MATH 114 QUIZ 11
13 DECEMBER 2016

Solve the following two problems. Justify all of your work with clearly written mathematics. Some trigonometric identities are listed on the back.

(1) Prove the following identity:

$$\sin(\theta) = 1 - \frac{\cos(\theta)^2}{1+\sin(\theta)}$$

Starting with the right side and finding a common denominator, we have

$$1 - \frac{\cos(\theta)^2}{1+\sin(\theta)} = \frac{1+\sin(\theta)}{1+\sin(\theta)} - \frac{\cos(\theta)^2}{1+\sin(\theta)} = \frac{1+\sin(\theta)-\cos(\theta)^2}{1+\sin(\theta)}.$$

By the Pythagorean theorem, $\cos(\theta)^2 = 1 - \sin(\theta)^2$, so

$$\begin{aligned} 1+\sin(\theta)-\cos(\theta)^2 &= 1+\sin(\theta)-(1-\sin(\theta)^2) \\ &= \sin(\theta)+\sin(\theta)^2 = \sin(\theta)(1+\sin(\theta)). \end{aligned}$$

Thus,

$$\frac{1+\sin(\theta)-\cos(\theta)^2}{1+\sin(\theta)} = \frac{\sin(\theta)(1+\sin(\theta))}{1+\sin(\theta)} = \sin(\theta),$$

proving the identity.

Alternatively, we could *start* with the Pythagorean theorem:

$$1 - \frac{\cos(\theta)^2}{1+\sin(\theta)} = 1 - \frac{1-\sin(\theta)^2}{1+\sin(\theta)},$$

and $1-\sin(\theta)^2$ factors as $(1-\sin(\theta))(1+\sin(\theta))$, so

$$1 - \frac{1-\sin(\theta)^2}{1+\sin(\theta)} = 1-(1-\sin(\theta)) = \sin(\theta).$$

(2) Compute the exact value of $\sin(\sin^{-1}(\frac{-1}{3}) + \frac{\pi}{4})$. (Once your answer involves only arithmetic operations and/or square roots, you do not need to simplify further.)

By the angle addition identity for sine,

$$\sin(\sin^{-1}(\tfrac{-1}{3}) + \tfrac{\pi}{4}) = \sin(\sin^{-1}(\tfrac{-1}{3}))\cos(\tfrac{\pi}{4}) + \cos(\sin^{-1}(\tfrac{-1}{3}))\sin(\tfrac{\pi}{4}).$$

We compute the four terms individually:

$$\begin{aligned} \sin(\sin^{-1}(-1/3)) &= -1/3, \\ \cos(\pi/4) &= \frac{\sqrt{2}}{2}, \\ \sin(\pi/4) &= \frac{\sqrt{2}}{2}, \end{aligned}$$

and finally,

$$\cos(\sin^{-1}(-1/3)) = \sqrt{1-\sin(\sin^{-1}(-1/3))^2} = \sqrt{1-(-1/3)^2} = \sqrt{8/9} = \frac{\sqrt{8}}{3}.$$

So

$$\sin(\sin^{-1}(\tfrac{-1}{3}) + \tfrac{\pi}{4}) = \frac{-1}{3} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{8}}{3} \cdot \frac{\sqrt{2}}{2}.$$

If we feel like simplifying this further (which you don't have to), we get

$$-\frac{\sqrt{2}}{6} + \frac{\sqrt{16}}{6} = -\frac{\sqrt{2}}{6} + \frac{4}{6} = \frac{4-\sqrt{2}}{6}.$$