MATH 221 QUIZ 3 (SOLUTIONS)
SEPTEMBER 29, 2013

Solve the following two problems, showing all your work. Do not use theorems that haven't been covered in this class yet (such as L'Hôpital's rule).

(1) Using the properties of limits and the known limit $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$,

$$\begin{aligned}
\lim_{x\to 0} \frac{x \cdot \cos(x)}{x - 2\sin(x)} &= \lim_{x\to 0} \frac{x}{x} \cdot \frac{\cos(x)}{1 - 2\frac{\sin(x)}{x}} \\
&= \lim_{x\to 0} \frac{\cos(x)}{1 - 2\frac{\sin(x)}{x}} \\
&= \frac{\lim_{x\to 0} \cos(x)}{\lim_{x\to 0} \left(1 - 2\frac{\sin(x)}{x}\right)} \\
&= \frac{1}{1 - 2\lim_{x\to 0} \frac{\sin(x)}{x}} \\
&= \frac{1}{1 - 2 \cdot 1} = -1.
\end{aligned}$$

(2) Recall that the range of $\cos(x)$ is $[-1, 1]$. So for any positive x, the function

$$\frac{x-1}{x^2+1} \le \frac{x-1}{x^2+\cos(x)} \le \frac{x+\cos(x^2)}{x^2+\cos(x)} \le \frac{x+1}{x^2+\cos(x)} \le \frac{x+1}{x^2-1}.$$

Furthermore,

$$\lim_{x\to\infty} \frac{x-1}{x^2+1} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1 + \frac{1}{x^2}} = \frac{\lim_{x\to\infty}\left(\frac{1}{x} - \frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(1 + \frac{1}{x^2}\right)} = \frac{0-0}{1+0} = 0,$$

and

$$\lim_{x\to\infty} \frac{x+1}{x^2-1} = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{\lim_{x\to\infty}\left(\frac{1}{x} + \frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)} = \frac{0+0}{1-0} = 0.$$

So, by the sandwich theorem (or squeeze theorem, or pinch theorem),

$$0 = \lim_{x\to\infty} \frac{x-1}{x^2+1} \le \lim_{x\to\infty} \frac{x+\cos(x^2)}{x^2+\cos(x)} \le \lim_{x\to\infty} \frac{x+1}{x^2-1} = 0,$$

which implies that

$$\lim_{x\to\infty} \frac{x+\cos(x^2)}{x^2+\cos(x)} = 0.$$