

MA 129 - Homework - October 23, 2007

Integral Test and Comparison Test for Infinite Series

Basic Problems from Stewart: 8.3: 4, 8, 13 - 15, 19 - 21

More Difficult Problems From Stewart: 25, 27, 35, 37, 39, 40

EXTRA CREDIT: In Problem 37 you showed that the infinite series represented by a decimal expansion, i.e.

$$0.d_1d_2d_3 \cdots d_n \cdots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n} + \cdots$$

always converges. The goal of this extra credit problem is to **prove** the following result:

A number  $x$  is a rational number if and only if it's decimal expansion is eventually repeating, i.e. it's decimal expansion has the form

$$n.c_1c_2 \cdots c_m d_1d_2d_3 \cdots d_k d_1d_2d_3 \cdots d_k \cdots d_1d_2d_3 \cdots d_k \cdots$$

where  $n = [x]$  is the greatest integer less than or equal to  $x$  and the repeating part is  $d_1d_2d_3 \cdots d_k$ .

a) Show that a number which has an eventually repeating decimal expansion is a rational number.

First show that if  $.d_1d_2d_3 \cdots d_k d_1d_2d_3 \cdots d_k \cdots d_1d_2d_3 \cdots d_k$  is a repeating decimal then it is a rational number and give an explicit formula for it (simplified to a rational number in the form  $p/q$  for  $p, q$  integers).

Generalize your result to a decimal expansion which is eventually repeating, i.e. has the form  $.c_1c_2 \cdots c_m d_1d_2d_3 \cdots d_k d_1d_2d_3 \cdots d_k \cdots d_1d_2d_3 \cdots d_k \cdots$ .

b) Show a rational number must have a decimal expansion which is eventually repeating. This direction is more difficult, so some hints and useful formulas are given below.

b1) Let  $0 \leq x = p/q < 1$ . Show

$$d_n = [10^n x - (10^{n-1}d_1 + \cdots + 10d_{n-1})]$$

and

$$0 \leq 10^n x - (10^{n-1}d_1 + \cdots + 10d_{n-1} + d_n) < 1$$

b2) Write

$$10^n x = a_n + \frac{b_n}{q}$$

where  $a_n, b_n$  are positive integers and  $0 \leq b_n < q$ . Show

$$a_n = 10^{n-1}d_1 + \cdots + 10d_{n-1} + d_n$$

See back of this page for further hints.

So

$$10a_n - (10^n d_1 + \cdots + 10^2 d_{n-1} + 10d_n) = 0$$

Applying the first formula in b1) with  $n = n + 1$  show

$$d_{n+1} = \left[ \frac{10b_n}{q} \right]$$

Now use the fact that  $b_n$  has only the finite number of possibilities

$$b_n = 0, 1, \dots, q - 1$$

and formulas above to show that the decimal expansion must eventually repeat.

NOTE: There is no partial credit on the extra credit problem. In particular there is no credit for a) if you have not done b) also.