MATH 221 QUIZ 4 (SOLUTIONS)
OCTOBER 7, 2013

Solve the following two problems, showing all your work. Do not use theorems that haven't been covered in this class yet.

(1) Find the asymptotes (horizontal, vertical, and slanted) of the following function:

$$f(x) = \frac{2x^3}{x^2 - 1}$$

The vertical asymptotes occur when the denominator is 0:

$$x^2 - 1 = 0 \iff x^2 = 1 \iff x = \pm 1.$$

To find slant asymptotes, first we find

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - \frac{1}{x^2}} = 2.$$

So the slope of the slant asymptote in each direction is 2. To find the y-intercept, we must quantify the difference between $f(x)$ and $2x$ in the limit:

$$\begin{aligned}
\lim_{x \to \pm\infty} \big(f(x) - 2x\big) &= \lim_{x \to \pm\infty} \left(\frac{2x^3}{x^2 - 1} - 2x \right) \\
&= \lim_{x \to \pm\infty} \frac{2x^3 - 2x(x^2 - 1)}{x^2 - 1} \\
&= \lim_{x \to \pm\infty} \frac{2x^3 - 2x^3 + 2x}{x^2 - 1} \\
&= \lim_{x \to \pm\infty} \frac{2x}{x^2 - 1} \\
&= \lim_{x \to \pm\infty} \frac{\frac{2}{x}}{1 - \frac{1}{x^2}} = \frac{0}{1 - 0} = 0.
\end{aligned}$$

Thus, the slant asymptote in both the positive and negative directions is $y = 2x$. (Also, since there are slant asymptotes with non-zero slope, there aren't horizontal asymptotes.)

(2) Use the **limit definition of derivative** (NOT properties of derivatives such as the product/quotient/chain rules) to find the derivative of the following function:

$$g(x) = \sqrt{3x + 5}$$

Using the limit definition of derivative,

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(h)}{h} = \lim_{h\to 0} \frac{\sqrt{3(x+h)+5}-\sqrt{3x+5}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+5}-\sqrt{3x+5}}{h} \cdot \frac{\sqrt{3(x+h)+5}+\sqrt{3x+5}}{\sqrt{3(x+h)+5}+\sqrt{3x+5}} \\
&= \lim_{h\to 0} \frac{3(x+h)+5-(3x+5)}{h(\sqrt{3(x+h)+5}+\sqrt{3x+5})} \\
&= \lim_{h\to 0} \frac{3x+3h+5-(3x+5)}{h(\sqrt{3(x+h)+5}+\sqrt{3x+5})} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3(x+h)+5}+\sqrt{3x+5})} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+5}+\sqrt{3x+5}} \\
&= \frac{3}{\sqrt{3(x+0)+5}+\sqrt{3x+5}} \\
&= \frac{3}{2\sqrt{3x+5}}.
\end{aligned}$$