MATH 221 QUIZ 1, FALL 2013

Solve the following two problems, showing all your work.

(1) For which values of a and b does the graph of the function $f(x) = ax+b$ intersect the graph of the function $g(x) = x^2 - 2x + 1$ at exactly one point, and also pass through the point $(1, -1)$?

Answer: The line $y = ax + b$ passes through $(1, -1)$, so setting $y = -1$ and $x = 1$,

$$-1 = a \cdot 1 + b \implies b = -a - 1,$$

so we can write

$$y = ax - a - 1.$$

Setting y equal to $g(x)$ gives the points of intersection of $y = f(x)$ and $g(x)$:

$$ax - a - 1 = x^2 - 2x + 1 \implies x^2 - (a+2)x + a + 2 = 0,$$

so by the quadratic formula,

$$x = \frac{(a+2) \pm \sqrt{(a+2)^2 - 4(a+2)}}{2}.$$

We want to find when there is exactly one solution, that is, when the discriminant (the expression inside the square root) is zero:

$$0 = (a+2)^2 - 4(a+2) = a^2 + 4a + 4 - 4a - 8 = a^2 - 4 = (a+2)(a-2).$$

So $a = \pm 2$. Using $b = -a - 1$, we get two solutions:

$$y = 2x - 3 \qquad \text{or} \qquad y = -2x + 1.$$

(2) Let f be the function defined by the requirement that, for any t,

$$y = f(t) \iff \begin{array}{c} y \text{ is the largest of all} \\ \text{possible solutions of} \\ y^2 + 4t^2 = t^2 y + 4y. \end{array}$$

Find a formula for $f(t)$.

Answer: Move everything to one side:

$$y^2 - t^2 y - 4y + 4t^2 = 0$$

Now factor:

$$0 = y^2 - t^2 y - 4y + 4t^2 = y^2 - \left(t^2 + 4\right) y + 4t^2 = (y-4)\left(y - t^2\right).$$

So $y = 4$ or $t = t^2$. The value of $f(t)$ is whichever of the two solutions is larger, so

$$f(t) = \max(t^2, 4) = \begin{cases} t^2 & \text{if } t > 2 \text{ or } t < -2, \\ 4 & \text{if } -2 \leq t \leq 2. \end{cases}$$