corollary 1 *The set of indecomposable element in H, $I(H)$, is in one to one correspondence with the set of primitive elements in $H^\star$, $P(H^\star)$.*

Proof From the theorem 7, $P(H) \simeq I(H^\star)$. The statement of the corollary follows now from the isomorphism $(H^\star)^\star \simeq H$. ♠

Corollary 2 *If H is a connected, graded Hopf algebra of finite type, which is generated by the set of its indecomposable elements, then $H^\star$ is generated by the set of its primitive elements.*

Proof The result follows from the theorem 7 and from the isomorphism between H and $H^\star$. ♠

Remark 5 *We stated the previous results assuming that the Hopf algebra H is graded and of finite type. Actually such hypothesis are overstated: it is enough to demand that the dual of the Hopf algebra H is a Hopf algebra itself. As it has already been stressed, this is always true for finite dimensional Hopf algebras but in general it fails to be true for infinite dimensional Hopf algebras, unless we consider Hopf algebras which are graded and of finite type. Moreover, we need to observe that the form of the coproduct is fundamental to prove that the map Γ defined in theorem 7 is surjective. Such a coproduct is a consequence of the hypothesis that H is connected.*

Let us make one more observation:

Proposition 13 *If $(H, m, u, \Delta, \varepsilon, S)$ is a graded cocommutative (commutative) Hopf algebra of finite type, then $(H^\star, \Delta_*, \varepsilon_*, m_*, u_*, S_*)$ is a graded commutative (cocommutative) Hopf algebra of finite type.*

Proof The statement is an easy consequence of the definition of product and coproduct: if m is commutative $m^t = \Delta_*$ is cocommutative and if Δ is cocommutative, $\Delta^t = m_*$ will be commutative.♠

The following fundamental theorem has been proved by John Milnor and John Moore in 1965, and it represents one of the main tools of the present work:

Theorem 8 *(Milnor-Moore) If H is a connected, graded, of finite type cocommutative Hopf algebra, then: $H \simeq (\mathcal{U}(P(H))$ as a Hopf algebras.*

Instead to prove the theorem 8 in its full generality, we will state and prove a slightly weaker form of it:

Theorem 9 *If H is a cocommutative Hopf algebra generated by the space of primitive elements $P(H)$, then we have that: $H \simeq (\mathcal{U}(P(H))$.*

Before giving the proof of the theorem 9, let us clarify its statement and let make some observations:

1) An Hopf algebra H is primitively generated if and only if there exist a surjective map $p : \mathcal{U}(P(H)) \longrightarrow H$, i.e if H is primitively generated, then any element $x \in H$ can be written as a linear combination of products of

elements in $P(H)$.

2) If H is primitively generated then H is connected. This follows from the fact that $P(H) \subset \ker \varepsilon$ (see proposition 5).

Proof (theorem 9) Let $P(H)$ be the set of primitive elements in H. We already now that $P(H)$ is a sub Lie algebra of $L(H)$. By the hypothesis we have that there exists a surjective (Hopf algebras') map: $\pi : \mathcal{U}(P(H)) \longrightarrow H$. The statement will follow if we can show that such a map is also injective. Let $I = \ker \pi \subset \mathcal{U}(P(H))$ be a Hopf ideal and let us consider the filtration in I induced by the standard one in $\mathcal{U}(P(H))$: $I = \bigcup_{n \geq 0} I_n$, where $I_n = I \bigcap \mathcal{U}(P(H))_n$. Clearly we have that $I_0 = 0 = I_1$; suppose that $I \neq \{0\}$ and let $\xi \in \mathcal{U}(P(H))_m$ be an element of minimal degree in I. Then: $\Delta(\xi) - \xi \otimes 1 + 1 \otimes \xi \in \mathcal{U}(P(H))_{m-1} \otimes \mathcal{U}(P(H))_{m-1}$. Since m is of minimal degree, and π is a Hopf algebra morphism we need to conclude that ξ is primitive: $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$. But this contradicts that $m > 1$. So that $I = \{0\}$ and the map $\pi : \mathcal{U}(P(H)) \longrightarrow H$ is a bijection. ♠

Now we will state two corollaries to the theorems 8, 9.

Corollary 3 *If H is a connected graded, commutative Hopf algebra of finite type, then it is isomorphic, as a Hopf algebra, to the dual of the enveloping algebra of some Lie algebra.*

Proof As above remarked $H^\star$ is also connected and it is cocommutative. Then by the theorem 8, $H^\star \simeq \mathcal{U}(P(H^\star))$. ♠
Similarly we have:

Corollary 4 *If H is a graded commutative Hopf algebra, of finite type, generated by its indecomposable elements, then it is isomorphic, as a Hopf algebra, to the universal enveloping algebra of some Lie algebra.*

Proof From theorem 7 it follows that $H^\star$ is generated by the set of its primitive elements. From theorem 9 it follows that $H^\star \simeq \mathcal{U}(P(H^\star))$ so that the statement follows. ♠

Let $x \in H$ be an indecomposable (and homogeneous) element, and let $Z_x = \Gamma(x) \in P(H^\star)$, where Γ is the linear map defined in the theorem 7. The set of elements Z_x with $x \in I(H)$ (each of those is a primitive element in $H^\star$), is a linear form on H. We extend their action to the full algebra H via the following theorem:

Theorem 10 *For each Z_x as above, $\langle Z_x, z_1 z_2\rangle = \langle Z_x, z_1\rangle\varepsilon(z_2)+\langle Z_x, z_2\rangle\varepsilon(z_1)$. In particular, assuming that the Hopf algebra H is graded and of finite type, as a consequence of the proposition 11, each Z_x extends by zero to the full algebra H.*

Proof The proof goes as follows: $\langle Z_x, z_1 z_2\rangle = \langle Z_x, m(z_1 \otimes z_2)\rangle = \langle \Delta(Z_x), z_1 \otimes z_2\rangle = \langle Z_x \otimes 1_H^\star + 1_H^\star \otimes Z_x, z_1 \otimes z_2\rangle = \langle Z_x, z_1\rangle\varepsilon(z_2) + \langle Z_x, z_2\rangle\varepsilon(z_1)$, by the definition of $1_H^\star$. The last part of the statement follows now from the fact that H is connected. ♠

Definition 31 *The elements in the (full) dual of H are called characters of the Hopf algebra. Since $H^\star \subset \mathrm{Hom}_k(H, k)$, each element in the restricted dual is a character of H. Every character Z of H, whose extension to the full algebra H fulfills the condition expressed in theorem 10, $\langle Z_x, z_1 z_2\rangle = \langle Z_x, z_1\rangle\varepsilon(z_2) + \langle Z_x, z_2\rangle\varepsilon(z_1)$, is called infinitesimal character.*

So we can rephrase theorem 10 with the following proposition:

Proposition 14 *Each primitive element of the Hopf algebra $H^\star$, is an infinitesimal character for H.*

Chapter 3

The Hopf algebra of rooted trees

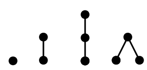
In this chapter, we first introduce the main definitions and the main properties of the Hopf algebra of rooted trees $\mathcal{H}_{rt}$. Such Hopf algebra turned out to be the algebraic backbone of the combinatorics behind the renormalization of perturbative quantum field theories, which ha been explored in the works of Alain Connes and Dirk Kreimer. Then we define the Lie algebra of the infinitesimal derivations of $\mathcal{H}_{rt}$, and finally, we give a description of the ladder Hopf algebra of rooted trees. The main references are [12] and [5].

3.1 Main definitions

Let us start with the definition of rooted tree:

Definition 32 *A (non planar) rooted tree t is a connected, simply connected one dimensional simplicial complex with a point base $*(t)$, which is called the root of the tree t, see example 16.*

Example 16 *Examples of rooted trees are given by:*



The root is the uppermost vertex.

Rooted trees will be denoted with the letter t (or T). For each rooted tree, the following sets can be defined: the set of edges $E(t)$, each of which will be assumed oriented, and the set of vertices $V(t)$ (we will assume the concepts of edge and of vertex as primitive). Each vertex is attached to at most one incoming (with respect to the orientation) edge. A vertex v will be called external if it is attached to one incoming and none outgoing edge, or if it is attached to exactly one outgoing and none incoming edge. Such an external edge is the root of the tree. In particular $V(t) = V^{\text{int}}(t) \sqcup V^{\text{int}}(t)$.

Example 17 *For example* $\text{card}\big(V(\bullet)\big) = 1$ *while* $E(\bullet)$ *is empty. For the tree* $\overset{\bullet}{\underset{\bullet}{|}}$ *we have* $\text{card}\big((V(\overset{\bullet}{\underset{\bullet}{|}})\big) = 2$, $V^{\text{int}}(\bullet)$ *is empty (i.e all the vertices are external) and* $\text{card}\big(E(\overset{\bullet}{\underset{\bullet}{|}})\big) = 2$.

Each rooted tree is then oriented away from the root. This orientation is induced by the orientation of the edges and by the property that each vertex has at most one incoming edge. A subtree of a given tree t is a connected and simply connected simplicial sub complex of t, whose orientation is compatible with the one of t.

Remark 6 *The presence of the distinguished vertex (the root), and the fact that for any other vertex there is only one incoming edge, has as a consequence that in any rooted tree loops are forbidden. In what follows, by tree will be meant a non-planar rooted tree.*

Definition 33 *We will denote the set of all trees by* $\mathcal{T}$. *The empty tree will be considered an element of* $\mathcal{T}$, *and it will be denoted with* $\mathbf{1}$. *Any subset of* $\mathcal{T}$ *is called a forest. In what follows,* $\mathcal{F}(\mathcal{T})$ *will denote the set of all forest with finite cardinality.*

Let $t \in \mathcal{T}$ with root $\{*(t)\}$:

Definition 34 *Two vertices,* $v_1, v_2 \in V(t)$, *are called path connected if it is possible to reach* v_2 *starting from* v_1 *in a way compatible with the orientation of* t.
In such a case, we can define a (unique) subtree of t, $P(v_1, v_2)$, *such that* $V^{\text{ext}}(P(v_1, v_2)) = \{v_1, v_2\}$ *and* $*(P(v_1, v_2)) = v_1$.

Remark 7 *The root is the only vertex that is path connected to each of the other vertices.*

The next goal is to define a structure of Hopf algebra on $\mathcal{F}(\mathcal{T})$. Such a Hopf algebra will be freely generated by the set of all trees. Let us start defining the algebra structure.

Multiplication and unit are defined as follows:

Definition 35

$$m : \mathcal{F}(\mathcal{T}) \otimes \mathcal{F}(\mathcal{T}) \longrightarrow \mathcal{F}(\mathcal{T})$$

,

$$t_{i_1} \cdots t_{i_n} \otimes t_{j_1} \cdots t_{j_m} \rightsquigarrow t_{i_1} \cdots t_{i_n} t_{j_1} \cdots t_{j_m},$$

where $t_{i_1} \cdots t_{i_n}$ and $t_{j_1} \cdots t_{j_m}$ are two forests with n and m trees. In particular, the product of two trees t_1 and t_2 will deliver a forest, $t_1 t_2$. Moreover, we define: $m(\mathbf{1}, t) = m(t, \mathbf{1}) = t$ for each $t \in \mathcal{T}$.
The unit is the linear map: $u : k \longrightarrow \mathcal{T} \subset \mathcal{F}(\mathcal{T}),\ 1 \rightsquigarrow \mathbf{1}$.

Graphically we will write:

$$m(\bullet \otimes \begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}) = \bullet \begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}$$

The following proposition is now trivial:

Proposition 15 $(\mathcal{F}(\mathcal{T}), m, u)$ *is a commutative algebra.*

Proof Commutativity and associativity follow immediately from the very definition of the multiplication m. ♠

Remark 8 *Since now on we will think of $\mathcal{F}(\mathcal{T})$ endowed of the algebra structure defined in the previous proposition 15. In particular, any given forest, $F = t_1 \cdots t_n \in \mathcal{F}(\mathcal{T})$, can be interpreted as the result of the iterated product of the trees t_1 up to t_n, so that the algebra $(\mathcal{F}(\mathcal{T}), m, u)$ is freely generated by set of all trees $\mathcal{T}$.*

The next definition is fundamental for what follows:

Definition 36 *An elementary cut for t, $\tilde{c}$, is a subset of $E(t)$ of cardinality equal to one. An admissible cut c, is a subset of $E(t)$ such that for each $v \in V(t)$, $E(P(*(t), v)) \cap c$ has cardinality less or equal to one. Obviously, each elementary cut is also an admissible cut. By definition, the empty cut and the total cut, are elementary.*

Example 18 *1) For the tree $\bullet$, there are only two admissible cuts: the empty and the total one. In particular, for any tree (non empty), the set of admissible cuts contains always two elements, which are the empty and the total one.*

2) The tree *has only one edge so that the set of admissible cut contains only three elements.*

3) The first non trivial examples are given by the trees: *and* *. For those trees, beside the total and the empty cuts, we have the following: for the first one, we have that each admissible cut has cardinality equal to one, in particular the set of admissible cuts contains two elements. The tree*  *is more interesting: in this case we have three possible admissible cuts, two of them with cardinality equal to one, one with cardinality equal to two.*

The notion of admissible cut c, allows to introduce a map

$$C : \mathcal{T} \longrightarrow \mathcal{F}(\mathcal{T}) \times \mathcal{T}, \;\; C(t) = (P_c(t), R_c(t)).$$

$R_c(t)$ is a distinguished tree such that $*(R_c(t)) = *(t)$ and $P_c(t) = \prod_i t_i$ is a forest whose cardinality is equal to card(c), and such that for any $t_i \in P_c(t)$, $*(t_i)$ is the incoming vertex of one of the edges in c (see examples below). Let us give some example which will clarify these important concepts.

Example 19 *In the following examples we will not consider the empty and the total cuts.*

1) $t =$ *. In this case, we have only one admissible cut $\tilde{c}$, and such has cardinality equal 1. Removing the edge we will get: $P_{\tilde{c}}($**$) = \bullet$, and similarly, $R_{\tilde{c}}($**$) = \bullet$. Beside the appearance $P_{\tilde{c}}(t)$ and $R_{\tilde{c}}(t)$, have a different meaning: by definition $R_{\tilde{c}}(t)$ coincide with $*($* *$)$.*

2) $t = \overset{\bullet}{\bullet\;\bullet}$. We have three admissible cuts, two of them with cardinality one and one with cardinality 2. For the first two cases we have that: $P_{\tilde{c}} = \bullet$ and $R_{\tilde{c}} = \overset{\bullet}{\bullet}$, while the third case give us $P_{\tilde{c}} = \bullet\bullet$ and $R_{\tilde{c}} = \overset{\bullet}{\bullet}$. As this example shows, in general $P_{\tilde{c}}(t)$ is not a tree but a forest.

Remark 9 *$\mathcal{T}$ is a naturally $\mathbb{Z}_{\geq 0}$ graded vector space freely generated by the trees. The grading is given as follows:* $\deg(t) = \mathrm{card}(V(t))$ and $\deg(\mathbf{1}) = 0$. *Then we can write: $\mathcal{T} = \bigoplus_{i \geq 0} \mathrm{T}_i$. Moreover, since $\dim_k(\mathrm{T}_i) < \infty$ for each i, $\mathcal{T}$ is a graded vector space of finite type.*

To define the coalgebra structure, let us first introduce the following maps:

1. $\varepsilon : \mathcal{T} \longrightarrow k$, such that:

$$\varepsilon(t) = 0, \;\; \varepsilon(\mathbf{1}) = 1,$$

and having the property: $\varepsilon \circ m(t_1 \otimes t_2) = \varepsilon(t_1)\varepsilon(t_2)$.

2. $\Delta : \mathcal{T} \longrightarrow \mathcal{F}(\mathcal{T}) \otimes \mathcal{T}$, such that:

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \;\; \Delta(t) = t \otimes \mathbf{1} + \mathbf{1} \otimes t + \sum_{\tilde{c}} P_{\tilde{c}}(t) \otimes R_{\tilde{c}}(t),$$

and having the property: $\Delta \circ m(t_1 \otimes t_2) = (m \otimes m)(\Delta t_1 \otimes \Delta t_2)$, which extends Δ to a map: $\mathcal{F}(\mathcal{T}) \longrightarrow \mathcal{F}(\mathcal{T}) \otimes \mathcal{F}(\mathcal{T})$.

Example 20

$$1)\ \Delta(\bullet) = \bullet \otimes \mathbf{1} + \mathbf{1} \otimes \bullet,$$

since the set of simple cuts is empty.

$$2)\ \Delta(\overset{\bullet}{\bullet}) = \overset{\bullet}{\bullet} \otimes \mathbf{1} + \mathbf{1} \otimes \overset{\bullet}{\bullet} + \bullet \otimes \bullet.$$

$$3)\ \Delta(\overset{\bullet}{\bullet\;\bullet}) = \overset{\bullet}{\bullet\;\bullet} \otimes \mathbf{1} + \mathbf{1} \otimes \overset{\bullet}{\bullet\;\bullet} + 2\, \bullet \otimes \overset{\bullet}{\bullet} + \bullet\bullet \otimes \bullet.$$

Now we can prove the following theorem:

Theorem 11 $(\mathcal{F}(\mathcal{T}), \Delta, \varepsilon)$ *is a coalgebra; i.e Δ is a coproduct and ε is a counit. Moreover they are compatible with the algebra structure defined in the proposition 15, so that $(\mathcal{F}(\mathcal{T}), m, \Delta, u, \varepsilon)$ is a bialgebra.*

Proof Since the compatibility of ε and Δ with the algebra structure is contained in the definition of these two maps, we are only left to show the counit property and the coassociativity of the map Δ. The counit property is trivially checked: $(\mathrm{Id}\otimes\varepsilon)\circ\Delta(t) = t = (\varepsilon\otimes\mathrm{Id})\circ\Delta(t)$, and $(\mathrm{Id}\otimes\varepsilon)\circ\Delta(\mathbf{1}) = \mathbf{1} = (\varepsilon\otimes\mathrm{Id})\circ\Delta(\mathbf{1})$ by the definition of Δ and ε. Now we need to show that Δ is coassociative, i.e: $(\Delta\otimes\mathrm{Id})\circ\Delta(t) = (\mathrm{Id}\otimes\Delta)\circ\Delta(t)$ for each tree. Let us start introducing the following operator:

$$L : \mathcal{F}(\mathcal{T}) \longrightarrow \mathcal{T};$$
$$L(t_1\cdots t_n) = t,$$

where t is the tree obtained joining the roots of $t_1, ..., t_n$ to a common vertex:

Example 21

$L(\bullet\bullet) =$ .

We now state the following lemma. For its proof, as well as for the relevance of the operator L and for its cohomological interpretation, we refer to [**?**].

Lemma 1 *For each $A \in \mathcal{F}(\mathcal{T})$,*

$$\Delta\circ L(A) = L(A)\otimes 1 + (Id\otimes L)\Delta(A).$$

Following [**?**], we will prove the associativity by an inductive argument. Let us define $\mathcal{F}_m$ the sub algebra of $\mathcal{F}(\mathcal{T})$, which is generated by the trees of degree less of equal to m (see remark 9). Clearly we have that $\mathcal{F}(\mathcal{T}) = \bigcup_n \mathcal{F}_n$. The induction will be performed on the degree of the filtration $\mathcal{F}(\mathcal{T}) : \cdots\mathcal{F}_m \subset \mathcal{F}_{m+1} \subset \cdots$. The restriction of Δ to $\mathcal{F}_1$ is clearly associative: $\Delta(\bullet) = \bullet\otimes\mathbf{1} + \mathbf{1}\otimes\bullet$. Let us suppose that associativity holds up to degree m, and let us prove that it holds also in degree $m+1$. In particular let us take T of degree $m+1$. This tree can be written as $L(A) = L(t_1\cdots t_l)$, where $\deg(t_j) \leq m$. Then, using the previous lemma and the induction, we can write the following chain of equalities:

$$(Id\otimes\Delta)\Delta(T) = (Id\otimes\Delta)\Delta(L(A)) = (Id\otimes\Delta)(L(A)\otimes 1 + (Id\otimes L)\Delta(A)) =$$

$$L(A)\otimes 1\otimes 1 + (Id\otimes\Delta)(A'\otimes\Delta(L(A''))) = L(A)\otimes 1\otimes 1 + A'\otimes L(A'')\otimes 1 +$$

$$+A'\otimes(Id\otimes L)\Delta(A'') = L(A)\otimes 1\otimes 1 + A'\otimes L(A'')\otimes 1 + (Id\otimes(Id\otimes L))A'\otimes\Delta(A'') =$$

$$= L(A) \otimes 1 \otimes 1 + A' \otimes L(A'') \otimes 1 + (Id \otimes (Id \otimes L))\Delta(A') \otimes A'' =$$

$$= L(A) \otimes 1 \otimes 1 + A' \otimes L(A'') \otimes 1 + ((Id \otimes Id) \otimes L)\Delta(A') \otimes A'' =$$

$$= L(A) \otimes 1 \otimes 1 + A' \otimes L(A'') \otimes 1 + \Delta(A') \otimes L(A'') =$$

$$= (\Delta \otimes Id)(L(A) \otimes 1 + A' \otimes L(A'')) = (\Delta \otimes Id)\Delta(L(A)),$$

where we used the notation $\Delta(A) = A' \otimes A''$. ♠

Theorem 12 *The bialgebra $(\mathcal{F}(\mathcal{T}), m, \Delta, u, \varepsilon)$ is graded, of finite type and connected.*

Proof If we define $\deg(m(t_1 \otimes t_2)) = \mathrm{card}(V(t_1)) + \mathrm{card}(V(t_2))$, $(\mathcal{F}(\mathcal{T}), m, u)$ becomes a graded algebra. To show that the coproduct is compatible with the grading, it suffices to observe that for any tree t and any admissible cut $\tilde{c}$, $P_{\tilde{c}}(t) = \prod_j t_j$ (see remark 8), so that $\deg(P_{\tilde{c}}(t)) = \sum_j \mathrm{card}(V(t_j)) = \sum_j \deg(t_j)$. Now the statement follows from the equality: $\mathrm{card}(V(t)) = \mathrm{card}V(R_{\tilde{c}}(t)) + \sum_j \mathrm{card}V(t_j)$. ♠

To show that $(\mathcal{F}(\mathcal{T}), m, \Delta, u, \varepsilon)$ is actually a Hopf algebra, we are left to define the antipode map: $S : \mathcal{F}(\mathcal{T}) \longrightarrow \mathcal{F}(\mathcal{T})$. To this end, we state the following theorem:

Theorem 13 *The map $S : \mathcal{F}(\mathcal{T}) \longrightarrow \mathcal{F}(\mathcal{T})$, defined as:*

$$S(\mathbf{1}) = \mathbf{1};$$

$$S(t) = -t - \sum_{\tilde{c}} S(P_{\tilde{c}}(t)) R_{\tilde{c}}(t);$$

and extended to an algebra morphism: $S(t_i t_j) = S(t_i)S(t_j)$ for each t_i, t_j, defines the antipode for the bialgebra $(\mathcal{F}(\mathcal{T}), m, \Delta, u, \varepsilon)$.

Proof The proof follows from the definition of the antipode map for a bialgebra; this is a map $S \in \mathrm{Hom}(B, B)$ such that: $S \star \mathrm{id}_B = \mathrm{id}_B \star S = u \circ \varepsilon$, where $S \star \mathrm{id}_B = m \circ (S \otimes \mathrm{id}_B) \circ \Delta$ is the convolution product. If such a map S does exist, this is unique. Since the $\mathcal{F}(\mathcal{T})$ is connected, for any tree t we can write:

$$0 = m \circ (S \otimes \mathbf{1}) \circ \Delta(t) = m \circ (S \otimes \mathbf{1})\big(t \otimes \mathbf{1} + \mathbf{1} \otimes t + \sum_{\tilde{c}} P_{\tilde{c}}(t) \otimes R_{\tilde{c}}(t)\big),$$

From this we now get:

$$S(t) = -t - \sum_{\tilde{c}} S(P_{\tilde{c}}(t)) R_{\tilde{c}}(t).$$

♠

As it follows from the previous argument, we could also write the antipode for the bialgebra $(\mathcal{F}(\mathcal{T}), m, \Delta, u, \varepsilon)$, as:

$$S(t) = -t - \sum_{\tilde{c}} S(R_{\tilde{c}}(t)) P_{\tilde{c}}(t).$$

Remark 10 *The formula for the antipode follows directly as a consequence of $\mathcal{F}(\mathcal{T})$ of being connected.*

Remark 11 *From now on, the Hopf algebra of rooted trees $(\mathcal{F}(\mathcal{T}), \Delta, m, \varepsilon, u, S)$ will be indicated with $\mathcal{H}_{rt}$.*

Example 22 *Let us give some explicit calculation of the antipode for some simple tree.*

1) $S(\bullet) = -\bullet$.

2) $S($  $) = -$  $+ \bullet\bullet$.

3) $S($  $) = -$  $+ 2\bullet$  $- \bullet\bullet\bullet$.

Proposition 16 *The Hopf algebra of rooted trees $\mathcal{H}_{rt}$ is generated by the set of its indecomposable elements $I(\mathcal{H}_{rt}) = \mathcal{T}$.*

Proof Since $\mathcal{H}_{rt}$ is connected we have that:

$$I(\mathcal{H}_{rt}) = \ker \varepsilon / m(\ker\varepsilon \otimes \ker \varepsilon) = \bigoplus_{k>0} \big(H_k / \oplus_{l+m=k} H_l H_m\big),$$

where $H_l H_m$ is a short hand notation for $m(H_l \otimes H_m)$. ♠

Example 23 *In degree one there is only one indecomposable element, i.e* $\bullet$. *In particular we have that* $H_1 = \mathrm{span}_k\langle \bullet \rangle$.

In degree two we have only one indecomposable element, i.e , *and* $H_2 =$ $\mathrm{span}_k\langle$ $, \bullet\bullet \rangle$.

In degree three, we have two indecomposable elements:  , *and* $H_3 =$ $\mathrm{span}_k\langle$  , ,  $, \bullet\bullet\bullet \rangle$.

What has been shown in this section is summarized in the following theorem:

Theorem 14 $\mathcal{H}_{rt}$ *is a Hopf algebra,* $\mathbb{Z}_+$*-graded, connected and of finite type and it is generated by the set of indecomposable elements.*

3.2 Duality and Lie algebra of infinitesimal derivations

In the previous subsection we showed that $\mathcal{H}_{rt}$ is endowed with a structure of a graded Hopf algebra, connected and of finite type. Now we want to study the (restricted) dual Hopf algebra, $\mathcal{H}_{rt}^{\star} = \bigoplus_{i\geq 0} H_i^{*}$. We already know that $\mathcal{H}_{rt}^{\star}$ is a $\mathbb{Z}_{+}$-graded, and of finite type Hopf algebra (see proposition 13). Since $\mathcal{H}_{rt}$ is commutative and (freely) generated by the set of all trees, i.e by the indecomposable elements, we have the following theorem (see theorem 8):

Theorem 15 *$\mathcal{H}_{rt}^{\star}$ is a cocommutative, primitively generated Hopf algebra, so that it is isomorphic, as a Hopf algebra, to the universal enveloping algebra of the Lie algebra of its primitive elements $P(\mathcal{H}_{rt}^{\star})$.*

The set of primitive elements $P(\mathcal{H}_{rt}^{\star}) \subset \mathcal{H}_{rt}^{\star}$ is in one to one correspondence with the set of indecomposable elements $I(\mathcal{H}_{rt}) \subset \mathcal{H}_{rt}$; for each tree $t \in I(\mathcal{H}_{rt})$, we get a linear form $Z_t \in P(\mathcal{H}_{rt}^{\star})$. Since $\mathcal{H}_{rt}$ is graded, of finite type and connected, we have the following proposition, (see theorem 10):

Proposition 17 *Each Z_t extends by zero to an algebra derivation.*

Proof In fact: $\langle Z_t, m(t_1 \otimes t_2)\rangle = \langle Z_t, t_1\rangle \varepsilon(t_2) + \langle Z_t, t_2\rangle \varepsilon(t_1)$. But $\ker \varepsilon \simeq \bigoplus_{i>0} H_i$. ♠

Definition 37 *The derivation Z_t defined in the previous proposition, is called infinitesimal character of the Hopf algebra $\mathcal{H}_{rt}$.*

For any $t_1, t_2 \in \mathcal{H}_{rt}$ let Z_{t_1}, Z_{t_2} be the corresponding infinitesimal characters. The product between those is the one induced by the coproduct Δ defined in $\mathcal{H}_{rt}$: $\langle m_*(Z_{t_1} \otimes Z_{t_2}), t\rangle = \langle Z_{t_1} \otimes Z_{t_2}, \Delta(t)\rangle$.

In particular, we have the following proposition:

Proposition 18 *For Z_{t_1}, Z_{t_2} and t as above, we have:*

$$\langle m_*(Z_{t_1} \otimes Z_{t_2}), t\rangle = \sum_{\tilde{c}} \big(\langle Z_{t_1}, P_{\tilde{c}}(t)\rangle \langle Z_{t_2}, R_{\tilde{c}}(t)\rangle\big), \tag{3.1}$$

where $\tilde{c}$ runs over the set of all possible admissible cut of cardinality equal to one (i.e over the set of elementary cuts).

Proof In fact, for any give admissible cut c, the number of factors in $P_c(t)$ is equal to the cardinality of c. Since the linear form Z_{t_1} extends by zero to a derivation of the algebra $\mathcal{H}_{rt}$, the only terms which are not identically equal to zero in (3.1) are the ones labelled by admissible cuts with cardinality equal to one.♠

The Lie algebra structure defined on $P(\mathcal{H}^\star_{rt})$ is given by the bracket:

$$[Z_{t_1}, Z_{t_2}] = m_*(Z_{t_1} \otimes Z_{t_2} - Z_{t_2} \otimes Z_{t_1}).$$

The associativity of the multiplication m_* (which follows from the coassociativity of the coproduct Δ), ensures that such a bracket fulfills the Jacobi identity.

Remark 12 *The product m_* is also called convolution product.*

We will now analyze in greater details the Lie algebra $P(\mathcal{H}_{\mathrm{rt}})$.

We start with the following proposition:

Proposition 19 *The set of the infinitesimal characters is not closed with respect to the convolution product.*

Proof Let Z_{t_1}, Z_{t_2} be two infinitesimal derivations and let T_1, T_2 be two trees. Then: $\langle m_*(Z_{t_1} \otimes Z_{t_2}), m(T_1 \otimes T_2)\rangle = \langle Z_{t_1} \otimes Z_{t_2}, m(\Delta(T_1) \otimes \Delta(T_2))$. On the other hand, by the definition of the coproduct, we have: $\Delta(T_i) = T_i \otimes \mathbf{1} + \mathbf{1} \otimes T_i + \sum_i T'_i \otimes T''_i$, $i = 1, 2$, where T'_i and T''_i have degree greater than zero. So that: $m(\Delta(T_1) \otimes \Delta(T_2)) = T_1 \otimes T_2 + T_2 \otimes T_1 + \Sigma' \otimes \Sigma''$ where either Σ' or Σ'' are decomposable (product of indecomposable, i.e product of trees), so that they belong to the kernel of the linear forms Z_{t_1}, Z_{t_2}. From this, we can conclude:

$$\langle Z_{t_1} \otimes Z_{t_2}, m(\Delta(T_1) \otimes \Delta(T_2)) = \langle Z_{t_1}, T_1\rangle\langle Z_{t_2}, T_2\rangle + \langle Z_{t_2}, T_1\rangle\langle Z_{t_1}, T_2\rangle,$$

which is, in general, different by zero. ♠

Remark 13 *It is clear that the previous results hold for any Hopf algebra connected, graded of finite type, since the key for those results is the characterization of the coproduct, given in proposition 12.*

Example 24 *Let us calculate the convolution product between $Z_\bullet$ with itself. From the definition:*

$$m_*\big(Z_\bullet \otimes Z_\bullet\big) = \big(Z_\bullet \otimes Z_\bullet\big) \circ \Delta. \tag{3.2}$$

This is an element in $\mathcal{H}^\star_{rt}$, so that can be written as a linear combination of elements like Z_F, where F is a forest. The generators in degree less or equal to 3, of the homogeneous components of the restricted dual are the following:

$$H_1^* = \mathrm{span}_k\langle Z_\bullet\rangle;\, H_2^* = \mathrm{span}_k\langle Z_{\substack{\bullet\\|\\\bullet}}, Z_{\bullet\bullet}\rangle,\, H_3^* = \mathrm{span}_k\langle Z_{\substack{\bullet\\|\\\bullet\\|\\\bullet}}, Z_{\substack{\bullet\\ \bigwedge \\ \bullet\;\bullet}}, Z_{\bullet\bullet\bullet}, Z_{\bullet\substack{\bullet\\|\\\bullet}}\rangle.$$

Since m_* is grading preserving: $m_*\big(Z_\bullet \otimes Z_\bullet\big) \in H_2^*$, so that we can write:

$$m_*\big(Z_\bullet \otimes Z_\bullet\big) = a_1 Z_{\substack{\bullet\\|\\\bullet}} + a_2 Z_{\bullet\bullet}$$

for some a_1, a_2. It is easy to find such coefficients using the equality (3.2);

$$m_*\big(Z_\bullet \otimes Z_\bullet\big) = Z_{\substack{\bullet\\|\\\bullet}} + 2Z_{\bullet\bullet}.$$

Remark 14 *Using the previous argument, it is possible, at least in principle, to write each generator of $\mathcal{H}^\star_{rt}$ as linear combinations of products (with respect to m_*) of infinitesimal characters. In particular, from the previous example we have that:*

$$Z_{\bullet\bullet} = \frac{1}{2}\Big(m_*\big(Z_\bullet \otimes Z_\bullet\big) - Z_{\substack{\bullet\\|\\\bullet}}\Big).$$

In the case under examination, i.e for the Hopf algebra $\mathcal{H}_{rt}$, it is possible to introduce a new product, that we will indicate with $*$, with respect to which $P(\mathcal{H}^\star_{rt})$ turn out to be closed. Such a remarkable product will induce a Lie algebra structure on the vector space $P(\mathcal{H}^\star_{rt})$ that will coincide with the one defined via the convolution product.

Definition 38 *For any given $Z_{t_1}, Z_{t_2} \in P(\mathcal{H}^\star_{rt})$, define:*

$$Z_{t_1} * Z_{t_2} = \sum_{\tilde{c}} n(t; t_1, t_2) Z_t, \tag{3.3}$$

where $\tilde{c}$ runs over the set of all admissible cut of cardinality equal to one. The numerical coefficients $n(t; t_1, t_2)$, express the number of ways in which t can be decomposed by an elementary cut $\tilde{c}$ (i.e an admissible cut with cardinality equal to one), in such a way $P_{\tilde{c}}(t) = t_1$ and $R_{\tilde{c}}(t) = t_2$.

Example 25 *Let us calculate the $*$-product in some simple case:*

$$
\begin{aligned}
&1)\quad Z_{\bullet} * Z_{\bullet} = Z_{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}}\ ;\\
&2)\ Z_{\bullet} * Z_{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} = Z_{\begin{smallmatrix}\bullet\\ |\\ \bullet\\ |\\ \bullet\end{smallmatrix}} + 2\ Z_{\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet\end{smallmatrix}}\ ,\\
&3)\quad Z_{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} * Z_{\bullet} = Z_{\begin{smallmatrix}\bullet\\ |\\ \bullet\\ |\\ \bullet\end{smallmatrix}}\ ;
\end{aligned}
$$

As it is evident in the previous examples, the $*$-product is not commutative. Moreover, it is not associative as the next example shows:

Example 26

$$\left(Z_{\bullet} * Z_{\bullet}\right) * Z_{\bullet} = Z_{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} * Z_{\bullet} = Z_{\begin{smallmatrix}\bullet\\ |\\ \bullet\\ |\\ \bullet\end{smallmatrix}}\ ;$$

on the other hand:

$$Z_{\bullet} * \left(Z_{\bullet} * Z_{\bullet}\right) = Z_{\bullet} * Z_{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} = Z_{\begin{smallmatrix}\bullet\\ |\\ \bullet\\ |\\ \bullet\end{smallmatrix}} + 2\ Z_{\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet\end{smallmatrix}}\ .$$

Nevertheless, the $*$-product defined in (3.3) fulfills the following property:

Theorem 16 *For each triple of rooted trees t_1, t_2 and t_3 we have:*

$$Z_{t_1} * \left(Z_{t_2} * Z_{t_3}\right) - \left(Z_{t_1} * Z_{t_2}\right) * Z_{t_3} = Z_{t_2} * \left(Z_{t_1} * Z_{t_3}\right) - \left(Z_{t_2} * Z_{t_1}\right) * Z_{t_3}. \quad (3.4)$$

Proof The proof is a consequence of the following lemma. Let us define the following triple product:

$$A(t_1, t_2, t_3) = Z_{t_1} * \left(Z_{t_2} * Z_{t_3}\right) - \left(Z_{t_1} * Z_{t_2}\right) * Z_{t_3}. \quad (3.5)$$

Lemma 2

$$A(t_1, t_2, t_3) = \sum_{\tilde{c}} n(t_1, t_2, t_3; t) Z_t,$$

where $n(t_1, t_2, t_3; t)$ is the number of admissible cuts $\tilde{c}$ of t such that $R_{\tilde{c}}(t) = t_3$ and $P_{\tilde{c}}(t) = t_1 t_2$ and card $\tilde{c} = 2$.

Proof(of the Lemma) By the definition of the product $*$, we can write:

$$A(t_1, t_2, t_3) = \sum_t \big(\sum_{t'} n(t_2, t_3; t') n(t_1, t'; t) - n(t_1, t_2; t') n(t', t_3; t) \big) Z_t.$$

The first sum corresponds to the couples of elementary cuts $(\tilde{c}, \tilde{c}')$ where $\tilde{c}$ is a cut of the tree $R_{\tilde{c}}(t)$. Such a set of couples is the (disjoint) union of the set of couples $(\tilde{c}, \tilde{c}')$, with the property that $\tilde{c} \cup \tilde{c}'$ is still admissible for t, with the set of couples which do not have such a property. On the other hand, the second sum corresponds to the set of couples of admissible cuts $(\tilde{c}, \tilde{c}')$, such that $R_{\tilde{c}}(t) = t_3$ and $\tilde{c}'$ is cut of $P_{\tilde{c}'}(t)$. This means that for such couples we never have the case $\tilde{c} \cup \tilde{c}'$ is admissible for t. From this follows the lemma, since the difference representing $A(t_1, t_2, t_3)$ will count only the couple $(\tilde{c}, \tilde{c}')$ of admissible cuts such that $\tilde{c} \cup \tilde{c}'$ is still admissible. ♠

To end the proof of the theorem, it suffices to observe that:

$$A(t_1, t_2, t_3) = A(t_2, t_1, t_3),$$

which follows from the definition given in the equation (4.1). ♠

Definition 39 *A vector space V endowed with a product $*$ that fulfills the condition expressed in (3.4) is called a pre-Lie algebra.*

Remark 15 *The trilinear form A defined in (4.1) can be thought as a measure of the non-associativity of the product $*$ and it is called associator. Pre-Lie algebras are also known in the literature as left-symmetric or right-symmetric. The one we introduced is left symmetric since:*

$$A(t_1, t_2, t_3) = A(t_2, t_1, t_3).$$

Right-symmetric ones would be defined using the following associator:

$$A(t_1, t_2, t_3) = A(t_1, t_3, t_2).$$

Proposition 20 *If $(V, *)$ is a pre-Lie algebra then $L(V)$ is a Lie algebra; i.e the bracket $[\cdot, \cdot] : V \otimes V \longrightarrow V$, $[x, y] = x * y - y * x$, for each $x, y \in V$ fulfills the Jacobi identity.*

Proof We need to show that for each x, y and z in V, $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$. This follows after applying the definition of the bracket $[x,y] = x * y - y * x$, and from the property (3.4). ♠

On $P(\mathcal{H}_{rt}^{\star})$ there are defined two products: the convolution product $m_* = \Delta^t$, and the pre-Lie product (3.3). The first one does not close to a product for the vector space of the primitive elements $P(\mathcal{H}_{rt}^{\star})$ (see proposition 19) while, by its very definition, the pre-Lie product $*$ does close to such a product. Nevertheless, we have the following result:

Proposition 21 *The convolution product between Z_{t_1} and Z_{t_2}, $(Z_{t_1} \otimes Z_{t_2}) \circ \Delta$, is a linear form on $\mathcal{H}_{rt}$, whose restriction to the indecomposable elements $I(\mathcal{H}_{rt})$ coincides with the linear form $Z_{t_1} * Z_{t_2}$.*

Proof Let us consider the $t_1, t_2, t \in I(\mathcal{H}_{rt})$. The coproduct gives us $\Delta(t) = t \otimes \mathbf{1} + \mathbf{1} \otimes t + \sum_{\tilde{c}} P_{\tilde{c}}(t) \otimes R_{\tilde{c}}(t)$, where the sum is taken over the set of admissible cuts. The convolution between Z_{t_1} and Z_{t_2} will give us:

$$\big((Z_{t_1} \otimes Z_{t_2}) \circ \Delta\big)(t) = \sum_{\tilde{c}} \langle Z_{t_1}, P_{\tilde{c}}(t)\rangle \langle Z_{t_2}, R_{\tilde{c}}(t)\rangle. \tag{3.6}$$

Since Z_{t_1}, Z_{t_2} are infinitesimal characters, the the sum in (3.6) reduces to a sum over the elementary cuts, i.e:

$$\big((Z_{t_1} \otimes Z_{t_2}) \circ \Delta\big)(t) = \sum_{\tilde{c}} \langle Z_{t_1}, P_{\tilde{c}}(t)\rangle \langle Z_{t_2}, R_{\tilde{c}}(t)\rangle.$$

This sum represents the number of ways in which t can be decomposed, via an elementary cut $\tilde{c}$, as $P_{\tilde{c}} = t_1$ and $R_{\tilde{c}}(t) = t_2$; such a number is by definition $n(t; t_1, t_2)$ ♠

Moreover, we have that:

Theorem 17 *For each t_1 and t_2 in $I(\mathcal{H}_{rt})$,*

$$m_*(Z_{t_1} \otimes Z_{t_2} - Z_{t_2} \otimes Z_{t_1}) = Z_{t_1} * Z_{t_2} - Z_{t_2} * Z_{t_1},$$

so that the Lie algebras structures induced on $P(\mathcal{H}_{rt}^{\star})$ by these two products coincide.

Proof We already know that the restriction of $m_*(Z_{t_1} \otimes Z_{t_2})$ to the indecomposable elements in $\mathcal{H}_{rt}$ coincides with $Z_{t_1} * Z_{t_2}$, so that $m_*(Z_{t_1} \otimes Z_{t_2} - Z_{t_2} \otimes Z_{t_1})(t) = (Z_{t_1} * Z_{t_2} - Z_{t_2} * Z_{t_1})(t)$ for any $t \in I(\mathcal{H}_{rt})$. Therefore, we are left to prove that $m_*(Z_{t_1} \otimes Z_{t_2} - Z_{t_2} \otimes Z_{t_1})$ restricts to zero to any decomposable element in $\mathcal{H}_{rt}$. Let $T = T_1 T_2$ such an element, then $\Delta(T) = T_1 \otimes T_2 + T_2 \otimes T_1 + \sum \sigma \otimes \sigma'$ where σ and or σ' are decomposable. Applying $Z_{t_1} \otimes Z_{t_2} - Z_{t_2} \otimes Z_{t_1}$ to right hand side of the previous equality we get zero: in fact $(Z_{t_1} \otimes Z_{t_2} - Z_{t_2} \otimes Z_{t_1})$ is zero for each term of the sum $\sum \sigma \otimes \sigma'$ since Z_{t_1}, Z_{t_2} are infinitesimal derivations, and $(Z_{t_1} \otimes Z_{t_2} - Z_{t_2} \otimes Z_{t_1})$ is zero on $T_1 \otimes T_2 + T_2 \otimes T_1$ since the last one is symmetric in T_1, T_2 while the first one in antisymmetric in t_1, t_2. ♠

Remark 16 *The convolution product m_* is associative, while, as already remarked, the product $*$ is pre-Lie. Nevertheless,*

$$Z_{t_1} * Z_{t_2}|_{I(\mathcal{H}_{rt})} = m_*(Z_{t_1} \otimes Z_{t_2})|_{I(\mathcal{H}_{rt})}.$$

Let us clarify this apparent anomaly discussing one example. Let us consider the triple product of the infinitesimal character $Z_\bullet$ with itself.
Let us calculate first the triple products using the $*$-product:

$$Z_\bullet * \big(Z_\bullet * Z_\bullet\big) = 2Z_{\substack{\bullet\\ \bigwedge\\ \bullet\ \bullet}} + Z_{\substack{\bullet\\ \mid\\ \bullet\\ \mid\\ \bullet}}, \tag{3.7}$$

and

$$\big(Z_\bullet * Z_\bullet\big) * Z_\bullet = Z_{\substack{\bullet\\ \mid\\ \bullet\\ \mid\\ \bullet}}. \tag{3.8}$$

Using the convolution product, we get instead:

$$m_*\Big(Z_\bullet \otimes m_*\big(Z_\bullet \otimes Z_\bullet\big)\Big) = m_*\Big(\big(2Z_{\bullet\bullet} + Z_{\substack{\bullet\\ \mid\\ \bullet}}\big) \otimes Z_\bullet\Big) = 6Z_{\bullet\bullet\bullet} + 2Z_{\substack{\bullet\\ \bigwedge\\ \bullet\ \bullet}} + \tag{3.9}$$
$$+3Z_{\substack{\bullet\\ \mid\\ \bullet}\,\bullet} + Z_{\substack{\bullet\\ \mid\\ \bullet\\ \mid\\ \bullet}}$$

and similarly:

$$m_*\Big(m_*\big(Z_\bullet \otimes Z_\bullet\big) \otimes Z_\bullet\Big) = m_*\Big(Z_\bullet \otimes \big(2Z_{\bullet\bullet} + Z_{\substack{\bullet\\ \mid\\ \bullet}}\big)\Big) = 6Z_{\bullet\bullet\bullet} + 2Z_{\substack{\bullet\\ \bigwedge\\ \bullet\ \bullet}} + \tag{3.10}$$

$+3Z_{\substack{\bullet\\ \bullet\,\bullet}} + Z_{\substack{\bullet\\ \bullet\\ \bullet}}$

The computations (3.9) and (3.10) give us the same result (the convolution product is associative). In particular, if we restrict the results of such computations to the set of indecomposable elements $I(\mathcal{H}_{rt})$ we get the same infinitesimal character we get from the computation done in (3.7). The difference between (3.7) and (3.8) lies in the absence of the term $2Z_{\substack{\bullet\\ \bullet\;\bullet}}$. *Such a term is given in (3.9) by the product* $m_*\big(2Z_{\bullet\bullet} \otimes Z_{\bullet}\big)$, *i.e it is a consequence of the presence of the linear term* $Z_{\bullet\bullet}$, *which is not an infinitesimal character.*

3.3 The ladder tree Hopf algebra

We will now introduce a sub Hopf algebra of $\mathcal{H}_{rt}$, which is at the base of the present work.

Let us consider the sub set of all indecomposable elements in $\mathcal{H}_{rt}$ having each vertex with fertility equal to one. Let us indicate such a set with I_L.

Lemma 3 $I_L \bigcup \mathbf{1}$ *generates a sub Hopf algebra of the Hopf algebra* $\mathcal{H}_{rt}$.

Proof First let us observe that for each $n \in \mathbb{Z}_{>0}$ there is exactly one tree $t \in I_L$ such that $\deg t = n$. Let us call t_n, the tree in I_L of degree equal to n. Let us now call with $\mathcal{H}_L$ the algebra generated by I_L. To prove the statement it suffices to show that coproduct and antipode defined on $\mathcal{H}_{rt}$ restrict to a coproduct and to an antipode to $\mathcal{H}_L$; i.e we need to show:

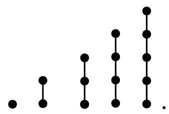
$$\Delta(\mathcal{H}_L) \subset \mathcal{H}_L \otimes \mathcal{H}_l;$$

and

$$S : \mathcal{H}_L \subset \mathcal{H}_L.$$

Both properties are easily checked once it is noticed the following: for each $t \in I_L$ each admissible cut has cardinality equal to one.

Example 27 *The following trees represent the generators for the ladder Hopf algebra up to degree equal to 5:*



From this observation we have that:

$$\Delta(t_n) = \sum_{k=0}^{n} t_{n-k} \otimes t_k; \tag{3.11}$$

and

$$S(t_n) = -t_n - \sum_{k=0}^{n} S(t_{n-k}) t_k.$$

♠

Corollary 5 *$\mathcal{H}_L$ is a graded, commutative and cocommutative Hopf algebra. The grading is induced by the one defined in $\mathcal{H}_{rt}$.*

Proof The commutativity does not need any comment. On the other hand:

$$\tau \circ \Delta(t_n) = \tau\Big(\sum_{k=0}^{n} t_{n-k} \otimes t_k\Big) = \sum_{k=0}^{n} t_k \otimes t_{n-k} = \sum_{j=0}^{n} t_{n-j} \otimes t_j = \Delta(t_n).$$

♠

Definition 40 *We will indicate with l_n, the generator of H_{L_n}. In particular we have that $l_0 = \mathbf{1}$.*

Proposition 22 *For any pair of ladder trees (l_n, l_m),*

$$Z_{l_m} * Z_{l_n}|_{I_L} = Z_{l_n} * Z_{l_m}|_{I_L}.$$

Proof The statement is equivalent to say that for each couple of non negative integer numbers (n, m), $n + m = m + n$. ♠

Chapter 4

Insertion elimination Lie algebras

This chapter is devoted to the introduction of a class of combinatorial Lie algebras, which can be defined using an underlying combinatorial Hopf algebras. Even if such Lie algebras can be defined for a large class of tree-like objects (e.g graphs), we will be only concerned with the Lie algebras associated to the Hopf algebra of rooted trees. The reference for material contained in this chapter is [7].

4.1 Derivations for the Hopf algebra of rooted trees

For any couple of trees (T, t) and any vertex $v \in V(T)$, we can define the following operation:

Definition 41 *Given the data $(T, t, v) \in \mathcal{H}_{rt} \times \mathcal{H}_{rt} \times V(T)$ we define a tree $T \cup_v t$ obtained gluing the $*(t)$ to v with a new edge.*

Example 28 *1) Let us take $T = t = \bullet$. In this case, the operation described above give us:*

$$T \cup_v t = \overset{\bullet}{\underset{\bullet}{|}}.$$

2) Let us consider $T = \overset{\bullet}{\underset{\bullet}{|}}$ and $t = \bullet$ and let us take $v = *(T)$. In this case we get:

$$T \cup_v t =$$ .

Remark 17 *Even if we will use the gluing operation mainly in the form we have just introduced it, i.e to glue a tree to another tree along a vertex, we can define a slightly more general operation, that consists in gluing a forest to a tree, along one of its vertices. Let us just give a simple example of such operation:* $T = \bullet$ *and let us take as forest* $F = \bullet\bullet$.

$$T \cup_v F =$$ 

It follows from the definition that:

$$\mathrm{card}\,\big(E(T \cup_v t)\big) = \mathrm{card}\,\big(E(T)\big) + \mathrm{card}\,\big(E(t)\big) + 1,$$

and that:

$$\mathrm{card}\,\big(V(T \cup_v t)\big) = \mathrm{card}\,\big(V(T)\big) + \mathrm{card}\,\big(V(t)\big).$$

Remark 18 *It is worthwhile to compare the gluing operation just introduced, with the one which is defined in [7]. In [7], to any triple* (T, t, v) *as defined above, it is associated the tree* $T\tilde{\cup}_v t$. *This tree is obtained by gluing the root of* t *to the vertex* v *of* T, *without inserting any new connecting edge. The difference between these two operations is exemplified in the following:*

1) For $T = t = \bullet$, *the new operation gives us:*

$$T\tilde{\cup}_v t = \bullet,$$

2) if $T =$, $t = \bullet$ *and* $v = *(T)$, *we get:*

$$T\tilde{\cup}_v t =$$.

It is clear that for such operations we have:

$$\mathrm{card}\,\big(E(T\tilde{\cup}_v t)\big) = \mathrm{card}\,\big(E(T)\big) + \mathrm{card}\,\big(E(t)\big),$$

and that:

$$\mathrm{card}\,\big(V(T\tilde{\cup}_v t)\big) = \mathrm{card}\,\big(V(T)\big) + \mathrm{card}\,\big(V(t)\big) - 1.$$

4.2 Insertion Lie algebra

We can define the first class of derivations naturally associated to $\mathcal{H}_{rt}$. First, let us consider the vector space freely generated by the set of the indecomposable elements $I(\mathcal{H}_{rt}) \subset \mathcal{H}_{rt}$ (we gave the same name to set of the indecomposable elements, such a choice should not cause any confusion). For each tree $t \in I(\mathcal{H}_{rt})$, let us introduce the linear form $D_t^+ \in \mathrm{Hom}_k\big(I(\mathcal{H}_{rt}), I(\mathcal{H}_{rt})\big)$, via the following:

Definition 42

$$D_t^+(T) = \sum_{v \in V(T)} T \cup_v t.$$

Moreover, for each couple of trees t_1, t_2 and $a_1, a_2 \in k$ we define:

$$D^+_{a_1 t_1 + a_2 t_2} = a_1 D^+_{t_1} + a_2 D^+_{t_2}.$$

Example 29

$$1)\ D^+_{\bullet}(\bullet) = \overset{\bullet}{\underset{\bullet}{|}}$$

$$2)\ D^+_{\bullet}\Big(\overset{\bullet}{\underset{\bullet}{|}}\Big) = \overset{\bullet}{\underset{\overset{\bullet}{\underset{\bullet}{|}}}{|}} + {}_{\bullet}\!\overset{\bullet}{\wedge}\!{}_{\bullet}\ .$$

We now extend D_t^+ to a derivation of $\mathcal{H}_{rt}$:

$$D_t^+\big(T_1, \cdots, T_n\big) = \sum_{i=1}^{n} T_1 \cdots D_t^+(T_i) \cdots T_n.$$

Remark 19 *The previous definition gives then a linear map:*

$$I(\mathcal{H}_{rt}) \rightsquigarrow \mathrm{Der}(\mathcal{H}_{rt}).$$

Following the remark (17), we can say that we have defined a linear map between $\mathcal{H}_{rt}$ and $\mathrm{Der}(\mathcal{H}_{rt})$.

Example 30 *Let us consider $F_1 = \bullet\bullet$ and $F_2 = \bullet\bullet$.*

$$D^+_{F_1}(F_2) = D^+_{\bullet\bullet}(\bullet\bullet) = \big(D^+_{\bullet\bullet}(\bullet)\big)\bullet + \bullet\ \big(D^+_{\bullet\bullet}(\bullet)\big) = 2\,{}_{\bullet}\!\overset{\bullet}{\wedge}\!{}_{\bullet}.$$

Let us now write a close formula for the composition of two operators $D^+_{t_1}$ and $D^+_{t_2}$:

Theorem 18 *For a given pair of trees* (t_1, t_2) *we have:*

$$D^+_{t_2} \circ D^+_{t_1} = A_{t_1 t_2} + D^+_{D^+_{t_2} t_1},$$

where $A_{t_1 t_2}$ *is symmetric (in* t_1, t_2*) linear map defined on each tree* T *by:*

$$A_{t_1 t_2}(T) = \sum_{v \neq \tilde{v} \in V(T)} (T \cup_v t_1) \cup_{\tilde{v}} t_2.$$

Proof Let us calculate:

$$D^+_{t_2} \circ D^+_{t_1}(T) = D^+_{t_2}\big(D^+_{t_1} T\big).$$

By definition:

$$D^+_{t_2}\big(D^+_{t_1} T\big) = \sum_{v \in V(D^+_{t_1} T)} \big(D^+_{t_1} T\big) \cup_v t_2.$$

But $V(D^+_{t_1} T) = \bigcup_{\tilde{v} \in T} V(T \cup_{\tilde{v}} t_1) = V(T) \cup V(t_1)$. So we can write:

$$D^+_{t_2}\big(D^+_{t_1} T\big) = \sum_{v \in V(T) \cup V(t_1)} D^+_{t_1}(T) \cup_v t_2 = \sum_{v \in V(T)} D^+_{t_1}(T) \cup_v t_2 + \sum_{v \in V(t_1)} D^+_{t_1}(T) \cup_v t_2 =$$

$$\sum_{v \in V(T)} \Big(\sum_{\tilde{v} \in V(T)} T \cup_{\tilde{v}} t_1\Big) \cup_v t_2 + \sum_{v \in V(t_1)} \Big(\sum_{\tilde{v} \in V(T)} T \cup_{\tilde{v}} t_1\Big) \cup_v t_2.$$

In the first double sum there is no dependence on t_1 and it represents all the possible way to glue the trees t_1, t_2, into the tree T, at the vertices $\tilde{v}, v \in V(T)$. In the second addendum, we can interchange the order of the sums:

$$\sum_{\tilde{v} \in V(T)} T \cup_{\tilde{v}} \Big(\sum_{v \in V(t_1)} t_1 \cup_v t_2\Big) = \sum_{\tilde{v} \in V(T)} T \cup_{\tilde{v}} (D^+_{t_2} t_1),$$

which is what we wanted to show. ♠

A close formula for the commutator $\big[D^+_{t_1}, D^+_{t_2}\big]$, is given in the following proposition:

Proposition 23

$$\big[D^+_{t_1}, D^+_{t_2}\big] = \sum_{v \in V(t_2)} D^+_{t_2 \cup_v t_1} - \sum_{v \in V(t_1)} D^+_{t_1 \cup_v t_2}.$$

Proof The proof of the statement follows from the theorem 18, noticing that the term $A^+_{t_1 t_2}$ is symmetric in t_1 and t_2. ♠

Example 31 *Let us consider the case of $D^+_{\bullet}$ and $D^+_{\substack{\bullet\\ |\\ \bullet}}$. Then:*

$$\left[D^+_{\bullet}, D^+_{\substack{\bullet\\ |\\ \bullet}}\right]$$

Definition 43 *Let us call $\mathcal{D}^+$ the Lie algebra of derivations of $\mathcal{H}_{rt}$ defined by the map $I(\mathcal{H}_{rt}) \ni t \rightsquigarrow D^+_t$. We will call the linear maps D^+_t insertion operators.*

Convention 1 *To the unit element $\mathbf{1}$ in $\mathcal{H}_{rt}$ will be associated the operator $D^+_{\mathbf{1}}$, whose action is given by the following formula:*

$$D^+_{\mathbf{1}}(T) = \sum_{v \in V(T)} T \cup_v \mathbf{1}.$$

$T \cup_v \mathbf{1}$ is the tree T at which it is attached the empty tree at the vertex v, so that:

$$T \cup_v \mathbf{1} = T.$$

From this we deduce:

$$D^+_{\mathbf{1}}(T) = \mathrm{card} V(T) T;$$

i.e the operator $D^+_{\mathbf{1}}$ is the grading operator for the algebra $\mathcal{H}_{rt}$.

Remember that the set of primitive elements in $\mathcal{H}^\star_{rt}$ is endowed with a Lie algebra structure induced by the $*$-product.
Let us define the linear map:

$$\Psi^+ : P(\mathcal{H}^\star_{rt}) \longrightarrow \mathcal{D}^+$$

$$Z_t \rightsquigarrow D^+_t.$$

Then:

Proposition 24 *The map Ψ^+ is a Lie algebra isomorphism.*

Proof Ψ^+ is clearly an isomorphism of vector spaces. Let us show that it preserve the Lie bracket. For $Z_{t_1}, Z_{t_2} \in P(\mathcal{H}^\star_{rt})$, we have

$$\big[Z_{t_1}, Z_{t_2}\big] = \sum_{T\in I(\mathcal{H}_{rt})} n(t_1,t_2;T)Z_T - \sum_{T\in I(\mathcal{H}_{rt})} n(t_2,t_1;T)Z_T,$$

so that:

$$\Psi^+\big(\big[Z_{t_1}, Z_{t_2}\big]\big) = \sum_{T\in I(\mathcal{H}_{rt})} n(t_1,t_2;T)D^+_T - \sum_{T\in I(\mathcal{H}_{rt})} n(t_2,t_1;T)D^+_T.$$

$n(t_1,t_2;T)$ is the number of ways in which we can decompose T with an elementary cut $\tilde{c}$ to obtain $P_{\tilde{c}}(T) = t_1$ and $R_{\tilde{c}}(T) = t_2$, or, equivalently is the number of times we can get T gluing t_1 to t_2. So that: $n(t_1,t_2;T)$ is the number of vertices $v \in V(t_2)$ such that $t_2 \cup_v t_1 = T$. We have a similar interpretation for the number $n(t_2,t_1;T)$, so that we can write:

$$\sum_{T\in I(\mathcal{H}_{rt})} n(t_1,t_2;T)D^+_T - \sum_{T\in I(\mathcal{H}_{rt})} n(t_2,t_1;T)D^+_T = \sum_{v\in V(t_2)} D^+_{t_2\cup_v t_1} - \sum_{v\in V(t_1)} D^+_{t_1\cup_v t_2}.$$

♠

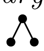
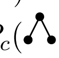
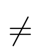
4.3 Elimination Lie algebra

Let us now introduce a second class of derivations for the Hopf algebra of rooted trees. For each $t \in I(\mathcal{H}_{rt})$ let us define:

$$D^-_t \in \mathrm{Hom}_k\big(I(\mathcal{H}_{rt}), I(\mathcal{H}_{rt})\big);$$
$$D^-_t(T) = \langle Z_t \otimes Id_{\mathcal{H}_{\mathrm{rt}}}, \Delta(T)\rangle.$$

Example 32 *Let us calculate:*

D^-().

It is enough to find the decomposition of the tree  *given by the subset of the elementary cuts. In this case, we have four possible elementary cuts. The empty one, the total one and the ones which correspond to* $R_c($$) =$ *and* $P_c($$) = \bullet$ *(there are two of them). Since in each of these cases* $P_c($$) \neq$ *,* D^-() $= 0$.

Lemma 4 *For each $t \in I(\mathcal{H}_{rt})$, D_t^- is a derivation for $\mathcal{H}_{rt}$.*

Proof Let T_1, T_2 rooted trees.

$$D_t^-(T_1T_2) = \langle Z_t \otimes \mathrm{Id}_{\mathcal{H}_{\mathrm{rt}}}, \Delta(T_1)\Delta(T_2)\rangle = \langle Z_t \otimes \mathrm{Id}_{\mathcal{H}_{rt}}, T_1 \otimes T_2 + T_2 \otimes T_1\rangle,$$

where the last equality is a consequence of Z_t being an infinitesimal character. ♠

Definition 44 *The derivations D_t^-, $t \in I(\mathcal{H}_{rt})$ are called eliminations operators. The set of such derivations will be denoted with $\mathcal{D}^-$.*

Convention 2 *To the unit element $\mathbf{1}$ in $\mathcal{H}_{rt}$ will be associated the operator $D_{\mathbf{1}}^-$. For each tree T we have:*

$$D_{\mathbf{1}}^-(T) = \sum_c \langle Z_{\mathbf{1}}, P_c(T)\rangle R_c(T) = T.$$

In other words, the only elementary cut which gives a non zero contribution is the empty cut, for which $P_c(T) = \mathbf{1}$ and $R_c(T) = t$.

Let us define the natural map:

$$\Psi^- : P(\mathcal{H}_{rt}^\star) \longrightarrow \mathcal{D}^-;$$

$$Z_t \rightsquigarrow D_t^-.$$

Now we can prove the following theorem:

Theorem 19 *The map Ψ^- is an anti-isomorphism of the Lie algebra of the infinitesimal characters with the Lie algebra $\mathcal{D}^-$.*

Proof For any two infinitesimal characters Z_{t_1}, Z_{t_2},

$$\big[Z_{t_1}, Z_{t_2}\big] = \sum_T \big(n(t_1, t_2; T) - n(t_2, t_1; T)\big) Z_T.$$

To prove the statement we need to show that:

$$\sum_T \big(n(t_1, t_2; T) - n(t_2, t_1; T)\big) D_T^- = -\big(D_{t_1}^- \circ D_{t_2}^- - D_{t_2}^- \circ D_{t_1}^-\big). \qquad (4.1)$$

It is clear that RHS and LHS of (4.1) act by zero on every t which does not contain $T = t_1 \cup t_2$ or $T = t_2 \cup t_1$ as a subtree. On the other hand for any tree t which has $T = t_1 \cup t_2$ as a subtree: $D^-_{t_1} \circ D^-_{t_2}(t) = n(t_2, t_1; T)$ and for an tree t having $T = t_2 \cup t_1$ as a subtree: $D^-_{t_2} \circ D^-_{t_1}(t) = n(t_1, t_2; T)$.♠

As an immediate consequence of the previous theorem, we get a closed formula for the commutator between two elimination operators.

Proposition 25 *For $t_1, t_2 \in I(\mathcal{H}_{rt})$, we have:*

$$\left[D^-_{t_1}, D^-_{t_2}\right] = \sum_t \big(n(t_2, t_1; t) - n(t_2, t_1; t)\big) D^-_t .$$

Example 33 *Let us consider $t_1 = \bullet$ and $t_2 = \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}$. Let us first calculate:*

$$\left[Z_{\bullet}, Z_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}}\right] = 2Z_{\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}} .$$

The commutator between the corresponding elimination operators is given by:

$$\left[D^-_{\bullet}, D^-_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}}\right] = -2D^-_{\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}}$$

We can rephrase what we have seen in the previous two subsections of this chapter by saying that the Lie algebra of the infinitesimal characters comes endowed with two natural representations: the Lie algebra of insertion operators, $\mathcal{D}^+$, and the Lie algebra of the elimination operators $\mathcal{D}^-$. Both are algebras of derivations of the Hopf algebras of the rooted trees. We are now going to define a bigger algebra of derivations for $\mathcal{H}_{rt}$, that will contain both $\mathcal{D}^+$ and $\mathcal{D}^-$ as sub Lie algebras.

4.4 The insertion-elimination Lie algebra

We want now to put on the same ground insertion and elimination operators. To this end, we need to show that they actually close to a Lie algebra, i.e we need to show, that for any choice of t_1, t_2:

$$\left[D^-_{t_1}, D^+_{t_2}\right] = \sum_t \alpha(t_1, t_2; t) D^+_t + \sum_t \beta(t_1, t_2; t) D^-_t ;$$

for $t \in I(\mathcal{H}_{rt})$ and $\alpha(t_1, t_2; t), \beta(t_1, t_2; t)$, numerical coefficients which depend, for each t, only on t_1, t_2.
To find out what the ingredients of the right hand side of the previous formula are, we need the following theorem:

Theorem 20 $\big[D^-_{t_1}, D^+_{t_2}\big](T) =$

$$\sum_{v\in V(T)} \sum_{c\in E(t_2)} \langle Z_{t_2}, P_c(t_2)\rangle T \cup_v R_c(t_2) + \sum_{c\in E(T)} \sum_{v\in P_c(T)} \langle Z_{t_1}, P_c(T) \cup_v t_2\rangle R_c(T).$$

Note that all the admissible cuts are elementary (i.e with cardinality equal to one).

Proof Let us calculate separately the terms:

$$D^-_{t_1} \circ D^+_{t_2}(T) \ \text{ and } \ D^+_{t_2} \circ D^-_{t_1}(T).$$

The latter gives us:

$$D^+_{t_2}\circ D^-_{t_1}(T) = D^+_{t_2}\Big(\sum_{c\in E(T)} \langle Z_{t_1}, P_c(T)\rangle R_c(T)\Big) = \sum_{c\in E(T)} \langle Z_{t_1}, P_c(T)\rangle \Big(\sum_{v\in R_c(T)} R_c(T) \cup_v t_2\Big).$$

The former is instead:

$$D^-_{t_1}\circ D^+_{t_2}(T) = D^-_{t_1}\Big(\sum_{v\in V(T)} T\cup_v t_2\Big) = \sum_{v\in V(T)} \Big(\sum_{c\in E(T\cup_v t_2)} \langle Z_{t_1}, P_c(T\cup_v t_2)\rangle R_c(T\cup_v t_2)\Big).$$

For each vertex $v \in V(T)$, the sum

$$\sum_{c\in E(T\cup_v t_2)} \langle Z_{t_1}, P_c(T \cup_v t_2)\rangle R_c(T \cup_v t_2)$$

can be rewritten as:

$$\sum_{c\in E(T)} \langle Z_{t_1}, P_c(T \cup_v t_2)\rangle R_c(T \cup_v t_2) + \sum_{c\in \tilde{E}(t_2)} \langle Z_{t_1}, P_c(t_2)\rangle T \cup_v R_c(t_2) + \langle Z_{t_1}, t_2\rangle T,$$

where with $\tilde{E}(t_2)$ we indicated the entire set of elementary cuts for t_2, but the empty and the total one, the latter being represented by the term $\langle Z_{t_1}, t_2\rangle T$.

The sum $\sum_{c\in E(T)}\langle Z_{t_1}, P_c(T\cup_v t_2)\rangle R_c(T\cup_v t_2)$ can be further decomposed in two pieces as follows:

$$\sum_{c\in E(T),\ v\in P_c(T)} \langle Z_{t_1}, P_c(T)\cup_v t_2\rangle R_c(T) + \sum_{c\in E(T),\ v\in R_c(T)} \langle Z_{t_1}, P_c(T)\rangle R_c(T)\cup_v t_2.$$

Putting all these together we will write:

$$\begin{aligned} D^-_{t_1}\circ D^+_{t_2}(T) = & \sum_{v\in V(T)} \Big(\sum_{c\in E(t_2)} \langle Z_{t_1}, P_c(t_2)\rangle T\cup_v R_c(t_2)\Big) \\ & + \sum_{c\in E(T)} \Big(\sum_{v\in P_c(T)} \langle Z_{t_1}, P_c(T)\cup_v t_2\rangle R_c(T)\Big) \\ & + \sum_{c\in E(T)} \Big(\sum_{v\in R_c(T)} \langle Z_{t_1}, P_c(T)\rangle R_c(T)\cup_v t_2\Big), \end{aligned} \tag{4.2}$$

where in the first summand of the equation (4.2), $E(t_2)$ denotes, with a light notational abuse, the set of elementary cuts for t_2, but the empty cut. Taking now the difference between the results obtained we get the proof. ♠

Remark 20 *From the previous theorem, we have that for any given tree T, the non zero contributions to $\big[D^-_{t_1}, D^+_{t_2}\big](T)$ comes from the subtrees t of T such that $t_1 = t\cup_v t_2$ (contained in $\sum_{c\in E(T),\ v\in P_c(T)}\langle Z_{t_1}, P_c(T)\cup_v t_2\rangle R_c(T)$), and from the subtrees t of t_2 such that $t = t_1$, ($\sum_{v\in V(T)}\big(\sum_{c\in E(t_2)}\langle Z_{t_1}, P_c(t_2)\rangle T\cup_v R_c(t_2)\big)$). This suggests the following definition: for each triple of trees t_1, t_2, t,*

- *$\alpha(t_1, t_2; t)$ is number of ways we can write $R_c(t_2) = t$ and $P_c(t_2) = t_1$, where $c\in E(T)$;*
- *$\beta(t_1, t_2; t)$ is the number of times we can write $t_1 = t\cup_v t_2$ for $v\in V(t)$.*

Corollary 6

$$\big[D^-_{t_1}, D^+_{t_2}\big] = \sum_t \alpha(t_1, t_2; t) D^+_t + \sum_t \beta(t_1, t_2; t) D^-_t.$$

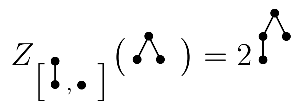
Proof The proof follows from the definition of α and β, from the proof of the theorem 20 and from the previous remark 20. ♠

We will now define a larger class of derivations for the Hopf algebra of rooted trees. These derivations will form a Lie algebra, which contains $\mathcal{D}^+$ and $\mathcal{D}^-$ as sub Lie algebras. For each pair of trees, let us define the linear operator $Z_{[t_1,t_2]}$, such that:

$$Z_{[t_1,t_2]}(T) = \sum_{c \in E(T)} \langle Z_{t_2}, P_c(T) \rangle R_c(T) \cup_{v_c} t_1.$$

Let us give a wordy definition of such an action: the linear map acts as the zero operator on on each tree T for which does not exist any elementary cut c such that $P_c(T) = t_2$. If such an elementary cut does exist, we will write $T = R_c(T) \cup_{v_c} P_c(T)$. In this case $Z_{[t_1,t_2]}$ will eliminate $P_c(T)$ and it will glue at the vertex v_c, the tree t_1. The final result will be the tree $R_c(T) \cup_{v_c} t_2$.

Example 34



Using our usual convention about the identification of the empty tree with the unit $\mathbf{1}$ of the Hopf algebra $\mathcal{H}_{rt}$, we have that:

$$\mathcal{D}^+ = \{Z_{[t_1,t_2]} |\ t_2 = \mathbf{1}\} \text{ and } \mathcal{D}^- = \{Z_{[t_1,t_2]} |\ t_1 = \mathbf{1}\}.$$

Lemma 5 *Each $Z_{[t_1,t_2]}$ is a derivation of the algebra $\mathcal{H}_{rt}$.*

Proof The proof is again a simple consequence of the definition of the action of the operators $Z_{[t_1,t_2]}$. ♠

Definition 45 *The operators $Z_{[t_1,t_2]}$ are called insertion elimination operators. They form an algebra with respect the composition, wich will be denoted with $\mathcal{L}$.*

A close formula for the commutator between two insertion elimination operators is given in the following theorem;

Theorem 21 *[7] For any quadruple t_1, t_2, t_3, t_4, we have:*

$$\left[Z_{[t_1,t_2]}, Z_{[t_3,t_4]}\right] = Z_{[Z_{[t_1,t_2]}(t_3),t_4]} - Z_{[t_3,Z_{[t_2,t_1]}(t_4)]} - Z_{[Z_{[t_3,t_4]}(t_1),t_2]} + \\ + Z_{[t_1,Z_{[t_4,t_3]}(t_2)]} - \delta^K_{t_2,t_3} Z_{[t_1,t_4]} + \delta^K_{t_1,t_4} Z_{[t_2,t_3]}, \tag{4.3}$$

where δ^K is the usual Kronecker delta.

Proof We refer to the reference [7] for the proof of this statement as for further information about the general insertion elimination Lie algebra. ♠

Chapter 5

The ladder Lie algebra

In this chapter, we introduce the ladder insertion elimination Lie algebra $\mathcal{L}_L$. This is a particular insertion elimination Lie algebra, which can be realized as a sub Lie algebra of derivations of the ladder Hopf algebra $\mathcal{H}_L$. The main definition will be followed by a detailed analysis of the structure of such a combinatorial Lie algebra. In particular, the relation of $\mathcal{L}_L$ with a well known infinite dimensional Lie algebra will be analyzed. The chapter will end with some remarks about the cohomology of $\mathcal{L}_L$. The main references for material contained in this chapter are [19], [20] where the results about the structure of the Lie algebra $\mathcal{L}_L$ were first proved.

5.1 Motivations and generalities

In this section, we will focus on the sub Hopf algebra of $\mathcal{H}_{rt}$ generated by the ladder trees. Such a Hopf algebra has been introduced in the subsection 3.3, to which we will refer for notations and definitions.
Let us introduce the following notation:

Definition 46 *For any pair of non negative integer numbers n, m define:*

$$\Theta(n-m) = 1 \text{ if } n \geq m, \text{ and } 0 \text{ otherwise.}$$

Let start with the following lemma:

Lemma 6 *For each ladder tree l_n, $D^-_{l_n} \in \mathrm{Hom}_k(I_L, I_L)$. Moreover:*

$$D^-_{l_n}(l_m) = \Theta(m-n) l_{m-n}.$$

Proof For any $l_m \in I_L$ we have, (3.11):

$$\Delta(l_m) = \sum_{k=0}^{m} l_k \otimes l_{m-k};$$

from which we deduce:

$$D^-_{l_n}(l_m) = \sum_{k=0}^{m} \langle Z_{l_n}, t_k \rangle l_{m-k}.$$

The formula is evident. ♠

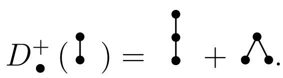
Definition 47 *Let us call $\mathcal{D}^-_L$, the vector space of the elimination operators, whose elements are $D^-_{l_n}$. Moreover, for each l_n, let us indicate with D^-_n the operator $D^-_{l_n}$.*

Lemma 7 *The restriction of $\mathcal{D}^-_L$ to the vector space I_L, is a commutative sub Lie algebra of $\mathcal{D}^-$.*

Proof This follows by direct inspection of the commutator between two such operators, or by using proposition 22 and by observing that $Z_{l_n} \rightsquigarrow D^-_n$ is an (anti) isomorphism of Lie algebras. ♠

Let us consider the vector space of the insertions operators $D^+_{l_n}$, with l_n ladder tree. Let us call this space $\mathcal{D}^+_L$. It is clear that in general, the insertion operator $D^+_{l_n}$ is not a linear map between I_L and itself:

Example 35



To define a class of insertion operators which maps I_L into itself we need to modify the gluing operation. For a general tree t, the only distinguished vertex is its root, $*(t)$. On the other hand, for ladder trees we also have another distinguished vertex, which is the one opposite to the root (remember that trees are supposed to be oriented). Therefore, it makes sense to define for the ladder trees the following gluing operation:

$$l_n \cup^L_{v_n} l_m = l_{m+n};$$

The wordy definition of this operation is as follows:
given the tree l_n whose vertex opposite to its root is v_n, and a ladder tree l_m, we get the ladder tree l_{m+n} by gluing the root of l_m to the vertex v_n of l_n via a new edge.

Example 36 *1)* $l_1 = \bullet$ *and* $l_2 =$.

$$l_1 \cup^L_{v_1} l_2 =$$.

2) $l_1 =$ *and* $l_2 = \bullet$.

$$l_1 \cup^L_{v_2} l_2 =$$.

For each ladder tree l_n we can now define the following insertion operator: D^+_n which will act on I_L as follows:

$$D^+_n(l_m) = l_m \cup^L_{v_m} l_n.$$

Lemma 8 *For each pair of ladder trees l_n, l_m,*

$$\left[D^+_n, D^+_m\right] = 0.$$

Proof The statement follows directly from the definition. ♠

We will then extend such a class of linear operators to a class of derivations of the Hopf algebra $\mathcal{H}_L$ in the obvious way. The following proposition summarizes what we have discussed in the present section:

Proposition 26 *From ladder Hopf algebra $\mathcal{H}_L$ we can define two classes of derivations, $\mathcal{D}^+_L$ and $\mathcal{D}^-_L$, whose elements represent the (ladder) insertion and respectively, elimination operators. Moreover, they are both commutative Lie algebras of derivations, for the $\mathcal{H}_L$.*

Remark 21 *In the ladder case, the composition between insertion operators and between elimination operators is associative, as it follows trivially from the definition. In the general case are pre-Lie and not associative.*

Following what has been done in the previous section, we can now introduce the ladder insertion elimination Lie algebra.

Definition 48 *For each pair of ladder trees l_n, l_m, we define the linear operator $Z_{n,m}$, whose action on the ladder tree l_k is given by the following:*

$$Z_{n,m}(l_k) = \Theta(k-m)l_{k-m+n}$$

We will adopt the following convention: the empty ladder tree, which is the unit in $\mathcal{H}_L \subset \mathcal{H}_{rt}$ will be denoted with l_0. In particular we have the following identifications:

$$Z_{n,0} = D_n^+ \text{ and } Z_{0,n} = D_n^-; \forall \ n \in \mathbb{Z}_{>0}.$$

The operator $Z_{0,0}$ will coincide with the identity in $\mathcal{H}_L$.

We extend the linear map $Z_{n,m}$, to a derivation of $\mathcal{H}_L$ in the obvious way. Let us now make a couple of preliminary observations.

Lemma 9 *The composition between insertion elimination operators is not commutative;*

$$\left[Z_{n,m}, Z_{l,s}\right] \neq 0.$$

Proof

$$Z_{n,m} \circ Z_{l,s}(l_k) = \Theta(k-s)Z_{n,m}(l_{k-s+l}) = \Theta(k-s+l-m)\Theta(k-s)l_{k-s+l-m+n}, \tag{5.1}$$

and similarly

$$Z_{l,s} \circ Z_{n,m}(l_k) = \Theta(k-m)Z_{l,s}(l_{k-m+n}) = \Theta(k-m+n-s)\Theta(k-m)l_{k-m+n-s+l}. \tag{5.2}$$

For general quadruples, the last terms of the previous equalities are in general different; it is enough that $k < s$, and $k \geq m$ and $s \leq (n-m+k)$. ♠

The second observation is contained in the following lemma:

Lemma 10 *The composition between the ladder insertion elimination operators is associative:*

$$Z_{r,t} \circ \left(Z_{n,m} \circ Z_{l,s}\right)(l_k) = \left(Z_{r,t} \circ Z_{n,m}\right) \circ Z_{l,s}(t_k),$$

for each $n, m, l, s, r, t, k \in \mathbb{Z}_{\geq}0$.

Proof The proof is similar to the one of the previous lemma. ♠

Let us give a close formula for the commutator between two insertion elimination operators:

Theorem 22 *For any pair of derivations $Z_{n,m}, Z_{l,s}$, we have that:*

$$\begin{aligned}\big[Z_{n,m}, Z_{l,s}\big] &= \Theta(l-m)Z_{l-m+n,s} - \Theta(s-n)Z_{l,s-n+m} \\ &\quad -\Theta(n-s)Z_{n-s+l,m} + \Theta(m-l)Z_{n,m-l+s} \\ &\quad -\delta_{m,l}Z_{n,s} + \delta_{n,s}Z_{l,m}, \end{aligned} \tag{5.3}$$

where:

$$\begin{cases} \Theta(l-m) & = 0 \ \text{ if } l < m, \\ \Theta(l-m) & = 1 \ \text{ if } l \geq m \end{cases} \tag{5.4}$$

and where $\delta_{n,m}$ is the usual Kronecker delta:

$$\begin{cases} \delta_{n,m} & = 1 \ \text{ if } m = n, \\ \delta_{n,m} & = 0 \ \text{ if } n \neq m. \end{cases} \tag{5.5}$$

Proof Let us first consider the equation (5.1). For this equation we have two cases:

1) $m > l$, from which it follows $s - l + m > s$. This implies $\Theta\big(k - (s - l + m)\big)\Theta(k - s) = \Theta\big(k - (s - l + m)\big) = \Theta(m - l)\Theta\big(k - (s - l + m)\big)$. This is equivalent to write:

$$Z_{n,m} \circ Z_{l,s} = \Theta(m-l)Z_{n,s-l+m}.$$

2) $l > m$, from which it follows $s - l + m < s$. Under this assumption we can write $\Theta\big(k - (s - l + m)\big)\Theta(k - s) = \Theta(k - s) = \Theta(l - m)\Theta(k - s)$. Then:

$$Z_{n,m} \circ Z_{l,s} = \Theta(l-m)Z_{n-m+l,s}.$$

The case $l = m$ will get a non zero contribution from 1) and 2) equal to $Z_{n,s}$, this explain the presence of the term $-\delta_{m,l}Z_{n,s}$ in equation (5.3). The term $Z_{l,s} \circ Z_{n,m}$ can be treated in a completely analogous fashion. ♠

Corollary 7 *The commutator defined in (5.3) fulfills the Jacobi identity.*

Proof The statement is a consequence of the theorem 22 and lemma 10. ♠

Definition 49 *We will call ladder insertion elimination Lie algebra, $\mathcal{L}_L$, the Lie algebra generated by the symbols $Z_{n,m}$ with $n, m \in \mathbb{Z}_{geq0}$, endowed with the bracket defined in (5.3).*

Remark 22 *Note that we could have deduced the formula (5.3) from the general formula (4.3) where, instead of the general insertion operation, we should have used the one defined for the ladder rooted trees. From this remark, we deduce that we cannot think of $\mathcal{L}_L$ as sub Lie algebra of $\mathcal{L}$.*

We can now state the first properties of the Lie algebra $\mathcal{L}_L$:

Corollary 8 1) *$\mathcal{L}_L$ is $\mathbb{Z}$-graded Lie algebra:*

$$\mathcal{L}_L = \oplus_{i\in\mathbb{Z}} l_i$$

where each for each $Z_{n,m} \in l_i$, $deg(Z_{n,m}) = i = n - m$ and $dim_{\mathbb{C}}\ l_i = +\infty$;
2) *$\mathcal{L}_L$ has the following decomposition:*

$$\mathcal{L}_L = L^+ \oplus L^0 \oplus L^-;$$

where $L^+ = \oplus_{n>0} l_n$, $L^- = \oplus_{n<0} l_n$ and $L^0 = l_0$.

Proof The proof follows from the very definition of a graded Lie algebra and from the formula (5.3). We recall here the definition of graded Lie algebra: a Lie algebra $\mathfrak{g}$ is G-graded (where G is any abelian group) if $\mathfrak{g} = \oplus_{i\in G}\mathfrak{g}_i$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. ♠

We conclude this section with the following proposition:

Proposition 27 *Each element $Z_{n,m} \in \mathcal{L}_L$ can be written in the following form:*

$$Z_{n,m} = [Z_{n,0}, Z_{0,m}] + \Theta(n-m) Z_{n-m,0} + \Theta(m-n) Z_{0,m-n} - \delta_{n-m,0} Z_{0,0}. \quad (5.6)$$

Proof The statement follows trivially applying formula (5.3) to the elements $Z_{n,0}$ and $Z_{0,m}$ for $n > m$, $n < m$ and $n = m$. ♠

Remark 23 *The previous proposition is equivalent to the following* (*vector space*) *decomposition of the Lie algebra* $\mathcal{L}_L$*:*

$$\mathcal{L}_L = [\mathcal{D}, \mathcal{D}] \oplus \mathcal{D};$$

where we define:

$$\mathcal{D} = \mathfrak{a}_+ \oplus \mathfrak{a}_- \oplus \mathbb{C}$$

and $\mathfrak{a}_+ = span_{\mathbb{C}}\langle Z_{n,0} : n > 0\rangle$, $\mathfrak{a}_- = span_{\mathbb{C}}\langle Z_{0,n} : n > 0\rangle$ *and* $\mathbb{C}$ *is the trivial Lie algebra generated by* $Z_{0,0}$. *In fact* $\mathfrak{a}_+$ *and* $\mathfrak{a}_-$ *are commutative sub algebras of* $\mathcal{L}_L$ *and* $Z_{0,0}$ *is a central element.*

5.2 The structure of the Lie algebra $\mathcal{L}_L$

We begin with two statements whose proofs are collected at the end of this section.

Theorem 23 *The center of the Lie algebra* $\mathcal{L}_L$ *has dimension one and it is generated by the element* $Z_{0,0}$.

Theorem 24 l^0 *is a maximal abelian sub-algebra of* $\mathcal{L}_L$.

In what follows, we will show that the Lie algebra $\mathcal{L}_L$ is not simple. Let us introduce the following:

Definition 50

$$\mathfrak{gl}_+(\infty) = span_{\mathbb{C}}\langle E_{i,j} : Z_{i,j} - Z_{i+1,j+1} | \; i, j \in \mathbb{Z}_{\geq 0}\rangle.$$

We have:

Proposition 28 1) $[E_{i,j}, E_{r,k}] = E_{i,k}\delta_{j,r} - E_{r,j}\delta_{k,i}$;
2) $\mathfrak{gl}_+(\infty)$ *is an ideal in* $\mathcal{L}_L$.

Proof The proof of 1) and 2) is a simple but tedious application of the commutator formula (5.3). ♠

We can now define the quotient Lie algebra:

$$C = \mathcal{L}_L/\mathfrak{g}l_+(\infty)$$

and consider the exact sequence:

$$0 \longrightarrow \mathfrak{g}l_+(\infty) \longrightarrow \mathcal{L}_L \xrightarrow{\pi} C \longrightarrow 0. \tag{5.7}$$

It is now clear that to have a better understanding of the Lie algebra $\mathcal{L}_L$ we need to study carefully the structure of the Lie algebra C. The crucial ingredient will be the following proposition:

Proposition 29

$$\mathfrak{g}l_+(\infty) = [\mathcal{L}_L, \mathcal{L}_L].$$

Proof Let us prove the two inclusions.
$\mathfrak{g}l_+(\infty) \subset [\mathcal{L}_L, \mathcal{L}_L]$ since from the definition of $\mathfrak{g}l_+(\infty)$:

$$E_{i,j} = Z_{i,j} - Z_{i+1,j+1} = [Z_{i,0}, Z_{0,j}] - [Z_{i+1,0}, Z_{0,j+1}],$$

where the second equality follows from the formula (5.6).
To show the other inclusion, i.e $[\mathcal{L}_L, \mathcal{L}_L] \subset \mathfrak{g}l_+(\infty)$, it suffices to observe that for any two generators, say $Z_{h,p}$ and $Z_{r,q}$, of $\mathcal{L}_L$ we have that their commutator is given by the difference between two elements having same degree (see formula (5.3)), say $Z_{n,m}$ and $Z_{l,s}$, such that $n - m = l - s$.

Under the hypothesis that $k = n - m = l - s > 0$ and that $s > m$ (the other cases are completely analogous), we can write their difference as follows:

$$\begin{aligned} Z_{n,m} - Z_{l,s} &= Z_{m+k,m} - Z_{s+k,s} = Z_{m+k,m} - Z_{m+k+1,m+1} + \\ &\quad + Z_{m+k+1,m+1} - .... - Z_{s+k-1,s-1} + Z_{s+k-1,s-1} - Z_{s+k,s}, \end{aligned}$$

which expresses the difference between $Z_{n,m}$ and $Z_{l,s}$ as finite linear combination of elements in $\mathfrak{g}l_+(\infty)$. ♠

In particular, we can rephrase the previous proposition as follows:

Lemma 11 *Two generators $Z_{n,m}$ and $Z_{l,s}$ are $\mathfrak{g}l_+(\infty)$-equivalent if and only if they have the same degree, i.e:*

$$Z_{n,m} \sim Z_{l,s} \iff deg(Z_{m,n}) = deg(Z_{l,s}).$$

Proof Using the same argument we used to prove proposition 29, we can conclude that if $Z_{n,m}$ and $Z_{l,s}$ have the same degree, then they are equivalent.
Suppose now that the difference between $Z_{n,m}$ and $Z_{l,s}$ can be written as a (finite) linear combination of elements in $\mathfrak{g}l_+(\infty)$ and also that $n-m \neq l-s$ (w.l.o.g. we can assume that $n-m > 0$ and that $l-s > 0$).
Under these assumptions, and from formula (5.6), it follows also that: $Z_{n-m,0} - Z_{l-s,0} = \sum_{finite} a_i E_{p_i,q_i}$. But this has as a consequence that each of these two elements are finite linear combinations of (homogeneous) elements in $\mathfrak{g}l_+(\infty)$. Accordingly we can write: $Z_{n-m,0} = \sum_{finite} c_i E_{r_i,k_i}$ and $Z_{l-s,0} = \sum_{finite} c_i E_{t_i,v_i}$. Rewriting the right hand side of each of those two equalities in terms of the generators $Z_{n,m}$, it follows that such equations can not hold. ♠

From the proposition 29 it follows that C is a (maximal) commutative Lie algebra coming from a quotient of $\mathcal{L}_L$.
Let us now introduce a set of (natural) generators for C. Since the set

$$\langle Z_{n,m} |\ n, m \in \mathbb{Z}_{\geq 0} \rangle$$

is a basis for $\mathcal{L}_L$ and since $\pi : \mathcal{L}_L \longrightarrow C$ is a surjection, it follows that:

$$\langle \overline{Z}_{n,m} = \pi(Z_{n,m}) |\ n, m \in \mathbb{Z}_{\geq 0} \rangle$$

is a set of generators for C. Moreover, from the lemma 11 it follows that when $n > m$, then $Z_{n,m} \sim Z_{n-m,0}$, when $m > n$, then $Z_{n,m} \sim Z_{0,m-n}$ and, finally, when $n = m$, then $Z_{n,m} \sim Z_{0,0}$.
So defining $Z_n = \overline{Z}_{n,0}$, $Z_{-n} = \overline{Z}_{0,n}$ for $n > 0$ and $Z_0 = \overline{Z}_{0,0}$, we get

$$C = span_{\mathbb{C}} \langle Z_n | n \in \mathbb{Z} \rangle.$$

The fact that such elements are also linearly independent (i.e they form a basis for C) follows easily from lemma 11. We now want to look more closely at the exact sequence (5.7). In particular, we will prove the following result:

Theorem 25 *The exact sequence (5.7) does not split, i.e the Lie algebra $\mathcal{L}_L$ is not the semi-direct product of the Lie algebra $\mathfrak{g}l_+(\infty)$ with the (commutative) Lie algebra C.*

From the exact sequence (5.7) and from what we explained above, we can conclude that the Lie algebra $\mathcal{L}_L$ is a *non-abelian* extension of the commutative Lie algebra C by the Lie algebra $\mathfrak{gl}_+(\infty)$. Let us explain with some care the meaning of such a statement (for more details we refer to the chapter 6 of the present work, where we collect a general overview of this subject).
For time being we will state what is needed for the applications to the Lie algebra $\mathcal{L}_L$. Let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{e}$ be Lie algebras.

Definition 51 *We will say that the Lie algebra $\mathfrak{e}$ is an extension of the Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{h}$, if $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{e}$ fit into the following exact sequence:*

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0, \tag{5.8}$$

Moreover, we will say that two such extensions, $\mathfrak{e}$ and $\mathfrak{e}'$, are equivalent, if and only if $\mathfrak{e}$ and $\mathfrak{e}'$ are isomorphic as Lie algebras.

Let $Der(\mathfrak{h})$ be the Lie algebra of derivations of $\mathfrak{h}$, $\alpha', \alpha \in \mathrm{H}om_{\mathbb{C}}(\mathfrak{g}, \mathrm{D}er(\mathfrak{h}))$ and $\rho', \rho \in \mathrm{H}om_{\mathbb{C}}(\Lambda^2\mathfrak{g}, \mathfrak{h})$. On the set of the couples (α, ρ) introduced above, we define the following equivalence relation:

$$(\alpha, \rho) \sim (\alpha', \rho') \iff \exists\, b \in Hom_{\mathbb{C}}(\mathfrak{g}, \mathfrak{h})$$

such that:

$$\alpha'(x).\xi = \alpha(x).\xi + [b(x), \xi]_{\mathfrak{h}},$$
$$\rho'(x, y) = \rho(x, y) + \alpha(x).b(y) - \alpha(y).b(x) - b([x, y]_{\mathfrak{g}}) + [b(x), b(y)]_{\mathfrak{h}}.$$

Then, we have that:

Theorem 26 1) *The classes of isomorphism of the extensions of the Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{h}$ given by the exact sequence (5.8), are in one-to-one correspondence with the equivalence classes $[(\alpha, \rho)]$, such that:*

$$[\alpha(x), \alpha(y)]_{Der(\mathfrak{h})}.\xi - \alpha([x, y]_{\mathfrak{g}}).\xi = [\rho(x \wedge y), \xi]_{\mathfrak{h}};$$
$$\sum_{\text{cyclic}} \big(\alpha(x).\rho(y, z) - \rho([x, y]_{\mathfrak{g}}, z)\big) = 0;$$

for every $x, y, z \in \mathfrak{g}$ and $\xi \in \mathfrak{h}$.
2) *The Lie algebra structure induced by the datum (α, ρ), on the vector space $\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{g}$, is given by:*

$$[(\xi_1, x_1), (\xi_2, x_2)]_{\mathfrak{e}} = ([\xi_1, \xi_2]_{\mathfrak{h}} + \alpha(x_1).\xi_2 - \alpha(x_2).\xi_1 + \rho(x_1, x_2), [x_1, x_2]_{\mathfrak{g}}). \tag{5.9}$$

We apply this result to our setting, where $\mathfrak{g} = C$ and $\mathfrak{h} = \mathfrak{g}l_+(\infty)$.
The exact sequence (5.7) tells us that we have:

$$\mathcal{L}_L \simeq \mathfrak{g}l_+(\infty) \oplus C,$$

where such a splitting holds in the category of vector spaces. We first prove that:

Proposition 30 *The Lie algebra structure on $\mathcal{L}_L$, given by the bracket* (5.3)*, corresponds to the couple (α, ρ) defined by:*

$$\alpha(Z_n).(E_{i,j}) = \Theta(n) \sum_{k\geq 0} (E_{n+k,j}\delta_{i,k} - E_{i,k}\delta_{n+k,j}) +$$

$$\Theta(-n) \sum_{k\geq 0} (E_{k,j}\delta_{k+n,i} - E_{i,k+n}\delta_{j,k})$$

for $n \neq 0$ and

$$\alpha(Z_0) \equiv 0;$$

while:

$$\rho(Z_n, Z_m) = 0$$

if $n, m \geq 0$ or $n, m \leq 0$ and

$$\rho(Z_n, Z_{-m}) = \sum_{k=0}^{m-1} E_{n-m+k,k}$$

if $n > m$, and:

$$\rho(Z_n, Z_{-m}) = \sum_{k=0}^{m-1} E_{k,m-n+k},$$

if $n < m$.

Proof The proof follows comparing formula (5.3) with formula (5.9). ♠

We now remark that:

Lemma 12 *Given:*

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0, \tag{5.10}$$

as in (5.8), any splitting $s : \mathfrak{g} \longrightarrow \mathfrak{e}$ (at the vector space level) of the previous exact sequence, induces a map $\alpha_s \in Hom_{\mathbb{C}}\big(\mathfrak{g}, Der(\mathfrak{h})\big)$, via the following:

$$\alpha_s(X).\xi = [s(X), \xi];$$

for each $X \in \mathfrak{g}$ and each $\xi \in \mathfrak{h}$.

Proposition 31 *The map $\alpha \in Hom_{\mathbb{C}}\big(C, Der(\mathfrak{g}l_+(\infty))\big)$, defined in proposition* 30*, is induced by the linear map $s : C \longrightarrow \mathcal{L}_L$, which is defined as follows:*

$$s(Z_n) = \Theta(n)Z_{n,0} + \Theta(-n)Z_{0,n} - \delta_{n,0}Z_{0,0}. \tag{5.11}$$

Proof The map s defined in formula (5.11) is a section of the projection $\pi : \mathcal{L}_L \longrightarrow C$ defined by the exact sequence (5.7). In other words, $s \in Hom_{\mathbb{C}}(C, \mathcal{L}_L)$ such that $s \circ \pi = Id_C$. From the lemma (12) we know that such a section s induces a linear map:

$$\alpha_s : C \longrightarrow Der(\mathfrak{g}l_+(\infty)),$$

defined by:

$$\alpha_s(x).\xi = [s(x), \xi]_{\mathcal{L}_L}.$$

It is now easy to check that this map is the same as the one defined in the proposition 30. ♠

We are now almost ready to prove theorem 25. We only need to remark the following. From theorem 26 we have that a given extension (α, ρ) of the Lie algebra C by the Lie algebra $\mathfrak{g}l_+(\infty)$ will split, i.e will be equivalent to a semi-direct product of the these two Lie algebras, if and only if $(\alpha, \rho) \sim (\alpha', 0)$, or, in other words, if and only if α' is a morphism of Lie algebras. Theorem 26 tells us that this is equivalent to ask for the existence of a linear map $b : C \longrightarrow \mathfrak{g}l_+(\infty)$, such that $s + b : C \longrightarrow \mathcal{L}_L$ is a morphism of Lie algebras. Moreover, since we are working with the category of graded Lie algebras, the map b has to be grade preserving. In conclusion, to prove theorem 25, we are left to show that such a map b does not exist.

Proof(Theorem 25) Suppose we can define a linear map $b : C \longrightarrow \mathfrak{gl}_+(\infty)$ such that $s + b : C \longrightarrow \mathcal{L}_L$ is a morphism of (graded) Lie algebras. That means that we can find elements $\sum_{i=1}^{M} a_{h_i} E_{h_i+1,h_i} \in \mathfrak{gl}_+(\infty)$ and $\sum_{i=1}^{N} b_{k_j} E_{k_j,k_j+1} \in \mathfrak{gl}_+(\infty)$ such that $b(Z_1) = \sum_{i=1}^{M} a_{h_i} E_{h_i+1,h_i}$, $b(Z_{-1}) = \sum_{i=1}^{N} b_{k_j} E_{k_j,k_j+1}$ and furthermore

$$0 = [(s+b)(Z_1), (s+b)Z_{-1}] = [Z_{1,0} + \sum_{i=1}^{M} a_{h_i} E_{h_i+1,h_i}, Z_{0,1} + \sum_{j=1}^{N} b_{k_j} E_{k_j,k_j+1}].$$

We can calculate such a commutator by re-writing each of the terms $E_{i,j}$ in the sums in terms of the generators $Z_{n,m}$, and applying to such terms the brackets given in formula (5.3). The result, written in terms of the generators $E_{i,j}$, takes the form:

$$-E_{0,0} + \sum_{j=1}^{N} b_{k_j}(E_{k_j+1,k_j+1} - E_{k_j,k_j}) + \sum_{i=1}^{M} a_{h_i}(1 + b_{h_i})(E_{h_i+1,h_i+1} - E_{h_i,h_i}) = 0.$$

The right hand side of the previous sum can be reorganized in term of the summands $E_{j+1,j+1} - E_{j,j}$ as follows:

$$-\sum_{i\geq 0}^{L} \phi_j(E_{j+1,j+1} - E_{j,j}),$$

where L is the biggest between N and M and the ϕ_j's are coefficients.
Then we have that:

$$E_{0,0} = \sum_{i\geq 0}^{L} \phi_j(E_{j+1,j+1} - E_{j,j}) = -\phi_0 E_{0,0} + \sum_{j\geq 0}(\phi_{j+1} - \phi_j)E_{j,j} + \phi_L E_{L+1,L+1},$$

that clearly give us a contradiction. ♠

Proof of Theorems 23 and 24

In this subsection we give the proofs of the theorems 23 and 24.

We observe that the Lie algebra $\mathcal{L}_L$ has an obvious module:

Definition 52

$$\mathcal{S} = \bigoplus_{n\geq 0} \mathbb{C}t_n = \mathbb{C}[t_0, t_1, t_2, t_3.....].$$

We will assign a degree equal to k to the generator t_k for each $k \geq 0$. $\mathcal{L}_L$ acts on $\mathcal{S}$ via the following:

$$\begin{aligned} Z_{n,m}t_k &= 0 \;\; if \;\; m > k, \\ Z_{n,m}t_k &= t_{k-m+n} \;\; if \;\; m \leq k. \end{aligned} \tag{5.12}$$

In what follows we will indicate by $\mathcal{Z}(\mathcal{L}_L)$ the center of the Lie algebra $\mathcal{L}_L$.

Proof(Theorem 23). It is obvious that $\mathbb{C}Z_{0,0} \subset \mathcal{Z}(\mathcal{L}_L)$. Let us prove the other inclusion. Let us suppose that there is some element $\alpha \in \mathcal{L}_L$, not proportional to $Z_{0,0}$ and that belongs to the center of $\mathcal{L}_L$. W.l.o.g. we assume

$$\alpha = \sum_{i=1}^{k} a_i Z_{n_i,m_i} = \sum_{i:\, n_i,m_i \neq 0} b_i Z_{n_i,m_i} + \sum_{i:\, \tilde{n}_i \neq 0} c_i Z_{\tilde{n}_i,0} + \sum_{i:\; \tilde{m}_i \neq 0} d_i Z_{0,\tilde{m}_i}, \tag{5.13}$$

where all the $\tilde{n}_i$'s ($\tilde{m}_i$'s) are different from 0 and $\tilde{n}_i \neq \tilde{n}_j$ ($\tilde{m}_i \neq \tilde{m}_j$), if $i \neq j$, and $(n_i, m_i) \neq (n_j, m_j)$, if $i \neq j$.
We will prove that α, defined above, is equal to zero by showing that the coefficients b_i, c_i and d_i are all equal to zero. We will split the proof of this assertion into two lemmas.

Lemma 13 *If $\alpha \in \mathcal{Z}(\mathcal{L}_L)$, where α is defined as above, then $b_i = d_i = 0$ for each i.*

Proof Let us consider some element $Z_{n,0} \in \mathcal{L}_L$ such that $0 < n \leq min_i\{m_i, \tilde{m}_i\}$. Then using formula (5.3), we get:

$$[Z_{n,0}, \alpha] = \sum_i b_i [Z_{n,0}, Z_{n_i,m_i}] + \sum_i d_i [Z_{n,0}, Z_{0,\tilde{m}_i}] =$$

$$\sum_i b_i (Z_{n_i+n,m_i} - Z_{n_i,m_i-n}) + \sum_i d_i (Z_{n,\tilde{m}_i} - Z_{0,\tilde{m}_i-n}).$$

Note the all the $\tilde{m}_i$'s are different (and different from 0), while in the set of the m_i's (which are also all different from 0) we can have repetitions.

Let us now define the set $M \doteq \{m_1, ..., m_k, \tilde{m}_1, ..., \tilde{m}_r\}$, and let us consider the disjoint union:

$$M = M_1 \cup \ldots \cup M_s.$$

Each M_i corresponds to the set of all indices in M which are equal to some given index l_i, say. We remark once more that for each i $M_i \cap \{\tilde{m}_1, \cdots, \tilde{m}_r\}$ contains at most one element, since in the set $\{\tilde{m}_1, ..., \tilde{m}_r\}$ we do not have repetitions.
Let us now consider $p_1 = l_1 - n$ (which is ≥ 0, as a consequence of the condition we imposed on n), and let us also consider the corresponding element $t_{p_1} \in \mathcal{S}$. Since α belongs to $\mathcal{Z}(\mathcal{L}_L)$, and since $n > 0$, we have:

$$0 = [Z_{n,0}, \alpha](t_{p_1}) = -\Big(\sum_{i:\, m_i \in M_1} b_i t_{p_1 - m_i + n + n_i} + \sum_{i:\, \tilde{m}_i \in M_1} d_i t_{p_1 - \tilde{m}_i + n} \Big). \quad (5.14)$$

Remark 24 *We observe that all the indices in M_1 are equal to l_1 and that $p_1 = l_1 - n$. Moreover the n_i's in the first sum of the right hand side in formula (5.14) are all different (since by assumption we have that $(n_i, m_i) \neq (n_j, m_j)$ unless $i = j$ and in our case all the m_i belong to the class M_1). Finally, we notice that the last sum, if not equal to zero, contains only one term.*

Let us now suppose that $M_1 \cap \{m_1, \cdots, m_k\}$ and $M_1 \cap \{\tilde{m}_1, \cdots, \tilde{m}_r\}$ are both not empty (the cases where one, or both, of those intersections are empty, are completely analogous). From the previous remark it follows that:

$$0 = [Z_{n,0}, \alpha](t_{p_1}) = -\big(\sum_{i:\, m_i \in M_1} b_i t_{l_1 - n - l_1 + n + n_i} + \sum_{i:\, \tilde{m}_i \in M_1} d_i t_{l_1 - n - l_1 + n} \big) =$$

$$-\big(\sum_i b_i t_{n_i} + d_1 t_0 \big).$$

Since all the n_i in the first sum are different, we have that $d_1 = 0$ and $b_i = 0$ for each i.
We can apply the same argument to the sets M_2,...,M_s, to show that each of the coefficients b_i and c_i are equal to 0. ♠

From the lemma 13 we conclude that if $\alpha \in \mathcal{Z}(\mathcal{L}_L)$, α defined as in equation (5.13), then:

$$\alpha = \sum_i c_i Z_{n_i,0}.$$

To conclude the proof of the theorem 23, we need to show that:

Lemma 14 *If $\alpha \in \mathcal{Z}(\mathcal{L}_L)$ and $\alpha = \sum_i c_i Z_{n_i,0}$, then $c_i = 0$ for each i.*

Proof We first notice that we can suppose all $n_i \neq 0$ and $n_1 < n_2 \ldots$. Let us now consider some element $Z_{0,n}$, such that $n \geq \max_i\{n_i\}$. Since we suppose $\alpha = \sum_i c_i Z_{n_i,0}$ to be in the center of $\mathcal{L}_L$, we can write:

$$0 = [\alpha, Z_{0,n}] = \sum_i c_i [Z_{n_i,0}, Z_{0,n}] = \sum_i c_i (Z_{n_i,n} - Z_{0,n-n_i}).$$

By the hypothesis on n and the one on the n_i's, we conclude that all the c_i's are equal to zero. ♠

Proof (Theorem 24). Let us suppose that l^0 is not maximal abelian sub-algebra of $\mathcal{L}_L$, i.e that there exists $\mathcal{L}_L \ni \alpha \notin l^0$, $\alpha = \sum_{i=1}^n a_i Z_{n_i,m_i}$, such that:

$$[\alpha, Z_{k,k}] = 0, \ \forall \ k > 0.$$

Without loss of generality we can suppose that in each of (n_i, m_i)'s, $n_i \neq m_i$ (if no, $\alpha = \beta + \sum_i f_i Z_{n_i,n_i}$ and $[\beta, Z_{k,k}] = [\alpha, Z_{k,k}]$).

Such an element can be written as:

$$\alpha = \sum_{i:\, m_i \neq 0,\, n_i \neq 0} b_i Z_{n_i,m_i} + \sum_{i:\, \tilde{n}_i \neq 0} c_i Z_{\tilde{n}_i,0} + \sum_{i:\, \tilde{m}_i \neq 0} d_i Z_{0,\tilde{m}_i}. \tag{5.15}$$

Remark 25 *We note that in formula (5.15) all the n_i's and the m_i's are different from 0 and also that $\tilde{n}_i \neq \tilde{n}_j$ and $\tilde{m}_i \neq \tilde{m}_j$ for each $i \neq j$.*

We will prove that such element is identically equal to zero, showing that each of the coefficients in the equation (5.15) is equal to zero. We will divide the proof of this statement in two lemmas.

Lemma 15 *Given $\alpha \in l^0$, defined as in formula (5.15), we have that $c_i = d_i = 0$ for all i.*

Proof Let us fix integer k, $0 < k \leq \min_i \{n_i, m_i, \tilde{n}_i, \tilde{m}_i\}$. Then we get:

$$[\alpha, Z_{k,k}] = \sum_{i:\, m_i\neq 0, n_i\neq 0} b_i[Z_{n_i,m_i}, Z_{k,k}] + \sum_{i:\, \tilde{n}_i\neq 0} c_i[Z_{\tilde{n}_i,0}, Z_{k,k}] + \sum_{i:\, \tilde{m}_i\neq 0} d_i[Z_{0,\tilde{m}_i}, Z_{k,k}] =$$

$$\sum_{i:\, \tilde{n}_i\neq 0} c_i(Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}) + \sum_{i:\, \tilde{m}_i\neq 0} d_i(Z_{0,\tilde{m}_i} - Z_{k,k+\tilde{m}_i}),$$

since:

$$[Z_{n_i,m_i}, Z_{k,k}] = 0, \ \forall \ \{n_i, m_i\} \text{ such that } n_i \geq k, \ m_i \geq k,$$

$$[Z_{\tilde{n}_i,0}, Z_{k,k}] = Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0} \text{ if } 0 < k \leq \tilde{n}_i, \text{ and}$$

$$[Z_{0,\tilde{m}_i}, Z_{k,k}] = -Z_{k,\tilde{m}_i+k} + Z_{0,\tilde{m}_i} \text{ if } 0 < k \leq \tilde{m}_i.$$

Since α commutes with all the elements of the sub-algebra l^0, we have:

$$0 = \sum_{i:\, \tilde{n}_i\neq 0} c_i(Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}) + \sum_{i:\, \tilde{m}_i\neq 0} d_i(Z_{0,\tilde{m}_i} - Z_{k,k+\tilde{m}_i}).$$

But in the right hand side of the previous formula the first sum contains only elements of positive degree while the second sum contains only those of negative degree, thus the sum is equal to zero if and only if separately

$$\sum_{i:\, \tilde{n}_i\neq 0} c_i(Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}) = 0 \text{ and } \sum_{i:\, \tilde{m}_i\neq 0} d_i(Z_{0,\tilde{m}_i} - Z_{k,k+\tilde{m}_i}) = 0.$$

From this it follows that all c_i's and d_i's are equal to zero. Indeed, consider the sum containing the c_i's (the one contains the d_i's can be treated in the same way):

$$\sum_{i:\, \tilde{n}_i\neq 0} c_i(Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}) = 0.$$

Since $k \neq 0$ and since $\tilde{n}_i \neq \tilde{n}_j$ if $i \neq j$, all the elements $Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}$ are linearly independent. ♠

Summarizing, so far we have proved that if a given element α commutes with each of the elements in l^0, then:

$$\alpha = \sum_{i:\, n_i\neq 0, m_i\neq 0} b_i Z_{n_i,m_i}. \tag{5.16}$$

Lemma 16 *If* $\left[\alpha, l^0\right] = 0$*, with* α *defined as in (5.16), then all the* b_i*'s are equal to* 0*.*

Proof Let us decompose the element α in term of elements of positive and negative degree, i.e:

$$\alpha = \sum_i a_i Z_{n_i,m_i} = \sum_j \big(\sum_{i\geq 0} b_i Z_{r_i+s_j,r_i}\big) + \sum_j \big(\sum_{i\geq 0} c_i Z_{p_i,p_i+t_j}\big).$$

Remark 26 *We remark that in* α *elements of the same (negative or positive) degree could be present; as an example of such element (of positive degree) we can consider:*

$$\beta_j = \sum_i b_i Z_{r_i+s_j,r_i}, \textit{ for a given } j$$

or the element (of negative degree):

$$\gamma_j = \sum_i c_i Z_{p_i,p_i+t_j}, \textit{ for a given } j \ .$$

From the previous remark let us re-write α as:

$$\alpha = \sum_j \beta_j + \sum_j \gamma_j,$$

each $\beta_j \in L^+$ and each $\gamma_j \in L^-$.

Let us now consider some element $Z_{k,k} \in l^0$ and let us take the commutator of such element with α

$$[\alpha, Z_{k,k}] = \sum_j [\beta_j, Z_{k,k}] + \sum_j [\gamma_j, Z_{k,k}].$$

Since $\mathcal{L}_L$ is a graded Lie algebra and since deg $Z_{k,k} = 0$, we have that

$$\text{deg } [\beta_j, Z_{k,k}] = s_j \ , \ \forall j$$

and similarly

$$\text{deg } [\gamma_j, Z_{k,k}] = -t_j \ , \ \forall j.$$

Hence

$$[\alpha, Z_{k,k}] = 0 \iff [\beta_j, Z_{k,k}] = 0 \text{ and } [\gamma_j, Z_{k,k}] = 0 \ , \ \forall j.$$

We are left to prove that any homogeneous element commuting with all the elements in l^0 can not exist.

So, to fix ideas, let us now consider some element of positive degree s, say, $\beta = \sum_{i=1}^{l} a_i Z_{n_i+s,n_i}$ and let us suppose that

$$[\beta, Z_{k,k}] = 0 \ \forall\ k \geq 1. \tag{5.17}$$

Without loss of generality we can further assume that $0 < n_1 < n_2 < ... < n_k$ (that β fulfills the hypothesis is constrained by the assumptions given for the element α defined in formula (5.16), which translates for β into the condition $n_i \neq 0$). To conclude, it suffices to show that each of the a_i's of $\beta = \sum_{i=1}^{l} a_i Z_{n_i+s,n_i}$ is equal to 0. So let us consider $k = n_2$ in formula (5.17). Applying the formula (5.3) to this case, we get:

$$[\beta, Z_{k,k}] = a_1[Z_{n_1+s,n_1}, Z_{n_2,n_2}] + \sum_{i\geq 2} a_i[Z_{n_i+s,n_i}, Z_{n_2,n_2}] =$$

$$= a_1\big(Z_{n_2+s,n_2} - \Theta(n_2-n_1-s)Z_{n_2,n_2-s} - \Theta(n_1+s-n_2)Z_{n_1+s,n_1} + \delta_{n_1+s,n_2}Z_{n_2,n_1}\big),$$

since $\sum_{i\geq 2} a_i[Z_{n_i+s,n_i}, Z_{n_2,n_2}] = 0$.
By the previous formula and the hypothesis for the n_i's, we conclude that $[\beta, Z_{n_2,n_2}] = 0 \iff a_1 = 0$. Taking $k = n_3, n_4, ...$, and using the same argument, we can conclude that each of the a_i's is equal to zero. ♠

5.3 Cohomology of $\mathcal{L}_L$

The main result of this section is contained in the following theorem:

Theorem 27 *The cohomology groups $H^i(\mathcal{L}_L)$, i=1,2, are infinite dimensional. In particular, the Lie algebra $\mathcal{L}_L$ has infinite many non equivalent central extensions.*

In order to prove this statement, we need to use some results about the cohomology of the Lie algebra $\mathfrak{gl}(n)$ and of its infinite dimensional analogous $\mathfrak{gl}_+(\infty)$. The results we need are stated below, their will proofs are collected in appendix 2. For general information about cohomology of Lie algebras we refer to appendix 1.
For any given integer $n \geq 0$ we have defined the standard inclusion map:

$$i_n : \mathfrak{gl}(n) \hookrightarrow \mathfrak{gl}(n+1); \tag{5.18}$$

where by definition $\mathfrak{gl}(0) = (0) \hookrightarrow \mathfrak{gl}(1) \simeq \mathbb{C}$.
Let us start with the following classic result:

Theorem 28 *1). The cohomology ring of the Lie algebra $\mathfrak{gl}(n)$ is an exterior algebra in n generators of degree $1, 3, ..., 2n-1$:*

$$H^{\bullet}(\mathfrak{gl}(n)) = \Lambda[c_1, c_3, ..., c_{2n-1}];$$

2) for any given n, the (inclusion) map defined in formula (5.18) induces a map i_n^* in cohomology:

$$i_n^* : H^{\bullet}(\mathfrak{gl}(n+1)) \longrightarrow H^{\bullet}(\mathfrak{gl}(n)),$$

such that:

$$i_n^* : H^p(\mathfrak{gl}(n+1)) \longrightarrow H^p(\mathfrak{gl}(n))$$

is an isomorphism for $p \leq 2n-1$, and it maps to zero the top degree generator when $p = 2n+1$;

Now we need to relate the Lie algebra $\mathfrak{gl}_+(\infty)$, with the Lie algebra of the general linear group $\mathfrak{gl}(n)$.

Lemma 17 *The data $(\mathfrak{gl}(n), i_n)$, with i_n defined in (5.18), define a direct system of Lie algebras:*

$$\cdots \longrightarrow \mathfrak{gl}(n-1) \xrightarrow{i_{n-1}} \mathfrak{gl}(n) \xrightarrow{i_n} \mathfrak{gl}(n+1) \longrightarrow \cdots$$

Then:

$$\varinjlim \mathfrak{gl}(n) \simeq \mathfrak{gl}_+(\infty).$$

Proof The proof follows immediately from the definition 50. ♠

We can now state the following result about the cohomology of the Lie algebra $\mathfrak{gl}_+(\infty)$:

Corollary 9 *The cohomology ring of the Lie algebra $\mathfrak{gl}_+(\infty)$ is a (infinitely generated) exterior algebra having generators only in odd degree:*

$$H^{\bullet}(\mathfrak{gl}_+(\infty)) = \Lambda[c_1, c_3, ...].$$

We are now ready to prove the theorem 27:

Proof The following exact sequence follows from the Hochschild-Serre spectral sequence:

$$0 \longrightarrow H^1(C) \xrightarrow{i} H^1(\mathcal{L}_L) \xrightarrow{r} H^1(\mathfrak{gl}_+(\infty))^C \xrightarrow{t} H^2(C) \xrightarrow{i} H^2(\mathcal{L}_L) \longrightarrow H^2(\mathfrak{gl}_+(\infty).$$

Note that C is infinite dimensional vector space, so that $H^i(C) \simeq C$ for each i. From the knowledge of the cohomology of $\mathfrak{gl}_+(\infty)$ and from the previous exact sequence, the thesis follows. The statement about the central extensions is a consequence of the fact that those are in one to one correspondence with the elements of the group $H^2(\mathcal{L}_L)$, see section 7.2 proposition 37. ♠

We can add one more piece of information:

Proposition 32

$$H^1(\mathfrak{gl}_+(\infty))^C \simeq \mathbb{C}.$$

Proof Let us start by observing that

$$H^1(\mathfrak{gl}_+(\infty)) \simeq \big(\mathfrak{gl}_+(\infty)/[\mathfrak{gl}_+(\infty), \mathfrak{gl}_+(\infty)]\big)' \simeq \mathbb{C},$$

and identifying $[\mathfrak{gl}_+(\infty), \mathfrak{gl}_+(\infty)]$ with $\mathfrak{sl}_+(\infty)$, i.e with the Lie algebra of infinite matrices of finite rank, having trace equal to zero.
In particular, this implies that the only non trivial class $[\phi] \in H^1(\mathfrak{gl}_+(\infty))$ corresponds to a (closed) cochain $\phi \in C^1(\mathfrak{gl}_+(\infty))$ whose kernel is $\mathfrak{sl}_+(\infty)$.
Let us now define the action of the (abelian) Lie algebra C on $H^1(\mathfrak{gl}_+(\infty))$: for any $\phi \in C^1(\mathfrak{gl}_+(\infty))$ and $[Z] \in C \simeq \mathcal{L}_L/\mathfrak{gl}_+(\infty)$, define:

$$([Z].\phi)(\alpha) = \phi([Z+\beta, \alpha]), \tag{5.19}$$

where $Z \in \mathcal{L}_L$ and $\beta \in \mathfrak{gl}_+(\infty))$. On the other hand, since ϕ is a cocycle, we have that:

$$\phi([Z+\beta, \alpha]) = \phi([Z, \alpha]).$$

It is a simple calculation to show that $[\mathcal{L}_L, \mathfrak{gl}_+(\infty)] \subset \mathfrak{sl}_+(\infty)$ so that, from the hypothesis on ϕ, we conclude that $\phi([Z, \alpha]) = 0$, i.e:

$$[Z].\phi = 0,$$

or that ϕ is C-invariant. ♠

Remark 27 *In this remark we want to compare the Lie algebra* $\mathfrak{g}l_+(\infty)$ *with its finite dimensional analogous, e.g* $\mathfrak{gl}(n)$*. These two Lie algebras are not simple. In fact the Lie algebra of the matrices having trace equal to zero is a non trivial ideal in both cases (*$\mathfrak{s}l_+(\infty)$ *in the infinite dimensional case and* $\mathfrak{s}l(n)$ *in the finite dimensional case). Moreover, in both cases the quotient is the trivial Lie algebra, e.g* $\mathbb{C}$*. While in the finite dimensional case the quotient* $\mathfrak{g}l(n)/\mathfrak{s}l(n) \simeq \mathbb{C} \simeq \mathcal{Z}(\mathfrak{g}l(n))$*, where* $\mathcal{Z}(\mathfrak{g}l(n))$ *is the center of* $\mathfrak{g}l(n)$*, in the infinite dimensional case the quotient* $\mathfrak{g}l_+(\infty)/\mathfrak{s}l_+(\infty)$ *does not correspond to any ideal in* $\mathfrak{g}l_+(\infty)$*. In particular* $\mathcal{Z}(\mathfrak{g}l_+(\infty)) = \{0\}$*.*

Chapter 6

Extensions of Lie algebras

In this chapter, we will give a detailed overview of subject of the Lie algebras extensions with the goal to give full proofs of the statements contained in the section 5.2. The results contained in the present chapter, even if not original, are not easily founded in the literature. As a reference there is the classical [23], where the author describe the case of (non abelian) group extensions. For the Lie algebra case we refer to the (unpublished) paper [1]. On the other hand, the abelian case is completely standard and can be founded in any books about homological algebra. We will refer to [24].

6.1 Extension of Lie algebras

Let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{e}$ three Lie algebras.

Definition 53 *We will say that they fit into an exact sequence if there are morphisms of Lie algebras* $i : \mathfrak{h} \longrightarrow \mathfrak{e}$ *and* $p : \mathfrak{e} \longrightarrow \mathfrak{g}$ *such that* $\ker p = \operatorname{im} i$. *In such a case we will write:*

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\ i\ } \mathfrak{e} \xrightarrow{\ p\ } \mathfrak{g} \longrightarrow 0\cdot \tag{6.1}$$

Definition 54 *An exact sequence of Lie algebra:*

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\ i\ } \mathfrak{e} \xrightarrow{\ p\ } \mathfrak{g} \longrightarrow 0$$

is said a to be a split exact sequence, if there exist a $s \in \mathrm{Hom}_{\mathrm{Lie}}\big(\mathfrak{g}, \mathfrak{e}\big)$, *such that:* $p \circ s = \mathrm{id}_{\mathfrak{g}}$. *Such a morphism is called a section.*

Note that in general, an exact sequence of Lie algebras does not have any section. In what follows, we describe the obstructions to the existence of such a morphism. On the other hand, it is easy to show that any exact sequences of k-modules has always a section, i.e exact sequences of vector spaces are always split. This in particular implies that any exact sequence of Lie algebras is split as exact sequence of vector spaces.

Remark 28 *If $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{e}$ form an exact sequence of Lie algebras then $i(\mathfrak{h}) = \ker p$ so that $i(\mathfrak{h})$ is an ideal in $\mathfrak{e}$. Since i is injective, $i(\mathfrak{h}) \simeq \mathfrak{h}$, so that, with a little abuse of notation, we will think of $\mathfrak{h}$ itself as an ideal in $\mathfrak{e}$.*

Definition 55 *If the Lie algebras $\mathfrak{h}$, $\mathfrak{g}$ and $\mathfrak{e}$ fit into an exact sequence as in (6.1), we will say that the Lie algebra $\mathfrak{e}$ is an extension of the Lie algebra $\mathfrak{g}$ via the Lie algebra $\mathfrak{h}$.*

We want now to study in some detail the following problem:

- given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ describe, up to isomorphism, all the possible Lie algebra structure on the vector space $\mathfrak{e} = \mathfrak{g} \oplus \mathfrak{h}$, or equivalently, the set of all the possible extensions of the Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{h}$ (note that these two problems are actually equivalents since if $\mathfrak{e}$ is such an extension, $\mathfrak{e} \simeq \mathfrak{g} \oplus \mathfrak{h}$ as a k modules).

Definition 56 *We will indicate with $\mathrm{Ext}(\mathfrak{g}, \mathfrak{h})$ the set of all Lie algebra extensions of $\mathfrak{g}$ by $\mathfrak{h}$.*

Let us start with the exact sequence (6.1) and let us fix a section $s \in \mathrm{Hom}_k(\mathfrak{g}, \mathfrak{e})$.
Then we can write any element $X \in \mathfrak{e} = i(\xi) + s(\overline{x})$ for some $\xi \in \mathfrak{h}$ and $\overline{x} \in \mathfrak{g}$ (note that the elements in $\mathfrak{g}$ are actually equivalence classes). If X, Y are two elements in $\mathfrak{e}$, their bracket will be written as follows:

$$[X,Y]_{\mathfrak{e}} = [i(\xi)+s(\overline{x}), i(\eta)+s(\overline{y})]_{\mathfrak{e}} = [i(\xi), i(\eta)]_{\mathfrak{e}} + [i(\xi), s(\overline{y})]_{\mathfrak{e}} + [s(\overline{x}), i(\eta)]_{\mathfrak{e}} +$$
$$+ [s(\overline{x}), s(\overline{y})]_{\mathfrak{e}} = i\big([\xi,\eta]_{\mathfrak{h}}\big) + [i(\xi), s(\overline{y})]_{\mathfrak{e}} + [s(\overline{x}), i(\eta)]_{\mathfrak{e}} +$$
$$+ [s(\overline{x}), s(\overline{y})]_{\mathfrak{e}},$$

where the last equality follows since $i : \mathfrak{h} \longrightarrow \mathfrak{e}$ is a homomorphism of Lie algebras.

Let us now write the previous commutator in terms of the Lie bracket in $\mathfrak{g}$ and in $\mathfrak{h}$.
First, observe that the section s define an element α_s of the vector space $\mathrm{Hom}_k(\mathfrak{g}, \mathcal{D}(\mathfrak{h}))$ which is defined via the following:

$$\alpha_s : \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{h}); \tag{6.2}$$

$$x \rightsquigarrow \alpha_s(\overline{x}), \quad \alpha_s(\overline{x}).\xi = [s(\overline{x}), i(\xi)]_{\mathfrak{e}},$$

for each $\overline{x} \in \mathfrak{g}$. Moreover, the same section define the following linear map ρ_s:

$$\rho : \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{h}; \tag{6.3}$$

$$\overline{x} \wedge \overline{y} \rightsquigarrow \rho_s(\overline{x}, \overline{y}), \quad \rho_s(\overline{x}, \overline{y}) = [s(\overline{x}), s(\overline{y})]_{\mathfrak{e}} - s\big([\overline{x}, \overline{y}]_{\mathfrak{g}}\big),$$

for each $\overline{x}, \overline{y} \in \mathfrak{g}$. Using the maps (6.2) and (6.3), we can rewrite the commutator $[X, Y]_{\mathfrak{e}}$ as follows:

$$[X, Y]_{\mathfrak{e}} = i\big([\xi, \eta]_{\mathfrak{h}}\big) + \alpha_s(\overline{x}).\eta - \alpha_s(\overline{y}).\xi + \rho_s(\overline{x}, \overline{y}) + s\big([\overline{x}, \overline{y}]_{\mathfrak{g}}\big). \tag{6.4}$$

The following proposition follows now by direct calculation:

Proposition 33 *The antisymmetry and the Jacoby identity of the Lie bracket of $\mathfrak{e}$ force the maps (6.2) and (6.3) to fulfill the following identities:*

$$1) \ [\alpha_s(\overline{x}), \alpha_s(\overline{y})]_{\mathcal{D}(\mathfrak{h})} - \alpha_s([\overline{x}, \overline{y}]_{\mathfrak{g}}) = \mathrm{ad}^{\mathfrak{h}}_{\rho_s(\overline{x}, \overline{y})}; \tag{6.5}$$

$$2) \sum_{\text{cyclic } \{\overline{x}, \overline{y}, \overline{z}\}} \big(\alpha_s(\overline{x}).\rho(\overline{y}, \overline{z}) - \rho([\overline{x}, \overline{y}]_{\mathfrak{g}}, \overline{z})\big) = 0. \tag{6.6}$$

Note that the choice of a section s as above is equivalent to the choice of a representative in $\mathfrak{e}$ for each equivalence class $\overline{x} \in \mathfrak{g}$. Each of such a choice determine the datum (α_s, ρ_s) as explained above. For the time being, let us work will the following section:

$$s : \mathfrak{g} \longrightarrow \mathfrak{e} \tag{6.7}$$

$$[x] \longrightarrow x$$

to which will correspond the datum (α, ρ). We leave for later the question of how to compare data $(\alpha_{s'}, \rho_{s'})$ (α_s, ρ_s) coming from different sections.
Let us now suppose that we have two extensions of $\mathfrak{g}$ by $\mathfrak{h}$, say $\mathfrak{e}$ and $\mathfrak{e}'$. Note that $\mathfrak{e} \simeq \mathfrak{e}'$ as a vector spaces and in particular that $\mathfrak{e} \simeq \mathfrak{g} \oplus \mathfrak{h} \simeq \mathfrak{e}'$, being the same section (6.7) chosen for both.

Definition 57 *We will say that $\mathfrak{e}$ and $\mathfrak{e}'$ are two equivalents extensions of $\mathfrak{g}$ by $\mathfrak{h}$, if and only if there is an isomorphism of Lie algebra $\varphi : \mathfrak{e} \longrightarrow \mathfrak{e}'$ which makes commutative the following diagram:*

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \arrow[r, "i"] \arrow[d, equal] & \mathfrak{e} \arrow[r, "p"] \arrow[d, "\varphi", "\wr"'] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{h} \arrow[r, "i'"] & \mathfrak{e}' \arrow[r, "p'"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Remark 29 *More in general, we could say two extensions $N \simeq M_1 \oplus M_2 \simeq N'$ of the k-module M_2 by the k-module M_1 are equivalent if and only if there exists an isomorphism $\varphi \in \mathrm{Hom}_k(N, N')$, that makes the following diagram commutative:*

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "i"] \arrow[d, equal] & N \arrow[r, "p"] \arrow[d, "\varphi"] & M_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "i'"] & N' \arrow[r, "p'"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

To proceed we need the following lemma:

Lemma 18 *Let us suppose that M_1, M_2 are two vector spaces. The classes of equivalence of the extensions of M_2 by M_1 are in one to one correspondence with the isomorphism $\varphi : M_1 \oplus M_2 \longrightarrow M_1 \oplus M_2$, of the following form:*

$$\varphi(x, y) = (x + b(y), y),$$

where $b \in \mathrm{Hom}_k(M_2, M_1)$.

Proof Since $\varphi : M_1 \oplus M_2 \longrightarrow M_1 \oplus M_2$, it can be written as:

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

where

$a \in \mathrm{Hom}_k(M_1, M_1), d \in \mathrm{Hom}_k(M_2, M_2), c \in \mathrm{Hom}_k(M_1, M_2)$ and $b \in \mathrm{Hom}_k(M_2, M_1)$.

From the definition of equivalence of extensions it follows that $a = 1 \in \mathrm{Hom}_k(M_1, M_1)$, $d = 1 \in \mathrm{Hom}_k(M_2, M_2)$ and $c = 0$. ♠

Let us now go back the Lie algebra case. Let $\mathfrak{e}$ and $\mathfrak{e}'$ be two equivalent extensions, and let (α, ρ) and (α', ρ') be the corresponding data.

Theorem 29 *Then there exists $b \in \mathrm{Hom}_k(\mathfrak{g}, \mathfrak{h})$ such that:*

$$\rho'(x, y) = \rho(x, y) + \alpha(x).b(y) - \alpha(y).b(x) - b\big([x, y]_{\mathfrak{g}}\big) + \big[b(x), b(y)\big]_{\mathfrak{h}};$$

and

$$\alpha'(x) = \alpha(x) + \mathrm{ad}^{\mathfrak{h}}_{b(x)},$$

for every $x, y \in \mathfrak{g}$ and $\xi \in \mathfrak{h}$.

Proof Since the two extensions are equivalent, there exists a Lie isomorphism $\varphi : \mathfrak{e} \longrightarrow \mathfrak{e}'$, that via the isomorphism $\mathfrak{e} \simeq \mathfrak{h} \oplus \mathfrak{g} \simeq \mathfrak{e}'$ can be written as:

$$\varphi = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix};$$

see lemma 18. Now apply the definitions of the map α, ρ, α' and ρ', and the hypothesis that φ is a morphism of Lie algebras. ♠

To state the converse of the theorem 29 let us consider the subset $\mathcal{E} \subset \mathrm{Hom}_k(\mathfrak{g}, \mathcal{D}(\mathfrak{h})) \times \mathrm{Hom}_k(\bigwedge^2 \mathfrak{g}, \mathfrak{h})$, whose elements fulfill the identities given in (6.5) and (6.6):

$$[\alpha(x), \alpha(y)]_{\mathcal{D}(\mathfrak{h})} - \alpha_{[x,y]_{\mathfrak{g}}} = \mathrm{ad}_{\rho(x,y)};$$

$$\sum_{\text{cyclic }\{x,y,z\}} \big(\alpha(x)\rho(y, z) - \rho([x, y], z)\big) = 0.$$

Definition 58 *We say that* $(\alpha, \rho), (\alpha', \rho') \in \mathcal{E}$ *are equivalent if and only if there exist* $b \in \mathrm{Hom}_k(\mathfrak{g}, \mathfrak{h})$ *such that:*

$$\alpha'(x) = \alpha(x) + \mathrm{ad}^{\mathfrak{h}}_{b(x)}$$

and

$$\rho'(x, y) = \rho(x, y) + \alpha(x).b(y) - \alpha(y).b(x) - b([x, y]_{\mathfrak{g}}) + \big[b(x), b(y)\big]_{\mathfrak{h}}$$

The converse of theorem 29 is the following:

Theorem 30 *If the datum* (α, ρ) *associated to* $\mathfrak{e}$ *is equivalent to the datum* (α', ρ') *associated to* $\mathfrak{e}'$*, in the sense of the definition (58), then there exists an isomorphism of Lie algebras* $\varphi : \mathfrak{e} \simeq \mathfrak{h} \oplus \mathfrak{g} \longrightarrow \mathfrak{h} \oplus \mathfrak{g} \simeq \mathfrak{e}'$*.*

Therefore theorems 32 and 29 become:

Theorem 31

$$\mathrm{Ext}(\mathfrak{g}, \mathfrak{h}) \simeq \mathcal{E}/\sim .$$

Let us now describe how, for a given extension $\mathfrak{e}$, the datum (α_s, ρ_s) depends on the choice of the section s. To this end, we can use the same argument we used to study the equivalence classes of the extensions. In particular, we can take as the isomorphism φ the identity. In this case given two sections s, s' we will write:

$$\mathfrak{e} = \mathfrak{h} \oplus s(\mathfrak{g}) \simeq \mathfrak{h} \oplus s'(\mathfrak{g}) = \mathfrak{e},$$

where $\simeq$ is actually given by the identity. Then:

$$(\eta, s(\overline{x})) \rightsquigarrow (\eta, s'(\overline{x}) + \xi),$$

for some $\xi \in \mathfrak{h}$. From this we deduce that

Lemma 19 *The set of all section of* $p : \mathfrak{e} \longrightarrow \mathfrak{g}$ *is an affine space modelled on* $\mathrm{Hom}_k(\mathfrak{g}, \mathfrak{h})$*.*

Proof In fact $s - s' : \mathfrak{g} \longrightarrow \mathfrak{e}$ is the linear map $(s - s')(\overline{x}) = \xi$. ♠

We have now the following result:

Theorem 32 *For a given extension $\mathfrak{e}$ of the Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{h}$ and two sections $s, s' : e \longrightarrow g$, we have that:*

$$\rho_{s'}(x \wedge y) = \rho_s(x \wedge y) + \alpha_s(x).f(y) - \alpha_s(y).f(x) - f\big([x,y]_{\mathfrak{g}}\big) + \big[f(x), f(y)\big]_{\mathfrak{h}};$$

$$\alpha_{s'}(x).\xi = \alpha_s(x).y + \mathrm{ad}_{f(x)}.\xi,$$

for each $x, y \in \mathfrak{g}$, $\xi \in \mathfrak{h}$.

Proof The proof follows from the formulas (6.3), (6.2) and from the previous lemma 19. ♠

Let us now describe in terms of the (α, ρ) under which assumptions a given extension $\mathfrak{e}$ of $\mathfrak{g}$ by $\mathfrak{h}$ is split. We recall that by definition this means that we can find a section s of $p : \mathfrak{e} \longrightarrow \mathfrak{g}$, such that $s\big([x,y]_{\mathfrak{g}}\big) = \big[s(x), s(y)\big]_{\mathfrak{e}}$. Such a section, will provide us with α_s, and ρ_s as usual.
Let us start with the following observation:

Proposition 34 *The extension is split if and only if for any section $s : \mathfrak{g} \longrightarrow \mathfrak{e}$, there exists $f \in \mathrm{Hom}_k(\mathfrak{g}, \mathfrak{h})$ such that the section: $s' = s + f$ is a morphism of Lie algebras.*

Proof The equivalence of the statement of the proposition with the definition is almost a tautology; the first implication follows from the definition of split extension. Viceversa if the extension is split, then we can find a section $s \in \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{e})$, but any other section differs from this one by some element in $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{e})$.♠

Let $s \in \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{e})$ be a section for $p : \mathfrak{e} \longrightarrow \mathfrak{g}$.

Theorem 33

$$1.\ \rho_s \in \mathrm{Hom}_k\Big(\bigwedge^2 \mathfrak{g}, \mathfrak{h}\Big)$$

is identically zero;

$$2.\ \alpha_s \in \mathrm{Hom}_k\big(\mathfrak{g}, \mathcal{D}(\mathfrak{h})\big)$$

belongs to $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathcal{D}(\mathfrak{h}))$*, i.e it is a morphism of Lie algebras.*

Proof 1. follows from the definition of ρ_s:

$$\rho_s(x,y) = s([x,y]_{\mathfrak{g}}) - \big[s(x), s(y)\big]_{\mathfrak{e}}.$$

2. Follows from the definition of (6.2) and from 1. above:

$$\big[\alpha_s(x), \alpha_s(y)\big]_{\mathcal{D}(\mathfrak{h})} - \alpha_s([x,y]_{\mathfrak{g}}) = \mathrm{ad}^{\mathfrak{h}}_{\rho_s(x,y)};$$

which becomes:

$$\big[\alpha_s(x), \alpha_s(y)\big]_{\mathcal{D}(\mathfrak{h})} - \alpha_s([x,y]_{\mathfrak{g}}) = 0.$$

♠

Remark 30 *From 2) in the previous theorem, we conclude that* $\mathfrak{h} \in \mathfrak{e}$ *is a* $s(\mathfrak{g})$*-module. Nevertheless, we cannot say that* $\mathfrak{h}$ *is a* $\mathfrak{g}$ *module; in fact, the action of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *depends on* s*.*

Proposition 35 *Any representative the class of equivalence* $[(\alpha', \rho')] \in \mathcal{E}/\sim$ *which corresponds to a split extension can be written in the following form:*

$$\big(\alpha'(\cdot) + \mathrm{ad}^{\mathfrak{h}}_{f(\cdot)},\ \ \rho'(\cdot,\cdot) = \alpha \wedge f(\cdot,\cdot) - f([\cdot,\cdot]_{\mathfrak{g}}) + \big[f(\cdot), f(\cdot)\big]_{\mathfrak{h}}\big),$$

for some $f \in \mathrm{Hom}_k(\mathfrak{g}, \mathfrak{h})$*.*

6.1.1 Abelian extensions

Let us now describe as a particular case of the general theory outlined above, the extensions of a Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{h}$, for $\mathfrak{h}$ abelian Lie algebra. The main point of the following discussion will be to interpret such class of extensions of the Lie algebra $\mathfrak{g}$ in terms of the cohomology of of the same Lie algebra. We refer to the appendix 1 for the main notions about Lie algebra cohomology.
Let us start our discussion with the following proposition:

Proposition 36 *Any given an extension* $\mathfrak{e}$ *of* $\mathfrak{g}$ *by the abelian Lie algebra* $\mathfrak{h}$ *defines an action of* $\mathfrak{g}$ *on* $\mathfrak{h}$*; i.e* $\mathfrak{h}$ *is a* $\mathfrak{g}$*-module.*

Proof Let us take $[x] \in \mathfrak{g} \simeq \mathfrak{e}/\mathfrak{h}$ and let us define its action on $\xi \in \mathfrak{h}$ by: $[x].\xi = [x, \xi]$. This definition does not depend on the representative chosen, since any other representative of the same class will be of the form $x + \eta$, $\eta \in \mathfrak{h}$. ♠

The previous proposition can be rephrased by saying that the abelian extensions of a given Lie algebra are in one correspondence with the extension of $\mathfrak{g}$ by a $\mathfrak{g}$-module, with its structure of abelian Lie algebra. Therefore, given $\mathfrak{g}$ and M, a module over this Lie algebra, let us consider the set $\mathrm{Ext}(\mathfrak{g}, M)$, i.e the set of equivalence of exact sequences of Lie algebras:

$$0 \longrightarrow M \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \longrightarrow 0,$$

where, as in the previous section, two such exact sequences are equivalent if and only if there exists a morphism of Lie algebras $\varphi : \mathfrak{e} \xrightarrow{\simeq} \mathfrak{e}'$, which make the following a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & \mathfrak{e} \arrow[r, "p"] \arrow[d, "\varphi", "\wr"'] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "i'"] & \mathfrak{e}' \arrow[r, "p'"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

From the theorem 31, $\mathrm{Ext}(\mathfrak{g}, M) \simeq \mathcal{E}/\sim$, as sets, so that from the extension we get a couple $(\alpha, \rho) \in \mathrm{Hom}_k(\mathfrak{g}, \mathcal{D}(M)) \times \mathrm{Hom}_k(\bigwedge^2 \mathfrak{g}, M)$, such that:

$$[\alpha(x), \alpha(y)]_{\mathcal{D}(M)} - \alpha([x, y]_{\mathfrak{g}}) = \mathrm{ad}^M_{\rho(x,y)};$$

$$\sum_{\text{cyclic } \{x,y,z\}} \big(\alpha(x)\rho(y, z) - \rho([x, y], z)\big) = 0.$$

Since M is abelian, $\mathcal{D}(M) \simeq \mathrm{Hom}_k(M, M)$, and $\mathrm{ad}^M_\xi = 0$ for each $\xi \in M$, so that the first of the previous equations is nothing more than the assertion that M is a $\mathfrak{g}$ module; in fact it can be written as:

$$\alpha([x, y]_{\mathfrak{g}}) = [\alpha(x), \alpha(y)]_{\mathrm{Hom}(M,M)},$$

and the action of $\mathfrak{g}$ on M is defined by $\alpha \in \mathrm{Hom}_k(\mathfrak{g}, \mathrm{Hom}_k(M, M))$. The identity involving ρ, can be spell out as follows:

$$\rho([x, y], z)-\rho([x, z], y)+\rho([y, z], x)-\alpha(x).\rho(y, z)+\alpha(y).\rho(x, z)-\alpha(z).\rho(x, y) = 0,$$

which is equivalent to $\rho \in \mathrm{Hom}(\bigwedge^2 \mathfrak{g}, M)$ being a 2-cocycle with coefficients in M. All of this is equivalent to say that we have a map:

$$\Upsilon : \mathcal{E} \longrightarrow H^2(\mathfrak{g}, M);$$

$$(\alpha, \rho) \rightsquigarrow [\rho].$$

Theorem 34 *If $(\alpha, \rho) \sim (\alpha', \rho')$ then $\Upsilon(\alpha, \rho) = \Upsilon(\alpha', \rho')$. More precisely:*

$$\mathcal{E}/ \sim \simeq \mathrm{Ext}(\mathfrak{g}, M) \simeq H^2(\mathfrak{g}, M).$$

Proof If $(\alpha, \rho) \sim (\alpha', \rho')$, there exists $b \in \mathrm{Hom}_k(\mathfrak{g}, M)$ such that:

$$\rho'(x, y) = \rho(x, y) + \alpha(x).b(y) - \alpha(y).b(x) - b\big([x, y]_{\mathfrak{g}}\big) + \big[b(x), b(y)\big]_M;$$

and

$$\alpha'(x) = \alpha(x) + \mathrm{ad}^M_{b(x)},$$

see theorem 29. Since M is a commutative algebra we have that: $\alpha'(x) = \alpha(x)$ (i.e the modulo structure on M does not change), and:

$$\rho'(x, y) = \rho(x, y) + \alpha(x).b(y) - \alpha(y).b(x) - b\big([x, y]_{\mathfrak{g}}\big);$$

which means that:

$$(\rho' - \rho)(x, y) = -(d_1 b)(x, y),$$

where

$$d_1 : \mathrm{Hom}_k(\mathfrak{g}, M) \longrightarrow \mathrm{Hom}_k(\bigwedge^2 \mathfrak{g}, M);$$

$$(d_1 f)(x, y) = f([x, y]) - x.f(y) + y.f(x), \ \forall\, x, y \in \mathfrak{g},$$

see (7.2). This prove that the map Υ is well behaved with respect to the relation $\sim$, and induces $\Upsilon' : \mathcal{E}/ \sim \longrightarrow H^2(\mathfrak{g}, M)$. The injectivity and surjectivity are trivially checked. ♠

Example 37 *(Semidirect product)*
An abelian extension of the Lie algebra $\mathfrak{g}$ by M, is called semidirect product if it is a split extension.

Theorem 35 *The semidirect product of $\mathfrak{g}$ with M, $\mathfrak{e} = M \rtimes \mathfrak{g}$, corresponds to the trivial class in $H^2(\mathfrak{g}, M)$.*

Proof *The 2-cocycle ρ is trivial.* ♠

Example 38 *(Central extensions)*
The central extensions form a particular class of abelian extensions. These are the ones for which $i(M)$ is contained in the center of $\mathfrak{e}$. For the one dimensional central extensions, we have the following proposition:

Proposition 37 *The classes of equivalence of one dimensional central extensions of $\mathfrak{g}$:*

$$0 \longrightarrow \mathbb{C} \xrightarrow{\quad i \quad} \mathfrak{e} \xrightarrow{\quad p \quad} \mathfrak{g} \longrightarrow 0$$

are parameterized by the second cohomology group of $\mathfrak{g}$ with trivial coefficients; i.e $H^2(\mathfrak{g}, \mathbb{C})$.

Chapter 7

Appendix 1

In this chapter we first collect some elementary facts about homological algebra and then, we overview the main features of the cohomology theory which is associated to any Lie algebra. References for the present chapter are [24] and [8] for a detailed and extensive analysis the cohomology of Lie algebras.

7.1 Some elementary homological algebra

Let R be a ring , and M_1, M_2, andM_3 R-modules. Let $i : M_1 \longrightarrow M_2$ and $p : M_2 \longrightarrow M_3$ two morphisms of R-modules.

Definition 59 *We say that the morphism i and p form an exact sequence if i is injective, p is surjective and* $\ker p = \operatorname{im} i$. *In such a case we will write:*

$$0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{p} M_3 \longrightarrow 0$$

If M is endowed with a morphism $\partial : M \longrightarrow M$ such that $\partial^2 = 0$ we will call (M, ∂) a differential module.
Let us define the spaces $Z(M, \partial) \equiv \ker \partial$ and $B(M, \partial) \equiv \operatorname{im} \partial$

The following proposition is trivially checked.

Proposition 38 *$Z(M, \partial)$ and $B(M, \partial)$ are submodules of M.*

Definition 60 *The elements of $Z(M, \partial)$ and $B(M, \partial)$ are called respectively cocycles and coboundaries of the differential module, (M, ∂) whose elements are called cochains.*

From the condition $\partial^2 = 0$ we deduce that $B(M,\partial) \subset Z(M,\partial)$, so that we can define the quotient $H(M,\partial) = Z(M,\partial)/B(M,\partial)$.

Definition 61 *The $H(M,\partial) = Z(M,\partial)/B(M,\partial)$ is called the cohomology group of the differential module (M,∂).*

Remark 31 *$H(M,\partial) = Z(M,\partial)/B(M,\partial)$ gives a measure of how the sequence:*

$$0 \longrightarrow Z(M,\partial) \longrightarrow M \longrightarrow B(M,\partial) \longrightarrow 0$$

is far from being exact.

Given (M,∂) and (M',∂') two differential modules, we will say that a morphism:

$$\phi : M \longrightarrow M'$$

is a morphism of differential modules if and only if it commutes with the differentials, i.e:

$$\phi \circ \partial = \partial' \circ \phi.$$

Proposition 39 *If $\phi : (M,\partial) \longrightarrow (M',\partial')$ is a morphism of differential modules then: $\phi\big((Z,\partial)\big) \subset (Z',\partial')$, and $\phi\big((B,\partial)\big) \subset (B',\partial')$.*

Proof The proof is evident from the definition. ♠

From the previous proposition it follows that given a morphism ϕ of differential modules we have induced a map between the cohomology groups:

$$\tilde{\phi} : H(M) \longrightarrow H(M'),$$

$$[x] \rightsquigarrow [\phi(x)].$$

Moreover, we have that:

Proposition 40 *Let:*

$$0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

be an exact sequence of differential modules. Then, there exists a morphism $\delta : H(M'') \longrightarrow H(M')$, called connecting morphism, and an exact triangle in cohomology:

$$\begin{array}{ccc} H(M'') & \xleftarrow{\delta} & H(M') \\ & {\scriptstyle\tilde{\psi}}\nwarrow \quad \swarrow{\scriptstyle\tilde{\phi}} & \\ & H(M) & \end{array}$$

For complexes of R-modules, the previous result becomes:

Definition 62 *A complex of R-modules is a $\mathbb{Z}$-graded differential module $(M^\bullet = \bigoplus_{i\in\mathbb{Z}} M_i, \partial)$, such that the differential is a graded morphism of degree 1, i.e: $\partial_i(M_i) \subset M_{i+1}$, for each $i \in \mathbb{Z}$. In this case we will write:*

$$\cdots \longrightarrow M_{i-1} \xrightarrow{\partial_{i-1}} M_i \xrightarrow{\partial_i} M_{i+1} \longrightarrow \cdots$$

To a given complex of R-modules $(M^\bullet, \partial)$, we can associate its cohomology, i.e for each $n \in \mathbb{Z}$ we can define the n-th group of cohomology as $H^n(M^\bullet) = Z^n(M^\bullet)/B^n(M^\bullet)$, where $B^n(M^\bullet) \subset M_n$ is by definition $\partial_{n-1}(M_{n-1})$ and $Z^n(M) = \ker \partial_n$.

The notion of morphism of R-modules is expressed by:

Definition 63 *A morphism between two complexes of R modules, $(M^\bullet, \partial_M)$ and $(N^\bullet, \partial_N)$, $\psi : M^\bullet \longrightarrow N^\bullet$, is a collection of morphisms $\psi_i : M_i \longrightarrow N_i$ such that the following diagrams commute for each $i \in \mathbb{Z}$:*

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_{m-2}} & M_{i-1} & \xrightarrow{\partial_{i-1}} & M_i & \xrightarrow{\partial_i} & M_{i+1} & \xrightarrow{\partial_{i+1}} & \cdots \\ & & \downarrow{\scriptstyle\psi_{i-1}} & & \downarrow{\scriptstyle\psi_i} & & \downarrow{\scriptstyle\psi_{i+1}} & & \\ \cdots & \xrightarrow{\partial_{m-2}} & N_{i-1} & \xrightarrow{\partial_{i-1}} & N_i & \xrightarrow{\partial_i} & N_{i+1} & \xrightarrow{\partial_{i+1}} & \cdots \end{array}$$

For complexes of module, the proposition 40 becomes:

Proposition 41 *Let*

$$0 \longrightarrow M^\bullet \xrightarrow{\psi} N^\bullet \xrightarrow{\phi} P^\bullet \longrightarrow 0$$

be an exact sequence of complexes of R-modules. Then, for each $i \in \mathbb{Z}$ there exist a connecting morphism $\delta_i : H^i(P^\bullet) \longrightarrow H^{i+1}(M^\bullet)$ and a long exact sequence in cohomology:

$$\cdots \longrightarrow H^i(M) \xrightarrow{\tilde{\psi}_i} H^i(N) \xrightarrow{\tilde{\phi}_i} H^i(P) \xrightarrow{\delta_i} H^{i+1}(M) \longrightarrow \cdots$$

7.2 Cohomology of Lie algebras

Let $\mathfrak{g}$ a Lie algebra over the field k (see subsection 2.3). Let M be a k-module, i.vector space over the field k, and let consider the Lie algebra $L(\mathrm{End}(M))$ (see example 8) the Lie algebra defined on the associative algebra $\mathrm{End}(M)$.

Definition 64 *We say that M is a $\mathfrak{g}$-module, if there is:*

$$\phi : \mathfrak{g} \longrightarrow L(\mathrm{End}(M)),$$

morphism of Lie algebras (see definition 14). In particular, we have that for each $x, y \in \mathfrak{g}$, $\phi(x) \in \mathrm{End}(M))$ and $\phi[x,y] = \phi(x) \circ \phi(y) - \phi(y) \circ \phi(x)$. We will say that the morphism ϕ defines an action of the Lie algebra $\mathfrak{g}$ on the k-module M, and that M is a (linear) representation of $\mathfrak{g}$ via ϕ. As a notational remark, for each $x \in \mathfrak{g}$ and $m \in M$, we will indicate with $x.m$ the value of $\phi(x)$ on m, when it is clear from the context what is the morphism ϕ.

Let us give some examples:

Example 39 *(trivial module)*
The ground field k is endowed with a $\mathfrak{g}$ module structure by the following: $x.\alpha = 0$ for each $\alpha \in k$ and $x \in \mathfrak{g}$. k endowed of such a structure of $\mathfrak{g}$ module is called the trivial module.

Example 40 *Every Lie algebra $\mathfrak{g}$ can be thought as a $\mathfrak{g}$ module. In fact for each $x \in \mathfrak{g}$, the map:* $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$, *defined by: $x \rightsquigarrow \mathrm{ad}_x(\cdot) = [x, \cdot] \in$* $\mathrm{End}(\mathfrak{g})$ *is a morphism of Lie algebras: as a consequence of the Jacobi identity we have:*

$$\mathrm{ad}_{[x,y]} = \big[[x,y],\cdot\big] = \big[x,[y,\cdot]\big] + \big[y,[x,\cdot]\big].$$

The action of $\mathfrak{g}$ defined by the morphism ad *is called the adjoint action.*

Definition 65 *The action of a Lie algebra $\mathfrak{g}$ on a module $\mathfrak{g}$-module M is called faithful is the map ϕ defining the action is injective.*

Example 41 *The Lie algebra $\mathfrak{sl}_2$, defined in the example 9, has a faithful representation on the k-module $k^2 = k \oplus k$:*

$$h \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; x \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; \text{and } y \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If M_1 and M_2 are two $\mathfrak{g}$ modules then:

Proposition 42 *$M_1 \oplus M_2$, $M_1 \otimes M_2$, $\mathrm{Hom}_k(M_1, M_2)$ are $\mathfrak{g}$ modules. In particular, the exterior algebra and the symmetric algebras generated by a $\mathfrak{g}$ module, are $\mathfrak{g}$ modules.*

Proof *We need to define the action of $\mathfrak{g}$ on each of the above k-modules. For each $x \in \mathfrak{g}$, $m_1 \in M_1$ and $m_2 \in M_2$ we define: 1) $x.(m_1, m_2) = (x.m_1, x.m_2)$, 2) $x.(m_1 \otimes m_2) = (x.m_1) \otimes m_2 + m_1 \otimes (x.m_2)$, and: 3) $(x.f)(m_1) = f(x.m_1)$, for any given $f \in \mathrm{Hom}(M_1, M_2)$. The action of $\mathfrak{g}$ of the exterior algebra generated by M, $\bigwedge^\bullet M$, and on the symmetric algebra, $\bigodot^\bullet M$, are defined accordingly. It is easy to show that 1), 2) and 3) define a structure of $\mathfrak{g}$ module.* ♠

For any $\mathfrak{g}$-module M we can define the k-module of $\mathfrak{g}$ invariant elements in M, by the following:

$$\mathrm{Inv}_{\mathfrak{g}}(M) = \{m \in M |\, x.m = 0, \, \forall x \in \mathfrak{g}\}. \tag{7.1}$$

The following proposition is clear:

Proposition 43 $\mathrm{Inv}_{\mathfrak{g}}(M)$ *is a $\mathfrak{g}$-submodule.*

We can now introduce the complex we need to compute the cohomology of any given Lie algebra. Let $\mathfrak{g}$ a Lie algebra and M any $\mathfrak{g}$-module.

Definition 66 *The complex of k-modules associated to $\mathfrak{g}$ and M is $(C^\bullet, d)$ where $C^\bullet = \bigoplus_{q \in \mathbb{Z}} C^q(\mathfrak{g}, M) = \mathrm{Hom}_k(\bigwedge^q \mathfrak{g}, M)$, is the space of q-antisymmetric linear form with values in M and $d : C^\bullet \longrightarrow C^\bullet$ is the linear map defined by:*

$$dc(x_1, \cdots, x_{q+1}) = \sum_{1 \le s < t \le q+1} (-1)^{s+t-1} c([x_s, x_t], x_1, \cdots, \hat{x}_s, \cdots, \hat{x}_t, \cdots, x_{q+1}) +$$

$$+\sum_{1\leq s\leq q+1}(-1)^s x_s.c(x_1,\cdots,\hat{x}_s,\cdots,x_{q+1}), \tag{7.2}$$

where $c \in C^q(\mathfrak{g}, M)$ and $x_1, \cdots, x_{q+1} \in \mathfrak{g}$.
We complete the definition by imposing that $C^i(\mathfrak{g}, M) = 0$ for every $i < 0$ and requiring that $d_i = 0$ when $i < 0$.

It is clear from the formula (7.2), that $d : C^q(\mathfrak{g}, M) \longrightarrow C^{q+1}(\mathfrak{g}, M)$. To prove that $(C^\bullet, d)$ defines a complex of k modules we need to check that:

Proposition 44 *The linear map $d : C^\bullet \longrightarrow C^\bullet$ defined in (7.2) is a differential, $d \circ d = 0$, i.e: $d_{q+1} \circ d_q = 0$ for every $q \in \mathbb{Z}$.*

Proof We need to show that for any given q and $\phi \in C^q(\mathfrak{g}, M)$, $d_{q+1} \circ d_q : C^q(\mathfrak{g}, M) \longrightarrow C^{q+2}(\mathfrak{g}, M)$ is identically equal to zero. Using the formula (7.2), we can write:

$$d_{q+1}\circ d_q\phi(x_1,\cdots,x_{q+2}) = \sum_{1\leq s<t\leq q+2}(-1)^{s+t-1}d_q\phi([x_s,x_t],\cdots,\hat{x}_s,\cdots,\hat{x}_t,\cdots,x_{q+2})+$$

$$\sum_{1\leq s\leq q+2}(-1)^s x_s.d_q\phi(x_1,\cdots,\hat{x}_s,\ldots,x_{q+2}).$$

The proof of the statement goes as follows: we prove that for any three indexes i, j, k, such that $1 \leq i < j < k \leq q+2$, the previous formula produces terms that cancel out. In particular, we can group the summand as follows:
1)

$$(-1)^{i+j-1}d\phi([x_i,x_j],\ldots,\hat{x}_i,\ldots,\hat{x}_j,\cdots)+(-1)^{i+k-1}d\phi([x_i,x_k],\ldots,\hat{x}_i,\ldots,\hat{x}_k,\cdots)+$$

$$+(-1)^{j+k-1}d\phi([x_j,x_k],\ldots,\hat{x}_j,\ldots,\hat{x}_k,\cdots).$$

Applying one more time the formula (7.2), we will get:

$$(-1)^{i+j-1}(-1)^{k-1+1-1}\phi(\big[[x_i,x_j],x_k\big],\ldots)+(-1)^{i+k-1}(-1)^{j+1-1}\phi(\big[[x_i,x_k],x_j\big],\ldots)+$$

$$+(-1)^{j+k-1}(-1)^{i+1+1-1}\phi(\big[[x_j,x_k],x_i\big],\ldots).$$

So that:

$$(-1)^{i+j+k}\phi(\big[[x_i,x_j],x_k\big]+[[x_k,x_i],x_j\big]+[[x_j,x_k],x_i\big],\cdots)=0.$$

2)

$$(-1)^{i+j-1}d\phi([x_i,x_j],\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_{q+2})+(-1)^i x_i.d\phi(x_1,\cdots,\hat{x}_i,\cdots,x_{q+2})+$$

$$+(-1)^j x_j.d\phi(x_1,\cdots,x_{q+2}).$$

Using again formula (7.2), we obtain:

$$(-1)^{i+j-1}(-1)[x_i,x_j]\phi(\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots)+(-1)^i(-1)^{j-1}x_i.x_j\phi(\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots)+$$

$$(-1)^i(-1)^j x_j.x_i.\phi(\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots)=(-1)^{i+j}\big([x_i,x_j]-x_i.x_j+x_j.x_i\big).\phi=0;$$

3) finally, from $(-1)^{i+j-1}d\phi([x_i,x_j],\ldots,\hat{x}_i,\ldots,\hat{x}_j,\cdots)$, we get the term:

$$(-1)^{i+j-1}(-1)^{k-1}x_k.\phi([x_i,x_j],\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,\hat{x}_k,\cdots)$$

and from the term: $(-1)^k x_k.d\phi(\cdots,\hat{x}_k,\cdots)$, we will get:

$$(-1)^k(-1)^{i+j-1}x_k\phi([x_i,x_j],\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,\hat{x}_k,\cdots).$$

♠

Definition 67 *We will call cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in M, the cohomology of the complex $(C^\bullet, d)$.*

Let us give some examples and the interpretation of the cohomology groups for a Lie algebra in some particular cases.

Example 42 *($\mathrm{H}^0(\mathfrak{g},M)$)*
Since $C^{-1}(\mathfrak{g},M)=0$, the 0-th cohomology group of the Lie algebra $\mathfrak{g}$ with coefficients in M is the same that $\{\ker d_0 : C^0(\mathfrak{g},M)\simeq M \longrightarrow C^1(\mathfrak{g},M)\}$. This means that $\phi\in C^0(\mathfrak{g},M)$ represents an element in the 0-th cohomology group, if and only if: $d_0\phi(x)\equiv 0$, for each $x\in\mathfrak{g}$. By formula (7.2), this condition is equivalent to: $x.\phi\equiv 0$, i.e that $\phi\in\mathrm{Inv}_{\mathfrak{g}}(M)$ (see formula (7.1)).

Example 43 *(trivial module k)*
Suppose that $M = k$ is the trivial module. In this case the formula (7.2) becomes:

$$dc(x_1, \cdots, x_{q+1}) = \sum_{1 \leq s < t \leq q+1} (-1)^{s+t-1} c([x_s, x_t], x_1, \cdots, \hat{x}_s, \cdots, \hat{x}_t, \cdots, x_{q+1}), \tag{7.3}$$

since the action of $\mathfrak{g}$ on the module k is given by $x.\alpha = 0$ for each $x \in \mathfrak{g}$ and $\alpha \in k$.

Example 44 *For the cohomology with trivial coefficients, we have that $H^0(\mathfrak{g}, k) \simeq k$, since $H^0(\mathfrak{g}, k) \simeq \ker d_0 \simeq \mathrm{Inv}_{\mathfrak{g}}(k) \simeq k$. The first cohomology group has the following significance: $d_0 : C^0(\mathfrak{g}) \longrightarrow C^1(\mathfrak{g})$ is trivial, therefore: $H^1(\mathfrak{g}) = \{\ker d_1 : C^1(\mathfrak{g}) \longrightarrow C^2(\mathfrak{g})\}$. This implies that:*

$$H^1(\mathfrak{g}) = \{\phi \in C^1(\mathfrak{g}) | d_1\phi(x, y) = 0, \; \forall \, x, y \in \mathfrak{g}\}.$$

Using formula (7.3), we get:

$$0 = d_1\phi(x, y) = \phi([x, y]).$$

In other words, the first cohomology group of $\mathfrak{g}$, with trivial coefficients, is in one to one correspondence with the linear forms on $\mathfrak{g}$, which are identically zero on the derived algebra $[\mathfrak{g}, \mathfrak{g}]$, i.e:

$$H^1(\mathfrak{g}) \simeq \Big(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]\Big)^*.$$

7.2.1 Derivations

Definition 68 *An endomorphism ϕ of a Lie a algebra $\mathfrak{g}$ is a derivation of the Lie algebra if and only if:*

$$\phi[x, y] = [\phi(x), y] + [x, \phi(y)], \; \forall x, y \in \mathfrak{g}.$$

Let us denote the set of all derivation of $\mathfrak{g}$, with $\mathcal{D}(\mathfrak{g})$. $\mathcal{D}(\mathfrak{g})$ is clearly a vector space. In particular we have:

Lemma 20 *$\mathcal{D}(\mathfrak{g})$ is a sub Lie algebra of the Lie algebra of the endomorphisms of $\mathfrak{g}$.*

Moreover, we have that:

Proposition 45 *We have a Lie morphism:*

$$\Gamma : \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g})$$

$$x \rightsquigarrow \mathrm{ad}_x.$$

Proof We need only to check that for each $x \in \mathfrak{g}$, $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ is a derivation of $\mathfrak{g}$. This follows easily from the Jacobi identity.
♠

Definition 69 *The derivations of* $\mathfrak{g}$ *belonging to the image of the map* $\Gamma : \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g})$ *are called inner derivations,* $\mathrm{Inn}(\mathfrak{g})$. *The element of the quotient space* $\mathrm{Out}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$, *are called outer derivations.*

The 1-st cohomology group of $\mathfrak{g}$ with coefficients in $\mathfrak{g}$ has the following interpretation.

Proposition 46 $H^1(\mathfrak{g}, \mathfrak{g}) \simeq \mathrm{Out}(\mathfrak{g})$.

Proof From the definition $H^1(\mathfrak{g}, \mathfrak{g}) = Z^1(\mathfrak{g}, \mathfrak{g})/B^1(\mathfrak{g}, \mathfrak{g})$, where $Z^1(\mathfrak{g}, \mathfrak{g}) = \{\ker d_1 : C^1(\mathfrak{g}, \mathfrak{g}) \longrightarrow C^2(\mathfrak{g}, \mathfrak{g})\}$ and $B^1(\mathfrak{g}, \mathfrak{g}) = \{\mathrm{im}\, d_0 : C^0(\mathfrak{g}, \mathfrak{g}) \longrightarrow C^1(\mathfrak{g}, \mathfrak{g})\}$. If $\phi \in Z^1(\mathfrak{g}, \mathfrak{g})$, $0 = d_1\phi(x, y) = \phi([x, y]) - x.\phi(y) + y.\phi(x)$, so that $\phi \in \mathcal{D}(\mathfrak{g})$. On the other hand $C^0(\mathfrak{g}, \mathfrak{g}) \simeq \mathfrak{g}$. ♠

Chapter 8

Appendix 2

In this appendix, we state and prove with some details some facts about the cohomology of the Lie algebra of the general linear group. The references for the material of the chapter are [8] and the original paper [4].

8.1 Cohomology of the lie algebra $\mathfrak{g}l(n)$

In this appendix we will work over the field of complex numbers. Let us start fixing the notation. We recall that $\mathfrak{g}l(n)$ is the Lie algebra of the general linear group $Gl(n)$. The generators for such Lie algebra are $E_{i,j}$, $i, j \in \{1, \cdots, n\}$ and the relations are given by: $[E_{i,j}, E_{k,l}] = E_{i,l}\delta_{j,k} - E_{k,j}\delta_{l,i}$. Let us also introduce the Lie algebra of the $n \times n$ skew hermitian matrices $\mathfrak{u}(n) = \{M \in \mathfrak{g}l(n) | M^\dagger = -M\}$, and the corresponding Lie group $U(n)$.

Let us start with the following general result about the topological structure of a Lie group:

Theorem 36 *[10] In a Lie group with a finite number of connected components there always exists a maximal compact subgroups. If K is one of them, then any compact subgroup of G is conjugate to a subgroup of K, and in particular any two maximal compact subgroups are conjugate. Furthermore, G is homeomorphic to $K \times \mathbb{R}^m$ for some m.*

Therefore, all the topological information about G are contained in K.

Definition 70 *Let $\mathfrak{g}$ is a real Lie algebra, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ its complexification. The complexification of the Lie group G is the connected and simply connected Lie group $G_{\mathbb{C}}$ whose Lie algebra is $\mathfrak{g}_{\mathbb{C}}$.*

Since every complex matrix A can be expressed uniquely as the sum $M + iN$ of two skew hermitian matrices, it follows that:

Lemma 21 *The complexification of the Lie algebra $\mathfrak{u}(n)$ is isomorphic to $\mathfrak{gl}(n)$, in other words: $U(n)_{\mathbb{C}} \simeq Gl(n)$*

In particular $U(n)$ sit inside $Gl(n)$ as a maximal compact subgroup.
We can now state the following result about the cohomology of the Lie algebra of the general linear group:

Theorem 37 *1). The cohomology ring of the Lie algebra $\mathfrak{gl}(n)$ is an exterior algebra in n generators of degree $1, 3, ..., 2n-1$:*

$$H^{\bullet}(\mathfrak{gl}(n)) = \Lambda[c_1, c_3, ..., c_{2n-1}];$$

2) for any given n, the (inclusion) map defined in formula (5.18) induces a map i_n^* in cohomology:

$$i_n^* : H^{\bullet}(\mathfrak{gl}(n+1)) \longrightarrow H^{\bullet}(\mathfrak{gl}(n)),$$

such that:

$$i_n^* : H^p(\mathfrak{gl}(n+1)) \longrightarrow H^p(\mathfrak{gl}(n))$$

is an isomorphism for $p \leq 2n-1$, and it maps to zero the top degree generator when $p = 2n+1$;

Proof From the previous discussion, it suffices to calculate the cohomology ring of the Lie algebra $\mathfrak{u}(n)$, which is the Lie algebra of the Lie group $U(n) = \{A \in Gl(n, \mathbb{C}) | AA^{\dagger} = \mathrm{Id}\}$. Since $U(n) = S^1 \times SU(n)$, (as a topological space), and since the Hochschild-Serre complex of a given Lie algebra is quasi-isomorphic to the de-Rham complex of the associated Lie group, the statement of the theorem will be proved once we prove that $H^{\bullet}(SU(n)) = \Lambda[c_3, ..., c_{2n-1}]$. The proof of the statement follows by induction on the rank n, and it is based on the cellular decomposition of the Lie group $SU(n)$. The (standard) action of $SU(n)$ on $\mathbb{C}^n$, has non trivial isotropy group which is isomorphic to $SU(n-1)$. Such an action define the principal bundle:

$$\begin{tikzcd} SU(n-1) \arrow[r] & SU(n) \arrow[d, "\pi"] \\ & S^{2n-1} \end{tikzcd}$$

where S^{2n-1} is the $2n-1$ dimensional sphere. It is now clear how to proceed: the bundle is locally trivial, with fiber $SU(n-1)$. The base has a cellular decomposition $S^{2n-1} = * \cup D^{2n-1}$ and the bundle is trivial when restricted to the the cell D^{2n-1}. The induction will start for $n = 3$, the case $n = 1$ being trivial and the case $n = 2$ being obvious. Let us give a glimpse for the first step of the induction. For $n = 3$:

$$\begin{tikzcd} SU(2) \arrow[r] & SU(3) \arrow[d, "\pi"] \\ & S^{5} \end{tikzcd}$$

where the fiber $SU(2) \simeq S^3$. The cellular decomposition of the of the bundle is given by the product: $(* \cup D^3) \times D^5$, where the first term is the contribution of the fiber and the second the one of the base. From this and from the cellular decomposition of the fiber above the point $* \in S^5$, we get the following cellular decomposition of the total space:

$$c^0 \cup c^3 \cup c^5 \cup c^8,$$

i.e it can be decomposed using four cells, one in dimension 0, one in dimension 3, one in dimension 5 and one in dimension 8. Now we use the following classical result about the ring cohomology of a Lie group:

Theorem 38 *(Borel) The cohomology ring of a Lie group is an exterior algebra.*

From this and the previous cellular decomposition we are forced to conclude that:

$$H^0(SU(3), \mathbb{R}) \simeq H^3(SU(3), \mathbb{R}) \simeq H^5(SU(3), \mathbb{R}) \simeq H^8(SU(3), \mathbb{R}) \simeq \mathbb{R},$$

and the ring $H^\bullet(SU(3), \mathbb{R})$ will be generated by the duals of c^3 and c^5, say y^3, y^5 such that $y^3 y^5 = -y^5 y^3 \neq 0$.

♠

Remark 32 *The argument used to prove the first of the assertions of the theorem stated above has more a pedagogical content than a rigorous one. Using such an inductive argument makes the other statements fairly clear. A more rigorous approach would be the use of the Leray-Serre spectral sequence for the fibration described above.*

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