GENERAL INSTRUCTIONS: Use definitions and theorems we proved in class to explain your answers to all the questions below.

1) (16 points) We have the following polynomial identity for any integer \( n > 1 \):

Use this identity to show that if \( a, n \) are integers greater than one and \( a^n - 1 \) is prime, then \( a = 2 \) and \( n \) is prime.

Answer: If \( a > 2 \), letting \( x = a \) in the above gives

\[
a^n - 1 = (a - 1)(a^{n-1} + a^{n-2} + \cdots + a + 1)
\]

hence \((a - 1)|(a^n - 1)\). Since \( a > 2 \Rightarrow a - 1 > 1 \) this is a non-trivial factor and \( a > 2 \) implies \( a^n - 1 \) is not a prime.

If \( n \) is not prime then we can write \( n = mn' \) with \( 1 < m, n' < n \). Then \( a^n = (a^m)^{n'} \).

So letting \( x = a^m \) and \( n = n' \) in the identity gives

\[
a^n - 1 = (a^m)^{n'} - 1 = (a^m - 1)((a^m)^{n'-1} + (a^m)^{n'-2} + \cdots + a^m + 1)
\]

hence \( a^m - 1|a^n - 1 \). Since \( m > 1 \Rightarrow a^m - 1 > 1 \) and \( m < n \Rightarrow a^m - 1 < a^n - 1 \) we have this is a non-trivial factor and \( a^n - 1 \) is not prime.

Hence if \( a^n - 1 \) is prime we must have \( a = 2 \) and \( n \) is prime.
2) (16 points) Show every integer greater than 11 is the sum of two composite numbers.
   
   Hint: Consider the cases of $n$ even and $n$ odd separately.

   Answer:

   **Case 1:** $n > 11$ is even.

   We know that every even integer $> 2$ is composite. In particular 4 is composite. We also know that $(even) - (even) = (even)$. Hence $n - 4$ is even. $n > 11 \Rightarrow (n - 4) > 7 > 2$. So $(n - 4)$ is an even composite number. So we have

   $$n = 4 + (n - 4)$$

   **Case 2:** $n > 11$ is even.

   We know that $(odd) - (odd) = (even)$. The smallest odd composite number is 9. So $n - 9$ is even and $n > 11 \Rightarrow (n - 9) > 2$. Hence $(n - 9)$ is an even number $> 2$ hence it is composite. So we have

   $$n = 9 + (n - 9)$$
3) (16 points)
   a) Show the product of two integers of the form $3k + 1$ is also of that form.
      
      Answer:
      
      $$(3k + 1)(3k' + 1) = 9kk' + 3k + 3k' + 1 = 3(3kk' + k + k') + 1$$
      
      $k, k'$ integers implies $(3kk' + k + k')$ is an integer, so the product has form $3k + 1$.
   
   b) Show
      
      $2^n$
      
      has the form $3k + 1$, $k \in \mathbb{Z}$ for every integer $n \geq 1$.
      
      Answer: We prove this by induction.
      
      It is true for $n = 1$ since for $n = 1$ the value is $2^2 = 4 = 3 \cdot 1 + 1$.
      
      Assume the statement is true for $n$. We wish to show it is true for $n + 1$. We have
      
      $$2^{n+1} = 2^{n \cdot 2} = \left(2^n\right)^2 = 2^n \cdot 2^n$$
      
      By the inductive hypothesis $2^n$ has the form $3k + 1$, so by a) it’s square also has the form $3k + 1$.
   
   c) Show
      
      $3 \mid 2^n + 5$
      
      for every integer $n \geq 1$.
      
      Answer: By b) we can write
      
      $2^n = 3k + 1$
      
      for some integer $k$. Hence
      
      $$2^n + 5 = (3k + 1) + 5 = 3k + 6 = 3(k + 2)$$
      
      Hence this shows $3|2^n + 5$. 
4) (16 points)

a) Find the least common multiple \([111, 303]\) of 111 and 303.

Answer: We note that 111 and 303 are both divisible by 3. If we factor them we get

\[
\begin{align*}
111 &= 3 \cdot 37 \\
303 &= 3 \cdot 101
\end{align*}
\]

We note that 37 and 111 are both prime (since \([\sqrt{37}] = 6 \) and 37 is not divisible by 2, 3, 5 and \([\sqrt{101}] = 10 \) and 101 is not divisible by 2, 3, 5, 7). Hence we can write their prime factorizations as

\[
\begin{align*}
111 &= 3^1 \cdot 37^1 \cdot 101^0 \\
303 &= 3^1 \cdot 37^0 \cdot 101^1
\end{align*}
\]

and by properties of the least common multiple we have

\[
[111, 303] = 3 \cdot 37 \cdot 101 = 11,211
\]

b) Find another pair of positive integers \(a, b\) with \(a, b > 1\) with the property that their least common multiple \([a, b] = [111, 303]\).

c) Describe all the pairs of positive integers \(a, b\) with the property that their least common multiple \([a, b] = [111, 303]\).

Answer b) and c): If we factor \(a, b\) and consider the formula for the least common multiple in terms of their factorizations, we see that no other primes than 3, 37, 101 can divide \(a\) or \(b\) since then the least common multiple would have that prime in its factorization. So we can write

\[
\begin{align*}
a &= 3^{e_1} \cdot 37^{e_2} \cdot 101^{e_3} \\
303 &= 3^{f_1} \cdot 37^{f_2} \cdot 101^{f_3}
\end{align*}
\]

But then the formula for the least common multiple of \(a\) and \(b\) is

\[
[a, b] = 3^{\max(e_1, f_1)} \cdot 37^{\max(e_2, f_2)} \cdot 101^{\max(e_3, f_3)}
\]

Hence we must have \(\max(e_i, f_i) = 1\) for \(i = 1, 2, 3\). Any choice of \(a, b\) satisfying this will work, i.e. \(a = 37 \cdot b = 3 \cdot 37 \cdot 101 = 11,211\).
5)(16 points)
a) If the greatest common divisor \((a, b)\) of \(a\) and \(b\) satisfies \((a, b) = 1\) show \((3a, 3b) = 3\).

Answer: Let \(d = (3a, 3b)\). Since \(3|3a\) and \(3|3b\) we have, by properties of the greatest common divisor, that \(d \geq 3\). Since \((a, b) = 1\) there are integers \(x, y\) with \(ax + by = 1\). Multiplying this by 3 we have

\[(3a)x + (3b)y = 3\]

Since the greatest common divisor \(d\) is the smallest positive number that can be written as a linear combination of \(3a\) and \(3b\) we get from this equation that \(d \leq 3\). Combining the inequalities give \(d = (3a, 3b) = 3\).

b) If \((a, b) = 1\) show \((a + 2b, 2a + b) = 1\) or 3

Answer: Let \(d = (a + 2b, 2a + b)\). Since \(d|a + 2b\) and \(d|2a + b\) it divides any linear combination of them. In particular we have

\[d \mid 2(a + 2b) - (2a + b) = 3b\]

\[d \mid (-1)(a + 2b) + 2(2a + b) = 3a\]

Hence \(d|3a\) and \(d|3b\). By properties of the greatest common divisor we have that \(d|(3a, 3b)\). But by a) \((3a, 3b) = 3\) so we get that \(d|3\). The only divisors of 3 are 1 and 3, so \(d = 1\) or \(d = 3\).
6) (16 points) Find all integer solutions of the system of linear diophantine equations:

\[
\begin{align*}
x + y + z &= 100 \\
x + 8y + 50z &= 212
\end{align*}
\]

Answer: \( x + y + z = 100 \Rightarrow x = 100 - y - z \). Insert this into the second equation to get

\[
(100 - y - z) + 8y + 50z = 7y + 49z = 212
\]

So we are reduced to solving \( 7y + 49z = 212 \). Since \( (7, 49) = 7 \) and \( 7|212 = 7 \cdot 16 \) there are solutions. First we find a solution to

\[
7x + 49z = 7
\]

\( x = 1 \), \( y = 0 \) is one solution, since \( 7 \cdot 1 + 49 \cdot 0 = 7 \). We multiply this by 16 to get

\[
7 \cdot 16 + 49 \cdot 0 = 16 \cdot 7 = 212
\]

So \( y_0 = 16 \), \( z_0 = 0 \) is one solution. All solutions \( y, z \) are then

\[
y = 16 + \frac{49}{7} \cdot t = 16 + 7 \cdot t \\
z = 0 - \frac{7}{7} \cdot t = -t \quad t \in \mathbb{Z}
\]

Substituting these back into \( x = 100 - y - z \) we get \( x = 100 - (16 + 7t) - (-t) = 84 - 6t \). So all solutions are

\[
x = 84 - 6t \\
y = 16 + 7t \\
z = -t \quad t \in \mathbb{Z}
\]