MA 124 - Spring '09 - Exam 2 - Prof. Meuser

ANSWERS

1) (16 points) Determine whether the improper integral

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4}\, dx$$

converges or diverges. Write any necessary limits needed to determine your answer, and explain why those limits exist or do not exist using properties of limits.

Answer: By definition the improper integral is equal to

$$\lim_{a\to-\infty} \int_a^0 \frac{x}{1+x^4}\, dx + \lim_{b\to\infty} \int_0^b \frac{x}{1+x^4}\, dx$$

We now find the indefinite integral

$$\int \frac{x}{1+x^4}\, dx$$

by noting $x^4 = (x^2)^2$ and letting $u = x^2$. Then $du = 2x\ dx$ and the integral becomes

$$\frac{1}{2}\int \frac{1}{1+u^2}\, du \ = \frac{1}{2}\tan^{-1} u = \frac{1}{2}\tan^{-1} x^2$$

Inserting this back into the limits we get the improper integral

$$= \lim_{a\to-\infty} \frac{1}{2}\tan^{-1} x^2\Big|_a^0 + \lim_{b\to\infty} \frac{1}{2}\tan^{-1} x^2\Big|_0^b$$

$$= \frac{-1}{2}\lim_{a\to-\infty} \tan^{-1} a^2 + \frac{1}{2}\lim_{b\to\infty} \tan^{-1} b^2$$

By considering the graph of $y = \tan x$ which means $\tan^{-1} y = x$ we see that

$$\lim_{a\to-\infty} \tan^{-1} a^2 = \lim_{b\to\infty} \tan^{-1} b^2 = \frac{\pi}{2}$$

So the improper integral converges to the value

$$\frac{-1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2} = 0$$

2) (16 points) What can you say about the convergence of divergence of the infinite series $\sum a_n$ in each of the following cases and why?

a) $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{2}$
b) $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$
c) $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 2$

Answer:

a) Since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ the series $\sum a_n$ converges (absolutely) by the ratio test.
b) Since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$ we cannot conclude anything because this is applying the ratio test and a limit of 1 gives no information about the convergence or divergence of the series $\sum a_n$.
c) Since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$ the series $\sum a_n$ diverges by the ratio test.

3) (16 points) The sum of the first 50 terms, S_{50}, is used to estimate the sum, S, of the convergent series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

a) Estimate the error $|R_{50}| = |S - S_{50}|$ using the Alternating Series Estimation Theorem.
b) Is $S_{50} > S$ or $S_{50} < S$? Explain.

Answer:

a) We know

$$|R_{50}| < b_{51} = \frac{1}{51}$$

by the Alternating Series Estimation Theorem. As a decimal we have $1/51 = .019\cdots$ so rounding up we get $|R_{50}| < .02$.

b) Since S_{50} ends with the term $(-1)^5 1/50 = -1/50$ we get $S_{50} < S$ since the partial sums of an alternating series alternate above and below the sum S of the series and $S_{50} < S$ since the last term of S_{50} is negative.

4) (16 points) Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

converges or diverges. Explain your answer and make clear which test you are using.

Answer:
Apply the integral test. Consider the associated improper integral

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln x}\, dx$$

To find the indefinite integral

$$\int \frac{1}{x \ln x}\, dx$$

we let $u = \ln x$. Then $du = (1/x)dx$ and we get

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1}{u}\, du = \ln u = \ln(\ln x)$$

Inserting this in the limit gives

$$\lim_{b \to \infty} \ln(\ln x)\Big|_2^b = \lim_{b \to \infty} \ln(\ln b) - \ln(\ln 2)$$

Since as $b \to \infty$, $\ln b \to \infty$ we get

$$\lim_{b \to \infty} \ln(\ln b) = \infty$$

so the improper integral diverges.

Hence the series diverges by the integral test.

Note: Ratio test cannot be used on this series since one will get

$$\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = 1$$

Comparison test (either basic or limit) with a p-series will not work with this series either.

5) (16 points)

a) Determine whether the **sequence**

$$a_n = \frac{n}{2n+1}$$

converges or diverges. If it converges, find the limit. Explain your answer.

b) Determine whether the **series**

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{2n+1}$$

converges or diverges. Explain your answer.

Answer:

a)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}$$

so the sequence converges to 1/2.

b) Since $\lim_{n\to\infty} a_n \neq 0$ the series diverges by the n-th term test for divergence.

6)

a) Determine whether the **sequence**

$$a_n = (-1)^n \frac{\sqrt{n}}{1+n}$$

converges or diverges. If it converges, find the limit. Explain your answer.

b) Determine whether the **series**

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+n}$$

converges absolutely, converges but not absolutely or diverges. Explain your answer and make clear any tests you are applying.

Answer:

a) Consider

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{\sqrt{n}}{1+n}$$

Dividing top and bottom by n we get

$$\lim_{n\to\infty} \frac{\sqrt{n}}{1+n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}+1} = \frac{0}{0+1} = 0$$

Since $\lim_{n\to\infty} |a_n| = 0$ we have $\lim_{n\to\infty} a_n = 0$ by the absolute value property. So the sequence converges to zero.

b1) In order to determine whether or not the series converges absolutely we must consider

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n}$$

We can apply the limit comparison test with $\sum b_n = \sum 1/\sqrt{n}$. We first find

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+n} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{n}{1+n} = 1$$

Since this limit is $\neq 0$, ∞ is is OK to compare. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

is a p-series with $p = 1/2 < 1$ we have that $\sum b_n$ diverges. Hence $\sum |a_n|$ diverges by the limit comparison test. Therefore our original series $\sum a_n$ does **not** converge absolutely.

Note: If one applies the ratio test here one will get a limit of one, which gives no info.

b2) Our original series $\sum a_n$ is an alternating series equal to $\sum(-1)^n b_n$ with

$$b_n = \frac{\sqrt{n}}{1+n}$$

We have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{1+n} = 0$$

as was shown in a). We now ask whether or not $b_{n+1} < b_n$ for all n, i.e. is

$$\frac{\sqrt{n+1}}{2+n} < \frac{\sqrt{n}}{1+n}$$

for all n. It is difficult to show this inequality directly so we instead let

$$f(n) = b_n = \frac{\sqrt{n}}{1+n}$$

and find the derivative $f'(n)$

$$f'(n) = \frac{1}{2}\frac{n^{1/2}(\frac{1}{n} - 1)}{(1+n)^2}$$

Since $1/n - 1 < 0$ for $n > 1$ and everything else in the expression is positive we have $f'(n) < 0$ for $n > 1$ which means that $f(n)$ is decreasing which means that $b_{n+1} = f(n+1) < f(n) = b_n$. Hence $\sum a_n$ converges by the Alternating Series Test, but does not converge absolutely by b1).

b3) The series $\sum a_n$ does not diverge since b2) shows it converges, however it does not converge absolutely by b1).