MA 123 - Exam 3 - Sample Questions from a Previous Exam - Prof. Meuser

1) Find

$$\lim_{x\to\infty} \frac{e^{x/3}}{x^3}$$

ANSWER: Since $\lim_{x\to\infty} e^{x/3} = \infty$ and $\lim_{x\to\infty} x^3 = \infty$ we can apply L'Hopital's rule to get

$$\lim_{x\to\infty} \frac{e^{x/3}}{x^3} = \lim_{x\to\infty} \frac{\frac{1}{3}e^{x/3}}{3x^2} = \lim_{x\to\infty} \frac{e^{x/3}}{9x^2}$$

Since $\lim_{x\to\infty} e^{x/3} = \infty$ and $\lim_{x\to\infty} 9x^2 = \infty$ we can apply L'Hopital's rule again to get

$$\lim_{x\to\infty} \frac{e^{x/3}}{9x^2} = \lim_{x\to\infty} \frac{\frac{1}{3}e^{x/3}}{18x} = \lim_{x\to\infty} \frac{e^{x/3}}{54x}$$

Since $\lim_{x\to\infty} e^{x/3} = \infty$ and $\lim_{x\to\infty} 54x = \infty$ we can apply L'Hopital's rule again to get

$$\lim_{x\to\infty} \frac{e^{x/3}}{54x} = \lim_{x\to\infty} \frac{\frac{1}{3}e^{x/3}}{54} = \lim_{x\to\infty} \frac{e^{x/3}}{162}$$

We finally get

$$\lim_{x\to\infty} \frac{e^{x/3}}{162} = \infty$$

So our original limit is equal to ∞.

2) A fence 8 ft. tall is parallel to a tall building at a distance of 4 ft. from the building. Find the length of the shortest ladder that will go from the ground over the fence to the side of the building.

ANSWER: A picture representing the situation is a small right triangle inside a larger right triangle with hypotenuse equal to the length of the ladder. The smaller triangle has a vertical side of 8 ft. Let $x =$ the distance from the base of the fence to the base of the ladder. Let y be the vertical distance from the ground to the top of the ladder. Then the smaller right triangle has a horizontal side of x and a vertical side of 8. The larger right triangle has a horizontal side of $x+4$, a vertical side of y, and hypotenuse of length l which is equal to the length of the ladder.

We will find the value of x that minimizes l. Note that the value of x that minimizes l will be the same as the value of x that minimizes l^2. So we will let $L = l^2$. (This makes computations easier since it avoids square roots and their derivatives).

We have the relation

$$l^2 = L = (x+4)^2 + y^2$$

We wish to write L as a function of x alone. From similar triangles we have

$$\frac{y}{8} = \frac{x+4}{x} \Rightarrow y = 8(\frac{x+4}{x})$$

Hence we get

$$L = (x+4)^2 + 64\frac{(x+4)^2}{x^2} = (x+4)^2(1+\frac{64}{x^2})$$

Computing dL/dx and factoring x^3 from the denominator give

$$\frac{dL}{dx} = \frac{(x+4)(2x^3-512)}{x^3}$$

Thus dL/dx is undefined for $x = 0$ and $dL/dx = 0$ when $x = -4$ or $2x^3 - 512 = 0$ which gives $x = 4 \cdot 2^{2/3}$. Since neither $x = 0$ or $x = -4$ are possible values the only value to consider is $x = 4 \cdot 2^{2/3}$. By considering the sign of $f'(x)$ we see that $x = 4 \cdot 2^{2/3}$ is a relative minimum. Since the possible range of x is $0 \leq x \leq \infty$ and $x = 0$ gives a vertical ladder, there are no endpoint values to consider so $x = 4 \cdot 2^{2/3}$ gives the ladder of minimum length.

Substituting $x = 4 \cdot 2^{2/3}$ into the expression for L and simplifying gives

$$L = 16(1+2^{(2/3)})^3 \Rightarrow l = 4(1+2^{(2/3)})^{(3/2)} \approx 16.65 \text{ ft.}$$

3) If

$$f'(x) = 2x - \frac{3}{x^4} \quad \text{for } x > 0 \quad \text{and} f(1) = 3$$

Find $f(x)$.

ANSWER: We calculate the indefinite integral

$$\int 2x - \frac{3}{x^4}\ dx = 2\int x\ dx - 3\int x^{-4}\ dx$$

Using the rule

$$\int x^n\ dx = \frac{1}{(n+1)} x^{n+1} + C \quad \text{for } n \neq -1$$

gives

$$\int 2x - \frac{3}{x^4}\ dx = 2(\frac{1}{2})x^2 - 3\frac{1}{-3}x^{-3} + C = x^2 + x^{-3} + C = x^2 + \frac{1}{x^3} + C$$

So $f(x) = x^2 + 1/x^3 + C$ for some constant C. Substituting $x = 1$ we get $f(1) = 2 + C$. Since we were given $f(1) = 3$ we get that $C = 1$, hence

$$f(x) = x^2 + \frac{1}{x^3} + 1$$

4) Consider the area under $y = 4 - x^2$ and above the x-axis from $x = -2$ to $x = 2$. Using $n = 4$ rectangles, find:

a) An underestimate of this area.

b) An overestimate of this area. In each case make clear what rectangles you are using.

ANSWER: If one draws a sketch one gets a downward pointing parabola with vertex at $(0, 4)$ and x-intercepts of $x = -2$ and $x = 2$. For a) and b) the sub-intervals have length $\Delta x = (2 - (-2))/4 = 1$. Thus the sub-intervals are $[-2, -1], [-1, 0], [0, 1], [1, 2]$.

a) For an underestimate in each sub-interval take the height of the rectangle to be the smallest value of $f(x)$ for x in the sub-interval. Since the base of each rectangle is of length 1 the sum of the areas of the rectangles is:

$$f(-2) + f(-1) + f(1) + f(2) = 0 + 3 + 3 + 0 = 6$$

b) For an overestimate in each sub-interval take the height of the rectangle to be the largest value of $f(x)$ for x in the sub-interval. Since the base of each rectangle is of length 1 the sum of the areas of the rectangles is:

$$f(-1) + f(0) + f(0) + f(1) = 3 + 4 + 4 + 3 = 14$$

5) Sketch a region in the plane whose area is given by

$$\int_0^4 \sqrt{16 - x^2}\ dx$$

and evaluate the definite integral by interpreting it as an area.

ANSWER: With $y = \sqrt{16 - x^2}$ we have

$$y^2 = 16 - x^2 \Rightarrow x^2 + y^2 = 16$$

This is the equation of a circle with center $(0, 0)$ and radius 4. The area represented by the definite integral is the intersection of this circle with the first quadrant and is (1/4) the total area of the circle, which is $\pi 4^2$. Hence

$$\int_0^4 \sqrt{16 - x^2}\ dx = \frac{1}{4}(\pi 4^2) = 4\pi$$

6) Find

$$\int x(x^2+2)^2\ dx$$

ANSWER: Since

$$(x^2+2)^2 = (x^2)^2 + 2\cdot 2x^2 + 4 = x^4 + 4x^2 + 4$$

we get

$$x(x^2+2)^2 = x^5 + 4x^3 + 4x$$

So

$$\int x(x^2+2)^2\ dx = \int x^5 + 4x^3 + 4x\ dx = \int x^5\ dx + 4\int x^3\ dx + 4\int x\ dx$$

Using the rule

$$\int x^n\ dx = \frac{1}{(n+1)}x^{n+1} + C \quad \text{for } n \neq -1$$

gives our answer

$$\int x(x^2+2)^2\ dx = \frac{1}{6}x^6 + 4\frac{1}{4}x^4 + 4\frac{1}{2}x^2 + C = \frac{1}{6}x^6 + x^4 + 2x^2 + C$$

7) Use Newton's method to find a root of the equation

$$x^3 - x - 1 = 0$$

with $x_1 = 1$. Find the approximations x_2, x_3 and x_4.

Let $f(x) = x^3 - x - 1$. Newton's method has the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We have $f'(x) = 3x^2 - 1$ We get $f(x_1) = f(1) = -1$ and $f'(x_1) = f'(1) = 2$ so

$$x_2 = 1 - \frac{(-1)}{2} = \frac{3}{2} = 1.5$$

Continuing with $x_2 = 1.5$ and applying the formula we get $x_3 = 1.3478$ and $x_4 = 1.3252$ (rounded to four decimal places). So

$$x_2 = 1.5 \ , \ x_3 = 1.3478 \ , \ x_4 = 1.3252$$