MATH 221 QUIZ 6
OCTOBER 28, 2013

Solve the following two problems, showing all your work. Do not use theorems that haven't been covered in this class yet.

(1) A cube increases in size so that the edge length of the cube increases at a constant rate of 7 mm/s (millimeters per second). How fast is the volume of the cube changing when the edge length is 40 mm?

Let $\ell = \ell(t)$ denote the edge length at time t, and $V = V(t)$ the volume of the cube at time t. Then $V = \ell^3$, so, implicitly differentiating,

$$\frac{dV}{dt} = \frac{d}{dt}\ell^3 = 3\ell^2\frac{d\ell}{dt}.$$

By assumption, $\frac{d\ell}{dt} = 7$ mm/s, so when $\ell = 40$ mm, $\frac{dV}{dt} = 3 \cdot 40^2 \cdot 7\ \mathrm{mm}^3/\mathrm{s}$.

(2) Determine the intervals where the following function is increasing and where it is decreasing:

$$f(x) = \frac{x+1}{x^2+1}$$

First, we find the derivative:

$$f'(x) = \frac{1 \cdot (x^2+1) - 2x \cdot (x+1)}{(x^2+1)^2} = \frac{-x^2-2x+1}{(x^2+1)^2}.$$

Since $x^2 + 1 \neq 0$ for any real number x, $f(x)$ has no vertical asymptotes, so $f(x)$ can only change from increasing to decreasing or vice versa when $f'(x) = 0$. Let us find all such points: if $f'(x) = 0$, then

$$\frac{-x^2-2x+1}{(x^2+1)^2} = 0,$$

so, multiplying by the denominator,

$$-x^2 - 2x + 1 = 0.$$

This is a quadratic equation, so by the quadratic formula,

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot (-1) \cdot 1}}{2 \cdot (-1)} = -1 \pm \frac{\sqrt{8}}{2} = -1 \pm \sqrt{2}.$$

These are the critical (stationary) points of f.

To determine whether f is increasing or decreasing, we check whether f is positive or negative in the intervals $(-\infty, -1-\sqrt{2})$, $(-1-\sqrt{2}, -1+\sqrt{2})$, and $(-1+\sqrt{2}, +\infty)$. Since $1 < \sqrt{2} < 2$, values of $x = -3, -1, 1$ will suffice:

$$f'(-3) = \frac{-(-3)^2 - 2(-3) + 1}{\left((-3)^2+1\right)^2} = \frac{-9+6+1}{(9+1)^2} < 0,$$

$$f'(-1) = \frac{-(-1)^2 - 2(-1) + 1}{\left((-1)^2+1\right)^2} = \frac{-1+2+1}{(1+1)^2} > 0,$$

$$f'(1) = \frac{-1^2 - 2(1) + 1}{\left(1^2+1\right)^2} = \frac{-1-2+1}{(1+1)^2} < 0.$$

So f is decreasing on $(-\infty, -1-\sqrt{2})$, increasing on $(-1-\sqrt{2}, -1+\sqrt{2})$, and decreasing on $(-1+\sqrt{2}, +\infty)$.