MATH 221 QUIZ 2
SEPTEMBER 23, 2013

Solve the following two problems, showing all your work.

(1) Using the epsilon-delta (ε-δ) definition of limit, prove that

$$\lim_{x \to 2} \frac{x}{3-x} = 2.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Let $\delta > 0$ be some value that we will choose later. For any x such that

$$0 < |x - 2| < \delta,$$

we want to show that

$$\left| \frac{x}{3-x} - 2 \right| < \varepsilon.$$

To do so, observe that

$$\left| \frac{x}{3-x} - 2 \right| = \left| \frac{x}{3-x} - \frac{2(3-x)}{3-x} \right| = \left| \frac{x - 2(3-x)}{3-x} \right| = \left| \frac{x - 6 + 2x}{3-x} \right|$$
$$= \left| \frac{3x-6}{3-x} \right| = \left| \frac{3(x-2)}{3-x} \right| = 3 \cdot |x-2| \cdot \frac{1}{|3-x|}.$$

By assumption, $|x-2| < \delta$. Also, if $\delta \leq \frac{1}{2}$, then

$$|x-2| < \frac{1}{2} \implies \frac{3}{2} < x < \frac{5}{2} \implies -\frac{3}{2} > -x > -\frac{5}{2} \implies \frac{3}{2} > 3 - x > \frac{1}{2},$$

hence

$$\frac{1}{|3-x|} = \frac{1}{3-x} < 2.$$

Therefore, applying these bounds, we obtain

$$\left| \frac{x}{3-x} - 2 \right| = 3 \cdot |x-2| \cdot \frac{1}{|3-x|} < 3 \cdot \delta \cdot \frac{1}{|3-x|} < 3 \cdot \delta \cdot 2 = 6\delta \leq \varepsilon,$$

assuming for the last step that $\delta \leq \frac{\varepsilon}{6}$. Combining the two assumptions we made of δ, we see that choosing

$$\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{6}\right)$$

makes our assumptions valid. This concludes the proof. □

(2) Find the following limit (using any methods you've learned so far from lecture, homework, or the course text):

$$\lim_{x \to 3} \frac{\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{x}}}{x - 3}.$$

To find the limit, observe that, for any positive $x \neq 3$,

$$\begin{aligned}
\frac{\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{x}}}{x-3} &= \frac{\left(\frac{\sqrt{x}-\sqrt{3}}{\sqrt{3}\cdot\sqrt{x}}\right)}{x-3} \\
&= \frac{\sqrt{x}-\sqrt{3}}{(x-3)\sqrt{3x}} \\
&= \frac{\left(\sqrt{x}-\sqrt{3}\right)\left(\sqrt{x}+\sqrt{3}\right)}{(x-3)\sqrt{3x}\left(\sqrt{x}+\sqrt{3}\right)} \\
&= \frac{|x| - |3| - \sqrt{3x} + \sqrt{3x}}{(x-3)\sqrt{3x}\left(\sqrt{x}+\sqrt{3}\right)} \\
&= \frac{x-3}{(x-3)\sqrt{3x}\left(\sqrt{x}+\sqrt{3}\right)} \\
&= \frac{1}{\sqrt{3x}\left(\sqrt{x}+\sqrt{3}\right)}.
\end{aligned}$$

Since the function

$$f(x) = \frac{1}{\sqrt{3x}\left(\sqrt{x}+\sqrt{3}\right)}$$

is continuous at $x = 3$, we now can take the limit as follows:

$$\begin{aligned}
\lim_{x\to 3} \frac{\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{x}}}{x-3} &= \lim_{x\to 3} \frac{1}{\sqrt{3x}\left(\sqrt{x}+\sqrt{3}\right)} = \frac{1}{\sqrt{3\cdot 3}\left(\sqrt{3}+\sqrt{3}\right)} \\
&= \frac{1}{3\cdot 2\cdot\sqrt{3}} = \frac{1}{6\sqrt{3}} = \frac{\sqrt{3}}{18}.
\end{aligned}$$