MA 124 - Spring '09 - Exam 1 - Prof. Meuser

ANSWERS

1) (16 points) Find the derivative of

$$f(x) = \int_x^{x^2} \frac{e^t}{t}\, dt$$

State any rules and theorems you are applying and where you are applying them.

Answer: First write

$$f(x) = \int_x^0 \frac{e^t}{t}\, dt\ + \int_0^{x^2} \frac{e^t}{t}\, dt$$
$$= -\int_0^x \frac{e^t}{t}\, dt\ + \int_0^{x^2} \frac{e^t}{t}\, dt$$

Let $g(x)$ be the first integral and $h(x)$ the second integral.

Then $g'(x) = \frac{e^x}{x}$ by FTC I . To find $h'(x)$ we let $u = x^2$. Then

$$h(x) = h(u) = \int_0^u \frac{e^t}{t}\, dt$$

Use the chain rule

$$\frac{dh}{dx} = \frac{dh}{du} \cdot \frac{du}{dx}$$

$$\frac{dh}{du} = \frac{e^u}{u} \text{ by FTC I and } \frac{du}{dx} = 2x$$

Inserting in chain rule, writing in terms of x and simplifying gives :

$$\frac{dh}{dx} = 2\frac{e^{x^2}}{x}$$

Since $f'(x) = -g'(x) + h'(x)$ we get

$$f'(x) = \frac{2e^{x^2} - e^x}{x}$$

2) (16 points) If $f(x)$ is continuous and

$$\int_1^4 f(x)\ dx = 9$$

Find the value of

$$\int_1^2 x f(x^2)\ dx$$

You must explain your answer.

Answer: Let $u = x^2$. Then $xdx = (1/2)du$. When $x = 1$, $u = 1$ and when $x = 2$, $u = 4$. Using the u-substitution rule for definite integrals give

$$\int_1^2 x f(x^2)\ dx = \frac{1}{2}\int_1^4 f(u)\ du$$

$$\int_1^4 f(x)\ dx = 9 \text{ implies } \int_1^4 f(u)\ du = 9$$

so our answer is 9/2.

3) (16 points) Find the value of the definite integral

$$\int_1^e (\ln x)^2 \; dx$$

Simplify your answer.

Answer: Apply integration by parts with

$$u = (\ln x)^2 \quad dv = dx$$

Then

$$du = 2(\ln x) \cdot \frac{1}{x} \; dx \quad v = x$$

and we get

$$\int_1^e (\ln x)^2 \; dx = x(\ln x)^2|_1^e - 2\int_1^e \ln x \; dx$$

Apply parts again with

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} \; dx \quad v = x$$

Hence $\int \ln x \; dx = x \ln x - x$ and substituting in previous we get

$$\int_1^e (\ln x)^2 \; dx = x(\ln x)^2|_1^e - 2(x \ln x - x)|_1^e$$

Evaluating we get

$$e(\ln e)^2 - 2e \ln e + 2e - ((\ln 1)^2 - 2\ln 1 + 2)$$

Since $\ln 1 = 0$ and $\ln e = 1$ substituting these values and simplifying gives the answer $e - 2$.

4) (16 points) Find the indefinite integral

$$\int \frac{3x^2+2x-2}{(x-1)(x^2+x+1)}\ dx$$

Answer: Apply partial fractions. Since x^2+x+1 is an irreducible quadratic the correct partial fractions form is

$$\frac{3x^2+2x-2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

Multiplying both sides by the denominator and finding the coefficients of each degree term on the right gives

$$3x^2+2x-2 = (A+B)x^2 + (A-B+C)x + (A-C)$$

Setting the coefficients of each degree equal on the left and right and solving for A, B, C gives

$$A = 1 \quad B = 2 \quad C = 3$$

Hence

$$\int \frac{3x^2+2x-2}{(x-1)(x^2+x+1)}\ dx = \int \frac{1}{x-1}\ dx + \int \frac{2x+3}{x^2+x+1}\ dx$$

The simple u-substitution $u = x-1$ gives that the first integral equals $\ln|x-1|$ (the absolute value is necessary since $x-1$ can be negative).

For the second integral we complete the square of x^2+x+1 to get

$$x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$$

Substituting this in the integral and then making the simple u-substitution $u = x+(1/2)$ and noting $x = u-(1/2)$ gives a u-integral of

$$\int \frac{2u+2}{u^2+\frac{3}{4}}\ du$$

Split this up into the sum of the integrals

$$\int \frac{2u}{u^2+\frac{3}{4}}\ du \ + \ 2\int \frac{1}{u^2+\frac{3}{4}}\ du$$

For the first integral letting $v = u^2+(3/4)$ turns the integral into $\int dv/v = \ln|v|$ and rewriting in terms of x gives $\ln(x^2+x+1)$ (the absolute value is unnecessary since $x^2+x+1>0$ for all x.

For the second integral using

$$\int \frac{1}{u^2+a^2}\ du = \frac{1}{a}\tan^{-1}(\frac{u}{a})$$

gives

$$\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2u}{\sqrt{3}}) = \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}})$$

Inserting the values of both integrals into the above gives the answer of:

$$\ln|x-1| + \ln(x^2+x+1) + \frac{4}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + C$$

5) (16 points) Find the value of the definite integral

$$\int_1^e \frac{\sqrt{1+(\ln x)^2}}{x}\ dx$$

using the u-substitution rule for **definite** integrals and one of the following formulas:

$$\int \frac{\sqrt{a^2+u^2}}{u}\ du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$$

$$\int \sqrt{a^2+u^2}\ du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln(u+\sqrt{a^2+u^2}) + C$$

Simplify your answer.

Answer: Let $u = \ln x$. Then $du = (1/x)dx$ and when $x = 1$ $u = \ln 1 = 0$ and when $x = e$ $u = \ln e = 1$ so using u-substitution formula for definite integrals gives

$$\int_0^1 \sqrt{1+u^2}\ du$$

Using the second formula with $a = 1$ we get

$$\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2})\Big|_0^1$$

Evaluating we get (using $\ln 1 = 0$) the answer

$$\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1+\sqrt{2})$$

6) (16 points) Determine whether or not the following limit exists and find it's value if it does exist.

$$\lim_{x\to\infty} \frac{\sin x}{x}$$

You must explain your answer using properties of limits in order to receive credit.

Answer: The limit does exist and equals 0.
The reason is that since $-1 \leq \sin x \leq 1$ for all x we can divide this by x to get

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

Since

$$\lim_{x\to\infty} \frac{-1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

by the Squeeze theorem we also have

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$