

Complex Exponential, Logarithm, and Powers

Definition: For a complex number  $z$  we define

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

This series converges for all  $z \in \mathbf{C}$  as can be seen by applying the ratio test.

The identity

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

holds, as one can show by multiplying the series for  $e^{z_1}$  and  $e^{z_2}$ .

We have if  $z = x + iy$  then  $e^z = e^x e^{iy}$  and if one expands the series for  $e^{iy}$  into even and odd powers of  $y$  one sees that

$$e^{iy} = (\cos y + i \sin y)$$

(This is done in the book).

Note that  $|e^z| = e^x > 0$ . So  $e^z$  is never zero. Note however that unlike  $e^x$  it can be negative. In particular  $e^{i\pi} = \cos \pi + i \sin \pi = -1$  And in fact  $e^z$  can equal any non-zero complex number as can be seen by solving  $e^z = w = |w|(\cos \theta + i \sin \theta)$ .

Similarly, one can also define:

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

These series also converge for all  $z \in \mathbf{C}$  as can be seen by applying the ratio test.

If one expands the series for  $e^{iz}$  into even and odd powers of  $z^n$  one gets the identity

$$e^{iz} = \cos z + i \sin z$$

(The argument is the same for  $z$  complex as for  $y$  real above).

We also note that since the power series of  $\cos z$  has only even powers of  $z$ , and the power series of  $\sin z$  has only odd powers of  $z$ , that we obtain the identities:

$$\cos(-z) = \cos z$$

$$\sin(-z) = -\sin z$$

Thus  $e^{-iz} = \cos z - i \sin z$  and by adding  $e^{iz}$  and  $e^{-iz}$  and solving for  $\cos z$  and  $\sin z$  one also obtains the identities

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

In defining a complex logarithm one would like it to satisfy the same properties as logarithm in the real numbers, that is if  $x, y$  are real numbers

$$e^y = x \Leftrightarrow y = \ln x$$

If we try to define the logarithm of a complex number  $z$ , by

$$e^w = z \Leftrightarrow w = \ln z$$

we encounter a problem. If we write  $w = u + iv$  and  $z = |z|(\cos \theta + i \sin \theta)$  we get that

$$e^u = |z| \Rightarrow u = \ln |z|$$

but also that  $\cos v + i \sin v = \cos \theta + i \sin \theta$  which tells us that

$$v = \theta + n \cdot 2\pi \quad \text{for any } n \in \mathbf{Z}$$

I.e. there are infinitely many  $w$  that satisfy  $e^w = z$ ! However if we allow multi-valued functions we can (and do) define for  $z = |z|(\cos \theta + i \sin \theta) \neq 0$  the complex logarithm

$$\log(z) = \ln |z| + i(\theta + n \cdot 2\pi) \quad \text{for any } n \in \mathbf{Z}$$

One calls any single value satisfying the above a logarithm of  $z$ . In order to define a function one can (and does) choose a specific value of  $z$  by defining

$$\text{Log}(z) = \ln |z| + i\theta \quad -\pi < \theta \leq \pi, \quad z \neq 0$$

This is called the **principal value of the logarithm**.

$\text{Log}(z)$  has a power series expansion analogous to the one for  $\ln x$ , i.e.

$$\text{Log}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n \quad \text{for } |z-1| < 1$$

Recall that for  $a$  and  $x$  real numbers with  $a > 0$ ,  $a^x$  is defined by:

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

To define  $a^z$  for  $a, z$  complex numbers with  $a \neq 0$  we similarly define

$$a^z = e^{z \log a}$$

Since  $\log a$  is multiple-valued, so is  $a^z$  in general. One can define a single value of  $a^z$  by choosing the specific value  $e^{z \text{Log } a}$  which is called the **principal value of  $a^z$** .

## Exercises on Complex Exponential, Logarithm, and Powers

1) Show

- a)  $\exp(2 \pm 3\pi i) = -e^2$
- b)  $\exp(z + \pi i) = -\exp(z)$

2)

- a) Show that for any complex number  $w \neq 0$ ,  $e^z = w$  has at least one solution  $z$ .
- b) Find the solutions of  $e^z = -2$ .
- c) Find the solutions of  $e^z = 1 + \sqrt{3}i$ .

3) Show

- a)  $\text{Log}(-ei) = 1 - (\pi/2)i$
- b)  $\text{Log}(1 - i) = (1/2) \ln 2 - (\pi/4)i$
- c)  $\log e = 1 + i(n \cdot 2\pi) \quad n \in \mathbf{Z}$
- d)  $\log i = (2n + (1/2))\pi i \quad n \in \mathbf{Z}$
- e)  $\log(-1 + \sqrt{3}i) = \ln 2 + 2(n + (1/3))\pi i \quad n \in \mathbf{Z}$

4) Show that the set of values of  $\log(i^2)$  is **not** the same as the set of values of  $2 \log i$ .

5) Show that

a)

$$(1 + i)^i = \exp\left(\frac{i}{2} \ln 2\right) \exp\left(-\frac{\pi}{4} + 2n\pi\right) \quad n \in \mathbf{Z}$$

- b)  $(-1)^{1/\pi} = e^{(2n+1)i} \quad n \in \mathbf{Z}$

6) Find the principal value of

- a)  $i^i$
- b)  $(1 - i)^{4i}$

7) Using the definition of  $a^z$  for  $a, z$  complex show

- a) If  $z = n$  where  $n$  is an integer then  $a^n$  is a single value, which is the usual definition of  $a^n$ .
- b) If  $z = m/n$  where  $m, n$  are integers then  $a^{m/n}$  is the set of values  $b^m$  where  $b$  is an  $n$ -th root of  $a$  (which is the definition you are already familiar with).
- c) If  $a$  is real and  $z = b$  is also real with  $b$  irrational, then  $a^b$  has infinitely many values, even if  $a > 0$ . (I.e. this definition includes more values than the one for the real numbers alone).