MA 124 - Spring '09 - Exam 3 - ANSWERS- Prof. Meuser

1) (16 points) Find the radius **and** the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \sqrt{n}\, x^n$$

Explain any tests you are using to show convergence or divergence, including at the end-points of the interval.

ANSWER: Applying the ratio test we get

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot |x| = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot |x|$$

Since $\lim_{n\to\infty}(1/n) = 0$ we get a limit of $|x|$.

Hence our series converges for $|x| < 1$. So the radius of convergence is equal to 1. The endpoint $x = 1$ gives the infinite series $\sum_{n=0}^{\infty} \sqrt{n}$ which diverges by the n-th term test. The endpoint $x = -1$ gives the infinite series $\sum_{n=0}^{\infty}(-1)^n\sqrt{n}$ which also diverges by the n-th term test.

Hence the interval of convergence is $(-1, 1)$.

2) (16 points) Find the Maclaurin series for

$$f(x) = \frac{x}{4+x^2}$$

and its radius of convergence.

NOTE: Your answer must be written in the form $\sum_{n=0}^{\infty} c_n x^n$ and include a general formula for all the coefficients c_n.

ANSWER: We first rewrite $f(x)$ as

$$\frac{x}{4+x^2} = \frac{x}{4\left(1+\frac{x^2}{4}\right)} = \frac{x}{4} \cdot \frac{1}{\left(1+\frac{x^2}{4}\right)}$$

Hence starting with the series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \, x^n \qquad \text{for} \quad |x| < 1$$

we can

a) substitute $x^2/4$ for x.

b) multiply by $x/4$ to get the series for $f(x)$.

For a), since

$$\left(\frac{x^2}{4}\right)^n = \frac{x^{2n}}{4^n}$$

we get for a) the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{4^n} x^{2n}$$

which converges for

$$|\frac{x^2}{4}| < 1 \Rightarrow \frac{|x|^2}{4} < 1 \Rightarrow |x|^2 < 4 \Rightarrow |x| < 2$$

For b), since

$$\frac{x}{4} \cdot \frac{x^{2n}}{4^n} = \frac{x^{2n+1}}{4^{n+1}}$$

we get for b) and our original $f(x)$ the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{n+1}} x^{2n+1}$$

Since multiplying by $x/4$ doesn't change the radius, the radius of convergence is equal to 2.

3) (16 points)

a) If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all x, what is the formula for c_n ? Why?

b) Let $f(x) = \sin(x^2)$. We know:

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \; x^{4n+2}$$

Use this and your answer in a) to find $f^{(10)}(0)$.

TIP: Be careful which coefficients you are using!

ANSWER:

a)

$$c_n = \frac{f^{(n)}(0)}{n!}$$

This is because if $f(x)$ is equal to a power series $\sum c_n x^n$ then that series is the Maclaurin series which has the above formula for the coefficients.

b) From a) we know that

$$\frac{f^{(10)}(0)}{10!}$$

is the coefficient of x^{10}. In the give series we see that the coefficient of x^{10} occurs for $n = 2$, and hence

$$\frac{f^{(10)}(0)}{10!} = (-1)^2 \frac{1}{(2 \cdot 2 + 1)!} = \frac{1}{5!}$$

Hence

$$f^{(10)}(0) = \frac{10!}{5!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30,240$$

4) (16 points) Use the binomial series to find the Maclaurin series of

$$f(x) = \frac{x^2}{(1-x)^3}$$

a) First give an answer **without** simplifying the binomial coefficients.
b) Next simplify the binomial coefficients to find a general formula for **all** of the coefficients.
c) Use your answer in b) to find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+3}}$$

ANSWER:
a) We can rewrite $f(x)$ as

$$f(x) = x^2(1+(-x))^{-3}$$

Starting with the binomial series for $k = -3$

$$(1+x)^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} x^n$$

we can
a1) substitute $-x$ for x.
a2) multiply by x^2 to get the series for $f(x)$. Since $(-x)^n = (-1)^n x^n$ we get for a1)

$$\sum_{n=0}^{\infty} (-1)^n \binom{-3}{n} x^n$$

Since $x^2 \cdot x^n = x^{n+2}$ we get for a2) and $f(x)$

$$\sum_{n=0}^{\infty} (-1)^n \binom{-3}{n} x^{n+2}$$

b) We now simplify $\binom{-3}{n}$ by computing for $n = 0, 1, 2, \cdots$ until we see a pattern. We get

$$\binom{-3}{0} = 1 \ , \ \binom{-3}{1} = -3 \ \binom{-3}{2} = \frac{3 \cdot 4}{2!}$$

$$\binom{-3}{3} = \frac{(-3)(-4)(-5)}{3!} = \frac{-3 \cdot 4 \cdot 5}{3!} \ \binom{-3}{4} = \frac{(-3)(-4)(-5)(-6)}{4!} = \frac{3 \cdot 4 \cdot 5 \cdot 6}{4!}$$

For the sign we get positive for n even and negative for n odd so $(-1)^n$. We get $n!$ in the denominator.

For the top we have the product of the integers from 3 up to $(n+2)$ (which works for $n \geq 1$). This can also be written as

$$\frac{(n+2)!}{2}$$

Combining the above we get the general formula

$$\binom{-3}{n} = (-1)^n \frac{(n+2)!}{2 \cdot n!} = (-1)^n \frac{(n+2)(n+1)}{2} \quad \text{for } n \geq 1$$

and checking $n = 0$ we see the formula works for $n = 0$ also.

Substituting this into a) and since $(-1)^n \cdot (-1)^n = (-1)^{2n} = 1$ we get answer

$$f(x) = \frac{x^2}{(1-x)^3} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)(n+1)}{2} x^{n+2}$$

c) If we substitute $x = 1/2$ into the above series, since

$$\left(\frac{1}{2}\right)^{n+2} = \frac{1}{2^{n+2}}$$

we get

$$\sum_{n=0}^{\infty} (-1)^n \frac{(n+2)(n+1)}{2} \cdot \frac{1}{2^{n+2}} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)(n+1)}{2^{n+3}}$$

which is the series in the question. Hence the above series equals the value $f(1/2)$ and

$$f\left(\frac{1}{2}\right) = \frac{(\frac{1}{2})^2}{(1-(\frac{1}{2}))^3} = \frac{(\frac{1}{2})^2}{(\frac{1}{2})^3} = \frac{1}{(\frac{1}{2})} = 2$$

5) (16 points)

a) Find the Taylor polynomial $T_3(x)$ of degree 3 for $f(x) = xe^{-2x}$ at $a = 0$.

Hint: You do **not** have to compute the derivatives if you know how to use a formula you have been given.

b) Use Taylor's inequality to estimate the accuracy of

$$xe^{-2x} \approx T_3(x)$$

when $|x| \le 0.1$.

You can use

$$f^{(4)}(x) = e^{-2x}(16x - 32)$$

and your calculator.

b1) Find a value of M with $|f^{(4)}(x)| \le M$ for $|x| \le 0.1$. Give reasons why your choice of M is correct.

b2) Now use your choice of M to get an explicit **upper bound** for $|R_3(x)|$ which should first be written as a fraction. After writing it as a fraction you can give the **upper bound** as a decimal with **two** (i.e. not more than two) non-zero decimal digits. Show all work to get your answers.

ANSWER a): Use

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We can get the Taylor series for $f(x)$ by:

a1) Substitute $-2x$ for x.

a2) Multiply by x.

a1) Since $(-2x)^n = (-1)^n 2^n x^n$ we get

$$e^{-2x} = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!} x^n$$

a2) Since $x \cdot x^n = x^{n+1}$ we get

$$xe^{-2x} = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!} x^{n+1}$$

Expanding out to the degree 3 term (which occurs for $n = 2$) we get:

$$T_3(x) = x - \frac{2}{1!}x^2 + \frac{2^2}{2!}x^3 = x - 2x^2 + 2x^3$$

ANSWER b):

b1) If graph $f^{(4)}(x)$ one sees that the largest value of $|f^{(4)}(x)|$ occurs when $x = -0.1$ and that $|f^{(4)}(-0.1)| = 41.039\cdots < 42$. So we can take $M = 42$.

b2) Taylor's Inequality tells us

$$|R_3(x)| \leq \frac{M|x|^4}{4!}$$

With $M = 42$ and since $|x| < 0.1$ we get

$$|R_3(x)| \leq \frac{42(.1)^4}{4!} \leq 1.75 \times 10^{-4} = .000175$$

Rounding up to two significant digits we get

$$|R_3(x)| \leq .00018$$

6) (16 points)

a) Put $z = \sqrt{3} + i$ and $w = 1 + \sqrt{3}i$ in polar form $r(\cos\theta + i\sin\theta)$ where you give the values of θ corresponding to z and w.
Then find the polar forms of zw and z/w where you give their values of θ.

b) Use DeMoivre's Theorem:

$$z^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$$

to find the value of $w^6 = (1 + \sqrt{3}i)^6$ in both polar form and $a + bi$ form.

c) Find the cube roots of i in both polar form and $a + bi$ form.

c) Give the value of

$$e^{i\frac{\pi}{3}}$$

in the form $a + bi$ where any values of $\cos\theta$ and $\sin\theta$ are given.

IMPORTANT NOTE

You should check your work to all of the above problems **CAREFULLY**. In particular you should be sure that you have followed the instructions and your answers are in the form requested.