

ANSWERS

1) Use the formula

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

to find  $f'(x)$  for

$$f(x) = \frac{1}{x-1}$$

Show all work.

Answer: Using the definition we have

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h}$$

Simplifying this expression we get

$$f'(x) = \lim_{h \rightarrow 0} \frac{-h}{h(x+h-1)(x-1)}$$

Cancelling the  $h$  from the numerator and the denominator we get

$$f'(x) = \lim_{h \rightarrow 0} \frac{-1}{(x+h-1)(x-1)}$$

We can substitute  $h = 0$  into this expression because it is a rational function to get

$$f'(x) = \frac{-1}{(x-1)^2}$$

2) Consider the function  $f(x) = 1 - |x|$ .

a) Which of the following properties holds ?

i)  $\lim_{x \rightarrow 0} f(x) = f(0)$  ii)  $\lim_{x \rightarrow 0} f(x)$  exist but  $\neq f(0)$  iii)  $\lim_{x \rightarrow 0} f(x)$  does not exist.

Explain your answer.

b) Sketch the graph of  $f'(x)$ . Clearly indicate which points are included or excluded.

Answer:

a) (i) holds because since

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

we get

$$f(x) = \begin{cases} 1 - x & x \geq 0 \\ 1 + x & x < 0 \end{cases}$$

Then

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 1 - x = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 1 + x = 1$$

Since  $f(0) = 1$  we get (i) is the answer. I.e.  $f(x)$  is continuous at  $x = 0$ .

b) From the formula for  $f(x)$  we get

$$f'(x) = \begin{cases} -1 & x > 0 \\ 1 & x < 0 \end{cases}$$

However at the point  $x = 0$  we must apply the definition. We get

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{(1-h) - 1}{h} = \lim_{h \rightarrow 0^+} \frac{-h}{h} = -1$$

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{(1+h) - 1}{h} = \lim_{h \rightarrow 0^-} \frac{h}{h} = 1$$

Since the limits are not the same from the left and the right we get that  $f'(0)$  does not exist.

Hence the graph of  $f'(x)$  is the line  $y = -1$  on  $(-\infty, 0)$  and the line  $y = 1$  on  $(0, \infty)$ . The points  $(0, -1)$  and  $(0, 1)$  are NOT included since  $f'(0)$  does not exist.

3) Find the following limits using properties of limits. Give reasons for your answer in each case.

a)

$$\lim_{x \rightarrow 0} \frac{(2+x)^2 - 4}{x}$$

b)

$$\lim_{x \rightarrow \pi/2} \frac{\sec x \cos x}{x}$$

c) If  $\lim_{x \rightarrow a} f(x) = 3$  and  $\lim_{x \rightarrow a} g(x) = 2$  find

$$\lim_{x \rightarrow a} \frac{f(x)^2 - 2g(x)}{g(x) - f(x)}$$

Answer:

a) Expanding out and simplifying the numerator we get the limit equals

$$\lim_{x \rightarrow 0} \frac{4x + x^2}{x} = \lim_{x \rightarrow 0} \frac{x(4+x)}{x}$$

We can cancel the  $x$  to get

$$\lim_{x \rightarrow 0} 4 + x = 4$$

since we can substitute  $x = 0$  into this expression since it is a polynomial.

b) We can use  $\sec x = 1/\cos x$  to simplify the numerator. Then we simply have

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

c) We can write the limit as

$$\frac{(\lim_{x \rightarrow a} f(x))^2 - 2 \lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} g(x) - \lim_{x \rightarrow a} f(x)} = \frac{3^2 - 2 \cdot 2}{2 - 3} = -5$$

since all limits exist and the denominator is not equal to zero.

4) Let

$$f(x) = \begin{cases} x + 1 & x \leq 0 \\ 1 - x^2 & 0 < x \leq 2 \\ -x + 1 & x > 2 \end{cases}$$

Find the following one-sided limits.

- a)  $\lim_{x \rightarrow 0^-} f(x)$
- b)  $\lim_{x \rightarrow 0^+} f(x)$
- c)  $\lim_{x \rightarrow 2^-} f(x)$
- d)  $\lim_{x \rightarrow 2^+} f(x)$
- e) At what points is  $f(x)$  not continuous? On what intervals is  $f(x)$  continuous?

Answer:

- a)  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x + 1 = 1$
- b)  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 1 - x^2 = 1$
- c)  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 1 - x^2 = -3$
- d)  $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} -x + 1 = -1$
- e)  $f(x)$  is not continuous at  $x = 2$  since  $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$ .  $f(x)$  is continuous at  $x = 0$  since  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$ .  $f(x)$  is continuous on the intervals  $(-\infty, 0)$ ,  $(0, 2)$ ,  $(2, \infty)$  because it is defined by polynomials on those intervals and polynomials are continuous. So the only value where  $f(x)$  is not continuous is  $x = 2$ . However we also have that  $\lim_{x \rightarrow 2^-} f(x) = f(2)$  so we can say that it is continuous from the left at  $x = 2$ . Thus the intervals where  $f(x)$  is continuous are  $(-\infty, 2]$ ,  $(2, \infty)$ .

- 5) A particle moves along a line with its position  $s(t)$  at time  $t$  given by  $s(t) = \sqrt{t}$ .
- What is its average velocity between  $t = 0$  and  $t = 2$  ?
  - Use a formula with limits to find its instantaneous velocity at  $t = 1$ .

Answer:

a)

$$\frac{s(2) - s(0)}{2 - 0} = \frac{\sqrt{2}}{2}$$

b)

$$\lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h}$$

Multiply top and bottom by  $\sqrt{1+h} + 1$  to get

$$\lim_{h \rightarrow 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{1+h} + 1)}$$

We can now cancel the  $h$  to get

$$\lim_{h \rightarrow 0} \frac{1}{(\sqrt{1+h} + 1)}$$

We can now substitute  $h = 0$  since roots of polynomials are continuous where defined and the denominator is not equal to zero to get answer of  $1/2$ .

6) Find the following limits using properties of limits. Give reasons for your answer in each case.

a)

$$\lim_{x \rightarrow 2} \frac{1 - x}{(x - 2)^2}$$

b)

$$\lim_{x \rightarrow \infty} \frac{x + 3}{\sqrt{9x^2 + 1}}$$

Answer:

a) As  $x$  approaches 2,  $1 - x$  approaches  $-1$ . As  $x$  approaches 2,  $(x - 2)^2$  approaches 0 and is also greater than zero, i.e. is a small positive number.  $-1$  over a small positive number gets larger and larger in absolute value but is negative, so the limit is equal to  $-\infty$ .

b) Rewrite the limit by dividing top and bottom by  $x$ . Since  $x = \sqrt{x^2}$  we can bring  $x^2$  under the square root in the denominator to get

$$\lim_{x \rightarrow \infty} \frac{1 + \frac{3}{x}}{\sqrt{\frac{9x^2 + 1}{x^2}}} = \lim_{x \rightarrow \infty} \frac{1 + \frac{3}{x}}{\sqrt{9 + \frac{1}{x^2}}}$$

Since  $\lim_{x \rightarrow \infty} \frac{3}{x} = \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$  and the limit of the denominator is not zero we have that the above limit can be written

$$\frac{1 + \lim_{x \rightarrow \infty} \frac{3}{x}}{\sqrt{9 + \lim_{x \rightarrow \infty} \frac{1}{x^2}}} = \frac{1 + 0}{\sqrt{9 + 0}} = \frac{1}{3}$$