Math 631 Notes
Algebraic Geometry

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Fall 2012*

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*Last revised December 21, 2016.

1 Algebraic sets, affine varieties, and the Zariski topology

List of topics:

(1) Algebraic sets

(2) Hilbert basis theorem

(3) Zariski topology

1.1 Algebraic sets

Fix a field k. Consider k^N, the set of N-tuples in k.

Definition 1.1. An *affine algebraic subset* of k^N is the common zero locus of a collection of polynomials in $k[x_1, \ldots, x_N]$.

That is: Fix $S \subseteq k[x_1, \ldots, x_N]$ any subset. Then

$$\mathbb{V}(S) = \left\{ p = (\lambda_1, \ldots, \lambda_N) \in k^N \;\middle|\; f(p) = 0 \;\forall f \in S \right\}.$$

Example 1.2. (1) Lines in $\mathbb{R}^2$: $\mathbb{V}(y - mx - b) \subseteq \mathbb{R}^2$.

(2) Rational points on a cone (arithmetic geometry): $\mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{Q}^3$

(3) All linear subspaces of k^N are affine algebraic sets.

(4) $\mathbb{V}(\det(x_{ij}) - 1) = \mathrm{SL}_n(\mathbb{C}) = \{n \times n \text{ matrices } /\mathbb{C} \text{ of } \det 1\} \subseteq \mathbb{C}^{n^2}$

(5) $\mathfrak{sl}_2(\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \;\middle|\; \text{trace} = 0 \right\} \subseteq \mathbb{R}^{2\times 2}$

(6) Point in k^N: $\{(a_1, \ldots, a_N)\} = \mathbb{V}(x_1 - a_1, \ldots, x_N - a_N)$.

(7) $\mathbb{V}(x, y) = (0, 0) = \mathbb{V}\left(\{x^n + y, y^{n+17}\}_{n \in \mathbb{N}_{\geq 30}}\right) \subseteq \mathbb{R}^2$

Remark 1.3. $S \subseteq T \subseteq k[x_1, \ldots, x_N] \implies \mathbb{V}(S) \supseteq \mathbb{V}(T)$.

1.2 Hilbert basis theorem

Theorem 1.4 (Hilbert basis theorem)**.** *Every affine algebraic set in k^N can be defined by* finitely many *polynomials.*

Proof requires a lemma:

Lemma 1.5. *Let $\{f_\lambda\}_{\lambda \in \Lambda} \subseteq k[x_1, \ldots, x_N]$ and let $I \subseteq k[x_1, \ldots, x_N]$ be the ideal generated by the $\{f_\lambda\}_{\lambda \in \Lambda}$. Then $\mathbb{V}(S) = \mathbb{V}(I)$.*

Proof. We know $\mathbb{V}(S) \supseteq \mathbb{V}(I)$. Take $p \in \mathbb{V}(S)$. We want to show that given any $g \in I$, we have $g(p) = 0$.

Take $g \in I$, so $g = r_1 f_1 + \cdots + r_t f_t$, where $f_i \in S$ and $r_i \in k[x_1, \ldots, x_N]$. So

$$g(p) = r_1(p) f_1(p) + \cdots + r_t(p) f_t(p) = 0$$

since $f_i(p) = 0$ for $i = 1, \ldots, t$. Hence $p \in \mathbb{V}(I)$. □

Proof of Theorem 1.4. Take any $S \subseteq k[x_1, \ldots, x_N]$, $I = \langle S \rangle$ ideal generated by S. We have $\mathbb{V}(S) = \mathbb{V}(I)$ by Lemma 1.5. But every ideal in a polynomial ring in finitely many variables is finitely generated. Hence

$$\mathbb{V}(S) = \mathbb{V}(I) = \mathbb{V}(g_1, \ldots, g_t),$$

where $g_1, \ldots, g_t$ generate I. □

Remark 1.6 (Algebra black box).

- R is *Noetherian* if every ideal is f.g.
- Thm: R Noetherian $\implies$ $R[x]$ Noetherian.
- $k[x_1, \ldots, x_{N-1}][x_N] \cong k[x_1, \ldots, x_N]$, use induction.

1.3 Zariski topology

Definition 1.7 (topology)**.** A *topology* on a set X is a collection of distinguished subsets, called *closed sets*, satisfying:

(1) $\varnothing$ and X are closed.

(2) An arbitrary intersection of closed sets is closed.

(3) A finite union of closed sets is closed.

Example 1.8. (1) On $\mathbb{R}$, the Euclidean topology.

(2) On $\mathbb{R}$, *cofinite*: closed sets are finite sets, and $\mathbb{R}, \varnothing$.

Definition 1.9 (Zariski topology)**.** The *Zariski topology* on k^N is defined as the topology whose closed sets are affine algebraic sets.

1.3.1 Proof that affine algebraic sets form closed sets on a topology on k^N

(1) $\varnothing = \mathbb{V}(1)$, $k^N = \mathbb{V}(0)$.

(2) WTS: $\{V_\lambda\}$ closed sets $\implies$ $\bigcap_{\lambda\in\Lambda} V_\lambda$ closed. Write $V_\lambda = \mathbb{V}(I_\lambda)$. Then

$$\bigcap_{\lambda\in\Lambda} V_\lambda = \bigcap_{\lambda\in\Lambda} \mathbb{V}(I_\lambda) = \mathbb{V}\Big(\bigcup_{\lambda\in\Lambda} I_\lambda\Big) = \mathbb{V}\Big(\sum_{\lambda\in\Lambda} I_\lambda\Big).$$

(3) WTS: Finite union of closed sets are closed. By induction, suffices to show $\mathbb{V}(f_1, \dots, f_t) \cup \mathbb{V}(g_1, \dots, g_s)$ is an algebraic set.

Note:

$$\mathbb{V}(f_1, \dots, f_t) \cup \mathbb{V}(g_1, \dots, g_s) = V\left(\{f_i g_j\}_{\substack{i \in \{1,\dots,t\} \\ j \in \{1,\dots,s\}}}\right).$$

Proof on quiz.

Example 1.10. Zariski topology on k^1 is the cofinite topology. Since $k[x]$ is a PID,

$$V = \mathbb{V}(\langle f_1, \dots, f_t \rangle) = \mathbb{V}(f) = \{\text{roots of } f\},$$

which is finite if $f \neq 0$.

2 Ideals, Nullstellensatz, and the coordinate ring

Today:

(1) ideal of V

(2) Hilbert's Nullstellensatz

(3) Regular functions

(4) coordinate ring

2.1 Ideal of an affine algebraic set

Affine algebraic subset of k^N:

$$V = \mathbb{V}\left((f_1, \dots, f_t)\right) \subseteq k^N.$$

Consider the map

$$\begin{aligned} \{\text{ideals in } k[x_1, \dots, x_N]\} &\to \left\{\text{(affine) algebraic subsets of } k^N\right\} \\ I &\mapsto \mathbb{V}(I). \end{aligned}$$

Note 2.1.

- This map is order reversing: $I \subseteq J \implies \mathbb{V}(J) \subseteq \mathbb{V}(I)$.
- Surjective.
- Not injective: e.g., $(x, y), (x^2, y^2)$.

Remark 2.2 (algebra). R commutative ring, $I \subseteq R$ any ideal.

Definition 2.3. The *radical* of I is the ideal

$$\operatorname{Rad} I = \left\{ f \in R \;\middle|\; f^N \in I \text{ for some } N \right\}.$$

- Sanity check: show this is an ideal.

- I is *radical* if $\operatorname{Rad} I = I$.

Lemma 2.4. *Let $I \subseteq k[x_1, \ldots, x_N]$. Then*

$$\mathbb{V}(I) = \mathbb{V}(\operatorname{Rad} I).$$

Proof. $I \subseteq \operatorname{Rad} I \implies \mathbb{V}(\operatorname{Rad} I) \subseteq \mathbb{V}(I)$.

So take $p \in \mathbb{V}(I) \subseteq k^N$. Need to show $\forall f \in \operatorname{Rad} I$ that $f(p) = 0$. We have $f \in \operatorname{Rad} I \implies f^N \in \operatorname{Rad} I$, so

$$\big(f(p)\big)^N = f^N(p) = 0 \implies f(p) = 0. \qquad \square$$

Now is the map $I \mapsto \mathbb{V}(I)$ injective?

Example 2.5. $(x^2 + y^2) \in \mathbb{R}[x, y]$.

$$\mathbb{V}(x, y) = (0, 0) = \mathbb{V}(x^2 + y^2) \subseteq \mathbb{R}^2.$$

We have 2 radical ideals defining the same algebraic set.

Definition 2.6. Let $V \subseteq k^N$ be an affine algebraic set. The *ideal of V* is

$$\mathbb{I}(V) = \left\{ f \in k[x_1, \ldots, x_N] \;\middle|\; f(p) = 0 \ \forall p \in V \right\}.$$

Note 2.7. $\mathbb{I}(V)$ is a radical ideal, and is the largest ideal defining V.

Proposition 2.8. $V = \mathbb{V}(\mathbb{I}(V))$.

Proof. Say $V = \mathbb{V}(I)$. Since $I \subseteq \mathbb{I}(V)$, we have $\mathbb{V}(\mathbb{I}(V)) \subseteq \mathbb{V}(I) = V$.

Take $p \in V$. Need to show $\forall g \in \mathbb{I}(V)$ that $g(p) = 0$, which is true by definition of $\mathbb{I}(V)$. $\square$

This shows that $\mathbb{I}$ is a right inverse of $\mathbb{V}$.

Example 2.9. Going back to our previous example, we should really view $\mathbb{V}\left(x^2 + y^2\right)$ in $\mathbb{C}^2$ rather than $\mathbb{R}^2$:

$$\mathbb{V}\left(x^2 + y^2\right) = \mathbb{V}\big((x + iy)(x - iy)\big) = \mathbb{V}(x + iy) \cup \mathbb{V}(x - iy).$$

2.2 Hilbert's Nullstellensatz

Theorem 2.10 (Hilbert's Nullstellensatz)**.** *Let $k = \overline{k}$ (i.e., assume k is algebraically closed). There is an order-reversing bijection*

$$\begin{aligned} \{\textit{radical ideals in } k[x_1, \ldots, x_N]\} &\longleftrightarrow \left\{\textit{affine algebraic subsets of } k^N\right\} \\ I &\mapsto \mathbb{V}(I) \\ \mathbb{I}(V) &\leftarrow\!\shortmid V. \end{aligned}$$

Remark 2.11. Points in affine space k^N correspond to maximal ideals in the polynomial ring $k[x_1, \ldots, x_N]$.

2.3 Irreducible spaces

Definition 2.12. A topological space X is *irreducible* if X is not the union of two nonempty proper closed sets.

Example 2.13. The cofinite topology on $\mathbb{R}$ is irreducible.

2.4 Sept. 10 warmup

- Draw $\mathbb{V}(xy, xz) \subseteq \mathbb{R}^3$.
- Prove Lemma: For $I, J \subseteq k[x_1, \ldots, x_N]$,
$$\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J).$$

Proof 1. $I \cap J \subseteq I, J \implies \mathbb{V}(I) \cup \mathbb{V}(J) \subseteq \mathbb{V}(I \cap J)$.

Take $p \in \mathbb{V}(I \cap J)$. Need $p \in \mathbb{V}(I)$ or $\mathbb{V}(J)$. If $p \notin \mathbb{V}(I)$, then $\exists f \in I$ such that $f(p) \neq 0$. Now: $\forall g \in J$, look at $fg \in IJ$. Because $p \in \mathbb{V}(I \cap J)$,
$$f(p)g(p) = (fg)(p) = 0,$$
hence $g(p) = 0$ and $p \in \mathbb{V}(J)$. □

Proof 2. $\mathbb{V}(I \cap J) = \mathbb{V}\left(\sqrt{I \cap J}\right) = \mathbb{V}\left(\sqrt{IJ}\right) = \mathbb{V}(IJ) = \mathbb{V}(I) \cup \mathbb{V}(J)$. □

2.5 Some commutative algebra

R commutative ring.

- I, J radical $\implies$ $I \cap J$ radical.
- $\mathfrak{p} \subseteq R$ is *prime* $\iff$ $R/\mathfrak{p}$ is a domain $\iff$ if $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.
- If R is Noetherian, I radical, then
$$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$$
uniquely, where the $\mathfrak{p}_i$ are prime (irredundant).

2.6 Review of Hilbert's Nullstellensatz

The mappings $\mathbb{I}$ and $\mathbb{V}$ are mutually inverse, giving us an order-reversing bijection
$$\left\{\text{affine algebraic subsets of } k^N\right\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \{\text{radical ideals of } k[x_1, \ldots, x_N]\}.$$

$$\begin{aligned}
k^N &\longleftrightarrow 0 \\
\varnothing &\longleftrightarrow (1) = k[x_1, \ldots, x_N] \\
\{\text{points}\} &\longleftrightarrow \{\text{maximal ideals}\} \\
(a_1, \ldots, a_N) &\longleftrightarrow (x_1 - a_1, \ldots, x_N - a_N) \\
\{\text{irreducible algebraic sets}\} &\longleftrightarrow \operatorname{Spec} k[x_1, \ldots, x_N] = \{\text{prime ideals}\}
\end{aligned}$$

2.7 Irreducible algebraic sets

Definition 2.14. An algebraic set $V \subseteq k^N$ is *irreducible* if it cannot be written as the union of two *proper* algebraic sets contained in V. [If $V = V_1 \cup V_2$, then $V = V_1$ or $V = V_2$.]

Exercise 2.15. $\mathbb{V}(I)$ is irreducible $\iff$ I is prime, where I is radical.

Observation 2.16. $I \subseteq k[x_1, \ldots, x_N]$ radical (k not necessarily algebraically closed), write $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$, where $\mathfrak{p}_i$ are prime (*unique*!).

$$\mathbb{V}(I) = \mathbb{V}(\mathfrak{p}_1) \cup \cdots \cup \mathbb{V}(\mathfrak{p}_t)$$

are the (unique) *irreducible components* of $\mathbb{V}(I)$.

The point is:

Proposition 2.17. *Every algebraic set in k^N is a union of its irreducible components.*

2.8 Aside on non-radical ideals

We also have $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I \cup J)$. However, $I \cup J$ is not usually an ideal, and $I + J$ is not necessarily radical.

Non-radical ideals lead into scheme theory:

$$\mathbb{V}(y - x^2) \cap \mathbb{V}(y) = \mathbb{V}(y - x^2, y) = \mathbb{V}(y, x^2).$$

We should somehow keep track of the multiplicity.

3 Regular functions, regular maps, and categories

3.1 Regular functions

Fix $V \subseteq k^N$ algebraic set, $k = \overline{k}$.

Definition 3.1. A function $V \to k$ is *regular* if it agrees with the restriction to V of some polynomial function on the ambient k^N.

Proposition–Definition 3.2. The set of all regular functions on V has a natural ring structure (where addition and multiplication are the functional notions). This is the *coordinate ring* of V, denoted $k[V]$.

Example 3.3. On k^N, $k[k^N] = k[x_1, \ldots, x_N]$.

Remark 3.4. (1) $k = \overline{k} \implies k$ is infinite.

(2) If k is infinite, then there is no ambiguity in the word "polynomial".

Example 3.5. Consider $\mathbb{V}(y - x^2) \subseteq \mathbb{R}^2$. This is the set of all points (t, t^2). The function "y" outputs the y-coordinate (projection to y-axis), and "x^2" is the *same function* in V.

Example 3.6. Consider $\mathbb{V}(xy - 1) \subseteq \mathbb{Q}^2$. Is $\frac{1}{y}$ regular?

Yes: $\frac{1}{y} = x$ on $\mathbb{V}(xy - 1)$.

Observation 3.7. The restriction map gives a natural ring surjection

$$\begin{aligned} k[x_1, \dots, x_N] &\to k[V] \\ \varphi &\mapsto \varphi\big|_V \end{aligned}$$

whose kernel is $\mathbb{I}(V)$. In particular,

$$k[V] \cong \frac{k[x_1, \dots, x_N]}{\mathbb{I}(V)}.$$

3.2 Properties of the coordinate ring

The coordinate ring $k[V]$ has the following properties:

(1) $k[V]$ is a f.g. k-algebra generated by the images of $x_1, \dots, x_N$.

(2) *reduced* (the only nilpotent element is 0)

(3) domain $\iff$ V is irreducible.

(4) The maximal ideals of $k[V]$ correspond to points of V (need $k = \overline{k}$).

Note 3.8 (commutative algebra). Maximal ideals in $k[V] \cong k[x_1, \dots, x_N]/\mathbb{I}(V)$ correspond to maximal ideals in $k[x_1, \dots, x_N]$ containing $\mathbb{I}(V)$. By the Nullstellensatz, these correspond to points on V.

3.3 Regular mappings

Definition 3.9. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets. A *regular mapping* of affine algebraic sets

$$\varphi : V \to W$$

is any mapping φ which agrees with a polynomial map Ψ on the ambient $k^n \to k^m$:

$$x = (x_1, \dots, x_n) \xmapsto{\Psi} \big(\Psi_1(x), \dots, \Psi_m(x)\big),$$

where Ψ_i are polynomials.

Note 3.10. If $W = k$, then a regular map is a regular function.

Note 3.11. We can describe a regular map $V \xrightarrow{\varphi} W \subseteq k^m$ by giving *regular functions* $\varphi_1, \dots, \varphi_m \in k[V]$:

$$p \mapsto \big(\varphi_1(p), \dots, \varphi_m(p)\big) \in W \subseteq k^m.$$

Example 3.12.

$$\begin{aligned} k &\to \mathbb{V}(y - x^2) \subseteq k^2 \\ t &\mapsto (t, t^2) \end{aligned}$$

is a regular map from k to $\mathbb{V}(y - x^2)$.

The projection

$$\begin{aligned} \mathbb{V}(y - x^2) \subseteq k^2 &\to k \\ (x, y) &\mapsto x \end{aligned}$$

is the inverse to the map $t \mapsto (t, t^2)$.

Definition 3.13. An *isomorphism* of affine algebraic sets is a *regular map* $V \xrightarrow{\varphi} W$ which has a *regular map* $W \xrightarrow{\psi} V$ inverse: $\psi \circ \varphi = \mathrm{id}_V$ and $\varphi \circ \psi = \mathrm{id}_W$.

Example 3.14. Let $V_1, V_2 \subseteq k^n$ be linear subspaces (defined by some collection of linear polynomials). Then $V_1 \cong V_2$ as algebraic sets $\iff \dim V_1 = \dim V_2$.

Example 3.15 (diagonal map). Give $k^n \times k^n$ coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$.

$$\begin{aligned} k^n &\xrightarrow{\Delta} k^n \times k^n \\ p &\mapsto (p, p) \end{aligned}$$

Image is the "diagonal"

$$D = \mathbb{V}(x_1 - y_1, \ldots, x_n - y_n) \subseteq k^n \times k^n.$$

The map $k^n \xrightarrow{\Delta} D \subseteq k^n \times k^n$ is an isomorphism of affine algebraic sets.

Example 3.16. $X, Y \subseteq k^n$ algebraic sets. View $X \subseteq k^n$ with coordinates $x_1, \ldots, x_n$ and $Y \subseteq k^n$ with coordinates $y_1, \ldots, y_n$.

$$\begin{tikzcd} k^n \arrow[r, "\Delta"] & k^n \times k^n \\ X \cap Y \arrow[u, phantom, "\cup|"] \arrow[r, "\cong", "p \mapsto (p,p)"'] & (X \times Y) \cap D \arrow[u, phantom, "\cup|"] \end{tikzcd}$$

3.4 Category of affine algebraic sets

Key idea: The category of affine algebraic sets over $k = \overline{k}$ is "the same" (anti-equivalence, duality) as the category of f.g. *reduced* k-algebras.

Point: Given a regular map $V \xrightarrow{\varphi} W$ of affine algebraic sets, there is a *naturally induced* k-algebraic homomorphism $k[W] \xrightarrow{\varphi^*} k[V]$ given for $g \in k[W]$, $W \xrightarrow{g} k$ by

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[rr, bend right, "g \circ \varphi"'] & W \arrow[r, "g"] & k \end{tikzcd}$$

$$x = (x_1, \ldots, x_n) \mapsto \big(\varphi_1(x), \ldots, \varphi_m(x)\big) \mapsto g\big(\varphi_1(x), \ldots, \varphi_m(x)\big) \in k[V],$$

where $\varphi_1, \ldots, \varphi_m$ are polynomials in $x_1, \ldots, x_n$.

Theorem 3.17. *For $k = \overline{k}$, there is an anti-equivalence*[1] *of categories*

$$\left\{ \begin{matrix} \textit{affine algebraic sets over} \\ k \textit{ with regular maps} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{f.g. reduced } k\textit{-algebras} \\ \textit{with } k\textit{-algebra} \\ \textit{homomorphisms} \end{matrix} \right\}$$

$$V \mapsto k[V]$$

$$(V \xrightarrow{\varphi} W) \mapsto \begin{pmatrix} k[W] \xrightarrow{\varphi^*} k[V] \\ g \mapsto g \circ \varphi \end{pmatrix}$$

$$k^n \supseteq \mathbb{V}(I) \leftarrow\!\shortmid R \cong \frac{k[x_1, \ldots, x_n]}{I}.$$

Proof.

Note 3.18. The assignment $V \mapsto k[V]$ is functorial: Given

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "h"'] & W \arrow[r, "g"] & X, \end{tikzcd}$$

there is f^*, g^*, h^* and a commutative diagram

$$\begin{tikzcd} k[V] & k[W] \arrow[l, "f^*"'] & k[X], \arrow[l, "g^*"'] \arrow[ll, bend left, "h^*"] \end{tikzcd}$$

i.e., $(g \circ f)^* = f^* \circ g^*$. (Make sure this is *obvious* to you.)

Problem: Given a reduced, f.g. k-algebra R, how to cook up V?

Fix a k-algebra presentation for R:

$$R = \frac{k[x_1, \ldots, x_n]}{I}.$$

Because R is reduced, I is radical. Let

$$V = \mathbb{V}(I) \subseteq k^n.$$

By the Nullstellensatz, $\mathbb{I}(\mathbb{V}(I)) = I$, so

$$k[V] \cong \frac{k[x_1, \ldots, x_n]}{\mathbb{I}(V)} = \frac{k[x_1, \ldots, x_n]}{I} = R.$$

What about homomorphisms of k-algebras?

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, equal] & S \arrow[d, equal] \\ k[y_1, \ldots, y_m]/I \arrow[r, "\varphi"] & k[x_1, \ldots, x_n]/J \end{tikzcd}$$

Let $\varphi_i = \varphi(y_i) \in k[V]$ for $i = 1, \ldots, m$. This uniquely defines φ.

[1] An *anti-equivalence* of categories C, D is an equivalence of C and the opposite category D^{op}.

Need to construct

$$
\begin{aligned}
k^n \supseteq \mathbb{V}(J) &\xrightarrow{\Psi} \mathbb{V}(I) \subseteq k^m \\
x = (x_1, \dots, x_n) &\mapsto \big(\varphi_1(x), \dots, \varphi_m(x)\big).
\end{aligned}
$$

We have that Ψ is a map $V \to k^m$. Need to check that

(1) the image is in W,

(2) $\Psi^* = \varphi$.

To check

$$\big(\varphi_1(x), \dots, \varphi_m(x)\big) \in \mathbb{V}(I) = W,$$

take any $g \in I$. For any $x \in V$,

$$g\big(\varphi_1(x), \dots, \varphi_m(x)\big) = \varphi(g)(x) = 0.$$

We have that φ is represented by a map

$$
\begin{aligned}
k[y_1, \dots, y_m] &\to k[x_1, \dots, x_n] \\
y_i &\mapsto \varphi_i, \qquad\qquad i = 1, \dots, m.
\end{aligned}
$$

Because φ induces a map of the quotient ring

$$\frac{k[y_1, \dots, y_m]}{I} \xrightarrow{\varphi} \frac{k[x_1, \dots, x_n]}{J},$$

$\widetilde{\varphi}(g) \in J$ for any $g \in I$. In other words, $\widetilde{\varphi}(I) \subseteq J$.

Finally, it's easy to check that this functor is the inverse functor to $V \mapsto k[V]$. □

3.5 Sep. 14 quiz question

Consider $k \xrightarrow{\varphi} \mathbb{V}(y^2 - x^3) \subseteq k^2$ given by

$$t \mapsto \left(t^2, t^3\right).$$

Is this a regular map? Bijective? Isomorphism? Describe explicitly the induced φ^*.

Inverse:

$$
\begin{aligned}
(x, y) &\mapsto \frac{y}{x} \text{ if } x \neq 0, \\
(0, 0) &\mapsto 0.
\end{aligned}
$$

φ is an isomorphism $\iff$ φ^* is an isomorphism.

$$
\begin{aligned}
\varphi^* : \frac{k[x, y]}{(y^2 - x^3)} &\to k[t] \\
x &\mapsto t^2 \\
y &\mapsto t^3
\end{aligned}
$$

is not an isomorphism of k-algebras.

3.6 Convention on algebraic sets

From now on, affine algebraic sets $V \subseteq k^n = \mathbb{A}^n$ will be considered as topological spaces with the induced (subspace) Zariski topology.

The closed sets of V are $\widetilde{W} \cap V$, where $\widetilde{W} \subseteq k^n$ (affine algebraic set contained in V) is closed in k^n.

3.7 Hilbert's Nullstellensatz and the Zariski topology

Assume $k = \overline{k}$. Fix $V \subseteq \mathbb{A}^n$ affine algebraic set.

$$\{\text{closed sets in } V\} \longleftrightarrow \{\text{radical ideals in } k[V]\}$$
$$W \mapsto \mathbb{I}(W) = \left\{ f \in k[V] \;\middle|\; f(p) = 0 \ \forall p \in W \right\}$$
$$V \supseteq \left\{ p \in V \;\middle|\; f(p) = 0 \ \forall f \in I \right\} = \mathbb{V}(I) \leftarrow\!\shortmid I$$

Proof. Follows immediately from the Nullstellensatz in $\mathbb{A}^n$:

$$\{\text{affine algebraic sets in } V\} \longleftrightarrow \{\text{radical ideals in } k[x_1, \ldots, x_n] \text{ containing } \mathbb{I}(V)\}$$
$$\longleftrightarrow \left\{ \text{radical ideals in } \frac{k[x_1, \ldots, x_n]}{\mathbb{I}(V)} \right\} = \{\text{radical ideals in } k[V]\}.$$

□

4 Rational functions

[Caution: Despite the name, *not functions*!]

4.1 Function fields and rational functions

Fix affine algebraic set V. Assume V is irreducible, equivalently, $k[V]$ is a domain.

Definition 4.1. The *function field* of V is the fraction field of $k[V]$, denoted $k(V)$.

Example 4.2. Let $V = \mathbb{A}^n$, $k[V] = k[x_1, \ldots, x_n]$. Then

$$k(V) = k(x_1, \ldots, x_n),$$

i.e., rational functions.

Definition 4.3. A *rational function* on V is an element $\varphi \in k(V)$. I.e., φ is an *equivalence class* f/g, where $f, g \in k[V]$, $g \neq 0$. Here,

$$\frac{f}{g} \sim \frac{f'}{g'} \iff fg' = gf'$$

as elements of $k[V]$.

Example 4.4. In $\mathbb{V}(xy - z^2) \subseteq \mathbb{A}^3$, x/z is a rational function. Moreover, z/y is the *same* rational function:

$$\frac{x}{z} \sim \frac{z}{y}$$

because $xy = z^2$ on V.

Example 4.5. $k[V] \subseteq k(V)$ always, by the map $f \mapsto f/1$.

4.2 Regular points

Definition 4.6. A rational function $\varphi \in k(V)$ is *regular* at $p \in V$ if it admits a representation $\varphi = f/g$ where $g(p) \neq 0$.

Definition 4.7. The *domain of definition* of $\varphi \in k(V)$ is the locus of all points $p \in V$ where φ is regular.

Example 4.8. In $\mathbb{V}(xy - z^2) \subseteq \mathbb{A}^3$ again, $(0, 1, 0)$ is in the domain of definition of $\frac{x}{z} = \frac{z}{y}$.

Remark 4.9. We can evaluate a rational function at any point of its domain of definition.

Proposition 4.10. *The domain of definition of fixed $\varphi \in k(V)$ is a nonempty open subset of V.*

Proof. Fix $\varphi \in k(V)$. Write $\varphi = \frac{f}{g}$, where $g \neq 0$, $f, g \in k[V]$.

Since $g \neq 0$ on V, $\exists p \in V$ such that $g(p) \neq 0$. So p is in $U =$ the domain of definition of φ, so $U \neq \varnothing$.

Take any $q \in U$. So I can write $\varphi = \frac{h_1}{h_2}$, where $h_2(q) \neq 0$. Now $U' := V - \mathbb{V}(h_2) \subseteq V$ is an open subset of V, and $q \in U' \subseteq U$. $\square$

4.3 Sheaf of regular functions on V

Let V be an irreducible affine algebraic set. Assign to any open set $U \subseteq V$ the ring $\mathcal{O}_V(U)$ of all rational functions on V regular at every $p \in U$.

Exercise 4.11. $\mathcal{O}_V(U)$ is a k-algebra (because the constant functions are regular on every open set) and a domain.

Whenever $U_1 \subseteq U_2$ is an inclusion of open sets, there is an induced ring-map

$$\begin{aligned} \mathcal{O}_V(U_2) &\to \mathcal{O}_V(U_1) \\ \varphi &\mapsto \varphi\big|_{U_1}. \end{aligned}$$

Note 4.12. If $U = V$, we have two definitions of "ring of regular functions on V".

$$\begin{aligned} k(V) \supseteq \mathcal{O}_V(V) &\supseteq k[V] \\ \frac{f}{1} &\hookleftarrow f \end{aligned}$$

Theorem 4.13. *For V irreducible affine algebraic set, $k[V] = \mathcal{O}_V(V)$.*

Proof. Take $\varphi \in \mathcal{O}_V(V)$. For any $p \in V$, there is a representation $\varphi = \frac{f_p}{g_p}$ such that $g_p(p) \neq 0$.

Consider the ideal $\mathfrak{a} \subseteq k[V]$ generated by the $\{g_p\}_{p \in V}$.

Note 4.14. $\mathbb{V}(\mathfrak{a}) \subseteq V$ is *empty*, so by the Nullstellensatz, $1 \in \operatorname{Rad}(\mathfrak{a}) \implies 1 \in \mathfrak{a}$.

So we can write

$$1 = r_1 g_1 + \cdots + r_t g_t$$

for some $g_i = g_{p_i}$ in $k[V] \subseteq k(V)$, $r_i \in k[V]$. Hence

$$\varphi = r_1 \varphi g_1 + \cdots + r_t \varphi g_t.$$

But $\varphi g_i = f_i$, so

$$\varphi = r_1 f_1 + \cdots + r_t f_t \in k[V].$$

□

5 Projective space, the Grassmannian, and projective varieties

5.1 Projective space

Fix k. Let V be a vector space over k.

Definition 5.1. The *projective space* of V, denoted $\mathbb{P}(V)$, is the set of 1-dimensional subspaces of V.

We denote $\mathbb{P}^n_k = \mathbb{P}(k^{n+1})$.

Example 5.2. $\mathbb{P}^1_k = \mathbb{P}(k^2) = \{$1-dimensional subspaces of $k^2\} = \{$lines through $(0,0)$ in $k^2\}$.

We can use stereographic projection onto a fixed reference line to view $\mathbb{P}^1 = k \cup \{\infty\}$ as a line with a point at infinity.

Specifically, $\mathbb{P}^1_{\mathbb{R}}$ is homeomorphic to a circle, and $\mathbb{P}^1_{\mathbb{C}}$ is the Riemann sphere.

Example 5.3. $\mathbb{P}^2_k = \mathbb{P}(k^3) = k^2 \sqcup \mathbb{P}^1_k$.

5.2 Homogeneous coordinates

In $\mathbb{P}^n_k$, represent each point $p = [a_0 : a_1 : \cdots : a_n]$ by choosing a basis for it (i.e., choose any non-zero point in the corresponding line through origin in k^{n+1}). At least some $a_i \neq 0$, and $[b_0 : \cdots : b_n]$ represents the same point in $\mathbb{P}^n$ iff $\exists k \neq 0$ such that

$$(kb_0, \ldots, kb_n) = (a_0, \ldots, a_n). \tag{5.1}$$

Another way to think of $\mathbb{P}^n_k$ is as $(k^{n+1} \setminus \{0\})/\sim$, where two points in k^{n+1} are equivalent iff (5.1) holds.

Note 5.4. If $k = \mathbb{R}$, this gives $\mathbb{P}^n_{\mathbb{R}}$ a natural (quotient) topology, and similarly if $k = \mathbb{C}$.

Exercise 5.5. $\mathbb{P}^n$ is compact in that Euclidean topology.

In these coordinates, we have an open cover

$$\mathbb{P}^n_k = \bigcup_{j=0}^{n} U_j,$$

where $U_j = \{[x_0 : \cdots : x_n] \mid x_j \neq 0\} \cong k^n$ are the *standard charts*.

Think of fixing one chart: $U_0 \subset \mathbb{P}^n_k$. Consider U_0 to be the "finite part", and $\mathbb{P}^n \backslash U_0 = \mathbb{P}^{n-1}$ the "part at infinity".

Exercise 5.6. (1) If $k = \mathbb{R}$, then $\mathbb{P}^n_{\mathbb{R}}$ is a smooth manifold.

(2) If $k = \mathbb{C}$, then $\mathbb{P}^n_{\mathbb{C}}$ is a complex manifold.

(3) For any k, the transition functions induced by the standard cover are regular functions.

5.3 More about projective space

Exercise 5.7. In $k^n \hookrightarrow \mathbb{P}^n$, consider a line with "slope" $(a_1, a_2, \ldots, a_n)$, i.e., parametrize as

$$\left\{ \begin{pmatrix} a_1 t \\ \vdots \\ a_n t \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \;\middle|\; t \in k \right\}.$$

Show that there is a unique point in $\mathbb{P}^n$ "at infinity" on this line, with coordinates $[0 : a_1 : \cdots : a_n]$.

Example 5.8. In $\mathbb{R}^n \hookrightarrow \mathbb{P}^2_{\mathbb{R}}$, consider two parallel lines, with one passing through the origin and (a, b). These two parallel lines both approach the point $[0 : a : b]$ in $\mathbb{P}^2$.

Example 5.9. Look at $\mathbb{V}(xy - 1) \subseteq \mathbb{R}^2 \subseteq \mathbb{P}^2$. In $\mathbb{P}^2$, we can "add in" two points at ∞ on the hyperbola, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. We get a closed connected curve!

5.4 Projective algebraic sets

$\mathbb{P}^n$ = one-dimensional subspaces in k^{n+1}. We have homogeneous coordinates $[x_0 : \cdots : x_n]$.

Look at $F \in k[x_0, \ldots, x_n]$.

Caution 5.10. F is *not* a function on $\mathbb{P}^n$ unless it is constant!

However, if F is *homogeneous*, then it makes sense to ask whether or not $F(p) = 0$ for a point $p \in \mathbb{P}^n$.

Lemma 5.11. *If $F \in k[x_0, \ldots, x_n]$ is homogeneous of degree d, then*

$$F(tx_0, \ldots, tx_n) = t^d F(x_0, \ldots, x_n).$$

Proof. Write

$$F = \sum_{|I|=d} a_I x_0^{i_0} \ldots x_n^{i_n}, \qquad a_I \in k.$$

Check for each monomial. $\square$

Definition 5.12 (projective algebraic set)**.** A *projective algebraic subset* of $\mathbb{P}^n_k$ is the common zero set of a collection of *homogeneous* polynomials in $k[x_0, \ldots, x_n]$.

Example 5.13. $V = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$ is a cone; it consists of a set of lines through the origin.

In the chart $U_x = \{[1 : y : z]\}$, the equation for $V \cap U_x = \mathbb{V}(1 + y^2 - z^2) \subseteq k^2$ is a hyperbola. In the chart U_z, $V \cap U_z = \mathbb{V}(x^2 + y^2 - 1) \subseteq k^2$ is a circle.

5.5 Projective algebraic sets, continued

Let $\{F_\lambda\}_{\lambda \in \Lambda} \subseteq k[x_0, \ldots, x_n]$ be a collection of *homogeneous* polynomials.

Note 5.14. The affine algebraic set $V = \mathbb{V}\left(\{F_\lambda\}_{\lambda\in\Lambda}\right) \subseteq \mathbb{A}^{n+1}$ is *cone-shaped*, i.e., $\forall p \in V$, the line through p and the origin is in V.

Example 5.15 (Linear subspaces). Say $W \subseteq k^{n+1}$ is a sub-vector space. Then

$$\mathbb{P}(W) = \text{one-dimensional subspaces of } W = \mathbb{P}(k^{n+1}) = \mathbb{P}^n.$$

Note 5.16. $\mathbb{P}(W) = \mathbb{V}(L_1, \ldots, L_t) \subseteq \mathbb{P}^n$, where $L_i = \sum_{j=0}^n a_{ij} x_j$ are a set of linear functionals in V^* which define W.

Example 5.17 (Some special cases). W is one-dimensional $\implies \mathbb{P}(W)$ is a point.

W is 2-dimensional $\implies \mathbb{P}(W)$ is a line in $\mathbb{P}^n$.

In general, if W is $(d+1)$-dimensional, then $\mathbb{P}(W)$ is a d-hyperplane in $\mathbb{P}^n$.

If W has codimension 1 in V, then $\mathbb{V}(L) = \mathbb{P}(W) \subseteq \mathbb{P}(V) = \mathbb{P}^n$ is called a *hyperplane* in $\mathbb{P}^n$.

Fact 5.18. Every projective algebraic set in $\mathbb{P}^n$ is defined by finitely many homogeneous equations.

Note 5.19. As in the affine case,

$$\begin{aligned}\mathbb{V}\left(\{F_\lambda\}_{\lambda\in\Lambda}\right) &= \mathbb{V}\left(\langle F_\lambda\rangle_{\lambda\in\Lambda}\right) = \mathbb{V}(\text{any set of (homogeneous) generators for } \langle F_\lambda\rangle_{\lambda\in\Lambda}) \\ &= \mathbb{V}\left(\operatorname{Rad}\langle F_\lambda\rangle_{\lambda\in\Lambda}\right).\end{aligned}$$

Definition 5.20 (homogeneous ideal)**.** An ideal $I \subseteq k[x_0, \ldots, x_n]$ is *homogeneous* if it admits a set of generators consisting of homogeneous polynomials.

Example 5.21. $I = (x^3 - y^2, y^2 - z, z)$ is homogeneous because $I = (x^3, y^2, z)$.

Fact 5.22. The projective algebraic sets form the closed sets of a topology on $\mathbb{P}^n$, the *Zariski topology*.

5.6 The projective Nullstellensatz

Definition 5.23. The *homogeneous ideal* of a projective algebraic set $V \subseteq \mathbb{P}^n$ is the ideal $\mathbb{I}(V) \subseteq k[x_0, \ldots, x_n]$ generated by all *homogeneous polynomials* which vanish at every point of V.

Note 5.24. Given a homogeneous ideal $I \subseteq k[x_0, \dots, x_n]$, we can define both an affine algebraic set $\mathbb{V}(I) \subseteq k^{n+1}$ and a projective algebraic set $\mathbb{V}(I) \subseteq \mathbb{P}^n$. These have the same radical ideal in $k[x_0, \dots, x_n]$.

Fact 5.25. For any projective algebraic set $V \subseteq \mathbb{P}^n$,

$$\mathbb{V}(\mathbb{I}(V)) = V.$$

Theorem 5.26 (Projective Nullstellensatz)**.** *Only when $k = \overline{k}$:*

$$\{\textit{projective algebraic sets in } \mathbb{P}^n\} \longleftrightarrow \left\{ \begin{array}{c} \textit{radical homogeneous} \\ \textit{ideals in } k[x_0, \dots, x_n] \\ \textit{except for } (x_0, \dots, x_n) \end{array} \right\}.$$

We call $(x_0, \dots, x_n)$ the *irrelevant ideal*.

In general, the Zariski topology in $\mathbb{P}^n$ restricts to the Zariski topology in each affine chart:

$$\begin{aligned} \mathbb{P}^n \supseteq V &= \mathbb{V}\big(F_1(x_0, \dots, x_n), \dots, F_t(x_0, \dots, x_n)\big) \\ \supseteq V \cap U_i &= \mathbb{V}\big(F_0(t_0, \dots, 1, \dots, t_n), \dots, F_t(t_0, \dots, 1, \dots, t_n)\big), \end{aligned}$$

where the coordinates are given by

$$\begin{aligned} U_i &\to k^n \\ [x_0 : \dots : x_i : \dots : x_n] &\mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{i}, \dots, \frac{x_n}{x_i} \right). \end{aligned}$$

5.7 Projective closure

Definition 5.27. The *projective closure* of an affine algebraic set $V \subseteq \mathbb{A}^n$ is the closure of V in $\mathbb{P}^n$, under the standard chart embedding $\mathbb{A}^n = U_0 \hookrightarrow \mathbb{P}^n$.

Example 5.28. Consider $V = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2$:

$$\overline{V} = \overline{\mathbb{V}(xy - 1)} = \mathbb{V}(xy - z^2) \subseteq \mathbb{P}^2.$$

Look at $\overline{V} \cap U_z = V$.

Look at $\overline{V} \cap \{$"line at infinity"$\}$:

$$\overline{V} \cap \mathbb{V}(z) = \mathbb{V}(xy - z^2, z) = \mathbb{V}(xy, z) = \{[1 : 0 : 0], [0 : 1 : 0]\} \subseteq \mathbb{P}^2.$$

Definition 5.29. Given a polynomial $f \in k[x_1, \dots, x_n]$, its *homogenization* is the polynomial $F \in k[X_0, \dots, X_n]$ obtained as follows: If f has degree d, write

$$f = \sum a_I x_1^{i_1} \dots x_n^{i_n} = f_d + f_{d-1} + f_{d-2} + \dots + f_0,$$

where f_i is the homogeneous component of degree i. Then

$$F = f_d + X_0 f_{d-1} + \dots + X_0^2 f_{d-2} + \dots + X_0^d f_0.$$

Caution 5.30. Given $V = \mathbb{V}(f_1, \dots, f_t) \subseteq k^n$, the projective closure $\overline{V}$ in $\mathbb{P}^n$ is *not* necessarily defined by the homogenization of the f_i.

For example:

$$\left\{\left(t, t^2, t^3\right) \,\middle|\, t \in k\right\} \subseteq k^3 \hookrightarrow \mathbb{P}^3$$
$$\left(t, t^2, t^3\right) \mapsto [1 : t : t^2 : t^3] = \left[\frac{1}{t^3} : \frac{1}{t^2} : \frac{1}{t} : 1\right],$$

so it has exactly one point at infinity, $[0 : 0 : 0 : 1]$.

Consider $I = (z - xy, y - x^2)$.

Exercise 5.31. Show $\mathbb{V}(zw - xy, yw - x^2) \subseteq \mathbb{P}^3$ is *not* the projective closure of the twisted cubic.

6 Mappings of projective space

6.1 Example: Second Veronese embedding

$$\mathbb{P}^1 \xrightarrow{\nu_2} \mathbb{P}^2$$
$$[x : y] \mapsto \left[x^2, xy, y^2\right]$$

Check: $[x : y]$ and $[tx : ty]$ for any $t \in k$ have the same image:

$$[tx : ty] \mapsto \left[(tx)^2 : (tx)(ty) : (ty)^2\right] = \left[t^2x^2 : t^2xy : t^2y^2\right] = \left[x^2 : xy : y^2\right].$$

Also, if $x \neq 0$, then $\nu_2([x : y]) \in U_0$, and if $y \neq 0$, then $\nu_2([x : y]) \in U_2$.

This is called the "2nd Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$." In general, the *d-th Veronese map*

$$\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$$
$$[x : y] \mapsto \left[x^d : x^{d-1}y : yx^{d-1} : y^d\right]$$

Look at ν_2 in *charts* of $\mathbb{P}^1 = U_x \cup U_y$:

$$\mathbb{A}^1 \to U_y = \left\{[x : y] \,\middle|\, y \neq 0\right\} \subset \mathbb{P}^1$$
$$t \mapsto [t : 1]$$
$$\frac{x}{y} \leftarrow [x : y]$$

We have

$$U_y \xrightarrow{\nu_2} U_2 = \mathbb{A}^2$$
$$[x : 1] \mapsto \left[x^2 : x : 1\right]$$
$$\mathbb{A}^1 \to \mathbb{A}^2$$
$$t \mapsto \left(t^2, t\right).$$

This is a *regular mapping* of $\mathbb{A}^1 \to \mathbb{A}^2$.

6.2 Geometric definition

Thinking *geometrically* of $\mathbb{P}^1$ as covered by two copies of $\mathbb{A}^1$, this map ν_2 is a *regular mapping* on each chart.

This is the idea *in general* of a "regular mapping of varieties".

6.3 Example: The twisted cubic

This is the third Veronese mapping:

$$\begin{aligned}
\nu_3 : \mathbb{P}^1 &\to \mathbb{P}^3 \\
[x : y] &\mapsto \left[x^3 : x^2y : xy^2 : y^3\right] \\
\mathbb{A}^1 = U_x &\to U_0 = \{[1 : x : y : z]\} = \mathbb{A}^3 \\
t = \frac{y}{x} &\mapsto \left[1 : t : t^2 : t^3\right] = \left(t, t^2, t^3\right)
\end{aligned}$$

6.4 Example: A conic in $\mathbb{P}^2$

$$\begin{aligned}
\mathbb{P}^2 \supseteq V = \mathbb{V}(xz - y^2) &\xrightarrow{\varphi} \mathbb{P}^1 \\
[x : y : z] &\mapsto \begin{cases} [x : y] & \text{if } x \neq 0, \\ [y : z] & \text{if } z \neq 0. \end{cases}
\end{aligned}$$

Note that if $x = z = 0$, then $y = 0$, so this case cannot occur.

What if $x \neq 0$ and $z \neq 0$? Then $y \neq 0$, so

$$[x : y] = \left[xy : y^2\right] = [xy : xz] = [y : z].$$

So φ is a well-defined map of *sets*.

Cover V by open sets, each identified with an affine algebraic set: $V \cap U_x$ and $V \cap U_z$.

$$\begin{aligned}
\mathbb{A}^2 \supseteq \mathbb{V}\left(\frac{z}{x} - \left(\frac{y}{x}\right)^2\right) = V \cap U_x &\xrightarrow{\varphi} \mathbb{P}^1 \\
[x : y : z] &\mapsto [x : y] \\
\left[1 : \frac{y}{x} : \frac{z}{x}\right] &\mapsto \left[1 : \frac{y}{x}\right] \\
[1 : t : s] &\mapsto [1 : t] \\
(t, s) &\mapsto t
\end{aligned}$$

So φ is projection onto the t-axis in U_x: regular in local charts. (Similar in every chart.)

6.5 Projection from a point in $\mathbb{P}^n$ onto a hyperplane

Fix any $p \in \mathbb{P}^n$ and any hyperplane $H \subseteq \mathbb{P}^n$ not containing p.

Example 6.1 (special case). Fix a point $p \in \mathbb{P}^2$ and a line $L \subseteq \mathbb{P}^2$ such that $p \notin L$.

Choosing coordinates, let $H = \mathbb{V}(x_0) = \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ and $p = [1 : 0 : \cdots : 0] \notin H$.

Definition 6.2. The *projection* from p to H is the map

$$\begin{aligned} \Pi_p : \mathbb{P}^n - \{p\} &\to \mathbb{P}^{n-1} = H \subseteq \mathbb{P}^n \\ x &\mapsto \overleftrightarrow{\ell p} \cap H, \end{aligned}$$

where $\overleftrightarrow{\ell p}$ is the unique line through p and x.

Question: How does this look in local charts on $\mathbb{P}^n$?

$$\begin{aligned} \mathbb{P}^n - \{[1 : 0 : \cdots : 0]\} &\xrightarrow{\Pi_p} \mathbb{P}^{n-1} = \mathbb{V}(x_0) \subseteq \mathbb{P}^n \\ U_0 \ni [1 : \lambda_1 : \cdots : \lambda_n] &\mapsto [\lambda_1 : \cdots : \lambda_n] \end{aligned}$$

We have

$$\ell = \left\{[1 : t\lambda_1 : \cdots : t\lambda_n] \;\middle|\; t \in k\right\} = \left\{\left[\tfrac{1}{t}, \lambda_1 \ldots \lambda_n\right] \;\middle|\; t \in k\right\} \ni [0, \lambda_1, \ldots, \lambda_n].$$

If we had a chart where p was *at* infinity, it would look like "projection"

$$\begin{aligned} \mathbb{A}^n &\to \mathbb{A}^{n-1} \\ (x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_{n-1}) \end{aligned}$$

in the usual sense.

6.6 Homogenization of affine algebraic sets

Exercise 6.3. If $V \subseteq \mathbb{A}^n$ is an affine algebraic set with projective closure $\overline{V} \subseteq \mathbb{P}^n$, and if $\mathbb{I}(V) \subseteq k[x_1, \ldots, x_n]$ is the ideal of V, then $\mathbb{I}(\overline{V}) \subseteq k[x_0, \ldots, x_n]$ is generated by the homogenizations of *all* the elements of $\mathbb{I}(V)$.

Exercise 6.4 (purely topological). Let $V \subseteq \mathbb{P}^n$ be a projective algebraic set. Then V is irreducible if and only if $V \cap U_i$ is irreducible $\forall i = 0, \ldots, n$, the "standard affine cover" of V.

7 Abstract and quasi-projective varieties

7.1 Basic definition and examples

Definition 7.1. A *quasi-projective variety* is any irreducible, locally closed (topological) subspace of $\mathbb{P}^n$.

I.e., $W \subseteq \mathbb{P}^n$ is a *quasi-projective variety* by definition if $W = U \cap V$, where $U \subseteq \mathbb{P}^n$ is open and $V \subseteq \mathbb{P}^n$ is an irreducible projective set.

Example 7.2 (Some quasi-projective varieties).
(1) Irreducible affine algebraic sets are quasi-projective varieties:
$$V = \overline{V} \cap U_0 \subseteq \mathbb{A}^n = U_0 \subseteq \mathbb{P}^n.$$

(2) Irreducible projective algebraic sets.

(3) Open subsets of affine or projective varieties.

Example 7.3 (An abstract variety).

$$\begin{aligned}\mathfrak{M}_g &= \{\text{moduli space of compact Riemann surfaces}\}\\ &= \{\text{moduli space of smooth projective varieties}/\mathbb{C} \text{ of dimension } 1\}\end{aligned}$$

This is an abstract algebraic variety.

Theorem 7.4 (Fields medal, Deligne and Mumford). *$\mathfrak{M}_g$ is quasi-projective.*

Example 7.5 (Another moduli space). Lines in $\mathbb{P}^2 = \mathbb{P}(k^3)$ can be viewed as $\mathbb{P}\left((k^3)^*\right)$.

7.2 Quasi-projective varieties are locally affine

Proposition 7.6. *A quasi-projective variety W has a basis of open sets which are (homeomorphic to) affine algebraic sets.*

Proof. First $W = V \cap U$, where $U \subseteq \mathbb{P}^n$ is open and $V \subseteq \mathbb{P}^n$ is closed and irreducible. Then

$$W \cap U_i = (V \cap U \cap U_i) = (V \cap U_i) \cap (U \cap U_i) \subseteq V_i = V \cap U_i \subseteq U_i = \mathbb{A}^n,$$

and $(V \cap U_i) \cap (U \cap U_i)$ is an open subset in the affine variety V_i.

But an open subset of an affine variety has an open cover by affine charts:

$$V - \mathbb{V}(g_1, \ldots, g_r) = U \subseteq V \subseteq \mathbb{A}^n$$

for $g_i \in k[V]$, then

$$U = \bigcup_{i=1}^{r} \big(V - \mathbb{V}(g_i)\big).$$

□

7.3 The sheaf of regular functions

Fix a quasi-projective variety W. What is $\mathcal{O}_W$?

Definition 7.7. Let $U \subseteq W$ be any open set. A *regular function* on U is a function $\varphi : U \to k$ with the property that $\forall p \in U$, there exists an open affine set $p \in U' \subseteq U$ such that $\varphi|_{U'}$ is regular on U'.

Equivalently, $\varphi : U \to k$ is regular $\iff$ $\varphi|_{U \cap U_i}$ is regular on $U \cap U_i$ $\forall i = 0, \ldots, n$.[2]

Example 7.8. X_0, X_1 in $k[X_0, X_1, X_2]$ are *not* functions on $\mathbb{P}^2$.

But the *ratio* $\frac{X_1}{X_0}$ is a well-defined function on $\mathbb{P}^2 - \mathbb{V}(X_0) = U_0$.

[2] $W = \widetilde{U} \cap V \implies U \subseteq W$ is $\widetilde{\widetilde{U}} \cap \widetilde{U} \cap V = U$, and $(\widetilde{\widetilde{U}} \cap \widetilde{U} \cap V) \cap U_i$ is open in $V \cap U_i$, which is affine.

Example 7.9. $\varphi = \frac{X_j}{X_i} = t_j$ (the "j-th coordinate function") is a regular function on $\mathbb{P}^n \setminus \mathbb{V}(X_i) = U_i \longleftrightarrow k^n$ in coordinates $\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}$.

How does this look in U_κ? U_κ has coordinates $\frac{X_0}{X_\kappa}, \dots, \frac{X_n}{X_\kappa}$, denoted $t_0, \dots, \widehat{t_\kappa}, \dots, t_n$. Then

$$\varphi = \frac{X_j}{X_i} = \frac{X_j/X_\kappa}{X_i/X_\kappa} = \frac{t_j}{t_i}$$

is a rational function of the coordinates, regular on $U_\kappa \setminus \mathbb{V}(t_i) = U_i \cap U_\kappa$.

Remark 7.10. We get a *sheaf* $\mathcal{O}_W$ of regular functions on the quasi-projective variety W. To each $U \subseteq W$, assign $\mathcal{O}_W(U) =$ ring of regular functions on U.

Example 7.11. $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$. So if $n \geq 1$, then $\mathbb{P}^n$ is not affine!

7.4 Main example of regular functions in projective space

Let $F, G \in k[x_0, \dots, x_n]$ be homogeneous of the same degree. Then $\varphi = \frac{F}{G}$ is a well-defined functions on $\mathbb{P}^n \setminus \mathbb{V}(G)$:

$$\frac{F(tx_0, \dots, tx_n)}{G(tx_0, \dots, tx_n)} = \frac{t^d F(x_0, \dots, x_n)}{t^d G(x_0, \dots, x_n)} = \frac{F(x_0, \dots, x_n)}{G(x_0, \dots, x_n)}.$$

Moreover, φ is regular on $\mathcal{U} := [\mathbb{P}^n \setminus \mathbb{V}(G)]$.

We now check this. It suffices to check that $\varphi|_{\mathcal{U} \cap U_i}$ (for $i = 0, \dots, 1$) is regular on $U_i \cap \mathcal{U} \overset{\text{open}}{\subseteq} U_i = \mathbb{A}^n$.

Lemma 7.12. *If $F \in k[X_0, \dots, X_n]$ is homogeneous of degree d, then*

$$\frac{F}{X_i^d} = F\left(\frac{X_0}{X_i}, \frac{X_1}{X_i}, \dots, 1, \frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i}\right).$$

Proof. Comes down to checking for $X_0^{\alpha_0} \dots X_n^{\alpha_n}$ (with $\sum \alpha_i = d$):

$$\frac{X_0^{\alpha_0} \dots X_n^{\alpha_n}}{X_i^d} = \prod_{j=0}^{n} \left(\frac{X_j}{X_i}\right)^{\alpha_j}.$$

□

Now we have

$$\varphi\big|_{U_i} = \frac{F}{G} = \frac{F/x_i^d}{G/x_i^d} = \frac{F\left(\frac{x_0}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i}\right)}{G\left(\frac{x_0}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i}\right)} = \frac{f(t_0, \dots, \widehat{t_i}, \dots, t_n)}{g(t_0, \dots, \widehat{t_i}, \dots, t_n)}$$

is a rational function on $\mathbb{A}^n = U_i$, regular on $[\mathbb{A}^n \setminus \mathbb{V}(g)] = U_i \cap (\mathbb{P}^n \setminus \mathbb{V}(G))$. So φ is regular on $\mathcal{U}$. □

7.5 Morphisms of quasi-projective varieties

Definition 7.13. A *regular map* (or morphism in the category) of quasi-projective varieties $X \xrightarrow{\varphi} Y \subseteq \mathbb{P}^n$ is a well-defined map of sets such that $\forall x \in X$, writing $\varphi(x) \in Y \cap U_i \subseteq U_i = k^n$ for some i, there exists an open affine neighborhood U of $x \in U \subseteq X$ such that $\varphi(U) \subseteq U_i$ and φ restricts to a map

$$\begin{aligned} U &\to Y \cap U_i \subseteq U_i \\ z &\mapsto \big(\varphi_1(z), \ldots, \varphi_n(z)\big), \end{aligned}$$

where $\varphi_i \in \mathcal{O}_X(U)$.

Definition 7.14. An *isomorphism of varieties* is a regular map $X \xrightarrow{\varphi} Y$ which has a regular inverse $Y \xrightarrow{\psi} X$.

Example 7.15 (The d-th Veronese map). Let $m = \binom{n+d}{n} - 1$. Then the d-th *Veronese map* is defined by

$$\begin{aligned} \mathbb{P}^n &\xrightarrow{\nu_d} \mathbb{P}^m \\ [x_0 : \cdots : x_n] &\mapsto \left[x_0^d : x_0^{d-1}x_1 : \cdots : x_n^d\right], \end{aligned}$$

where the coordinates are all degree d monomials in $x_0, \ldots, x_n$.

Example 7.16 (Projection). $p \notin H =$ hyperplane in $\mathbb{P}^n$:

$$\begin{aligned} \mathbb{P}^n \setminus \{p\} &\to \mathbb{P}^{n-1} = H \\ [x_0 : \cdots : x_n] &\mapsto [x_1 : \cdots : x_n]. \end{aligned}$$

8 Classical constructions

8.1 Twisted cubic and generalization

Definition 8.1. The *twisted d-ic* in $\mathbb{P}^d$ is the image of $\mathbb{P}^1$ under the d-Veronese map

$$\begin{aligned} \mathbb{P}^1 &\xrightarrow{\nu_d} C_d \subseteq \mathbb{P}^d \\ [s : t] &\mapsto \left[s^d : s^{d-1}t : \cdots : st^{d-1} : t^d\right] = [x_0 : \cdots : x_d]. \end{aligned}$$

Fact 8.2. ν_d is an isomorphism $\mathbb{P}^1 \cong C_d$. The inverse map is

$$\begin{aligned} C_d &\to \mathbb{P}^1 \\ [x_0 : \cdots : x_d] &\mapsto \begin{cases} [x_0 : x_1] & \text{if } x_1 \neq 0, \\ [x_{d-1} : x_d] & \text{if } x_1 = 0. \end{cases} \end{aligned}$$

8.2 Hypersurfaces

Definition 8.3. A *hypersurface* in $\mathbb{P}^n$ of degree d is the zero set of one homogeneous polynomial of degree d.

Let $V = \mathbb{V}(F_d) \subseteq \mathbb{P}^n$, with F_d irreducible. Pick $p \notin V$.

$$\begin{tikzcd} \mathbb{P}^n \setminus \{p\} \arrow[r, "\Pi_p"] & \mathbb{P}^{n-1} \arrow[d, equal] \\ V \arrow[u, phantom, "\cup\!\mid"] \arrow[r, two heads, "\Pi_p"] & \mathbb{P}^{n-1} \end{tikzcd}$$

finite map, "generically" d-to-1.

Lemma 8.4. *Every line in $\mathbb{P}^n$ must intersect V at $\leq d$ points. ("Generically" exactly d points; strict inequality is possible due to multiplicity.)*

Proof.

$$\mathbb{V}(F_d) \cap \mathbb{V}(x_2, \ldots, x_n) = \mathbb{V}(F_d, x_2, \ldots, x_n) = \mathbb{V}(\overline{F_d}) \subseteq L = \mathbb{V}(x_2, \ldots, x_n) \subseteq \mathbb{P}^n$$

□

8.3 Segre embedding

Category of *quasi-projective* varieties:

Objects (irreducible) locally closed subspaces of $\mathbb{P}^n$ (all n) over fixed $k = \overline{k}$.

Morphisms Map of sets $\mathbb{P}^n \supseteq X \xrightarrow{\varphi} Y \subseteq \mathbb{P}^m$ such that on sufficiently small open subsets of $X_i = X \cap U_i \subseteq \mathbb{A}^n$, $\varphi|_U$ is a regular mapping into some chart of $\mathbb{P}^m$.

Is there a notion of *product* in this category?
Recall: For $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ affine algebraic sets,

$$X \times Y \subseteq \mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$$

is an affine algebraic set. But $\mathbb{P}^m \times \mathbb{P}^n \neq \mathbb{P}^{m+n}$, so we can't do a similar thing for projective algebraic sets.

Indeed, $\mathbb{P}^2 \setminus \mathbb{A}^2$ is one line at infinity, but

$$\left(\mathbb{P}^1 \times \mathbb{P}^1\right) \setminus \mathbb{A}^2 = \left\{\infty \times \mathbb{P}^1\right\} \cup \left\{\mathbb{P}^1 \times \infty\right\}$$

consists of *two* lines at infinity.

Goal 8.5. Put the structure of a quasi-projective variety (projective) on $\mathbb{P}^n \times \mathbb{P}^m$.
Want:

(1) $\sigma : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \Sigma \subseteq \mathbb{P}^?$, where Σ is a (*closed*) projective algebraic set, and σ is compatible with the identification $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{m+n} \xrightarrow{\sigma} \sigma(\mathbb{A}^{m+n})$ on each affine chart $U_i \times U_j = \mathbb{A}^n \times \mathbb{A}^m$.

(2) There should be *regular maps* $\Sigma \xrightarrow{\pi_1} \mathbb{P}^n$, $\Sigma \xrightarrow{\pi_2} \mathbb{P}^m$.

(3) (Linear space) $\times p \subseteq \mathbb{P}^n \times \mathbb{P}^m$ maps under σ to a *linear space* of the same dimension in $\mathbb{P}^?$.

Example 8.6.

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\xrightarrow{\sigma_{11}} \mathbb{P}^3 \\ ([x:y],[z:w]) &\mapsto [xz : xw : yz : yw] \end{aligned}$$

The image of σ_{11} is $\mathbb{V}(X_0X_3 - X_1X_2)$.

On $U_x \times U_z = \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$:

$$\begin{aligned} \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1 &\xrightarrow{\simeq} \mathbb{V}(xy - z) \subseteq \mathbb{A}^3 \\ \big((1,t),(1,s)\big) &\mapsto [1 : t : s : ts] \end{aligned}$$

Also,

$$\mathbb{P}^1 \times [a:b] \mapsto \Big\{[xa : xb : ya : yb] \;\Big|\; [x:y] \in \mathbb{P}^1\Big\} \subseteq \mathbb{P}^3 \subseteq \mathbb{P}(k^4)$$

is a line in $\mathbb{P}^3$ corresponding to the 2-dimensional subspace

$$\operatorname{span}\{(a,b,0,0),(0,0,a,b)\} \subset k^4.$$

This is the "*definition*" of $\mathbb{P}^1 \times \mathbb{P}^1$ as a quasi-projective variety.

Definition 8.7. The *Segre map* is

$$\begin{aligned} \mathbb{P}^n \times \mathbb{P}^m &\xrightarrow{\sigma_{nm}} \Sigma_{nm} \subseteq \mathbb{P}^{(n+1)(m+1)-1} \\ \big([x_0 : \cdots : x_n],[y_0 : \cdots : y_m]\big) &\mapsto \underbrace{\begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_0 & \dots & y_m \end{bmatrix}}_{(n+1)\times(m+1) \text{ matrix}} \in \mathbb{P}\big(\mathrm{Mat}_k(n+1,m+1)\big). \end{aligned}$$

Remark 8.8 (Linear algebra review). TFAE for any matrix A of size $d \times e$:

(1) The rows are all multiples of each other by a scalar.

(2) The columns are all multiples of each other by a scalar.

(3) A factors as $(d \times 1) \times (1 \times e)$.

(4) The rank of A is ≤ 1.

(5) All 2×2 subdeterminants of A are zero.

Writing the matrix coordinates as $\begin{bmatrix} z_{00} & \cdots & z_{0m} \\ \vdots & & \vdots \\ z_{n0} & \cdots & z_{nm} \end{bmatrix}$,

$$\Sigma_{nm} = \mathbb{V}\left(\text{determinant of } 2\times 2 \text{ minors of } \begin{bmatrix} z_{00} & \cdots & z_{0m} \\ \vdots & & \vdots \\ z_{n0} & \cdots & z_{nm} \end{bmatrix}\right).$$

The projections $\Sigma \xrightarrow{\pi_1} \mathbb{P}^n,\ \Sigma \xrightarrow{\pi_2} \mathbb{P}^m$ are given by

$$p = \begin{bmatrix} z_{ij} \end{bmatrix} \xmapsto{\pi_1} \text{any column of } p,$$

and likewise, π_2 takes any row. (This is well-defined because the matrix has rank 1.)

8.4 Products of quasi-projective varieties

Definition 8.9. If $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are quasi-projective varieties, then we define a quasi-projective variety structure on the set $X \times Y$ by identifying $X \times Y$ with its image under the appropriate Segre map σ_{nm}:

$$\sigma_{nm}(X \times Y) \subseteq \Sigma_{nm} \subseteq \mathbb{P}^{(n+1)(m+1)-1}$$

This gives $X \times Y$ a Zariski topology!

How do the closed sets look?

Definition 8.10. A polynomial $F \in k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is *bihomogeneous* if F is homogeneous separately in $x_0, \ldots, x_n$ (treating the y_i as scalars) and $y_0, \ldots, y_m$ (treating the x_i as scalars).

Example 8.11. The polynomial $x_0^5 y_1 y_2 - x_0 x_1 x_2^3 y_3^2$ is bihomogeneous of degree $(5, 2)$.
However, $x_0^7 - y_0^7$ is *not* bihomogeneous.

Note 8.12. If $F \in k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is bihomogeneous, then $\mathbb{V}(F) \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is well-defined.

Exercise 8.13. The closed sets of $\mathbb{P}^n \times \mathbb{P}^m$ are precisely the sets defined as the common zero set of a collection of *bihomogeneous* polynomials in $k[x_0, \ldots, x_n, y_0, \ldots, y_m]$.

Example 8.14. The Zariski topology on $\mathbb{P}^n \times \mathbb{A}^m$ with coordinates $k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ has closed sets exactly of the form

$$\mathbb{V}\left(\{F_\lambda(x_0, \ldots, x_n, y_1, \ldots, y_m)\}_{\lambda \in \Lambda}\right),$$

where F_λ is homogeneous in $x_0, \ldots, x_n$.

8.5 Conics

Definition 8.15. A *conic* in $\mathbb{P}^2$ is a hypersurface (curve) given by a single degree 2 homogeneous polynomial.

Three kinds:

Nondegenerate $\mathbb{V}(F) \subseteq \mathbb{P}^2$ such that F does not factor into 2 linear factors. (Showed in homework: changing coordinates, these are all the same.)

Degenerate, two lines $F = L_1 L_2$, where $\lambda L_1 \neq L_2$. Then $\mathbb{V}(F) = \mathbb{V}(L_1) \cup \mathbb{V}(L_2)$.

Think of this as the limit as $t \to 0$ of a family of nondegenerate conics

$$\{\mathbb{V}(xy - t)\}_{t \in k} \subseteq \mathbb{A}^2.$$

Degenerate, double line $F = L_1^2$. Then $\mathbb{V}(F) = \mathbb{V}(L_1^2)$.

Think of this as the limit as $t \to 0$ of a family of degenerate conics

$$\mathbb{V}(y(y - tx)) = \mathbb{V}(y) \cup \mathbb{V}(y - tx) \subseteq \mathbb{A}^2.$$

This line $\mathbb{V}(y^2)$ is one line "counted twice". This is a scheme, but not a variety.

Every conic is uniquely described by its equation $F \in [k[x, y, z]]_2$.[3]
Let $C \subseteq \mathbb{P}(k^3)$ be a conic. We have a correspondence

$$\begin{aligned} C = \mathbb{V}\left(Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2\right) &\longleftrightarrow [A : B : C : D : E : F] \\ \left\{\text{conics in } \mathbb{P}(k^3)\right\} &\longleftrightarrow \mathbb{P}\left(\operatorname{Sym}^2\left((k^3)^*\right)\right) = \mathbb{P}^5. \end{aligned}$$

Moreover, we have proper inclusions of closed subvarieties

$$D_2 = \{\text{double lines}\} \subsetneqq D_1 = \{\text{pairs of lines}\} \subsetneqq \left\{\text{all conics in } \mathbb{P}(k^3)\right\} = \mathbb{P}\left(\operatorname{Sym}^2\left((k^3)^*\right)\right).$$

As we will show on the homework, $D_2 \cong$ image of $\mathbb{P}^2$ under the Veronese map $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$.
This is the beginning of the study of moduli spaces.

8.6 Conics through a point

Fix $p \in \mathbb{P}^2$. Consider the set

$$\mathcal{C}_p = \left\{C \subseteq \mathbb{P}^2 \text{ conic in } \mathbb{P}^2 \text{ passing through } p\right\} \subsetneqq \mathbb{P}\left(\operatorname{Sym}^2\left((k^3)^*\right)\right) = \mathbb{P}^5.$$

This is a *hyperplane*. Indeed, write $p = [u : v : t]$. A conic

$$C = \mathbb{V}(\underbrace{Ax^2 + Bxy + \cdots + Fz^2}_{G})$$

passing through $p \iff G(p) = 0 \iff Au^2 + Buv + Cv^2 + Dut + Evt + Ft^2 = 0$, which is a linear equation L in the homogeneous coordinates A, B, C, D, E, F for $\mathbb{P}^5 = \mathbb{P}\left(\operatorname{Sym}^2\left((k^3)^*\right)\right)$. Thus,

$$\mathcal{C}_p = \mathbb{V}(L) \subseteq \mathbb{P}^5.$$

[3] $[k[x, y, z]]_2 = \operatorname{Sym}^2\left((k^3)^*\right)$ denotes the vector space of degree 2 homogeneous polynomials, i.e., the 2nd component of the graded ring $k[x, y, z]$.

Theorem 8.16 ("5 points determine a conic"). *Given $p_1, p_2, p_3, p_4, p_5 \in \mathbb{P}^2$ distinct points, there is a conic through all 5 points, unique if the points are in general position.*

If no three points are on the same line, then there is a unique nondegenerate *conic through them.*

9 Parameter spaces

9.1 Example: Hypersurfaces of fixed degree

Recall:

$$\begin{aligned}\left\{\text{conics in } \mathbb{P}^2\right\} &\longleftrightarrow \{\text{their homogeneous equations up to scalar multiple}\}\\ &\longleftrightarrow \mathbb{P}\left(\mathrm{Sym}^2\left((k^3)^*\right)\right) = \{\deg 2 \text{ homogeneous polynomials in 3 variables}\}/\text{scalars}\\ &= [k[x,y,z]]_2/\text{scalars} = \mathrm{Sym}^2\left((k^3)^*\right)/\text{scalars}\end{aligned}$$

Similarly:

$$\begin{array}{ccc}\{\text{hypersurface of degree } d \text{ in } \mathbb{P}^n\} & \longleftrightarrow & \{\text{their equations up to scalar multiple}\}\\ \| & & \| \\ \mathbb{V}(\underbrace{Ax_0^d + Bx_0^{d-1}x_1 + \cdots +}_{\text{"homog. degree } d \text{ in } x_0,\ldots,x_n\text{"}}) & & \mathbb{P}\left(\mathrm{Sym}^d\left((k^{n+1})^*\right)\right) = \mathbb{P}^{\binom{n+d}{n}-1}\end{array}$$

Note that these are not really varieties, since we remember the homogeneous equation.

9.2 Philosophy of parameter spaces

Philosophy: the set of hypersurfaces of degree d "is" in a natural way a *variety.* The subsets ("algebraically natural" subsets) are subvarieties.

The "good" properties will hold on *open* subsets of $\mathbb{P}^{\binom{n+d}{n}-1}$ (hopefully non-empty), and "bad" properties will hold on closed subsets of $\mathbb{P}^{\binom{n+d}{n}-1}$ (hopefully proper).

9.3 Conics that factor

Look in $\mathbb{P}\left(\mathrm{Sym}^2\left((k^3)^*\right)\right)$ = set of conics in $\mathbb{P}^2$. Does "$\mathbb{V}(G)$" $\longleftrightarrow [A : B : C : D : E : F]$ factor or not?

$$G = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$$

factors $\iff$

$$\det \begin{bmatrix} A & \frac{1}{2}B & \frac{1}{2}D \\ \frac{1}{2}B & C & \frac{1}{2}E \\ \frac{1}{2}D & \frac{1}{2}E & F \end{bmatrix} = 0.$$

The subset where the conic degenerates into 2 lines is

$$\mathbb{V}\left(\det\begin{bmatrix} A & \frac{1}{2}B & \frac{1}{2}D \\ \frac{1}{2}B & C & \frac{1}{2}E \\ \frac{1}{2}D & \frac{1}{2}E & F \end{bmatrix}\right).$$

Now we have

$$\begin{array}{ccc} \{\text{hypersurface of degree } d \text{ in } \mathbb{P}^n\} & \longleftrightarrow & \{\text{their equations up to scalar multiple}\} \\ & & \| \\ \cup\!\mid & & \mathbb{P}\left(\mathrm{Sym}^d\left((k^{n+1})^*\right)\right) = \mathbb{P}^{\binom{n+d}{n}-1} \\ & & \cup\!\mid \text{ closed} \\ \{\text{hypersurfaces whose equations factor}\} & \longleftrightarrow & X \end{array}$$

where $G = G_i G_{d-i}$ factors and

$$X = \bigcup_{i=1}^{\frac{d-1}{2}} X_i,$$

with $X_i =$ the subset of hypersurfaces of degree d where equation factors as $(\deg i)(\deg d - i)$.

Theorem 9.1. *The set of degree d hypersurfaces in $\mathbb{P}^n = \mathbb{P}(V)$ which are* not *irreducible (meaning: whose equations factor non-trivially) is a proper closed subset of $\mathbb{P}\left(\mathrm{Sym}^d(V^*)\right)$.*

Proof. It suffices to show each $X_i = \{G = G_i G_{d-i}\}$ is closed and proper. Consider

$$\begin{aligned} \mathbb{P}\left(\mathrm{Sym}^i(V^*)\right) \times \mathbb{P}\left(\mathrm{Sym}^{d-i}(V^*)\right) &\xrightarrow{\varphi} \mathbb{P}\left(\mathrm{Sym}^d(V^*)\right) \\ (G, H) &\mapsto GH, \end{aligned}$$

where G, H are homogeneous of degrees $i, d-i$, respectively, in $x_0, \ldots, x_n$.

Easy to check: φ is *regular* and image is X_i. Need to check closed (proper). $\square$

This follows from the following *big theorem*:

Theorem 9.2. *If V is* projective *and $V \xrightarrow{\varphi} Y$ is any regular map of quasi-projective varieties, then φ sends closed sets of V to closed sets of Y.*

Caution 9.3. Really need the hypothesis that the source variety is projective. E.g.:

$$\mathcal{U}_f = \mathbb{A}^n - \mathbb{V}(f) \overset{i}{\hookrightarrow} \mathbb{A}^n$$

regular map, image is open. Also, the hyperbola:

$$\begin{gathered} \mathbb{A}^2 \xrightarrow{\pi} \mathbb{A}^1 \\ (x, y) \mapsto x \\ \pi(\mathbb{V}(xy-1)) = \mathbb{A}^1 - \{0\}, \end{gathered}$$

which is not closed.

10 Regular maps of projective varieties

10.1 Big theorem on closed maps

Theorem 10.1. *If V is* projective *and $V \xrightarrow{\varphi} X$ is a regular map to X (any quasi-projective variety), then φ is* closed *(i.e., if $W \subseteq V$ is a closed subset of V, then $\varphi(W)$ is closed).*

Note 10.2. To prove the theorem, it suffices to show that $\varphi(V)$ is closed.

[If $W \subseteq V$ is closed (irreducible), then W is also projective. So $\varphi|_W : W \to X$ has the property that $\varphi|_W(W)$ is closed, thus $\varphi(W) = \varphi|_W(W)$ is closed.]

Corollary 10.3. *If V is projective, then $\mathcal{O}_V(V) = k$.*

Proof. Let $V \xrightarrow{\varphi} k \subseteq \mathbb{P}^1$ be a regular function. We can interpret $\varphi : V \to \mathbb{P}^1$ as a regular map. So the image is closed in $\mathbb{P}^1$ by Theorem 10.1.

Thus $\varphi(V)$ is either a finite set of points (or $\varnothing$) or $\varphi(V) = \mathbb{P}^1$. Since φ is an actual map into $k \subsetneqq \mathbb{P}^1$, $\varphi(V)$ must be a finite set of points. But V is irreducible, so $\varphi(V)$ is a single point. $\square$

10.2 Preliminary: Graphs

Fix any regular map of quasi-projective varieties $X \xrightarrow{\varphi} Y$.

Definition 10.4. The *graph* Γ_φ of $\varphi : X \to Y$ is the set

$$\big\{(x, y) \bigm| \varphi(x) = y\big\} \subseteq X \times Y.$$

Proposition 10.5. *Γ_φ is always closed in $X \times Y$.*

Proof. Step 1: Without loss of generality, $Y = \mathbb{P}^m$, since $X \xrightarrow{\varphi} Y \subseteq \mathbb{P}^m$, and we interpret φ as a regular map $X \to \mathbb{P}^m$. We have

$$\Gamma_\varphi \subseteq X \times Y \subseteq X \times \mathbb{P}^m,$$

and to show Γ_φ is closed in $X \times Y$, it suffices to show $\Gamma_\varphi \subseteq X \times \mathbb{P}^m$ is closed.

Step 2: Consider the regular map

$$\begin{aligned} \psi : X \times \mathbb{P}^m &\xrightarrow{(\varphi,\mathrm{id})} \mathbb{P}^m \times \mathbb{P}^m \\ (x, y) &\mapsto (\varphi(x), y)\,. \end{aligned}$$

Note 10.6. $\Gamma_\varphi = \psi^{-1}(\Delta)$, where $\Delta = \{(z, z) \mid z \in \mathbb{P}^m\}$ is the diagonal subset of $\mathbb{P}^m \times \mathbb{P}^m$, which is closed.

Because Δ is closed, so is Γ_φ. $\square$

10.3 Proof of Theorem 10.1

Fix $V \xrightarrow{\varphi} X$ regular map, V projective. Need to show $\varphi(V)$ is closed.

Let $\Gamma_\varphi \subseteq V \times X$ be the graph. Consider the projection

$$\Gamma_\varphi \subseteq V \times X \xrightarrow{\pi} X \supseteq \pi(\Gamma_\varphi) = \varphi(V),$$

which is a regular map. It suffices to prove that $\pi(\Gamma_\varphi)$ is closed.

Theorem 10.7. *If V is projective and X is quasi-projective, then the projection $V \times X \xrightarrow{\pi} X$ is closed.*

Proof of Theorem 10.7. First, using point-set topology arguments, reduces as follows:

(1) WLOG, $V = \mathbb{P}^n$.

(2) WLOG, X is affine.

(3) WLOG, $X = \mathbb{A}^m$.

Now:

$$\mathbb{P}^n \times \mathbb{A}^m \xrightarrow{\varphi} \mathbb{A}^m.$$

Put coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n$ and $y_1, \ldots, y_m$ on $\mathbb{A}^n$.

Want to show: Given closed $Z \subseteq \mathbb{P}^n \times \mathbb{A}^m$, that $\varphi(Z)$ is closed in $\mathbb{A}^m$. Write

$$Z = \mathbb{V}\big(g_1(x_0, \ldots, x_n, y_1, \ldots, y_m), \ldots, g_t(x_0, \ldots, x_n, y_1, \ldots, y_m)\big),$$

where g_i are homogeneous in $x_0, \ldots, x_n$ (but not in the y_i). What is the image of Z?

Note 10.8. $(\lambda_1, \ldots, \lambda_m) \in \mathbb{A}^m$ is in $\pi(Z)$ iff

$$\varnothing \neq \mathbb{V}\big(g_1(x_0, \ldots, x_n, \lambda_1, \ldots, \lambda_m), \ldots, g_t(x_0, \ldots, x_n, \lambda_1, \ldots, \lambda_m)\big) \subseteq \mathbb{P}^n$$

iff (by the projective Nullstellensatz)

$$\operatorname{Rad}\big(g_1(x, \lambda), \ldots, g_t(x, \lambda)\big) \not\supseteq (x_0, \ldots, x_n)$$

iff

$$\big(g_1(x, \lambda), \ldots, g_t(x, \lambda)\big) \not\supseteq (x_0, \ldots, x_n)^T \quad \forall T.$$

So we need to show: The set L_T of all $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{A}^m$ such that

$$(x_0, \ldots, x_n)^T \not\subseteq \big(g_1(x, \lambda), \ldots, g_t(x, \lambda)\big)$$

is closed. The image of $\pi(Z) \subseteq \mathbb{A}^m$ is

$$\bigcap_{T=1}^{\infty} L_T,$$

so it suffices to show that each $L_T \subseteq \mathbb{A}^m$ is closed.

Aside 10.9 (Converse). Let's consider the converse:

$$(x_0, \ldots, x_n)^T \subseteq \big(g_1(x, \lambda), \ldots, g_t(x, \lambda)\big) \text{ in } k[x_0, \ldots, x_n]$$

Look in degree T part of $k[x_0, \ldots, x_n]$:

$$[k[x_0, \ldots, x_n]]_T \subseteq [(g_1, \ldots, g_n)]_T$$

Basis here is $\{x_0^{i_0} \cdots x_n^{i_n}\}_{\sum i_k = T}$.

Spanning set for the σ-dimensional $[(g_1, \ldots, g_n)]$ = subvector space of degree T elements in $(g_1(x, \lambda), \ldots, g_t(x, \lambda))$:

$$\{g_J\} = \Big\{g_i(x, \lambda) \cdot x_0^{j_0} \cdots x_n^{j_n} \;\Big|\; \deg(g_i) = d_i,\ \sum j_\ell = T - d_i,\ i = 1, \ldots, t\Big\}.$$

Write a matrix with the coefficient x^I in g_J in the (IJ)-th spot. The coefficients are *polynomials* in $\lambda_1, \ldots, \lambda_m$. This is a basis iff the matrix is nondegenerate.

□

11 Function fields, dimension, and finite extensions

11.1 Commutative algebra: transcendence degree and Krull dimension

Fix $k \hookrightarrow L$ extension of fields.

- The *transcendence degree* of L/k is the maximum number of algebraically independent elements of L/k.
- Every maximal set of algebraically independent elements of L/k has the same cardinality.
- If $\{x_1, \ldots, x_d\}$ are a maximal set of algebraically independent elements, we call them a *transcendence basis* for L/k.
- If R is a finitely generated domain over k, with fraction field L, then the transcendence degree of L/k is equal to the Krull dimension of R.

11.2 Function field

Fix V affine variety.

Definition 11.1 (function field of an affine variety)**.** The function field of V, denoted $k(V)$, is the fraction field of $k[V]$.

Say $V - \mathbb{V}(g) = U_g = U \overset{\text{open}}{\subset} V$ for some $g \in k[V]$. Then

$$\begin{tikzcd} \mathcal{O}_V(V) \arrow[r, hook, "\text{rest.}"] \arrow[d, equal] & \mathcal{O}_V(U) \arrow[r, hook, "\text{rest.}"] & \mathcal{O}_V(U_g) \arrow[d, equal] \\ k[V] \arrow[rr, hook] & & k[V]\left[\frac{1}{g}\right] \end{tikzcd}$$

Note 11.2. Function fields of U_g and V are the *same* field.

Fix $V \subseteq \mathbb{P}^n$ projective variety.

Definition 11.3 (function field of a projective variety)**.** The function field of V, denoted $k(V)$, the function field of any $V \cap U_i$ (standard affine chart) such that $V \cap U_i \neq \varnothing$.

Question: Why is this independent of the choice of U_i?
$V_i = V \cap U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$ is an affine variety in $U_i = \mathbb{A}^n$. Then $k[V_i]$ is generated by (the restrictions of) the *actual* functions on U_i

$$\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i},$$

and likewise for $k[V_j]$. If $\frac{x_i}{x_j} = 0$ on $U_i \cap U_j \cap V$, then x_i vanishes on $U_i \cap U_j \cap V$, which implies that x_i vanishes on V and hence $V \cap U_i$ is empty. So we can write

$$\frac{x_k}{x_i} = \frac{x_k/x_j}{x_i/x_j},$$

thus $k[V_i] \subseteq k(V_j)$, hence $k(V_i) \subseteq k(V_j)$. By symmetry, $k(V_j) = k(V_i)$.

Definition 11.4 (function field of a quasi-projective variety)**.** The function field of a quasi-projective variety V is $k(\overline{V})$, where $\overline{V}$ is the closure of $V \subseteq \mathbb{P}^m$.

Equivalently, it is the function field of any $V \cap U_i$ (such that $V \cap U_i \neq \varnothing$) or indeed of *any* open affine subset of V.

11.3 Dimension of a variety

Definition 11.5. The *dimension* of a (quasi-projective) variety V/k is the transcendence degree of $k(V)$ over k.

By convention, the dimension of an algebraic set is the maximal dimension of any of its (finitely many) components.

Example 11.6.

- $\dim \mathbb{A}^n = n$
- $\dim \mathbb{P}^n = n$
- $\dim(X \times Y) = \dim X + \dim Y$
- All components of a hypersurface $\mathbb{V}(F) \subseteq \mathbb{P}^n$ have dimension $n - 1$.

Definition 11.7. A regular map $X \xrightarrow{\varphi} Y$ is *finite* if (in the affine case) the corresponding map of coordinate rings is an integral extension, or (in general) if the preimage of an affine cover of Y is affine and φ is finite on each affine chart.

Theorem 11.8. *If $X \xrightarrow{\varphi} Y$ is a regular map,* finite*, then* $\dim X = \dim Y$.

Proof. Reduce to the affine case: $X \xrightarrow{\varphi} Y$ finite $\iff$ $k[Y] \xrightarrow{\varphi^*} k[X]$ is an integral extension. □

11.4 Noether normalization

Take some $p \notin V$. Then

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, dashed, "\pi_p"] & \mathbb{P}^{n-1} \arrow[r, dashed, "\pi_{p_2}"] & \mathbb{P}^{n-2} \arrow[r, dashed] & \ldots \arrow[r, dashed] & \mathbb{P}^d \arrow[d, equal] \\
\cup\mathrm{I} & \cup\mathrm{I} & \cup\mathrm{I} & & \\
V \arrow[r, two heads] \arrow[rrrr, bend right, "\text{finite map}"] & V_1 \arrow[r, two heads] & V_2 \arrow[r, two heads] & \ldots \arrow[r, two heads] & \mathbb{P}^d
\end{tikzcd}$$

Theorem 11.9. *If $V \subseteq \mathbb{P}^n$ is a projective variety,* $\dim d$*, then there exists a projection $V \twoheadrightarrow \mathbb{P}^d$ (finite).*

Intersect with $U_0 = \mathbb{A}^n$:

$$V \cap \mathbb{A}^n \twoheadrightarrow V_1 \cap \mathbb{A}^n \twoheadrightarrow \ldots \twoheadrightarrow V_{n-d} \cap \mathbb{A}^n = \mathbb{A}^d.$$

This induces the pullback

$$\frac{k[x_1, \ldots, x_n]}{\mathbb{I}(V)} \xleftarrow{\text{finite int.}} k[y_1, \ldots, y_d],$$

where the y_i are linear in the x_i.

Theorem 11.10 (Noether normalization)**.** *Given a domain R, finitely generated over k (k infinite), there exists a transcendence basis $y_1, \ldots, y_d$ consisting of linear combinations of the generators for R.*

11.5 Dimension example

Recall: $\dim V =$ transcendence degree of $k(V)$ over k.

The dimension of a point is 0, since $k(\{p\}) = k$.

The dimension of the variety $\mathbb{V}(xy - zw) \subseteq \mathbb{A}^{2\times 2}$ of 2×2 matrices over k of determinant 0:

$$k[V] = \frac{k[x, y, z, w]}{(xy - zw)}$$

Observe that x, y, z is *not* a transcendence basis, because w is not integral over $k[x, y, z]$; indeed, it's not a finite map, because the preimage of the zero matrix under the projections $w \mapsto 0$ is infinite.

Claim 11.11. *Let $t = x - y$. Then $k[z, w, t] \overset{i}{\hookrightarrow} k[x, y, w, z]/(xy - zw)$, and z, w, t is a transcendence basis for $k(V)$ over k.*

Need: z, w, t are algebraically independent. [Means: If z, w, t satisfy some polynomial p with coefficients in k, then $p = 0$.]

Need: Check i is integral: Suffices to check x is integral over $k[z, w, t]$.

Note: $x^2 - tx - zw = 0$ in $k[x, y, z, w]/(xy - zw)$.

11.6 Facts about dimension

Fix V irreducible quasi-projective variety.

Fact 11.12. If $U \subseteq V$ is open and nonempty, then $\dim U = \dim V$.

Fact 11.13. If $Y \subsetneqq V$ is a proper closed subset, then $\dim Y < \dim V$.

Fact 11.14. Every component of a hypersurface $\mathbb{V}(F)$ in $\mathbb{A}^n$ (or $\mathbb{P}^n$) has dimension $n - 1$ (codimension 1).

Sketch of Fact 11.14. Pick $p \notin \mathbb{V}(F) \subseteq \mathbb{A}^n$, with F irreducible. Choose coordinates such that $p = (0, \ldots, 0, 1)$. So

$$f = x_n^d + a_1 x_n^{d-1} + \cdots + a_d,$$

where $a_i \in k[x_1, \ldots, x_{n-1}]$. Easy to see: $x_1, \ldots, x_{n-1}$ are a transcendence basis over k for

$$\frac{k(x_1, \ldots, x_n)}{(f)}.$$

□

Fact 11.15. Every codimension 1 subvariety of $\mathbb{A}^n$ (or $\mathbb{P}^n$) is a hypersurface.

Proof. Let $X \subsetneqq \mathbb{A}^n$ have codimension 1. Let $\mathbb{I}(X) \subsetneqq k[x_1, \ldots, x_n]$, which is prime by irreducibility. We need to show $\mathbb{I}(X)$ is principal.

Take any $F \in \mathbb{I}(X)$. Without loss of generality, F is irreducible. Then $(F) \subseteq \mathbb{I}(x)$, and if we have equality, then we are done. Otherwise,

$$\mathbb{V}(F) \supsetneqq \mathbb{V}(\mathbb{I}(X)) = X,$$

and since $\dim \mathbb{V}(F) = n - 1$, we have $\dim \mathbb{V}(\mathbb{I}(x)) < n - 1$. □

Fact 11.16. If $X \to Y$ is finite, then $\dim X = \dim Y$.

Fact 11.17. If $V \subseteq \mathbb{P}^n$ is projective, then V has $\dim d \iff V \xrightarrow{\pi} \mathbb{P}^d$ is a finite map to $\mathbb{P}^d$.

Fact 11.18. If we have a projection $\mathbb{P}^n \xrightarrow{\pi} \mathbb{P}^m$ from a linear space $\mathbb{V}(L_0, \ldots, L_m)$, then

$$[x_0 : \cdots : x_n] \mapsto [L_0 : \cdots : L_m]$$

gives a *finite map* when restricted to any projective variety $V \subseteq \mathbb{P}^n$, whose disjoint union forms a linear space $\mathbb{V}(L_0, \ldots, L_m)$.

11.7 Dimension of hyperplane sections

Definition 11.19. A *hyperplane section* of X is $X \cap H$, where $H = \mathbb{V}(a_0x_0 + \cdots + a_nx_n) \subseteq \mathbb{P}^n$ is a hyperplane.

Theorem 11.20. $\dim(X \cap H) = \dim X - 1$*, unless (of course)* $X \subseteq H$ *(in which case* $X \cap H = X$*).*

Proof

First: For any closed set $X = X_1 \cup \cdots \cup X_t$ (irreducible components of X) in $\mathbb{P}^n$, I can find a hyperplane H such that $\dim(X \cap H) < \dim X$, or more specifically,

$$X \cap H = (X_1 \cap H) \cup \cdots \cup (X_t \cap H),$$

and each $X_i \cap H \subsetneqq X_i$.

Claim 11.21. *Most hyperplanes H have this property!*

Lemma 11.22. *Fix any finite set of points $p_1, \ldots, p_t$ in $\mathbb{P}^n$. Then there exists a hyperplane H which does not contain* any *p_i.*

Proof of 11.22.

$$\begin{array}{ccccc} \{\text{hyperplanes on } \mathbb{P}^n = \mathbb{P}(V)\} & \longleftrightarrow & \mathbb{P}(V^*) & & \\ \cup\!\! \mid & & \cup\!\!\nmid & & \\ \{\text{hyperplanes through } p_i\} & \longleftrightarrow & H_{p_i} & = & \mathbb{V}(L_i) \end{array}$$

So

$$\{\text{hyperplanes not containing } p_1, \ldots, p_t\} = \mathbb{P}(V^*) \setminus \{\mathbb{V}(L_1) \cup \cdots \cup \mathbb{V}(L_t)\}. \qquad \square$$

Back to Theorem 11.20, we have

$$\begin{array}{ccccccccc} \mathbb{P}^n & \supseteq & \mathbb{V}(L_1) = H_1 & \supseteq & \mathbb{V}(L_1, L_2) = H_1 \cap H_2 & \supseteq & \ldots & \supseteq & \mathbb{V}(L_1, \ldots, L_d) \\ \cup\!\!\nmid & & \cup\!\!\nmid & & \cup\!\!\nmid & & & & \cup\!\!\nmid \\ X & \supsetneqq & X \cap H_1 & \supsetneqq & X \cap H_1 \cap H_2 & \supsetneqq & \ldots & \supsetneqq & X \cap H_1 \cap \cdots \cap H_d \\ \| & & \| & & \| & & & & \| \\ X_0 & & X_1 & & X_2 & & \ldots & & \varnothing \\ & & & & & & & & \\ d = \dim X_0 & > & \dim X_1 & > & \dim X_2 & > & \ldots & > & 0 \end{array}$$

Want to show the dimension drops by 1 each time. If not, after d steps, get $\varnothing$.

So the linear space $\mathbb{P}(W) = \mathbb{V}(L_1, \ldots, L_d) \cap X = \varnothing$. Project from $\mathbb{P}(W)$:

$$\mathbb{P}^n \xrightarrow{\pi} \mathbb{P}^{d-1}$$

$$[x_0 : \cdots : x_n] \mapsto [L_1(x) : \cdots : L_d(x)]$$

$$X \xrightarrow[\text{finite!}]{\pi} X'$$

$\implies \dim X = \dim X' \leq (d-1)$, a contradiction. Hence $\dim X = d$.

11.8 Equivalent formulations of dimension

$V \subseteq \mathbb{P}^n$ projective variety.

The *dimension* of V is any one of the following, which are equivalent:

(1) transcendence degree of $k(V)$ over k.

(2) the unique d such that $\exists$ finite map $V \twoheadrightarrow \mathbb{P}^d$.

(3) the unique d such that $V \cap H_1 \cap H_2 \cap \cdots \cap H_d$ is a finite set of points, where the H_i are generic linear subvarieties of codimension d.

(4) the length of the longest chain of proper irreducible closed subsets of V:

$$V = V_d \supsetneqq V_{d-1} \supsetneqq V_{d-2} \supsetneqq \cdots \supsetneqq V_1 \supsetneqq V_0 = \{\text{point}\}\,.$$

12 Families of varieties

12.1 Family of varieties (schemes)

(Not necessarily irreducible.)

Definition 12.1. A *family* is a surjective *morphism* (regular map) $X \xrightarrow{f} Y$ of variety.

The *base* (or *parameter space*) of the family is Y. The *members* are the *fibers* $\{f^{-1}(y)\}_{y \in Y}$.

Example 12.2. $X = \mathbb{V}(xy - z) \subseteq \mathbb{A}^3$,

$$\begin{aligned} \mathbb{V}(xy - z) &\xrightarrow{F} \mathbb{A}^1 \\ (x, y, z) &\mapsto z. \end{aligned}$$

Then

$$f^{-1}(\lambda) = \mathbb{V}(xy - \lambda) \subseteq \mathbb{A}^2 \times \{\lambda\}\,.$$

Example 12.3. Hyperplanes in $\mathbb{P}^n \longleftrightarrow \mathbb{P}\left((k^{n+1})^*\right)$ by the correspondence

$$H = \mathbb{V}(A_0X_0 + \cdots + A_nX_n) \longleftrightarrow \{A_0X_0 + A_1X_1 + \cdots + A_nX_n\}\,/\text{scalar values.}$$

12.2 Incidence correspondences

Consider the "incidence correspondence"

$$\mathscr{X} = \left\{(p, H) \,\middle|\, p \in H\right\} \subseteq \mathbb{P}^n \times \mathbb{P}^n = \mathbb{P}(V) \times \mathbb{P}(V^*).$$

Putting coordinates $[X_0, \ldots, X_n]$ on $\mathbb{P}(V)$ and $[A_0, \ldots, A_n]$ on $\mathbb{P}(V^*)$, we have

$$\begin{aligned} \mathscr{X} = \mathbb{V}(A_0X_0 + \cdots + A_nX_n) &\xrightarrow{\pi}\!\!\!\!\rightarrow (\mathbb{P}^n)^* \\ \pi^{-1}([A_0 : \cdots : A_n]) = \mathbb{V}(A_0X_0 + \cdots + A_nX_n) &\mapsto [A_0, \ldots, A_n] \end{aligned}$$

Theorem 12.4. *Let $X \overset{f}{\twoheadrightarrow} Y$ be a surjective regular map of varieties,* $\dim X = n$*,* $\dim Y = m$*. Then:*

(1) $n \geq m$*.*

(2) $\dim F \geq n - m$*, where F is any component of any fiber $f^{-1}(y) \subseteq X$ (with $y \in Y$).*

(3) There is a dense open set $U \subseteq Y$ such that $\forall y \in U$, $f^{-1}(y)$ has dimension $n - m$.

Corollary 12.5. *Let $X \xrightarrow{f} Y$ be a surjective regular map of* projective *algebraic sets. Assume Y is irreducible and all fibers are irreducible of the same dimension. Then X is also irreducible!*

Example 12.6 (Blowup). $B = \{(p, \ell) \mid p \in \ell\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$.

$$B = \big\{(p, \ell) \bigm| p \in \ell\big\} \xrightarrow{\pi} \mathbb{P}^1$$
$$\mathbb{A}^2 \times \ell \supseteq \mathbb{V}(ax - by) = \pi^{-1}(\ell) \mapsto \ell = [a : b].$$

Note that each of the fibers is 1-dimensional.

Now: B is dimension 2, and

$$B \xrightarrow{\pi} \mathbb{A}^2$$
$$(q, [a : b]) \mapsto q = (a, b) \in \mathbb{A}^2 - \{(0, 0)\}$$

is a "generic" fiber and has dimension $0 = 2 - 2$. But the fiber over $(0, 0)$ is $\mathbb{P}^1$, which has dimension 1. The dimension jumps!

12.3 Lines contained in a hypersurface

Q: Fix an (irreducible) hypersurface of degree d in $\mathbb{P}^3$. Does it have any lines on it?

A: For $d = 1$: $X = \mathbb{V}(L) \cong \mathbb{P}^2 \subseteq \mathbb{P}^3$ is covered by lines.

For $d = 2$: $X = \mathbb{V}(xy - wz) \cong \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ is covered by lines. Degenerate cone: $X = \mathbb{V}(x^2 + y^2 + z^2) \subseteq \mathbb{P}^3$ is also covered by lines, as is $\mathbb{V}(xy)$, the union of two planes.

Consider the incidence correspondence

$$\mathscr{X} = \big\{(\mathbb{V}(F), \ell) \bigm| \ell \subseteq \mathbb{V}(F)\big\} \subseteq \mathbb{P}\left(\mathrm{Sym}^d(k^4)^*\right) \times \mathrm{Gr}(2, 4),$$

where $\mathbb{P}\left(\mathrm{Sym}^d(k^4)^*\right)$ = parameter space of hypersurfaces of degree d in $\mathbb{P}^3$, and $\mathrm{Gr}(2, 4)$ = lines in $\mathbb{P}^3$ = 2-dimensional subspaces of k^4.

Take the projections

$$\mathscr{X} \xrightarrow{\pi} \mathbb{P}\left(\mathrm{Sym}^d(k^4)^*\right),$$
$$\mathscr{X} \xrightarrow{\nu} \mathrm{Gr}(2, 4).$$

Consider ν: Compute the fiber over ℓ. Without loss of generality, $\ell = \mathbb{V}(X_0, X_1) \subseteq \mathbb{P}^3$. Then $\nu^{-1}(\ell) = \mathbb{V}(F_d)$ such that

$$\mathbb{V}(X_0, X_1) \subseteq \mathbb{V}(F_d) \iff (X_0, X_1) \supseteq (F_d) = X_0 G_{d-1} + X_1 H_{d-1}.$$

The equation F_d has coefficients 0 on the terms $X_2^d, X_2^{d-1}X_3, \ldots, X_3^d$. So

$$\nu^{-1}(\ell) \subseteq \mathbb{P}\left(\mathrm{Sym}^d(k^4)^*\right)$$

is a linear subspace of codimension $d+1$. The dimension of the fiber is

$$\binom{d+3}{3} - 1 - (d+1).$$

Hence, the fibers are all irreducible of the same dimension.

Thus, by Corollary 12.5, $\mathscr{X}$ is irreducible of dimension $4+$ (fiber dimension).

12.4 Dimension of fibers

Theorem (12.4)**.** *Given a surjective regular map $X \xrightarrow{\varphi} Y$ of varieties, we have*

(1) $\dim X \geq \dim Y$

(2) $\dim F \geq \dim X - \dim Y$ *for F any component of any fiber $\varphi^{-1}(y)$*

(3) There is a nonempty open subset $U \subseteq Y$ where $\dim F = \dim X - \dim Y$.

We studied the incidence correspondence

$$\mathscr{X} = \left\{(X,\ell) \;\middle|\; \ell \subseteq X\right\} \subseteq \mathbb{P}\left(\mathrm{Sym}^d(k^4)^*\right) \times \mathrm{Gr}(2,4)$$

and its projections

$$X \xrightarrow{\pi_1} \mathbb{P}\left(\mathrm{Sym}^d(k^4)^*\right),$$
$$X \xrightarrow{\pi_2} \mathrm{Gr}(2,4).$$

We saw that π_2 is surjective.

The fiber of $\ell \in \mathrm{Gr}(2,4)$ is

$$\pi_2^{-1}(\ell) = \left\{(X,\ell) \;\middle|\; X \supseteq \ell\right\} = \{\text{surfaces of degree 2 containing } \ell\} \times \ell$$

and is $\cong$ a linear space in $\mathbb{P}(\mathrm{Sym}^d)$ of dimension $M-(d+1)$, where

$$M = \binom{d+3}{3} - 1 = \dim\left[\mathbb{P}\left(\mathrm{Sym}^d(k^4)^*\right)\right].$$

Study the other projection:

$$X \xrightarrow{\pi_1} \mathbb{P}\left(\mathrm{Sym}^d(k^4)^*\right) = \left\{\text{degree } d \text{ hypersurfaces in } \mathbb{P}^3\right\} \cong \mathbb{P}^M.$$

The fiber of $X \in \mathbb{P}\left(\mathrm{Sym}^d(k^4)^*\right)$ is

$$\pi_1^{-1}(X) = \left\{(X,\ell) \;\middle|\; \ell \subseteq X\right\} = X \times \{\text{lines on } X\}.$$

So $X \in \pi_1(\mathscr{X}) \iff X$ contains some line.

Consequence: If $d \geq 4$, then π_1 can't be surjective. "Most" surfaces of degree ≥ 4 contain no line: "The generic surface of degree $d \geq 4$ contains no line."

12.5 Cubic surfaces

What about $d = 3$?

$$\mathscr{X} \xrightarrow{\pi_1} \mathbb{P}\left(\mathrm{Sym}^3(k^4)^*\right) = \mathbb{P}^{19},$$

and $\dim \mathscr{X} = 19$. Two possibilities:

(1) π_1 is surjective $\iff$ generic fiber is dim 0. "The generic cubic contains finitely many lines."

(2) π_1 is not surjective $\iff$ there are cubic surfaces that don't contain lines, and the fibers are $\dim \geq 1$.

In fact, the former is what actually occurs; π_1 is surjective.

It suffices to find one cubic surface that contains finitely many lines:

$$X = \mathbb{V}(X_1X_2X_3 - X_0^3) \subseteq \mathbb{P}^3$$

Exercise 12.7. X contains exactly 3 lines, $\mathbb{V}(X_0, X_i)$ for $i = 1, 2, 3$.

The non-generic fibers have $\dim \geq 1$, so these cubics contain infinitely many lines.

It turns out that the subset of cubic surfaces containing only finitely many lines

$$\mathcal{U} \subseteq \mathbb{P}^{19} = \mathbb{P}\left(\mathrm{Sym}^3(k^4)^*\right)$$

consists exactly of the irreducible $X = \mathbb{V}(F)$.

Fact 12.8. $\pi_1 : \pi_1^{-1}(X) \to \mathcal{U}$ is finite of degree 27 over $\mathcal{U}$. On the subset of smooth cubic surfaces, this map is exactly 27-to-1.

13 Tangent spaces

- Intersection multiplicity $(V, \ell)_p$
- Tangent line
- Tangent space
- Smooth point

13.1 Big picture

To any point p on any variety V, we will define a vector space T_pV, the tangent space to V at p, such that

(1) Given any regular map

$$V \xrightarrow{\varphi} W$$
$$p \mapsto q,$$

we get an induced linear map of vector spaces

$$T_pV \xrightarrow{d_p\varphi} T_qW.$$

Goal: to define tangent space to a variety V at a point $p \in V$.

Since tangency is a local issue, assume $p = (0, \dots, 0) \in V \subseteq \mathbb{A}^n$ with V a closed affine algebraic set.

13.2 Intersection multiplicity

We work out an example in detail.

Example 13.1. Let $V = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$. We calculate the intersection multiplicity of V with $\ell = \{(at, bt) \mid t \in k\}$. The intersection $V \cap \ell$ is given by

$$\mathbb{V}\left((bt) - (at)^2\right) \subseteq \ell \subseteq \mathbb{A}^2.$$

Solving this:

$$\begin{aligned} bt - a^2t^2 &= 0 \\ t(b - a^2t) &= 0, \end{aligned}$$

so $t = 0$ or $t = \frac{b}{a^2}$. Hence the intersection points are $(0,0)$ and $\left(\frac{b}{a}, \left(\frac{b}{a}\right)^2\right)$.

We get a "double intersection" point when $b = 0$. Get that ℓ is tangent to V at $(0,0)$ because the intersection multiplicity is V and ℓ at $(0,0)$ is 2.

More precisely, we will see that ℓ has intersection multiplicity 1 for all ℓ except when ℓ is the x-axis, in which case the intersection multiplicity is 2.

Now we are ready to give a formal definition.

Definition 13.2. Let $p = \mathbf{0} \in V \subseteq \mathbb{A}^n$, and let $\mathbb{I}(V) = (F_1, \dots, F_r)$. Say

$$\ell = \left\{(a_1t, \dots, a_nt) \;\middle|\; t \in k\right\} \subseteq \mathbb{A}^n$$

is a line through $\mathbf{0}$. The *intersection multiplicity* of V and ℓ at p, denoted $(V, \ell)_p$, is the highest power of t which divides all the polynomials

$$\{F_i(a_1t, \dots, a_nt)\}_{i=1,\dots,r}.$$

Equivalently, look at the ideal of $k[t]$ generated by $\{F(a_1t, \dots, a_nt)\}$, where $F(x_1, \dots, x_n) \in \mathbb{I}(V)$. That ideal is generated by some polynomial

$$t^m(t - \lambda_1)^m_1 \cdots (t - \lambda_s)^{m_s}, \qquad \lambda_i \neq 0.$$

Then $(V, \ell)_{\mathbf{0}} = m$.

13.3 Tangent lines and the tangent space

Definition 13.3 (tangent line)**.** A line ℓ is *tangent to V at p* if $(\ell, V)_p \geq 2$.

Definition 13.4 (tangent space)**.** The *tangent space* to $V \subseteq \mathbb{A}^n$ at p, denoted T_pV, is the *set* of points $(a_1, \dots, a_n) \in \mathbb{A}^n$ lying on lines $\ell \subseteq \mathbb{A}^n$ which are tangent to V are p.

Example 13.5. Consider $V = \mathbb{V}(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$. Take a line through the origin

$$\ell = \left\{(at, bt) \mid t \in k\right\}.$$

The intersections are given by

$$(bt)^2 - (at)^2 - (at)^3 = t^2\left(b^2 - a^2 - a^3 t\right) = 0.$$

So the intersection multiplicity at the origin is 2. Note that *all* lines through $(0,0)$ are tangent:

$$T_{(0,0)}V = \mathbb{A}^2 = k^2.$$

In other words, tangent lines are not always a limit of secant lines.

Theorem 13.6. *Let $p \in V \subseteq \mathbb{A}^n$, where V is a (not necessarily irreducible) closed subset of $\mathbb{A}^n$. The tangent space T_pV is a* linear *algebraic variety in $\mathbb{A}^n$, and*

$$\dim T_pV \geq \dim_p V.$$

13.4 Smooth points

Definition 13.7. A point $p \in V$ is *smooth* if $\dim T_pV = \dim_p V$.

Proposition 13.8. *Say $\mathbf{0} \in V \subseteq \mathbb{A}^n$ and $\mathbb{I}(V) = (F_1, \ldots, F_r)$. Then*

$$T_{\mathbf{0}}V = \mathbb{V}(L_1, \ldots, L_r) \subseteq \mathbb{A}^n,$$

where $L_i = a_{i1}x_1 + \cdots + a_{in}x_n$ is the "degree 1 part" of F_i, i.e.,

$$F_i = L_i + F_i^{(2)} + F_i^{(3)} + \ldots,$$

where $F_i^{(j)}$ is homogeneous of degree j in $x_1, \ldots, x_n$.

Proof. We have $(a_1, \ldots, a_n) \in T_{\mathbf{0}}V \iff (a_1, \ldots, a_n) \in \ell$ which is tangent to V at $\mathbf{0} \iff \{(a_1t, \ldots, a_nt) \mid t \in k\}$ intersects V with multiplicity ≥ 2 at $\mathbf{0}$

$$\iff \{F_1(a_1t, \ldots, a_nt), \ldots, F_r(a_1t, \ldots, a_nt)\}$$

are divisible by t^2. Observe that

$$F_i(a_1t, \ldots, a_nt) = L_i(a_1t, \ldots, a_nt) + G_i(a_1t, \ldots, a_nt) = t \cdot L_i(a_1, \ldots, a_n) + G_i(a_1t, \ldots, a_nt),$$

and t^2 divides $G_i(a_1t, \ldots, a_nt)$. So

$$t^2 \mid F_i(at_1, \ldots, a_nt) \iff L_i(a_1, \ldots, a_n) = 0. \qquad \square$$

Example 13.9. In $V = \mathbb{V}(y - x^2) \subset \mathbb{A}^2$,

$$T_{(0,0)}V = \mathbb{V}(y) \subset \mathbb{A}^2.$$

Example 13.10. In $V = \mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{A}^2$,

$$T_{(0,0)}V = \mathbb{A}^2.$$

Remark 13.11 (Explicit computation of tangent spaces). To find $T_pV \subseteq \mathbb{A}^n$ for any p, center everything at $p = (\lambda_1, \dots, \lambda_n)$. Write all polynomials not in $(x_1, \dots, x_n)$, but in $(x_1 - \lambda_1, \dots, x_n - \lambda_n)$.

Use Taylor expansion at $p = (\lambda_1, \dots, \lambda_n)$:

$$\begin{aligned} F = F(p) &+ \underbrace{\left.\frac{\partial F}{\partial x_1}\right|_p (x_1 - \lambda_1) + \dots + \left.\frac{\partial F}{\partial x_n}\right|_p (x_n - \lambda_n)}_{\text{linear part around } p} \\ &+ \frac{1}{2} \left.\frac{\partial^2 F}{\partial x^2}\right|_p (x_1 - \lambda_1)^2 + \dots \\ &+ \left(\frac{1}{i_1!}\frac{\partial^{i_1}}{\partial x_1^{i_1}}\right) \cdots \left(\frac{1}{i_n!}\frac{\partial^{i_n}}{\partial x_n^{i_n}}\right) F\Bigg|_p (x_1 - \lambda_1)^{i_1} \cdots (x_n - \lambda_n)^{i_n}. \end{aligned}$$

Theorem 13.12. *$T_pV = \mathbb{V}(d_pF_1, \dots, d_pF_r) \subseteq \mathbb{A}^n$, where $\mathbb{I}(V) = (F_1, \dots, F_r)$.*

13.5 Differentials, derivations, and the tangent space

Definition 13.13. Fix $R = k[x_1, \dots, x_n]$, $p \in \mathbb{A}^n = k^n$. The "*differential* at p" is the map

$$\begin{aligned} k[x_1, \dots, x_n] &\xrightarrow{d_p} k[x_1, \dots, x_n] \\ g &\mapsto d_pg = \underbrace{\sum_{i=1}^n \left.\frac{\partial g}{\partial x_i}\right|_p (x_i - \lambda_i)}_{\text{linear form in } (x_i - \lambda_i)} \in [k[x_1 - \lambda_1, \dots, x_n - \lambda_n]]_1 . \end{aligned}$$

Caution: Not a ring map!

Fact 13.14. $d_p : R \to R$ is a k-linear *derivation*, meaning:

(1) k-linear: $d_p(f + g) = d_pf + d_pg$ and $d_p(\lambda f) = \lambda d_pf$ for all $f, g \in R,\ \lambda \in k$.

(2) $d_p(fg) = f(p)d_pg + g(p)d_pf$.

Last time: If

$$p \in V = \mathbb{V}(f_1, \dots, f_r) \subseteq \mathbb{A}^n, \qquad (f_1, \dots, f_r) = \mathbb{I}(V),$$

then

$$T_pV = \mathbb{V}(d_pf_1, \dots, d_pf_r) = \text{vector space in } k^n \text{ translated by } p \subseteq (T_p\mathbb{A}^n) = k^n,$$

where d_pf_i are linear forms in $(x_1 - \lambda_1, \dots, x_n - \lambda_n)$.

Why is this independent of choice of generators?

$$(g_1, \dots, g_t) = (f_1, \dots, f_r) = \mathbb{I}(V) \subseteq k[x_1, \dots, x_n]$$

Write $g_i = h_1 f_1 + \cdots + h_r f_r$ for some $h_j \in R$. Apply d_p:

$$d_p g_i = f_1(p) d_p h_1 + h_1(p) d_p f_1 + \cdots + f_r(p) d_p h_r + h_r(p) d_p f_r.$$

Since $p \in V$ and $f_i \in \mathbb{I}(V)$, we have $f_i(p) = 0$. So $d_p g_i$ is a linear combination of $d_p f_1, \ldots, d_p f_r$. Hence $d_p g_i \in (d_p f_1, \ldots, d_p f_r)$, as was to be shown.

We have a surjective map

$$\begin{aligned} k[x_1, \ldots, x_n] &\xrightarrow{d_p} (T_p \mathbb{A}^n)^* \\ x_i - \lambda_i &\mapsto x_i - \lambda_i. \end{aligned}$$

Note 13.15. $d_p(f) = d_p(f + \lambda)$. Replace f by $f - f(p)$:

$$d_p f = d_p\left(f - f(p)\right).$$

So we can restrict to the (still surjective) map on $\mathfrak{m}_p = (x_1 - \lambda_1, \ldots, x_n - \lambda_n) \subseteq k[x_1, \ldots, x_n]$:

$$\begin{aligned} \mathfrak{m}_p &\xrightarrow{d_p} (T_p \mathbb{A}^n)^* \\ x_i - \lambda_i &\mapsto x_i - \lambda_i. \end{aligned}$$

Say $g \in \mathfrak{m}_p$ is in the kernel of d_p. Write g out as a polynomial in $(x_1 - \lambda_1, \ldots, x_n - \lambda_n)$:

$$g = g(p) + d_p g + G,$$

where each monomial of G is of degree ≥ 2 in $(x_1 - \lambda_1, \ldots, x_n - \lambda_n)$.

Since $g \in \mathfrak{m}_p$, we have $g(p) = 0$. Moreover,

$$d_p g = 0 \iff g = G \in (x_1 - \lambda_1, \ldots, x_n - \lambda_n)^2.$$

So $\ker d_p = \mathfrak{m}_p^2$.

This gives us a *natural isomorphism*:

$$\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2} \xrightarrow[\simeq]{d_p} (T_p \mathbb{A}^n)^*.$$

Theorem 13.16. *For $p = (\lambda_1, \ldots, \lambda_n) \in V = \mathbb{V}(f_1, \ldots, f_r) \subseteq \mathbb{A}^n$ with $(f_1, \ldots, f_r) = \mathbb{I}(V)$, let*

$$\mathfrak{m}_p = \left\{ f : V \to k \;\middle|\; f(p) = 0 \right\} \subseteq k[V].$$

There is a natural surjective vector space map

$$\begin{aligned} \mathfrak{m}_p &\xrightarrow{d_p} (T_p V)^* \\ g = G\big|_V &\mapsto \left[d_p G\big|_{T_p V} : T_p V \to k \right], \qquad G \in k[x_1, \ldots, x_n], \end{aligned}$$

whose kernel is $\mathfrak{m}_p^2$.

Proof. Why is this well-defined?

Say $g = G|_V = H|_V$ for some $G, H \in k[x_1, \ldots, x_n]$. Need to check that $d_pG, d_pH \in (T_p\mathbb{A}^n)^*$ restrict to the *same* linear functional in $T_pV = \mathbb{V}(d_pf_1, \ldots, d_pf_r)$.

By considering $G - H$, say $G \in \mathbb{I}(V)$. Need to show that d_pG vanishes on T_pV, i.e., that $d_pG \in (d_pf_1, \ldots, d_pf_r)$.

We already showed that $G = H_1f_1 + \cdots + H_rf_r \implies d_pG \in (d_pf_1, \ldots, d_pf_r)$, provided $p \in V$. So we are done. $\square$

Conclusion:

$$(T_pV)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2$$

as a k-vector space for any $p \in V \overset{\text{closed}}{\subseteq} \mathbb{A}^n$.

13.6 The Zariski tangent space

Corollary 13.17. *Consider an isomorphism of affine algebraic sets*

$$\begin{aligned} V &\xrightarrow{\varphi} W \\ p &\mapsto q. \end{aligned}$$

Then we have an isomorphism

$$\begin{aligned} k[W] &\xrightarrow{\varphi^*} k[V] \\ \mathfrak{m}_p &\xrightarrow{\simeq} \mathfrak{m}_q \\ \mathfrak{m}_p^2 &\xrightarrow{\simeq} \mathfrak{m}_q^2. \end{aligned}$$

I.e., the tangent space is an *invariant* of the isomorphism class of the variety at p.

Definition 13.18. The *Zariski tangent space* at a point p of a quasi-projective variety V is $\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*$, where $\mathfrak{m}_p$ is the maximal ideal in the local ring of V at p.

Recall: $p \in V$ variety.

Definition 13.19. The *local ring of* V *at* p is

$$\mathcal{O}_{p,V} = \left\{\varphi \in k(V) \;\middle|\; \varphi \text{ is regular at } p\right\}.$$

It has unique maximal ideal

$$\mathfrak{m}_p = \left\{\varphi \in \mathcal{O}_{p,V} \;\middle|\; \varphi(p) = 0\right\}.$$

To compute $\mathcal{O}_{p,V}$, choose *any* affine open neighborhood of p, say $p \in U \subseteq V$. We have

$$\mathfrak{m}_p \subseteq k[U] = \mathcal{O}_V(U).$$

Then

$$\mathcal{O}_{p,V} = k[U]_{\mathfrak{m}_p} \supseteq \mathfrak{m}_p k[U]\mathfrak{m}_p.$$

This doesn't depend on the choice of U.

Note 13.20.

$$\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2} = \frac{\mathfrak{m}_p k[U]_{\mathfrak{m}_p}}{\left(\mathfrak{m}_p k[U]_{\mathfrak{m}_p}\right)^2}.$$

13.7 Tangent spaces of local rings

Definition 13.21. For *any local ring* $(R, \mathfrak{m})$ (e.g., $\mathbb{Z}_p, \mathbb{Z}_{(p)}[[x]], \widehat{\mathbb{Z}_p}$, convergent power series in $z_1, \ldots, z_r$ over C, etc.), define the *Zariski tangent space* as $(\mathfrak{m}/\mathfrak{m}^2)^*$. This is a vector space over the residue field $R/\mathfrak{m} = k$.

Theorem 13.22. *For any local ring,* $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq \dim R$.

Definition 13.23. A local ring $(R, \mathfrak{m})$ is *regular* if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim R$.

Example 13.24. If $R = \mathcal{O}_{p,V}$, where p is a point on a variety V, then

$$\left(\mathfrak{m}/\mathfrak{m}^2\right)^* = (T_pV),$$

the tangent space to V at p, $\dim_p T_pV \geq \dim_p V$. (Proof in Shafarevich!)
$\mathcal{O}_{p,V}$ is *regular* $\iff$ p is a smooth point of V.

Definition 13.25. (1) $p \in V$ is *smooth* $\iff$ $\dim T_pV = \dim_p V$. (In general, $\forall p \in V$, we have $\dim T_pV \geq \dim_p V$.)

(2) The *singular locus* of V is the set

$$\operatorname{Sing} V = \left\{p \in V \;\middle|\; p \text{ is not smooth}\right\} = \left\{p \in V \;\middle|\; \dim(T_pV) > \dim_p V\right\}.$$

Example 13.26. Since $\dim \mathbb{Z}_{(p)} = 1$ and $\dim(p)/(p^2) = 1$, $\mathbb{Z}$ "is" the coordinate ring of something like a variety which is smooth of dimension 1.

Example 13.27. Let $p \in (\lambda_1, \ldots, \lambda_n) \in \mathbb{A}^n$. Then

$$\dim(T_p\mathbb{A}^n) = \dim(k^n) = n,$$
$$\dim\left[\frac{(x_1 - \lambda_1, \ldots, x_n - \lambda_n)}{(x_1 - \lambda_1, \ldots, x_n - \lambda_n)^2}\right] = n.$$

I.e., $\mathbb{A}^n$ is smooth at all points.

Theorem 13.28. *The singular set of V (a* variety*) is a* proper *closed subset of V.*

Proof. We have $\operatorname{Sing} V \subseteq V$. To check that this is a proper closed set, it reduces immediately to the case where V is affine.

Assume $V = \mathbb{V}(f_1, \ldots, f_r) \subseteq \mathbb{A}^n$ with $(f_1, \ldots, f_r) = \mathbb{I}(V)$. For $p \in V$,

$$T_pV = \mathbb{V}(d_pf_1, \ldots, d_pf_r), \qquad \text{each } d_pf_i = \sum_{j=1}^{n}\left(\left.\frac{\partial f_i}{\partial x_j}\right|_p \left(x_j - x_j(p)\right)\right).$$

Equations $d_pf_1, \ldots, d_pf_r$ can be written as a matrix:

$$T_pV = \mathbb{V}\left(\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{bmatrix}_p \begin{bmatrix} x_1 - x_1(p) \\ x_2 - x_2(p) \\ \vdots \\ x_n - x_n(p) \end{bmatrix}\right) = \ker\left(\left.\left(\frac{\partial f_i}{\partial x_j}\right)\right|_p\right) \subseteq \mathbb{A}^n.$$

So

$$\dim T_p V = \dim \left(\ker(J_p|_p) \right) = n - \operatorname{rank}(J_p).$$

We have $p \in \operatorname{Sing} V \iff \dim T_p V > d \iff \operatorname{rank} \left(\frac{\partial f_i}{\partial x_j} \right) \Big|_p < n - d \iff (n-d) \times (n-d)$ subdeterminants of $\left(\frac{\partial f_i}{\partial x_j} \right)$ all vanish at p. Thus

$$\begin{aligned} \operatorname{Sing} V &= \left\{ p \in V \;\middle|\; (n-d) \times (n-d) \text{ minors of } \left(\frac{\partial f_i}{\partial x_j} \right) \text{ vanish at } p \right\} \\ &= \mathbb{V} \left(\text{codimension-sized minors of } \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{bmatrix} \right) \cap V. \end{aligned}$$

It remains to show that it is *proper*! $\square$

Example 13.29. Consider $V = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{C}^3$:

$$T_p V = \mathbb{V}\left(2x|_p(x - x(p)) + 2y|_p(y - y(p)) - 2z|_p(z - z(p))\right) \subseteq \mathbb{C}^3.$$

This defining equation is a linear function in $(x - \lambda_1, y - \lambda_2, z - \lambda_3)$, nonzero $\iff$ some $\frac{\partial f}{\partial x_i}$ is nonzero.

Hence, the dimension is 2 if $\lambda_1, \lambda_2, \lambda_3$ are not all zero, and dimension 3 otherwise:

$$\operatorname{Sing} V = V \cap \mathbb{V}\left(1 \times 1(2x, 2y, 2z)\right) = V \cap \mathbb{V}(x, y, z) = \{(0,0,0)\}.$$

14 Regular parameters

Read Shafarevich, II, §2, 2.1, 2.2, 2.3.

14.1 Local parameters at a point

Fix V variety, $p \in V$. Consider

$$\mathcal{O}_{p,V} = \left\{ \varphi \in k(V) \;\middle|\; \varphi \text{ is regular at } p \right\},$$

the local ring of V at p. The maximal ideal is $\mathfrak{m} \subset \mathcal{O}_{p,V}$, the regular functions vanishing at p.

Recall:

Definition 14.1. p is a smooth (or non-singular) point of V iff

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_p V$$

($\geq$ always holds).

Fix V variety of dimension d, $p \in V$ smooth point.

Definition 14.2. Say regular functions $u_1, \dots, u_d \in \mathfrak{m}_p$ in a neighborhood of $p \in V$ are *regular parameters* (or *local parameters*) at p if their images in $\mathfrak{m}/\mathfrak{m}^2$ are a basis for this vector space.

Example 14.3. If $p = (\lambda_1, \dots, \lambda_d) \in \mathbb{A}^d$, then $\{x_1 - \lambda_1, \dots, x_d - \lambda_d\}$ are local parameters at p.

Example 14.4. $p = (1, 0) \in V = \mathbb{V}(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$. The dimension is 1. Note that V is smooth (for $\operatorname{char}(k) \neq 2$):

$$\operatorname{Sing} V = V \cap \mathbb{V}(2x, 2y) = \mathbb{V}(x^2 + y^2 - 1, 2x, 2y) = \varnothing.$$

We have

$$\mathcal{O}_{p,V} = \frac{k[x,y]}{(x^2 + y^2 - 1)} \cdot (x - 1, y) \supseteq \mathfrak{m},$$

$\mathfrak{m}/\mathfrak{m}^2$ (dim 1) obviously spanned by $\{x - 1, y\}$. In $\mathcal{O}_{p,V}$,

$$(x - 1)(x + 1) = -y^2 \implies x - 1 = -\frac{1}{x + 1} y^2 \in \mathfrak{m}^2.$$

Thus y is a local parameter for V at $p = (1, 0)$, since $\overline{y}$ in $\mathfrak{m}/\mathfrak{m}^2$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$.

In other words, y generates $\mathfrak{m}$ as an $\mathcal{O}_{p,V}$-module.

14.2 Nakayama's lemma

Lemma 14.5 (Nakayama)**.** *Let $(R, \mathfrak{m})$ be a local Noetherian commutative ring, and let M be a finitely generated R-module. Every vector space basis for $M/\mathfrak{m}M$ over $R/\mathfrak{m}$ lifts to a (minimal) generating set for M as an R-module.*

We apply this to $R = \mathcal{O}_{p,V} \supseteq \mathfrak{m}$ and $M = \mathfrak{m}$: Every vector space basis $\overline{u_1}, \dots, \overline{u_d}$ for $\mathfrak{m}/\mathfrak{m}^2$ lifts to a (minimal) generating set $u_1, \dots, u_d$ for $\mathfrak{m}$.

14.3 Embedding dimension

Definition 14.6. The *embedding dimension* of a point p on a variety V (not necessarily smooth) is the dimension of $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Fact 14.7. The embedding dimension at p is $\geq$ the dimension at p, with equality $\iff$ p is a smooth point of V.

Theorem 14.8 (Transverse intersection)**.** *Let $u_1, \dots, u_d$ be local parameters at a smooth point $p \in V$. The subvariety $\mathbb{V}(u_i) \subseteq V$ is also smooth at p_j of codimension 1, and furthermore, $\mathbb{V}(u_{i_1}, \dots, u_{i_t}) \subseteq V$ is smooth at p of codimension t.*

Proof. We have $p \in V_i = \mathbb{V}(u_i) \subsetneq V$ and a ring map given by modding out by $\operatorname{Rad}(u_i)$,

$$\begin{array}{ccc} \mathcal{O}_{p,V_i} & \xleftarrow[\text{restriction}]{} \!\!\!\!\leftarrow & \mathcal{O}_{p,V} \\ \cup\!| & & \cup\!| \\ \overline{\mathfrak{m}}_{p,V_i} & \longleftarrow & \mathfrak{m}_{p,V}, \end{array}$$

and we have $\overline{\mathfrak{m}}_{p,V_i} = (\overline{u_1}, \overline{u_2}, \ldots, \overline{u_d})$ and $\mathfrak{m}_{p,V} = (u_1, \ldots, u_d)$. Since $\overline{u_i} = 0$, we have

$$d - 1 \leq \dim_p V_i \leq \dim T_p V_i = \dim \frac{\overline{\mathfrak{m}_p}}{\overline{\mathfrak{m}_p}^2} \leq d - 1.$$

Hence $d - 1 = \dim T_p V_i = \dim_p V_i$, so p is a smooth point of V_i.

Similarly, take $p \in V_I = \mathbb{V}(u_1, \ldots, u_t) \subseteq \mathbb{V}$. Then

$$\overline{\mathfrak{m}} = (\overline{u_1}, \ldots, \overline{u_d}) = (\overline{u_{t+1}}, \ldots, \overline{u_d}) \subseteq \mathcal{O}_{p,V_I}.$$

So

$$\dim_p V_i \leq \dim \frac{\overline{\mathfrak{m}}}{\overline{\mathfrak{m}}^2} \leq d - t \leq \dim_p V_I,$$

hence equality holds and we are done. □

Example 14.9. Let $p = (0,0) \in \mathbb{A}^2$. Then $\{y - x^2, x\}$ are local parameters at $(0,0)$, and are said to intersect transversely.

However, $\{y - x^2, y\}$ are *not* local parameters at $(0,0) \in \mathbb{A}^2$, and do not intersect transversely.

14.4 Transversal intersection at arbitrary points

For a point p (not necessarily smooth) on a variety V, and elements $u_1, \ldots, u_n \in \mathfrak{m} \subseteq \mathcal{O}_{p,V}$, the following are equivalent:

(1) $u_1, \ldots, u_n$ minimally generate $\mathfrak{m}$ (as an ideal of $\mathcal{O}_{p,V}$).

(2) The images $\overline{u_1}, \ldots, \overline{u_n}$ are a basis for $\mathfrak{m}/\mathfrak{m}^2$.

(3) Their differentials $d_p u_1, \ldots, d_p u_n$ are a basis for $(T_p V)^*$.

(4) The subspace of $T_p V$ defined by the zero set of the ($n = \dim T_p V$) linear functionals $d_p u_1, \ldots, d_p u_n$ is $\mathbf{0}$.

Fact 14.10. If p is smooth, then $n = \dim V$, and any set $\{u_1, \ldots, u_n\}$ satisfying these equivalent conditions is called a system of "local parameters at p".

In this case where p is smooth, these are equivalent to:

(5) The inclusion $k[u_1, \ldots, u_n]_{(u_1,\ldots,u_n)} \subseteq \mathcal{O}_{p,V}$ becomes an equality when we complete with respect to the maximal ideals $(u_1, \ldots, u_n) \subset k[u_1, \ldots, u_n]_{(u_1,\ldots,u_n)}$ and $\mathfrak{m} \subset \mathcal{O}_{p,V}$, and we get

$$k[[u_1, \ldots, u_n]] \cong \widehat{\mathcal{O}_{p,V}}.$$

14.5 Philosophy of power series rings

Philosophy: Fix $p \in V$, and let U be an affine patch containing p. Then

$$\mathcal{O}_V(U) \subseteq \mathcal{O}_{p,V} \hookrightarrow \widehat{\mathcal{O}_{p,V}},$$

where

- $\mathcal{O}_V(U)$ is the coordinate ring of an affine patch U containing p, "functions regular on U";
- $\mathcal{O}_{p,V}$ is "functions regular on some Zariski-open subset of V containing p";
- $\widehat{\mathcal{O}_{p,V}}$ is "functions on an even smaller (analytic, not Zariski) neighborhood of p".

For example, if $p = \mathbf{0} \in \mathbb{A}^n$, we have

$$R = k[x_1, \dots, x_n] \hookrightarrow k[x_1, \dots, x_n]\left[\frac{1}{x_1 - 1}\right] \hookrightarrow R_\mathfrak{m} = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)} \hookrightarrow k[[x_1, \dots, x_n]].$$

The ring $k[[x_1, \dots, x_n]]$ includes "functions" on an "even smaller" open neighborhood, including things like

$$\frac{1}{x_1 - 1} \mapsto -1 - x_1 - x_1^2 - x_1^3 - \dots$$

and

$$\text{“}e^{x_1}\text{”} = 1 + x_1 + \frac{x_1^2}{2!} + \frac{x_1^3}{3!} + \frac{x_1^4}{4!} + \dots$$

These inclusions induce maps of the spectrums in the opposite direction:

$$\text{“}\mathbb{A}^n\text{”} = \operatorname{Spec} k[x_1, \dots, x_n] \leftarrow \operatorname{Spec} R\left[\frac{1}{x_1 - 1}\right] = U_{x_1 - 1} \leftarrow \operatorname{Spec} R_\mathfrak{m} \leftarrow \operatorname{Spec} k[[x_1, \dots, x_n]].$$

14.6 Divisors and ideal sheaves

Theorem 14.11. *Let $Y \subseteq X$ be a codimension 1 subvariety of a* smooth *variety X. Then Y is* locally defined *by a vanishing of a single regular function on X at each point $p \in X$.*

More precisely: If Y is a codimension 1 subvariety of a smooth variety X, then $\forall p \in Y$, there exists an open (affine) neighborhood $p \in U \subseteq X$ such that ($p \in Y \cap U \subseteq U$ affine) the ideal

$$I_Y(Y \cap U) \subseteq k[U] = \mathcal{O}_X(U)$$

of $Y \cap U$ in U is principal.

Caution 14.12. Even if X is affine already, we can only expect Y to be *locally* defined by one equation.

There is an alternative (equivalent) formulation in terms of sheaves:

Definition 14.13. Fix a closed set W in a variety V. The *ideal sheaf* of W, denoted $\mathscr{I}_W$, assigns to each open $U \subseteq V$ the ideal

$$\mathscr{I}_W(U) = \left\{ f \in \mathcal{O}_V(U) \;\middle|\; f(p) = 0 \ \forall p \in W \right\} \subseteq \mathcal{O}_V(U).$$

Theorem 14.14. *If Y is a codimension 1 subvariety of a smooth variety X, then the* ideal sheaf *$\mathscr{I}_Y$ is locally principal in $\mathcal{O}_X$.*

This means: $\forall p \in X$, $\exists$ open affine neighborhood $U \ni p$ such that $\mathscr{I}_Y(U) \subseteq \mathcal{O}_X(U)$ is principal.

Remark 14.15. If $p \notin Y$, then $\exists U \ni p$ such that $Y \cap U = \varnothing$, so $\mathscr{I}_Y(U) = \mathcal{O}_X(U) = (1)$ is principal.

Equivalently, the condition that $\mathscr{I}_Y$ be locally principal means: $\forall p \in X$, the ideal $\mathscr{I}_{p,Y} \subseteq \mathcal{O}_{p,X}$ defined by

$$\begin{aligned}\mathscr{I}_{p,Y} &= \left\{ \varphi \in \mathcal{O}_{p,X} \;\middle|\; \begin{array}{l} \varphi \text{ has a representative } \frac{f}{g} \text{ where } f, g \in \mathcal{O}_X(U), \\ p \in U,\ g(p) \neq 0,\ f(q) = 0 \ \forall q \in Y \cap U \end{array} \right\} \\ &= \left\{ \varphi \in \mathcal{O}_{p,X} \;\middle|\; \varphi \text{ vanishes at all points of } Y \text{ in some neighborhood of } p \right\}\end{aligned}$$

is principal. This is called "the stalk at p" of the sheaf $\mathscr{I}_Y$. (Recall that $\mathcal{O}_{p,X}$ = the localization of $\mathcal{O}_X(U)$ at the maximal ideal $\mathfrak{m}_p \subseteq \mathcal{O}_X(U)$, where u is *any* open *affine* neighborhood of p.)

We have an inclusion of sheaves $\mathscr{I}_Y \subseteq \mathcal{O}_X$, which induces an inclusion of an ideal in a ring

$$\mathscr{I}_Y(U) \subseteq \mathcal{O}_X(U).$$

By localization at $\mathfrak{m}_p$, this induces

$$\mathscr{I}_Y(U)^e = \mathscr{I}_{p,Y} \subseteq \mathcal{O}_{p,X}.$$

Now we prove the theorem.

Proof of Theorem 14.14. Need to show: $\forall p \in X$, the ideal $\mathscr{I}_{p,Y} \subseteq \mathcal{O}_{X,p}$ is principal.

Step 1: $\mathcal{O}_{X,p}$ is a UFD. [More general theorem: Every regular local ring is a UFD.]

Sketch: $\mathcal{O}_{X,p}$ is a UFD $\iff$[4] $\widehat{\mathcal{O}_{X,p}}$ is a UFD $\iff$ $k[[u_1, \ldots, u_d]]$ is a UFD. Math 593 exercise: A is a UFD $\implies$ $A[[u]]$ is a UFD.

Step 2: Fix $p \in Y \subseteq X$, Y codimension 1 in X. Without loss of generality, X is affine. We have

$$I_Y \subseteq \mathfrak{m}_p \subseteq k[X] = \mathcal{O}_X(X).$$

Take any nonzero $h \in I_Y \subseteq \mathfrak{m}_p$. Look at the image of h in the UFD $\mathcal{O}_{X,p}$, and factor h into irreducibles

$$h = g_1^{a_1} \cdots g_r^{a_r} \in I_{Y,p},$$

[4] Shafarevich, Appendix §7

where $g_i \in \mathcal{O}_{X,p}$. Thus some $g_i \in I_{Y,p}$.

[Alternatively, pass to smaller open affine neighborhood U of p where each g_i is regular. Then
$$h = g_1^{a_1} \cdots g_r^{a_r} \in \mathscr{Y}(U),$$
which is a prime ideal in $\mathcal{O}_X(U)$, so $g_1 \in \mathscr{I}_Y(U)$.]

Because $g_i = g_1$ is irreducible in a UFD, it follows that (g_1) is a prime ideal of $\mathcal{O}_{X,p}$.

Consider: in U,
$$Y \cap U \subseteq \mathbb{V}(g_1) \subseteq U \subseteq X.$$
We have $\dim U = \dim X = d$ and $\dim \mathbb{V}(g_1) = d - 1$. If $Y \cap U \subset \mathbb{V}(g_1)$ is a proper inclusion, then $Y \cap U$ has dim $\leq d-2$, since a proper subset of an irreducible variety has smaller dimension. Hence $Y \cap U = \mathbb{V}(g_1)$. $\square$

Caution 14.16. The theorem can fail for non-smooth X. For example, consider
$$p = \mathbf{0} \in Y = \mathbb{V}(x, z) \subsetneqq X = \mathbb{V}(xy - zw) \subseteq \mathbb{A}^4.$$
We have $\dim Y = 2$ and $\dim X = 3$. See that
$$I_Y = (x, z) \subseteq k[X]_{(x,y,z,w)} = \frac{k[x,y,z,w]_{(x,y,z,w)}}{xy - zw}$$
cannot be generated by 1 polynomial. Note: $k[X]_{(x,y,z,w)}$ is *not* a UFD.

15 Rational maps

15.1 Provisional definition

Fix a variety V. A rational map $V \overset{\varphi}{\dashrightarrow} \mathbb{A}^n$ is given by rational functions coordinate-wise:
$$\begin{aligned} V &\dashrightarrow \mathbb{A}^n \\ x &\mapsto (\varphi_1(x), \ldots, \varphi_n(x)) \qquad \text{where } \varphi_i \in k(V). \end{aligned}$$
Note 15.1. Each φ_i is regular on some open (dense) subset U_i. So
$$\begin{tikzcd} V \arrow[r, dashed, "\varphi"] & \mathbb{A}^n \\ U \arrow[u, phantom, "\cup|"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$
is a *regular map* on $U = U_1 \cap \cdots \cap U_n$.

For
$$\begin{aligned} V &\overset{\varphi}{\dashrightarrow} \mathbb{P}^n \\ x &\mapsto [\varphi_0(x) : \cdots : \varphi_n(x)], \end{aligned}$$
take $\varphi_i \in k(V)$ and say φ_i has domain of definition U_i. This is regular on the dense open subset of V
$$\underbrace{U_0 \cap \cdots \cap U_n}_{U} \cap \left[(V \cap U) \setminus \mathbb{V}(\varphi_0|_U, \ldots, \varphi_n|_U)\right].$$

Example 15.2.

$$\begin{aligned}\mathbb{A}^2 &\overset{\varphi}{\dashrightarrow} \mathbb{P}^1\\ (x,y) &\mapsto [x:y] = \left[\frac{x}{y}:1\right] = \left[1:\frac{y}{x}\right].\end{aligned}$$

Defined on $\mathbb{A}^2 \setminus \{(0,0)\}$.

We can represent φ by $\varphi_{U_x} : U_x = \mathbb{A}^2 \setminus \mathbb{V}(x) \to \mathbb{P}^1$, and also by

$$\begin{aligned}\varphi_{\mathbb{A}^2\setminus\{(0,0)\}} : \mathbb{A}^2 \setminus \{(0,0)\} &\to \mathbb{P}^1\\ (x,y) &\mapsto [x:y].\end{aligned}$$

15.2 Definition of rational map

Definition 15.3. A *rational map* $X \overset{\varphi}{\dashrightarrow} Y$ between varieties is an *equivalence class* of regular maps $\left\{U \xrightarrow{\varphi_U} Y\right\}$ (with $U \subseteq X$ dense open subset), where

$$[U \xrightarrow{\varphi_U} Y] \sim [U' \xrightarrow{\varphi_{U'}} Y]$$

means φ_U and $\varphi_{U'}$ agree on $U \cap U'$ (or equivalently,

$$\varphi_U\big|_{\widetilde{U}} = \varphi_{U'}\big|_{\widetilde{U}}$$

for *any* dense open subset of $U \cap U'$).

Note 15.4. If two regular maps agree on some dense open set, then they agree everywhere they are both defined.

Proof sketch. Since regular maps are *locally* given by regular functions in coordinates, it suffices to check that if φ, φ' are *regular* functions $X \xrightarrow{\varphi} k$, $X \xrightarrow{\varphi'} k$ and $\varphi|_{\widetilde{U}} = \varphi'|_{\widetilde{U}}$, where $\widetilde{U} \subseteq X$ is an open dense set, then

$$(\varphi - \varphi') : X \to k$$

is regular. Its zero set contains $\widetilde{U}$ and is closed, hence the zero set contains $\overline{\widetilde{U}}$ = closure of $\widetilde{U}$ in X, so $\varphi - \varphi'$ is zero on X. Thus, $\varphi = \varphi'$ everywhere on X. ☐

In practice: A rational map is given by

$$\begin{aligned}X &\overset{\varphi}{\dashrightarrow} Y \subseteq \mathbb{P}^m\\ x &\mapsto [\varphi_0(x) : \cdots : \varphi_m(x)],\end{aligned}$$

where $\varphi_i \in k(X)$.

Definition 15.5. A rational map $\varphi : X \dashrightarrow Y$ is *regular* at $p \in X$ if φ admits a representative $U \xrightarrow{\varphi_U} Y$ such that $p \in U$.

The *domain of definition* of φ is the open subset of X where φ is regular. The *locus of indeterminacy* is the complement of the domain of definition.

15.3 Examples of rational maps

(1) A rational map $X \overset{\varphi}{\dashrightarrow} \mathbb{A}^1_k$ is the same as $\varphi \in k(X)$.

(2) Every regular map $X \to Y$ is a rational map. (The domain of definition is X, and the locus of indeterminacy is $\varnothing$.)

For example:

$$\begin{aligned} \mathbb{P}^1 &\dashrightarrow \mathbb{P}^3 \\ [s:t] &\mapsto \left[s^3 : s^2t : st^2 : t^3\right] = \left[1 : \frac{t}{s} : \left(\frac{t}{s}\right)^2 : \left(\frac{t}{s}\right)^3\right]. \end{aligned}$$

Note that $k(\mathbb{P}^1) = k\left(\frac{t}{s}\right)$.

(3) The map used in the blowup (to be studied in more detail later):

$$\begin{aligned} \mathbb{A}^2 &\dashrightarrow \mathbb{P}^1 \\ (x,y) &\mapsto \{\text{the line through } (x,y) \text{ and } (0,0)\} = [x:y] \end{aligned}$$

The locus of indeterminacy is $\{(0,0)\}$.

15.4 Rational maps, composition, and categories

Caution 15.6. A rational map is *not* a map!

In particular, we cannot always compose rational maps.

Example 15.7. Here's an example that shows why we can't compose rational maps:

$$\begin{aligned} \mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^3 & \overset{\psi}{\dashrightarrow} \mathbb{P}^3 \\ [s:t] \mapsto \left[s^3 : s^2t : st^2 : t^3\right] & \\ [w:x:y:z] & \mapsto \left[wz - xy : x^2 - wy : y^2 - xz\right] \end{aligned}$$

Caution 15.8. "$\psi \circ \varphi$" $= [0:0:0:0]$, which is nonsense.

Note 15.9. There is *no* category of varieties over k with rational maps as morphisms.

However, there is a category whose objects are algebraic varieties over k and whose morphisms are *dominant* rational maps.

Isomorphism in this category is birational equivalence.

15.5 Types of equivalence

Note 15.10. Birational equivalence is much weaker than isomorphism of varieties. For instance:

$$\begin{aligned} \mathbb{A}^2 \overset{\varphi}{\dashrightarrow} \mathbb{P}^2 & \overset{\varphi^{-1}}{\dashrightarrow} \mathbb{A}^2 \\ (x,y) \mapsto [x:y:1] & \\ [x:y:z] & \mapsto \left(\frac{x}{z}, \frac{y}{z}\right), \end{aligned}$$

so $\mathbb{A}^2$ and $\mathbb{P}^2$ are birationally equivalent. Also,

$$\begin{gathered}
\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\
[x : y : z] \mapsto ([x : z], [y : z]) \\
U_z \xrightarrow{\simeq} U_1 \times U_1,
\end{gathered}$$

so $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent.

In order of increasing strength and difficulty:

- Classify varieties up to birational equivalence
- Classify varieties up to isomorphism
- Classify varieties up to projective equivalence

It turns out that birational equivalence and isomorphism are the same for smooth projective curves, for which we have a complete classification.

15.6 Dimension of indeterminacy

Theorem 15.11. *If X is smooth and $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^n$ is a rational map, then the locus of indeterminacy has codimension ≥ 2 in X.*

Example 15.12.

$$\begin{gathered}
\mathbb{P}^2 \overset{\varphi}{\dashrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \\
[x : y : z] \mapsto ([x : z], [y : z])
\end{gathered}$$

The locus of indeterminacy $W \subseteq \mathbb{P}^2$ is either empty or dimension 0 (i.e., finite).

In fact, $W = \{[0 : 1 : 0], [1 : 0 : 0]\}$.

Corollary 15.13. *If X is a smooth curve and $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^m$ is a rational map, then φ is* regular *everywhere.*

Corollary 15.14. *If two smooth projective curves are birationally equivalent, then they are isomorphic.*

Proof. Say $X \sim Y$ are birationally equivalent. Then the rational map $X \overset{\varphi}{\dashrightarrow} Y \subseteq \mathbb{P}^m$ is a regular map $X \to Y$. Reversing roles of X and Y, $Y \overset{\varphi^{-1}}{\dashrightarrow} X \subseteq \mathbb{P}^n$ is also regular. So

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rr, bend right, "\mathrm{id}"'] & Y \arrow[r, "\varphi'"] & X, \end{tikzcd}$$

thus $X \cong Y$. $\square$

15.7 Dimension of indeterminacy, continued

Example 15.15. Let $X = \mathbb{V}\left(x_0^2 + \cdots + x_n^2\right) \subseteq \mathbb{P}^n$ (char $\neq 2$).

Pick any $p \in X$, project from it. Then we have

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, dashed, "\pi_p"] & \mathbb{P}^{n-1} \\
X \arrow[u, phantom, sloped, "\subseteq"] \arrow[ur, dashed, "\pi_p"'] &
\end{tikzcd}$$

and $X \overset{\pi_p}{\dashrightarrow} \mathbb{P}^{n-1}$ is a rational map.

Case 1: $\dim X = 1$ ($n = 2$): $X \overset{\pi_p}{\dashrightarrow} \mathbb{P}^1$ must be regular everywhere by Theorem 15.11.

So we have a map

$$\mathbb{P}^2 \supseteq \mathbb{V}(x^2 + y^2 - z^2) = X \xrightarrow{\pi_p} \mathbb{P}^1$$

which is regular everywhere, and fact is an isomorphism.

Case 2: $\dim X \geq 2$: The rational map is *not* regular everywhere. For $\dim X = 2$, we have

$$\begin{tikzcd}
\mathbb{P}^3 \arrow[dr, dashed] & \\
X \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, dashed, "\pi_p"] & \mathbb{P}^2 \\
X - \{p\} \arrow[u, phantom, sloped, "\subseteq"] \arrow[ur, "\text{regular}"'] &
\end{tikzcd}$$

The locus of indeterminacy is $\{p\}$. Codimension is $n - 1 = \dim X$.

Now we prove:

Theorem (15.11)**.** *If X is smooth, then the locus of indeterminacy of a rational map $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^n$ has codimension ≥ 2.*

Proof. Let X be smooth, $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^n$ a rational map, $W =$ locus of indeterminacy $\subseteq X$.

Then W is (locally at p) a *hypersurface*. For all sufficiently small affine open neighborhoods U of p, $U \cap W = \mathbb{V}(g) \subseteq U$, where $g \in \mathscr{O}_X(U)$. We have

$$\begin{aligned} X &\dashrightarrow \mathbb{P}^n \\ x &\mapsto [\varphi_0(x) : \cdots : \varphi_n(x)]\,, \end{aligned}$$

where $\varphi_i \in k(X) =$ fraction field of $k[U]$. Without loss of generality, $\varphi_i \in k[U]$.

Because $p \in W =$ locus of indeterminacy, we know $p \in \mathbb{V}(\varphi_0, \ldots, \varphi_n) \subseteq U$. Then

$$p \in W \cap U \subseteq \mathbb{V}(\varphi_0, \ldots, \varphi_n) \subseteq U \text{ affine.}$$

By the Nullstellensatz,

$$(g) = \mathscr{I}_W(U) \supseteq (\varphi_0, \ldots, \varphi_n),$$

so g divides each φ_i (in $k[U]$).

Note: $\mathscr{O}_{p,X}$ is a UFD, so we can factor $\varphi_0, \ldots, \varphi_n$ into irreducibles and cancel out any common factors. Thus, without loss of generality, the φ_i do not have a common factor! $\square$

15.8 Images and graphs of rational maps

Definition 15.16. The *image* of a rational map $X \xrightarrow{\varphi} Y$ is the closure in Y of the image of any representing regular map $U \xrightarrow{\varphi_U} Y$.

Check: This does not depend on the choice of φ_U. Indeed,

$$\overline{\varphi_U(U \cap U')} \subseteq \overline{\varphi_U(U)} = \overline{\varphi_{U'}(U')}.$$

Recall: The graph of a *regular* map $X \xrightarrow{\varphi} Y$ is the set

$$\Gamma_\varphi = \{(x, \varphi(x))\} \subseteq X \times Y.$$

This is a closed set isomorphic to X. (Check: vertical line test.)

Definition 15.17. The *graph* Γ_φ of a rational map $X \overset{\varphi}{\dashrightarrow} Y$ is the closure in $X \times Y$ of the graph of any representing regular map $U \xrightarrow{\varphi_U} Y$.

Check: This is independent of representative.

Note 15.18. Γ_φ is birationally equivalent to X.

Example 15.19.

$$\begin{aligned} \mathbb{A}^2 &\overset{\varphi}{\dashrightarrow} \mathbb{P}^1 \\ (x, y) &\mapsto \{\text{line through } (x, y) \text{ and } (0, 0)\} = [x : y]. \end{aligned}$$

Consider on $\mathbb{A}^2 - \mathbb{V}(x) = U_x \subseteq \mathbb{A}^2$. Then

$$\begin{aligned} U_x = \mathbb{A}^2 - (y\text{-axis}) &\to U_0 = \mathbb{A}^1 \hookrightarrow \mathbb{P}^1 \\ (x, y) &\mapsto \frac{y}{x} \to \left[1 : \frac{y}{x}\right] = [x : y], \end{aligned}$$

noting that $\frac{y}{x}$ is the slope of the line through $(0, 0)$ and (x, y).

16 Blowing up

16.1 Blowing up a point in $\mathbb{A}^n$

Choose coordinates so the point is $\mathbf{0}$. Let

$$B = \left\{(p, \ell) \;\middle|\; p \in \ell\right\} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

In coordinates,

$$\begin{aligned} B &= \left\{ \left((x_1, \ldots, x_n); [y_1 : \cdots : y_n]\right) \;\middle|\; \operatorname{rank} \begin{bmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{bmatrix} \leq 1 \right\} \\ &= \mathbb{V}\left(2 \times 2 \text{ minors of } \begin{bmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{bmatrix} \right) \\ &= \mathbb{V}\left(\left\{ x_i y_j - x_j y_i \;\middle|\; i \leq 1, j \leq n \right\} \right). \end{aligned}$$

Definition 16.1. The *blowup* of $\mathbb{A}^n$ at $\mathbf{0}$ is the variety

$$B = \{(p, \ell) \mid p \in \ell\} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$$

together with the projection $B \xrightarrow{\pi} \mathbb{A}^n$.

Note 16.2. (1) π is surjective, and one-to-one over $\mathbb{A}^n \setminus \{0\}$.

Also, π is *birational* (i.e., a birational equivalence) with rational inverse

$$\begin{aligned} \mathbb{A}^n \overset{\pi^{-1}}{\dashrightarrow} B \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1} \\ (x_1, \ldots, x_n) \mapsto \big((x_1, \ldots, x_n); [x_1 : \cdots : x_n]\big). \end{aligned}$$

(2) B is the graph of the rational map

$$\begin{aligned} \varphi : \mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1} \\ (x_1, \ldots, x_n) \mapsto [x_1 : \cdots : x_n], \end{aligned}$$

and $B \xrightarrow{\pi} A$ is projection to the "source".

Intuition again: B is "like $\mathbb{A}^n$" except at $\mathbf{0}$; we've removed $\mathbf{0}$ from $\mathbb{A}^n$ and replaced it by the set of all directions approaching the origin.

Proposition 16.3. *B is a smooth (irreducible) variety of the dimension n.*

Proof. We have $B \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1} \supseteq (\mathbb{A}^n \times U_i)$, where $U_i = \mathbb{A}^{n-1}$ is a standard affine chart. It suffices to check that each $B \cap (\mathbb{A}^n \times U_i)$ is smooth.

For simplicity, we do the case $i = n$.

Claim 16.4. $B \cap (\mathbb{A}^n \times \mathbb{A}^{n-1}) \xrightarrow{\simeq} \mathbb{A}^n$.

Observe that

$$\begin{aligned} B \cap (\mathbb{A}^n \times \mathbb{A}^{n-1}) &= \{(x_1, \ldots, x_n); [y_1 : \cdots : y_n] \mid y_n \neq 0,\ x_i y_j = x_j y_i\} \\ &= \left\{(x_1, \ldots, x_n); \left[\frac{y_1}{y_n} : \cdots : \frac{y_{n-1}}{y_n} : 1\right] \mid x_j = x_n \left(\frac{y_j}{y_n}\right)\right\} \end{aligned}$$

We have an isomorphism

$$\begin{aligned} B \cap U &\xrightarrow{\varphi} \mathbb{A}^n \\ \left((x_1, \ldots, x_n); \left[\frac{y_1}{y_n} : \cdots : \frac{y_{n-1}}{y_n} : 1\right]\right) &\mapsto \left(\frac{y_1}{y_n}, \ldots, \frac{y_{n-1}}{y_n}, x_n\right) \\ B \cap U &\xleftarrow{\varphi^{-1}} \mathbb{A}^n \\ \big((t_n t_1, \ldots, t_n t_{n-1}, t_n); [t_1 : \cdots : t_{n-1} : 1]\big) &\leftarrow\!\shortmid (t_1, \ldots, t_{n-1}, t_n). \end{aligned}$$

□

16.2 Resolution of singularities

Theorem 16.5 (Hironaka, 1964)**.** *If k has characteristic 0, then every affine variety V admits a resolution of singularities, i.e., $\exists$ smooth variety $\widetilde{V} \overset{\text{closed}}{\subseteq} \mathbb{A}^n \times \mathbb{P}^m$ such that the projection onto the first factor $\mathbb{A}^n \times \mathbb{P}^m \twoheadrightarrow \mathbb{A}^n$ is a birational map $\pi : \widetilde{V} \twoheadrightarrow V$ when restricted to $\widetilde{V}$.*

Furthermore, π is an isomorphism over $V \setminus \operatorname{Sing}(V)$. The fibers are all projective (over $\mathbb{C}$, all compact), i.e., π is a proper map.[5]

16.3 More about blowups

Recall: The blowup of $(0,0)$ in $\mathbb{A}^2$ is the graph of the rational map

$$\begin{aligned} \mathbb{A}^2 \overset{\varphi}{\dashrightarrow} \mathbb{P}^1 &= \text{lines through } (0,0) \text{ in } \mathbb{A}^2 \\ (x,y) &\mapsto [x:y] \end{aligned}$$

together with the projection onto the source

$$\{(p,\ell) \mid p \in \ell\} = B = \Gamma_\varphi \xrightarrow{\pi} \mathbb{A}^2.$$

Note 16.6. (1) The map π is a *projection*, birational. In fact, π is an isomorphism over the domain of definition of φ.

(2) The fiber over the locus of indeterminacy $\{(0,0)\}$ is

$$\{(0,0)\} \times \mathbb{P}^1 \overset{\text{closed}}{\subseteq} B \overset{\text{closed}}{\subseteq} \mathbb{A}^2 \times \mathbb{P}^1$$

is a smooth, codimension 1 subset of B.

What happens if we graph a different rational map?

$$\begin{aligned} \mathbb{A}^3 &\overset{\psi}{\dashrightarrow} \mathbb{P}^1 \\ (x,y,z) &\mapsto [x:y] = \text{normal line to } L = \text{the } z\text{-axis} \end{aligned}$$

This is an isomorphism on $\mathbb{A}^3 \setminus L$, and is birational on $\mathbb{A}^3$.

The fiber over the locus of indeterminacy L is $L \times \mathbb{P}^1 \subseteq \Gamma_\varphi$, which is a codimension 1 subvariety of Γ_φ.

This is called the *blowup of* $\mathbb{A}^3$ *at the line* L (or the blowup along the ideal (x,y)).

16.4 Blowing up in general

Definition 16.7. Let V be an affine variety, and let $f_0, \ldots, f_r$ be nonzero regular functions on V. The *blowup* of V along the ideal $(f_0, \ldots, f_r)$ is the graph of the rational map

$$\begin{aligned} V &\overset{\varphi}{\dashrightarrow} \mathbb{P}^r \\ x &\mapsto [f_0(x) : \cdots : f_r(x)] \end{aligned}$$

together with the projection

$$V \times \mathbb{P}^r \supseteq \widetilde{V} := \Gamma_\varphi \xrightarrow{\pi} V.$$

[5]The technical definition of "proper map" in algebraic geometry is more complicated, but agrees with the other definition over $\mathbb{C}$. In any case, π is a proper map in the algebraic geometry sense.

Definition 16.8 (projective map)**.** A *projective map* $X \xrightarrow{f} Y$ is a composition

$$\begin{tikzcd} X \arrow[r, hook, "\text{closed}"] \arrow[rr, bend right, "f"'] & Y \times \mathbb{P}^m \arrow[r, "\substack{\text{proj. onto}\\ \text{1st coord.}}"] & Y \end{tikzcd}$$

Remark 16.9. (1) Since φ is rational on $V - \mathbb{V}(f_0, \dots, f_r)$, $\pi : \widetilde{V} \to V$ is an isomorphism over $V - \mathbb{V}(f_0, \dots, f_r)$, i.e., is birational.

(2) This depends only on the ideal generated by $(f_0, \dots, f_r)$, not the choice of generators: Say $(f_0, \dots, f_r) = (g_0, \dots, g_m) \subseteq k[V]$. Then

$$\begin{tikzcd} V \times \mathbb{P}^r & & V \times \mathbb{P}^m \\ \Gamma_\varphi \arrow[u, phantom, "\cup\mid"] \arrow[rr, "\exists \text{ isomorphism}"] \arrow[dr, "\pi_1"'] & & \Gamma_{\varphi'} \arrow[u, phantom, "\cup\mid"] \arrow[dl, "\pi_2"] \\ & V & \end{tikzcd}$$

(3) If $(f_0, \dots, f_r)$ is radical, defines a subvariety $W \subseteq V$, then we also say "*blowup of V along W*".

If $W \subseteq V$ is *smooth*, then the blowup $\widetilde{V}$ "looks like" V with surgery performed: remove W, and replace it by all directions normal to W in V.

Example 16.10. Blowup of (x^2, y^2) in $\mathbb{A}^2$: The graph of

$$\mathbb{A}^2 \overset{\varphi}{\dashrightarrow} \mathbb{P}^1$$
$$(x, y) \mapsto [x^2 : y^2]$$

We have
$$\underset{(x,y)}{\mathbb{A}^2} \times \underset{[u:v]}{\mathbb{P}^1} \supseteq \mathbb{V}(uy^2 - vx^2) = \Gamma_\varphi \to \mathbb{A}^2.$$

So blowing up can sometimes make things "worse"!

16.5 Hironaka's theorem

Theorem 16.11 (Hironaka's theorem on resolution of singularities)**.** *Suppose* $\operatorname{char} k = 0$. *For any affine variety V, there exist $f_0, \dots, f_r \in k[V]$ such that the graph of the rational map*

$$V \overset{\varphi}{\dashrightarrow} \mathbb{P}^r$$
$$x \mapsto [f_0(x) : \cdots : f_r(x)]$$

is smooth. The map $\widetilde{V} = \Gamma_\varphi \xrightarrow{\pi} V$ is projective, birational, and an isomorphism over $V \setminus \operatorname{Sing} V$.

Furthermore, $\pi^{-1}(\operatorname{Sing} V)$ is a smooth, codimension 1 subvariety of $\widetilde{V}$.

17 Divisors

17.1 Main definitions

Fix an irreducible variety X.

Definition 17.1. A *prime divisor* on X is a codimension 1 irreducible (closed) subvariety of X.

A *divisor* D on X is a formal $\mathbb{Z}$-linear combination of prime divisors

$$D = \sum_{i=1}^{t} k_i D_i, \qquad k_i \in \mathbb{Z}.$$

Example 17.2. In $\mathbb{P}^2$, here are some prime divisors:

$$C = \mathbb{V}(xy - z^2) \subseteq \mathbb{P}^2, \qquad L_1 = \mathbb{V}(x), \qquad L_2 = \mathbb{V}(y).$$

Here are some divisors which are not prime: $2C, 2L_1 - L_2$.

Definition 17.3. We say a divisor $D = \sum_{i=1}^{t} k_i D_i$ is *effective* if each $k_i \geq 0$.

The *support* of D is the list of prime divisors occurring in D with non-zero coefficient.

The set of all divisors on X form a group $\mathrm{Div}(X)$, the free abelian group on the set of prime divisors of X.

The zero element is the *trivial divisor* $D = \sum 0 D_i$, and

$$\mathrm{Supp}(0) = \varnothing.$$

Example 17.4. Consider

$$\varphi = \frac{f}{g} = \frac{(t-\lambda_1)^{a_1} \cdots (t-\lambda_n)^{a_n}}{(t-\mu_1)^{b_1} \cdots (t-\mu_m)^{b_m}} \in k(\mathbb{A}^1) = k(t),$$

where $f, g \in k[t]$ (assume lowest terms).

The "divisor of zeros and poles" of φ is

$$\underbrace{a_1\{\lambda_1\} + a_2\{\lambda_2\} + \cdots + a_n\{\lambda_n\}}_{\text{(divisor of zeros)}} - \underbrace{b_1\{\mu_1\} - \cdots - b_1\{\mu_m\}}_{\text{(divisor of poles)}}.$$

Example 17.5. Let $\mathbb{A}^n = X$. A prime divisor is $D = \mathbb{V}(h)$, where $h \in k[x_1, \ldots, x_n]$ is irreducible. Write

$$\varphi = \frac{f}{g} = \frac{f_1^{a_1} \cdots f_n^{a_n}}{g_1^{b_1} \cdots g_m^{b_m}} \in k(\mathbb{A}^n) = k(x_1, \ldots, x_n),$$

where $f, g \in k[x_1, \ldots, x_n]$ and f_i, g_i irreducible, $a_i \in \mathbb{N}$.

Denoting the divisor of zeros and poles of φ by $\mathrm{div}(\varphi)$, we have

$$\mathrm{div}(\varphi) = a_1\mathbb{V}(f_1) + a_2\mathbb{V}(f_2) + \cdots + a_n\mathbb{V}(f_n) - b_1\mathbb{V}(g_1) - \cdots - b_m\mathbb{V}(g_m).$$

Note 17.6. Every divisor on $\mathbb{A}^n$ has the above form.

17.2 The divisor of zeros and poles

In general, on almost any X, we will associate to each $\varphi \in k(X) \setminus \{0\}$ some divisor, $\mathrm{div}(\varphi)$, "the divisor of zeros and poles", in such a way that the map

$$\begin{aligned} k(X)^* = k(x) \setminus \{0\} &\to \mathrm{Div}(X) \\ \varphi &\mapsto \mathrm{div}\,\varphi = \sum_{\substack{D \subseteq X \\ \text{prime}}} \nu_D(\varphi) \cdot D \end{aligned}$$

preserves the group structure on $k(X)^*$, i.e.,

$$(\varphi_1 \circ \varphi_2) \mapsto \mathrm{div}\,\varphi_1 + \mathrm{div}\,\varphi_2.$$

The image of this map will be the group of *principal* divisors:

$$P(X) \subseteq \mathrm{Div}(X)$$

The quotient $\mathrm{Div}(X)/P(X)$ is the *divisor class group* of X.

Remark 17.7. If X is smooth, then the divisor class group is isomorphic to the Picard group.

Remark 17.8. The kernel of $k(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(X)$ consists of $\varphi \in k(X)$ such that φ, φ^{-1} are both regular on X.

Remark 17.9. We will write

$$\mathrm{div}\,\varphi = \sum_{\substack{D \subseteq X \\ \text{prime}}} \nu_D(\varphi) \cdot D,$$

where $\nu_D(\varphi) = \mathrm{ord}_D(\varphi) =$ "order of vanishing of φ along D".

Example 17.10.

$$\varphi = \frac{x}{y} \in k(x, y) = k(\mathbb{A}^2)$$
$$\mathrm{div}(\varphi) = \sum_{\substack{D \subseteq \mathbb{A}^2 \\ \text{prime}}} \nu_D\left(\frac{x}{y}\right) D,$$

where $\nu_D\left(\frac{x}{y}\right)$ is 0 for all divisors D except for $L_1 = \mathbb{V}(x)$, where the order of vanishing is 1, and $L_2 = \mathbb{V}(y)$, where $\nu_{L_2}(\varphi) = -1$.

To define $\mathrm{div}(\varphi)$ for $\varphi \in k(X)^*$, we need to define $\nu_D(\varphi)$ for every every divisor D. We will do this under the following assumption: X is non-singular in codimension 1.[6] In this case, we have

$$\begin{gathered} X \supseteq X_{\mathrm{sm}} = X - \mathrm{Sing}\,X \\ \mathrm{Div}(X) \xrightarrow{\simeq} \mathrm{Div}(X_{\mathrm{sm}}) \\ \sum_i a_i D_i \mapsto \sum_i a_i (D_i \cap X_{\mathrm{sm}}). \end{gathered}$$

[6]This means that $X_{\mathrm{sing}} \subseteq X$ has codimension ≥ 2.

To get an idea of how this will work, assume X is smooth and affine, and let $\varphi \in k[X]$. Any prime divisor $D \subseteq X$ is locally principal, i.e., locally $D = \mathbb{V}(\pi)$.

"D is a zero of φ" means that $D \subseteq \mathbb{V}(\varphi)$, meaning $(\pi) \ni \varphi$. Look at the largest k such that $\varphi \in (\pi^k)$, i.e., $\varphi \in (\pi^k) \setminus (\pi^{k+1})$. This is $\nu_D(\varphi) = k$.

17.3 Order of vanishing

Goal: Define "order of vanishing" of $\varphi \in k(X) \setminus \{0\}$ along a prime divisor D, denoted $\nu_D(\varphi) \in \mathbb{Z}$.

This is done *only* under the assumption that X is non-singular in codimension 1 (i.e., $\operatorname{Sing} X$ has codimension ≥ 2).

Case 1

Say X is affine, $\varphi \in k[X]$, $D = \mathbb{V}(\pi)$ is a hypersurface defined by $\pi \in k[X]$.

We say "φ vanishes along D" provided that $D = \mathbb{V}(\pi) \subseteq \mathbb{V}(\varphi)$. So by the Nullstellensatz, $(\varphi) \subseteq (\pi)$. It could be that $\varphi \in (\pi^2)$ or (π^3) or some higher power.

Definition 17.11. The *order of vanishing* of φ along D, denoted $\nu_D(\varphi)$, is the unique integer $k \geq 0$ such that $\varphi \in (\pi^k) \setminus (\pi^{k+1})$.

Note 17.12. $\nu_D(\varphi) = 0 \implies \varphi \in (\pi^0) \setminus (\pi^1) = k[X] \setminus (\pi)$, i.e., φ does not vanish on all of D.

Can it be that $\varphi \in (\pi^k)\ \forall k$? If so, then $\varphi \in \bigcap_{k \geq 0} (\pi^k)$, which remains true after localizing at any prime ideal of $k[X]$ containing π (e.g., (π) itself).

Lemma 17.13. *If $(R, \mathfrak{m})$ is a Noetherian local ring, then*

$$\bigcap_{t \geq 0} \mathfrak{m}^t = 0.$$

Thus, if $\varphi \in \bigcap_{k \geq 0} (\pi^k)$, then $\varphi = 0$.

Note 17.14. ν_D has the following properties:

(1) $\nu_D(\varphi \cdot \psi) = \nu_D(\varphi) + \nu_D(\psi)$.

(2) If $\varphi + \psi \neq 0$, then $\nu_D(\varphi + \psi) \geq \min\{\nu_D(\varphi), \nu_D(\psi)\}$.

Case 1b

If φ is rational and $\varphi = \frac{f}{g}$, where $f, g \in k[X]$, define

$$\nu_D(\varphi) = \nu_D(f) - \nu_D(g).$$

Case 2

General case: $\varphi \in k(X) \setminus \{0\}$, $D \subseteq X$ arbitrary prime divisor.

Choose $U \subseteq X$ open affine such that

(a) U is smooth;

(b) $U \cap D \neq \varnothing$;

(c) D is a hypersurface: $D = \mathbb{V}(\pi)$ for some $\pi \in k[U] = \mathcal{O}_X(U)$.[7]

We have $\varphi \in k(X) = k(U)$. Define $\nu_D^U(\varphi)$ as in case 1.

Claim 17.15. *This doesn't depend on the choice of U.*

Proof. Say U_1, U_2 both satisfy conditions (a), (b), (c). To check $\nu_D^{U_1}(\varphi) = \nu_D^{U_2}(\varphi)$, it suffices to check $\nu_D^{U_1}(\varphi) = \nu_D^{U}(\varphi)$ for any $U \subseteq U_1 \cap U_2$ satisfying (a), (b), (c).

Fix $U_1 \supseteq U_2$. We have $\varphi \in (\pi^k) \setminus (\pi^{k+1})$ in $k[U_1] = \mathcal{O}_X(U_1)$, and after restricting to $k[U_2] = \mathcal{O}_X(U_2)$, the condition $\varphi \in (\pi^k) \setminus (\pi^{k+1})$ still holds. □

So define $\nu_D(\varphi)$ to be $\nu_D^U(\varphi)$ for any U.

17.4 Alternate definitions of order of vanishing

17.4.1 Alternate definition 1

Let $D \subseteq X$ be a prime divisor, $\varphi \in K(X)$. Pick any smooth point $x \in X$ such that $x \in D$. The local ring

$$\mathcal{O}_{x,X} = \left\{ \varphi \in k(X) \;\middle|\; \varphi \text{ is regular at } x \right\}$$

is a UFD. The equation of D in $\mathcal{O}_{x,X}$ is $= (\pi) \subseteq \mathcal{O}_{x,X}$, where π is an irreducible element in the UFD.

Writing $\varphi = \frac{f}{g}$ with $f, g \in \mathcal{O}_{x,X}$, φ factors uniquely as

$$\varphi = \pi^k \frac{f_1^{a_1} \cdots f_r^{a_r}}{g_1^{b_1} \cdots g_s^{b_s}}$$

with f_i, g_i irreducible. Then

$$\nu_D(\varphi) = \text{multiplicity of } \pi \text{ in the unique factorization in } \mathcal{O}_{x,X}.$$

17.4.2 Alternate definition 2

Let D be a prime divisor on X (non-singular in codimension 1). Look at the ring

$$\mathcal{O}_{D,X} = \left\{ \varphi \in k(X) \;\middle|\; \varphi \text{ is regular on some open } U \text{ such that } U \cap D \neq \varnothing \right\} = k[U]_{\mathscr{I}_D(U)},$$

the local ring of X along D. We have $U \supseteq D \cap U \neq \varnothing$ and $k[U] \supseteq \mathscr{I}_D(U)$.

Choose U satisfying (a), (b), (c). The maximal ideal of $\mathcal{O}_{D,X}$ is (π), generated by the single element π.

Observe that $\mathcal{O}_{D,X}$ is a local domain whose maximal ideal is *principal*, i.e., a *discrete valuation ring.*

[7]We can do this by our earlier theorem that a codimension 1 subvariety is locally a hypersurface.

Definition 17.16. A *discrete valuation ring* (DVR) is a Noetherian local domain with any of the following equivalent properties:

(1) It is regular of dimension 1.

(2) The maximal ideal is principal, (π).

(3) It is a UFD with one irreducible element, π.

(4) Every nonzero ideal is (π^t) for some $t \in \mathbb{Z}_{\geq 0}$.

(5) Normal of dimension 1.

Then we can define $\nu_D(\varphi) = t$, where t is obtained as follows: We have

$$\mathcal{O}_{D,X} \subseteq k(X).$$

Write $\varphi = \frac{f}{g}$, where $f, g \in \mathcal{O}_{D,X}$. Then

$$f = (\text{unit}) \cdot \pi^n, \;\; g = (\text{unit}) \cdot \pi^m,$$

and

$$\nu_D(\varphi) = n - m = t.$$

17.5 Divisors of zeros and poles, continued

Now we get a way to define a "*divisor of zeros and poles*" associated to every $\varphi \in k(X)$, namely,

$$\operatorname{div}(\varphi) = \sum_{\substack{D \subseteq X \\ \text{prime}}} \nu_D(\varphi) D.$$

To see that this is a *finite* sum: when X is affine, write $\varphi = \frac{f}{g}$, and observe that $\operatorname{div}\varphi$ has support contained in

$$\mathbb{V}(f) \cup \mathbb{V}(g) = (D_1 \cup \cdots \cup D_r) \cup (D_1' \cup \cdots \cup D_s'),$$

so finiteness of the sum follows from quasi-compactness of the Zariski topology.

17.6 Divisor class group, continued

Recall: For a variety X which is non-singular in codimension 1, we defined the "order of vanishing $\nu_D(\varphi)$ of $\varphi \in k(X)^*$ along a prime divisor D"; ν_D is the valuation of $k(X)$ associated with the DVR $\mathcal{O}_{D,X}$.

This gives a group homomorphism

$$\begin{aligned} (k(X))^* &\xrightarrow{\operatorname{div}} \operatorname{Div}(X) \\ \varphi &\mapsto \operatorname{div}(\varphi) = \sum_{\substack{D \subseteq X \\ \text{prime}}} \nu_D(\varphi) \cdot D. \end{aligned}$$

We defined the subgroup $P(X)$ of *principal divisors* to be the image of $\mathrm{div} : k(X)^* \to \mathrm{Div}(X)$.

The cokernel of $\mathrm{div} : k(X)^* \to \mathrm{Div}(X)$ is the *divisor class group* of X,

$$\mathrm{Cl}(X) = \frac{\mathrm{Div}(X)}{P(X)}.$$

Example 17.17. $\mathrm{Cl}(\mathbb{A}^n) = 0$.

Proposition 17.18. *$\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$, generated by the class of a hyperplane $H = \mathbb{V}(a_0x_0 + \cdots + a_nx_n)$.*

Definition 17.19. If $D_i = \mathbb{V}(G_i) \subseteq \mathbb{P}^n$ is a prime divisor, where G_i is an irreducible homogeneous polynomial in $k[x_0, \ldots, x_n]$, we define the degree of D_i to be the degree of G_i.

Proof of Proposition 17.18. We have a surjective homomorphism

$$\mathrm{Div}(\mathbb{P}^n) \xrightarrow{\deg} \mathbb{Z}$$
$$D = \sum_{i=1}^{t} k_iD_i \mapsto \sum k_i \deg D_i = \sum k_i \deg G_i.$$

Say $D = \sum_{i=1}^{t} k_i\mathbb{V}(G_i) \in \mathrm{Div}(\mathbb{P}^n)$ is in the kernel of $\deg : \mathrm{Div}(\mathbb{P}^n) \to \mathbb{Z}$. Then

$$\sum_{i=1}^{t} k_i\mathbb{V}(G_i) = \sum_{i=1}^{r} a_i\mathbb{V}(F_i) - \sum_{i=1}^{s} b_i\mathbb{V}(H_i) \overset{\deg}{\longmapsto} 0.$$

This is the divisor of zeros and poles of

$$\varphi = \frac{F_1^{a_1} \cdots F_r^{a_r}}{H_1^{b_1} \cdots H_s^{b_s}} = \prod_{i=1}^{t} G_i^{k_i} \in k(\mathbb{P}^n).$$

Therefore,

$$\mathrm{Cl}(\mathbb{P}^n) = \frac{\mathrm{Div}(\mathbb{P}^n)}{P(\mathbb{P}^n)} \cong \mathbb{Z}$$

by the first isomorphism theorem. □

Caution 17.20. There is no inherent notion of degree of a divisor on arbitrary X (though okay for $\mathbb{P}^n$, $\mathbb{A}^n$, curves).

17.7 Divisors and regularity

Theorem 17.21. *If X is smooth (or even just normal), then $\varphi \in k(X)^*$ is regular on X if and only if $\mathrm{div}\,\varphi$ is effective (denoted $\mathrm{div}\,\varphi \geq 0$).*

Remark 17.22. φ regular $\implies \mathrm{div}\,\varphi \geq 0$ is clear.

17.8 Commutative algebra digression

Let R be any domain, and let K be the fraction field.

Definition 17.23. The *normalization* of R is the integral closure of R in K. (This is a subring of K.)

We say R is *normal* if R is equal to its normalization $\widetilde{R}$.

We have the inclusion

$$R \hookrightarrow \widetilde{R} \subseteq K$$

into the integral closure.

Example 17.24. Consider the ring

$$R = \frac{k[x,y]}{y^2 - x^3}.$$

We have

$$\left(\frac{y}{x}\right)^2 - x = 0,$$

so $\frac{y}{x}$ is integral over R in the fraction field $\mathrm{Frac}(R)$. Can check that

$$R \hookrightarrow \widetilde{R} = \frac{k[x,y,z]}{(y^2 - x^3, xz - y)} \cong k\left[\frac{y}{x}\right] = k[t] \subseteq \mathrm{Frac}(R).$$

Note that normalizing gets rid of the singularity. The above inclusion induces a finite birational map of varieties.

Fact 17.25. Normality is a local property: R is normal $\iff$ $R_{\mathfrak{m}}$ is normal $\forall \mathfrak{m} \in \operatorname{mSpec} R$ $\iff$ $R_{\mathfrak{p}}$ is normal $\forall \mathfrak{p} \in \operatorname{Spec} R$.

This lets us make the following definition:

Definition 17.26. Let X be a variety. We say X is *normal* if any of the following equivalent conditions hold:

(1) For all points $x \in X$, the local ring $\mathcal{O}_{x,X}$ is normal.

(2) For all subvarieties $W \subseteq X$, $\mathcal{O}_{W,X}$ is normal.

(3) There exists an open affine cover $\{U_\lambda\}$ such that each $\mathcal{O}_X(U_\lambda) = k[U_\lambda]$ is normal.

(4) For every open affine $U \subseteq X$, $\mathcal{O}_X(U)$ is normal.

Fact 17.27. All smooth varieties are normal. If X is dimension 1, then X is smooth $\iff$ X is normal.

Fact 17.28. If a ring R is normal and $\mathfrak{p}$ is height[8] 1, then $R_{\mathfrak{p}}$ is a DVR.

Theorem 17.29. *Let R be a domain with fraction field K. Then*

$$\widetilde{R} = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \text{height } 1}} R_{\mathfrak{p}} \subseteq K.$$

[8]The *height* of a prime $\mathfrak{p} \in \operatorname{Spec} R$ is the Krull dimension of $R_{\mathfrak{p}}$.

Now we can prove the theorem from earlier:

Proof of Theorem 17.21. Say $\varphi \in k(X)$ and $\operatorname{div}\varphi \geq 0$. It suffices to check $\varphi|_U$, where U is affine open in X, is regular.

On U, we have $\varphi \in k(U) = k(X)$ with $\operatorname{div}_U \varphi \geq 0$. All $\nu_D(\varphi) \geq 0$, so $\varphi \in \mathcal{O}_{D,X}$ $\forall D$. Thus

$$\varphi \in \bigcap_{D \text{ prime in } U} \mathcal{O}_{D,X} = \bigcap_{\mathfrak{p} \text{ ht. } 1} R_{\mathfrak{p}} = R = \mathcal{O}_X(U). \qquad \square$$

17.9 Divisors and regularity, continued

Recall:

Theorem (17.21)**.** *Let φ be a nonzero rational function on a normal variety X. Then φ is regular on $X \iff \operatorname{div}\varphi$ is effective.*

E.g., on $\mathbb{P}^n$, there are no nonzero principal effective divisors (i.e., $\operatorname{div}\varphi \geq 0 \implies \varphi$ is regular on $\mathbb{P}^n \implies \varphi \in k \setminus \{0\}$).

More generally, for any U open in a normal variety X, the following are equivalent for $\varphi \in k(X)^*$:

(1) $\varphi \in k(X)$ is regular on U.

(2) φ has no poles on U.

(3) $\operatorname{div}\varphi$ on U is effective.

(4) $\nu_D(\varphi) \geq 0$ for all divisors D with $D \cap U \neq \varnothing$.

Also, the following are equivalent:

(1) $\operatorname{div}_U \varphi = 0$

(2) φ regular in U, φ^{-1} regular on U.

(3) $\varphi \in \mathcal{O}_X^*(U) =$ subgroup of invertible elements of the ring $\mathcal{O}_X(U)$.

Example 17.30. Let $X = \mathbb{P}^2$ and

$$\varphi = \frac{(x^2+y^2-z^2)^2}{x^3 y} \in k(\mathbb{P}^2).$$

Then

$$\operatorname{Supp}(\operatorname{div}\varphi) = C \cup L_1 \cup L_2 = \mathbb{V}(x^2+y^2-z^2) \cup \mathbb{V}(x) \cup \mathbb{V}(y),$$

and

$$\begin{aligned} \operatorname{div}_{\mathbb{P}^2} \varphi &= 2C - 3L_1 - L_2 \\ \operatorname{div}_{U_z} \varphi &= 2C - 3L_1 - L_2 \\ \operatorname{div}_{U_x} \varphi &= 2C - L_1 \\ \operatorname{div}_{U_x \cap U_y} \varphi &= 2C. \end{aligned}$$

Since $2C$ is effective, Theorem 17.21 implies that $\varphi \in \mathcal{O}_{\mathbb{P}^2}(U_x \cap U_y)$.

Also, denoting $U := U_x \cap U_y \cap U_{x^2+y^2-z^2}$, we have $\operatorname{div}_U \varphi = 0$, so $\varphi \in \mathcal{O}^*_{\mathbb{P}^2}(U)$.

18 Locally principal divisors

18.1 Locally principal divisors

Important idea: If X is smooth, then every divisor on X is *locally principal.*

Fix $D = \sum_{i=1}^{t} k_i D_i$ divisor on X, with X smooth.

Take any $x \in X$, and choose a neighborhood $U = U_x$ of x such that D_i is the vanishing set of some irreducible $\pi_i \in \mathcal{O}_X(U)$ (i.e., $\mathscr{I}_{D_i}(U) = (\pi_i)$, or equivalently, $D_i \cap U = \operatorname{div}_U \pi_i$).

On U, D is principal, and we have

$$D \cap U = \operatorname{div}_U(\pi_1^{k_1} \cdots \pi_t^{k_t}).$$

Example 18.1. In the setting of our previous example in $\mathbb{P}^2$, $D = 2C - L_1$ has degree 3, so it is *not* globally principal.

However, D is locally principal. Let

$$\varphi_1 = \frac{(x^2+y^2-z^2)^2}{x^4}, \qquad \varphi_2 = \frac{(x^2+y^2-z^2)^2}{xy^3}, \qquad \varphi_3 = \frac{(x^2+y^2-z^2)^2}{xz^3}.$$

Then

$$\operatorname{div}_{U_x} \varphi_1 = D \cap U_x, \qquad \operatorname{div}_{U_y} \varphi_2 = D \cap U_y, \qquad \operatorname{div}_{U_z} \varphi_3 = D \cap U_z.$$

Remark 18.2. On $U_x \cap U_y$, φ_1 and φ_2 have the *same* divisor C

$$\iff \operatorname{div}_{U_x \cap U_y} \varphi_1 = \operatorname{div}_{U_x \cap U_y} \varphi_2 \iff \operatorname{div}_{U_x \cap U_y}(\varphi_1/\varphi_2) = 0 \iff \frac{\varphi_1}{\varphi_2} \in \mathcal{O}_X^*(U_x \cap U_y).$$

Now we give the formal definition.

Definition 18.3. A *locally principal* (or *Cartier*) divisor on a variety X is described by the following data:

- $\{U_\lambda\}_{\lambda \in \Lambda}$ open cover of X,
- $\varphi_\lambda \in k(X) = k(U_\lambda)$ rational function on X

such that $\varphi_\lambda \cdot \varphi_\mu^{-1} \in \mathcal{O}_X^*(U_\lambda \cap U_\mu)$ for all $\lambda, \mu \in \Lambda$.

The corresponding (Weil[9]) divisor is the unique D such that on U_x, $D \cap U_\lambda = \operatorname{div}_{U_\lambda} \varphi_\lambda \; \forall \lambda$.

The set of all locally principal divisors on X forms a group $\operatorname{CDiv}(X) \subseteq \operatorname{Div}(X)$.

Remark 18.4. If $D_1 = \{U_\lambda, \varphi_\lambda\}$ and $D_2 = \{U_\mu, \psi_\mu\}$ are two collections of data describing two Cartier divisors, then their sum $D_1 + D_2$ is given by $\{U_\lambda \cap U_\mu, \varphi_\lambda \cdot \psi_\mu\}$.

Remark 18.5. The main advantage to locally principal divisors is that they can be pulled back under dominant regular morphisms.

Say $X \xrightarrow{f} Y$ is a dominant regular morphism, so we can identify $k(Y) \subseteq k(X)$ by f^*. So for $D \in \operatorname{CDiv}(Y)$, define f^*D as the Cartier divisor X whose local defining equations are the pullbacks of local defining equations for D.

In symbols, if $D = \{U_\lambda, \varphi_\lambda\}$, then

$$f^*D = \left\{f^{-1}(U_\lambda), f^*(\varphi_\lambda)\right\} = \left\{f^{-1}(U_\lambda), \varphi_\lambda \circ f\right\}.$$

[9] A *Weil divisor* is a formal $\mathbb{Z}$-linear combination of irreducible, codimension 1 subvarieties. This is the same kind of divisor we defined earlier.

18.2 The Picard group

Let X be a normal variety. Then we have

$$P(X) \subseteq \mathrm{CDiv}(X) \subseteq \mathrm{WDiv}(X) \stackrel{\text{def}}{=} \mathrm{Div}(X).$$

Definition 18.6. The *divisor class group* of X is $\mathrm{Cl}(X) = \mathrm{Div}(X)/P(X)$.

The *Picard group* of X is $\mathrm{Pic}(X) = \mathrm{CDiv}(X)/P(X)$.

18.3 Summary of locally principal divisors

Let D be a locally principal divisor on X (normal).

Then D is given by data $\{U_\lambda, \varphi_\lambda\}$, where the U_λ are open sets covering X and $\varphi \in k(X)^*$, and D is $\operatorname{div} \varphi_\lambda$ on U_λ:

$$D \cap U_\lambda = \operatorname{div}_{U_\lambda} \varphi_\lambda.$$

Example 18.7. $D =$ hyperplane $\mathbb{V}(x_0)$ on $X = \mathbb{P}^3$. This is *not* principal.

However, it is locally principal, being given by $\left\{\left(U_i, \frac{x_0}{x_i}\right)\right\}_{i=1}^{4}$.

Note 18.8. (1) The φ_λ are uniquely determined only up to multiplication by some φ having *no* zeros or poles on U_λ, or equivalently, any of the following:

- $\operatorname{div} \varphi = 0$
- $\varphi \in \mathcal{O}_X^*(U)$
- φ is a unit in $\mathcal{O}_X(U_\lambda)$.

(2) There is a relationship between φ_λ and φ_μ given by any of the following:

- $\operatorname{div} \varphi_\lambda = \operatorname{div} \varphi_\mu$ on $U_\lambda \cap U_\mu$
- $\operatorname{div} \varphi_\lambda - \operatorname{div} \varphi_\mu = 0$ on $U_\lambda \cap U_\mu$
- $\operatorname{div}(\varphi_\lambda/\varphi_\mu) = 0$ on $U_\lambda \cap U_\mu$.

(Or, if we don't assume X is normal, $\varphi_i/\varphi_j \in \mathcal{O}_X^*(U_i \cap U_j)$.)

18.4 Pulling back locally principal divisors

18.4.1 Case 1

Let $Y \xrightarrow{f} X$ be a *dominant* regular map.

Given $D \in \mathrm{CDiv}(X) =$ set of all locally principal divisors on X, think of D as given by $\{U_\lambda, \varphi_\lambda\}$. Then f^*D is given by $\{f^{-1}(U_\lambda), f^*(\varphi_\lambda)\}$. Then we think of f^*D as $\operatorname{div}(f^*\varphi_\lambda)$ on $f^{-1}(U_\lambda)$.

Note 18.9. Each $f^*\varphi_\lambda$ is a *nonzero* rational function on Y.

Note 18.10. $\mathrm{Supp}(f^*D) = f^{-1}(\mathrm{Supp}\, D)$.

Example 18.11. Let $V = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$, and consider $V \to \mathbb{A}^1,\ (x, y) \mapsto y$. Consider the divisor

$$D = 2p_1 - 3p_2 = \operatorname{div}\left(\frac{(t-1)^2}{(t-2)^3}\right) \in \operatorname{CDiv}(\mathbb{A}^1),$$

where $p_1 = 1$ and $p_2 = 2$ in $\mathbb{A}^1$. Then

$$\begin{aligned} f^*(D) &= \operatorname{div}_V f^*\left(\frac{(t-1)^2}{(t-2)^3}\right) = \operatorname{div}_V \frac{f^*(t-1)^2}{f^*(t-2)^3} = \operatorname{div}_V \frac{(t \circ f - 1)^2}{(t \circ f - 2)^3} \\ &= \operatorname{div}_V \frac{(y-1)^2}{(y-2)^3} = \operatorname{div}_V \frac{(x^2-1)^2}{(x^2-2)^3} = 2q_1 + 2q_1' - 3q_2 - 3q_2', \end{aligned}$$

where

$$\begin{aligned} q_1 &= (1, 1), & q_1' &= (-1, 1), \\ q_2 &= (\sqrt{2}, 2), & q_2' &= (-\sqrt{2}, 2). \end{aligned}$$

Note 18.12. $Y \xrightarrow{f} X$ is dominant $\iff$ on affine charts (say X, Y affine),

$$\begin{aligned} k[Y] &\leftarrow k[X] \\ g \circ f &\leftarrow\!\shortmid g \end{aligned}$$

is *injective.*

Think: $Y \xrightarrow{f} X$ yields a map $(\mathcal{O}_X \xrightarrow{f^*} \mathcal{O}_Y) = f^*\mathcal{O}_Y$, and the kernel is an ideal sheaf $\mathscr{I}_f$.

In the affine case, $Y \xrightarrow{f} X$ induces a map

$$k[X] \xrightarrow{f^*} k[Y]$$

with kernel I, and we have

$$\begin{tikzcd} k[Y] & k[X] \arrow[l, "f^*"'] \arrow[d, two heads] \\ & k[X]/I \arrow[ul] \end{tikzcd} \iff \begin{tikzcd} Y \arrow[r, "f"] \arrow[dr] & X \\ & W \arrow[u, hook] \end{tikzcd}$$

Example 18.13.

$$\begin{aligned} \mathbb{P}^1 &\xrightarrow{\nu} \mathbb{P}^3 \\ [s : t] &\mapsto \left[s^3 : s^2t : st^2 : t^3\right] \\ \left[\frac{s}{t} : 1\right] &\mapsto \left[\left(\frac{s}{t}\right)^3 : \left(\frac{s}{t}\right)^2 : \left(\frac{s}{t}\right) : 1\right]. \end{aligned}$$

Let $H = \mathbb{V}(x_0)$, corresponding to

$$\left\{(U_0, 1), \left(U_i, \frac{x_0}{x_i}\right)\right\}.$$

Can we pull back H under ν?

The pullback ν^*H is given by

$$\left\{\left(\nu^{-1}U_0, 1\right), \left(\nu^*U_3, \nu^*\left(\frac{x_0}{x_3}\right) = \left(\frac{s}{t}\right)^3\right)\right\},$$

so

$$\nu^*H = 3 \cdot P,$$

where $P = [0:1] \in \mathbb{P}^1$.

18.4.2 Case 2

Proposition 18.14. *If $Y \xrightarrow{f} X$ is a regular map, and $D \in \mathrm{CDiv}(X)$ such that $f(Y) \nsubseteq \operatorname{Supp} D$, then f^*D is defined exactly as before: If D is given by $\{U_\lambda, \varphi_\lambda\}$, then f^*D is given by*

$$\left\{f^{-1}(U_\lambda), f^*\varphi_\lambda\right\},$$

where the $f^*\varphi_\lambda$ are nonzero rational functions.

Proof. We have $f(Y) \nsubseteq \operatorname{Supp}(D) \iff Y \nsubseteq f^{-1}(\operatorname{Supp} D)$. Since $\operatorname{Supp} D$ consists of the zeros and poles of $\frac{h_\lambda}{g_\lambda} = \varphi_\lambda$ on U_λ, i.e., (zeros of h_λ) $\cup$ (zeros of g_λ). Then $f^{-1}(\operatorname{Supp} D)$ is the set of zeros of $(h_\lambda \circ f)$ and $(g_\lambda \circ f)$. $\square$

Example 18.15. Let $V = \mathbb{V}(y - x^2) \overset{f}{\subseteq} \mathbb{A}^2$ and $D = X - Y = \mathbb{V}(x) - \mathbb{V}(y) = \operatorname{div}\left(\frac{x}{y}\right)$ on $\mathbb{A}^2$. Then

$$f^*D = \operatorname{div} \frac{f^*(x)}{f^*(y)} = \operatorname{div} \frac{x}{y} = \operatorname{div} \frac{x}{x^2} = \operatorname{div} \frac{1}{x}.$$

We have $f^*D = f^*X - f^*Y$.

18.5 The Picard group functor

Theorem 18.16. *Let $X \xrightarrow{\varphi} Y$ be a regular map of varieties. There is a naturally induced (functorial) group homomorphism $\operatorname{Pic} Y \xrightarrow{\varphi^*} \operatorname{Pic} X$.*

In other words, there is a contravariant functor

$$\begin{aligned}\{\textit{varieties over } k\} &\to \mathbf{Ab}\\ X &\mapsto \operatorname{Pic} X.\end{aligned}$$

Example 18.17. The morphism

$$\begin{aligned}\mathbb{P}^1 &\xrightarrow{\nu} \mathbb{P}^3\\ [s:t] &\mapsto \left[s^3 : s^2t : st^2 : t^3\right]\end{aligned}$$

yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(\mathbb{P}^1) & \mathrm{Pic}(\mathbb{P}^3) \arrow[l] \\
\mathbb{Z}\cdot[p] \arrow[u, equal] \arrow[d, leftrightarrow, "\simeq"'] & \mathbb{Z}\cdot[H] \arrow[u, equal] \arrow[d, leftrightarrow, "\simeq"] \\
\mathbb{Z} & \mathbb{Z} \arrow[l, "3 \leftarrow\!\shortmid 1"']
\end{tikzcd}$$

Example 18.18. The d-th Veronese map $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$ induces

$$\begin{aligned} \mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^m) &\leftarrow \mathrm{Pic}(\mathbb{P}^N) = \mathbb{Z} \\ d &\mapsfrom 1. \end{aligned}$$

18.6 Moving lemma

Lemma 18.19. *Given any X, a Cartier divisor D on X, and a point $x \in X$, there exists a Cartier divisor D' such that $D \sim D'$ and $x \notin \operatorname{Supp} D$.*

Example 18.20. On $\mathbb{P}^2$, take $x = [1:0:0]$ and $D = H = \mathbb{V}(y)$. Note that $x \in \operatorname{Supp} D$.

By the moving lemma, there exists a divisor $D' \sim H$ such that $[1:0:0] \notin D'$. We can take $D' = \mathbb{V}(x)$. Here: $D' = D + \operatorname{div}\left(\frac{x}{y}\right)$.

Proof of moving lemma. Say D is given by data $\{U_i, \varphi_i\}$. Say $x \in U_1$.

Let D' be the divisor corresponding to data $\left\{U_i, \varphi_1^{-1}\cdot\varphi_i\right\}$. [Note: $D' \cap U_1 = \operatorname{div}_{U_1}(1)$ is empty, so $x \notin \operatorname{Supp} D'$.] Hence

$$D' = D + \operatorname{div}_x \varphi^{-1}.$$

□

Proof of Theorem 18.16. Let $X \xrightarrow{\varphi} Y$ be a morphism and D a locally principal divisor. We can define $\varphi^* D$ whenever $\operatorname{Supp} D \not\supseteq \varphi(X)$. Then we need to check also:

(1) $D_1 \sim D_2 \implies \varphi^* D_1 \sim \varphi^* D_2$

(2) $\varphi^*(D_1) + \varphi^*(D_2) = \varphi^*(D_1 + D_2)$

when we can define φ^*.

So: if we try to define $\varphi^*[D]$ where $\operatorname{Supp} D \supseteq \operatorname{im}\varphi$, simply use the moving lemma to replace D by D', where $x \notin \operatorname{Supp} D'$ (for any x we pick in φ). □

19 Riemann–Roch spaces and linear systems

19.1 Riemann–Roch spaces

Fix X normal, D any divisor. Consider the set

$$\mathscr{L}(D) = \left\{ f \in k(X)^* \;\middle|\; \operatorname{div}_X f + D \geq 0 \right\} \cup \{0\} \subseteq k(X).$$

Example 19.1. If $X = \mathbb{A}^1$ and $D = 2 \cdot p_0$ (where $p_0 = \mathbf{0}$ is the origin), then

$$\mathscr{L}(D) = \left\{ f \in k(t)^* \;\middle|\; \operatorname{div} f + 2p_0 \geq 0 \right\} \cup \{0\} = \left\{ \frac{1}{t^2} g(t) \;\middle|\; g(t) \in k[t] \right\}.$$

A function $f \in \mathscr{L}(D)$ can have zeros anywhere, but can't have any poles except at p_0, where a pole can be order 2 or less.

Definition 19.2. $\mathscr{L}(D)$ is the *Riemann–Roch space* of (X, D).

Remark 19.3. (I) $\mathscr{L}(D)$ is a vector space over k.

(II) Even better, $\mathscr{L}(D)$ is a module over $\mathcal{O}_X(X)$.
The proof follows from a basic fact about "order of vanishing" along prime divisors.

If D_i is a *prime* divisor on normal X, then

$$\nu_{D_i} : k(X)^* \to \mathbb{Z}$$

is a *valuation*, i.e.:

(I) $\nu_{D_i}(f + g) \geq \min\{\nu_{D_i}(f), \nu_{D_i}(g)\}$

(II) $\nu_{D_i}(fg) = \nu_{D_i}(f) + \nu_{D_i}(g)$.

To prove $\mathscr{L}(D)$ is a vector subspace of $k(X)$, observe that

$$f, g \in \mathscr{L}(D) \implies f + g \in \mathscr{L}(D),$$

and

$$\begin{aligned} \operatorname{div} f + D &\geq 0 \\ D + \sum_{D_i} \nu_{D_i}(g) \cdot D_i = \operatorname{div} g + D &\geq 0, \end{aligned}$$

hence $\operatorname{div}_X(f + g) \geq -D$, so if

$$D = \sum_{\substack{D_i \subseteq X \\ \text{prime}}} k_i D_i,$$

then for any D_i prime divisor,

$$\begin{aligned} \nu_{D_i}(f) &\geq -k_i \\ \nu_{D_i}(g) &\geq -k_i. \end{aligned}$$

Thus

$$\nu_{D_i}(f + g) \geq \min\{\nu_{D_i}(f), \nu_{D_i}(g)\} \geq -k_i \qquad \forall i,$$

whence

$$\operatorname{div}_X(f + g) \geq -D,$$

so $f + g \in \mathscr{L}(D)$. $\square$

Theorem 19.4. *If X is projective, then $\mathscr{L}(D)$ is a finite-dimensional vector space over k.*

Example 19.5. Say $D = 0$ and

$$\mathscr{L}(D) = \left\{ f \in k(x) \;\middle|\; \operatorname{div} f \geq 0 \right\} = \mathcal{O}_X(X).$$

If X is projective, then $\mathscr{L}(0)$ has dimension 1.

Denote $p_0 = [0 : 1]$ and $p_\infty = [1 : 0]$. Let $X = \mathbb{P}^1$ and $D = p_0 + p_\infty$. We have $k(\mathbb{P}^1) = k\left(\frac{x}{y}\right)$, and then

$$\begin{aligned} \mathscr{L}(D) &= \left\{ f\left(\frac{x}{y}\right) \;\middle|\; \operatorname{div} f + p_0 + p_\infty \geq 0 \right\} \\ &= \left\{ \frac{F_2(x,y)}{xy} \;\middle|\; F_2 \text{ degree 2 homogeneous} \right\}. \end{aligned}$$

A basis for this is

$$\left\{ \frac{x^2}{xy}, \frac{xy}{xy}, \frac{y^2}{xy} \right\} = \left\{ \frac{x}{y}, 1, \frac{y}{x} \right\},$$

so $\dim \mathscr{L}(D) = 3$.

19.2 Riemann–Roch spaces, continued

Let X be a normal variety, $D = \sum k_i D_i$ a divisor. The Riemann–Roch space

$$\mathscr{L}(D) = \left\{ f \in k(X)^* \;\middle|\; \operatorname{div} f + D \geq 0 \right\} \cup \{0\} \subseteq k(X)$$

consists of rational functions f such that

(1) f has no poles except possibly along D_i if $k_i > 0$ (order of pole up to $-k_i$), and

(2) f must have zeros along D_i if $k_i < 0$ (order of zero at least $-k_i$).

Remark 19.6.

- $\mathscr{L}(D)$ can be infinite-dimensional *or* finite-dimensional, though it is always finite-dimensional if X is projective.
- $\mathscr{L}(D)$ is a module over $\mathcal{O}_X(X)$.

Proposition 19.7. *If $D \sim D'$, then $\mathscr{L}(D) \cong \mathscr{L}(D')$ (natural isomorphism, not equality).*

Proof. We have $D - D' = \operatorname{div} f$ for some $f \in k(X)^*$. Consider

$$\begin{gathered} \left\{ g \;\middle|\; \operatorname{div} g + D \geq 0 \right\} = \mathscr{L}(D) \xrightarrow{\cdot f} \mathscr{L}(D') = \left\{ h \;\middle|\; \operatorname{div} h + D' \geq 0 \right\} \\ g \mapsto gf. \end{gathered}$$

Is $gf \in \mathscr{L}(D')$? Indeed, if $g \in \mathscr{L}(D)$, then $\operatorname{div} g + D \geq 0$, so

$$\operatorname{div}(gf) + D' = \operatorname{div} g + \operatorname{div} f + D' = \operatorname{div} g + D \geq 0.$$

The inverse map is multiplication by $\frac{1}{f}$. Thus, this is an isomorphism of k-vector spaces. (It is also a $\mathcal{O}_X(X)$-module isomorphism.) $\square$

Note 19.8. Each nonempty open set $U \subseteq X$ is a normal variety. Each divisor $D = \sum k_i D_i$ on X induces a divisor

$$D\big|_U = \sum_i k_i(D_i \cap U) = \text{“}D_i \cap U\text{”}.$$

Look at the Riemann–Roch space of $(U, D|_U)$.

Definition 19.9 (sheaf associated to D)**.** The sheaf $\mathcal{O}_X(D)$ associated to D is the sheaf assigning to each nonempty open set $U \subseteq X$ the Riemann–Roch space

$$\mathcal{O}_X(D)(U) = \text{the Riemann–Roch space of } \left(U, D\big|_U\right),$$

which is an $\mathcal{O}_X(U)$-module.

- This is a subsheaf of the constant sheaf $k(X)$.
- $\mathcal{O}_X(D)$ is a sheaf of $\mathcal{O}_X$-modules.
- If $D \sim D'$, then there is an isomorphism

  $$\mathcal{O}_X(D) \xrightarrow{\cdot f} \mathcal{O}_X(D')$$

  of $\mathcal{O}_X$-modules.

Example 19.10. If $D = 0$, then $\mathcal{O}_X(D) = \mathcal{O}_X$.

Example 19.11. Let $X = \mathbb{P}^1$ and $D = 2p_0 - p_\infty$ (where $p_0 = [0:1]$ and $p_\infty = [1:0]$). Then

$$\begin{aligned}\mathcal{O}_X(D)(\mathbb{P}^1) &= \left\{ f \in k(\mathbb{P}^1) \;\middle|\; \operatorname{div} f + 2p_0 - p_\infty \geq 0 \right\} \\ &= \left\{ \frac{y(ax+by)}{x^2} \;\middle|\; a, b \in k \right\}.\end{aligned}$$

If we restrict to $U_\infty = \mathbb{P}^1 \setminus \{[1:0]\}$, then using coordinates $t = \frac{x}{y}$,

$$\begin{aligned}\mathcal{O}_X(D)(U_\infty) &= \left\{ f \in k(\mathbb{P}^1) \;\middle|\; \operatorname{div}_{U_\infty} f + 2p_0 \geq 0 \right\} \\ &= \left\{ \frac{g}{t^2} \;\middle|\; g \in k[t] \right\}.\end{aligned}$$

Similarly, letting $s = \frac{y}{x} = t^{-1}$,

$$\begin{aligned}\mathcal{O}_X(D)(U_0) &= \left\{ f \in k(\mathbb{P}^1) \;\middle|\; \operatorname{div} f - p_\infty \geq 0 \right\} \\ &= \left\{ f \in k(s) \;\middle|\; f \in s \cdot k[s] \right\} \\ &= \left\{ t^{-1} \cdot k[t^{-1}] \right\} \cong \mathcal{O}_X(U_0),\end{aligned}$$

and

$$\mathcal{O}_X(D)(U_\infty \cap U_0) = \mathcal{O}_X(U_\infty \cap U_0) = k[t, t^{-1}].$$

Fact 19.12. If D is a *Cartier* divisor, then $\mathcal{O}_X(D)$ is a locally free, rank 1 $\mathcal{O}_X$-module (a submodule of $k(X)$).

Hint: If D is given by data $\{U_i, \varphi_i\}$, then

$$\mathcal{O}_X(D)(U_i) = \varphi_i^{-1} \cdot \mathcal{O}_X(U_i) \subseteq k(X).$$

19.3 Complete linear systems

Let X be a normal variety, $D = \sum k_i D_i$ a divisor.

Definition 19.13. The *complete linear system* $|D|$ is the *set* of all effective divisors D' on X such that $D \sim D'$.

Example 19.14. On $\mathbb{P}^2$ ($\operatorname{char} k \neq 3$), let

$$D = 3\mathbb{V}(x^3 + y^3 + z^3) - 7\mathbb{V}(x).$$

Then $|D| =$ the set of all *conics* on $\mathbb{P}^2$.

Proposition 19.15. *There is a natural map*

$$\begin{aligned} \mathscr{L}(D) - \{0\} &\to |D| \\ f &\mapsto \operatorname{div} f + D \end{aligned}$$

which induces a surjective map $\mathbb{P}(\mathscr{L}(D)) \twoheadrightarrow |D|$ *which is bijective if* X *is projective.*

Proof. Why surjective? If $D' \in |D|$, then $D' \geq 0$ and $D' \sim D$, i.e., $D' = D + \operatorname{div} f$ for some $f \in k(X)^*$. So

$$f \mapsto \operatorname{div} f + D = D'.$$

Why injective for projective X? Say $D_1, D_2 \in |D|$ such that

$$f, g \mapsto \operatorname{div} f + D.$$

Then $\operatorname{div}(f/g) = 0$, so $\frac{f}{g}$ is regular on X and hence is constant. □

19.4 Some examples

Example 19.16 (Case where the map is not injective). Consider $X = \mathbb{A}^1 - \{0\}\,,\ D = p = [1]$. Then

$$\mathscr{L}(D) = \left\{ f \in k(t) \;\middle|\; \operatorname{div} f + p \geq 0 \right\} = \frac{1}{(t-1)} \cdot k[t, t^{-1}],$$

and the natural map $\mathbb{P}(\mathscr{L}(D)) \to |D|$ is not injective.

Example 19.17. Let $L \subseteq \mathbb{P}^2$ be a line. Say $L = \mathbb{V}(x_0) \subseteq \mathbb{P}^2$. Then

$$\begin{aligned} |L| &= \left\{ \text{lines on } \mathbb{P}^2 \right\} \\ &= \mathbb{P}(\mathscr{L}(L)) = \mathbb{P}\left\{ f \in k(\mathbb{P}^2) \;\middle|\; \operatorname{div} f + L \geq 0 \right\} = \mathbb{P}\left\{ \frac{a_0 x_0 + a_1 x_1 + a_2 x_2}{x_0} \;\middle|\; a_i \in k \right\}. \end{aligned}$$

Note that $|L|$ is geometric, independent of choices, while $\mathscr{L}(L)$ depends on choice of line; if we choose a different line, we get a different (but isomorphic) subset of $k(\mathbb{P}^2)$.

Example 19.18. Let $C \subseteq \mathbb{P}^2$ be the conic $\mathbb{V}(F)$, where $F = x^2 + y^2 - z^2$. Then

$$\begin{aligned} \mathscr{L}(C) &= \left\{ f \in k(\mathbb{P}^2) \;\middle|\; \operatorname{div} f + C \geq 0 \right\} \\ &= \left\{ \frac{G(x,y,z)}{(x^2+y^2-z^2)} \;\middle|\; G \in [k[x,y,z]]_2 \right\}. \end{aligned}$$

This is a dimension 6 vector space. Basis:

$$\left\{ \frac{x^2}{F}, \frac{xy}{F}, \frac{y^2}{F}, \frac{xz}{F}, \frac{z^2}{F}, \frac{yz}{F} \right\}.$$

Map this to the linear system:

$$\begin{aligned} \mathscr{L}(C) &\to |C| = \left\{ \text{conics on } \mathbb{P}^2 \right\} \\ \frac{G}{F} &\mapsto \operatorname{div} \frac{G}{F} + C = \mathbb{V}(G) && \text{(as a scheme)} \end{aligned}$$

The linear system $|C|$ of conics on $\mathbb{P}^2$ corresponds to a map to projective space (up to choice of coordinates on that target):

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^5 \\ [x:y:z] &\mapsto \left[\frac{x^2}{F} : \frac{xy}{F} : \frac{y^2}{F} : \frac{xz}{F} : \frac{z^2}{F} : \frac{yz}{F} \right]. \end{aligned}$$

This is the Veronese 2-map.

Note that if we denote $L = \mathbb{V}(x)$, then $|C| = |2L|$, and the corresponding Riemann–Roch space is

$$\mathscr{L}(2L) = \left\{ \frac{G}{x^2} \;\middle|\; G \in [k[x,y,z]]_2 \right\},$$

which has a basis

$$\left\{ 1, \frac{y}{x}, \left(\frac{y}{x}\right)^2, \ldots, \frac{y^2}{x^2} \right\},$$

which is also dimension 6.

Note 19.19. The elements of the linear system $|C| = |2L|$ are the *pullbacks* of the *hyperplanes* in $\mathbb{P}^5$.

Multiplying by F, we can also describe this map as

$$\begin{aligned} \mathbb{P}^2 &\xrightarrow{\nu_2} \mathbb{P}^5 \\ [x:y:z] &\mapsto \left[x^2 : xy : y^2 : xz : z^2 : yz \right]. \end{aligned}$$

Look at the linear system $|H|$ on $\mathbb{P}^5$ of hyperplanes. Say

$$H = \mathbb{V}(a_0 x_0 + \cdots + a_5 x_5).$$

Then

$$\nu_2^* H = \mathbb{V}(a_0 x^2 + a_1 xy + \cdots + a_5 yz).$$

19.5 Linear systems

Definition 19.20. A *linear system* on X is a set of divisors (all effective, all linearly equivalent to each other) which corresponds to some (projective) linear space in some complete linear system $|D|$.

In other words: Fix D, and consider a subspace

$$V \subseteq \mathscr{L}(D) \twoheadrightarrow |D|\,.$$

Then we have a map $V \twoheadrightarrow \mathbb{P}(V) \subseteq |D|$. The image of $\mathbb{P}(V)$ is a linear system.

Example 19.21. In $\mathbb{P}^n$, take the set of lines through a point $p = [0 : \cdots : 0 : 1] \in \mathbb{P}^n$. Fix $H = \mathbb{V}(x_n)$. Call this set

$$\mathcal{V} = \mathbb{P}(V) = \left\{ f \;\middle|\; \operatorname{div} f + H \geq 0 \right\}.$$

Then

$$V = \left\langle \text{span of } \frac{x_0}{x_n}, \ldots, \frac{x_{n-1}}{x_n} \right\rangle \subseteq \mathscr{L}(H) = \left\langle \frac{x_0}{x_n}, \frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, 1 \right\rangle.$$

Definition 19.22. The *base locus* of a linear system $\mathcal{V}$ is the set

$$\operatorname{Bs} \mathcal{V} = \left\{ x \in X \;\middle|\; x \in \operatorname{Supp} D \;\forall D \in \mathcal{V} \right\}.$$

A linear system is *base point free* if $\operatorname{Bs} \mathcal{V} = \varnothing$.

The *fixed components* of a linear system are prime divisors D such that D appears in the support of every $D \in \mathcal{V}$ (i.e., divisors in the base locus).

Example 19.23. Fix $L_1 = \mathbb{V}(x) \subseteq \mathbb{P}^2$. Take the linear system $\mathcal{V}$ of conics in $\mathbb{P}^2$ which contain L_1. This consists of the unions of L_1 with another line, and the double line consisting of L_1 with multiplicity 2.

We have

$$\begin{aligned} |2L_2| \supseteq \mathcal{V} &\longleftrightarrow |L| \\ L_1 + L_2 &\longleftrightarrow L_2. \end{aligned}$$

A conic $C \subseteq \mathbb{P}^2$ contains $L_1 = \mathbb{V}(x)$ iff

$$I_C = (F) = (ax + by + cz)x \subseteq I_{L_1} = (x).$$

A basis for $\mathcal{F}$ is given by

$$\frac{x^2}{F}, \frac{xy}{F}, \frac{zx}{F}.$$

Map to projective space by

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2 \\ [x : y : z] &\mapsto \left[\frac{x^2}{F} : \frac{xy}{F} : \frac{xz}{F} \right] = [x : y : z], \end{aligned}$$

i.e., the identity map.

19.6 Linear systems and rational maps

Theorem 19.24. *Let X be normal (in practice, projective). There is a one-to-one correspondence*

$$\frac{\{\text{rational maps } X \dashrightarrow \mathbb{P}^n\}}{(\text{projective change of coordinates})} \longleftrightarrow \left\{ \begin{array}{c} n\text{-dimensional linear systems of divisors} \\ \text{on } X \text{ with no fixed component} \end{array} \right\}$$

$$\left[X \overset{\varphi}{\dashrightarrow} \mathbb{P}^n\right] \mapsto \{\text{pullback of hyperplane linear systems on } \mathbb{P}^n\}.$$

Example 19.25. Consider the map

$$\begin{aligned} \mathbb{P}^1 &\xrightarrow{\nu} \mathbb{P}^3 \\ [s:t] &\mapsto \left[s^3 : s^2 t : st^2 : t^3\right] \end{aligned}$$

and the linear system

$$|H| = \left\{\text{hyperplanes on } \mathbb{P}^3\right\} = \left\{\mathbb{V}(ax+by+cz+dw) \;\middle|\; [a:b:c:d] \in \mathbb{P}^3\right\}.$$

Then

$$\begin{aligned} \nu^* |H| &= \left\{\nu^*\big(\mathbb{V}(ax+by+cz+dw)\big) \;\middle|\; [a:b:c:d] \in \mathbb{P}^3\right\} \\ &= \left\{\mathbb{V}(as^3+bs^2t+cst^2+dt^3)\right\} \\ &= \left\{\text{complete linear system on } \mathbb{P}^1 \text{ of degree 3 divisors}\right\} = |3P|. \end{aligned}$$

Going back to the theorem, for any n-dimensional linear system $\mathcal{V}$ of divisors on X with no fixed component, let $|D|$ be a complete linear system such that $\mathcal{V} \subseteq |D|$. Then $\mathcal{V} = \mathbb{P}(V)$, where $V \subseteq \mathscr{L}(D)$ is $(n+1)$-dimensional. Send

$$\mathcal{V} \mapsto \left[\begin{array}{l} X \dashrightarrow \mathbb{P}^n \\ \;x \mapsto [\varphi_0(x) : \cdots : \varphi_n(x)] \end{array}\right],$$

where the φ_i are a basis for V.

Furthermore: the locus of indeterminacy of φ is the base locus of $\mathcal{V}$.

Example 19.26. In $\mathbb{P}^2$, fix a line L. Look at the linear system $\mathcal{W}_L \subseteq |C_3|$ (where $|C_3|$ is the 9-dimensional complete linear system of cubics in $\mathbb{P}^2$) of cubics that contain L. We have

$$L \subseteq C_3 \iff F_3 = x \cdot F_2,$$

where $F_2(x,y,z)$ is degree 2. So

$$\mathscr{L}(C_3) = \left\langle \frac{x^3}{F_3} : \frac{x^2y}{F_3} : \cdots : \frac{z^3}{F_3} \right\rangle \supseteq \left\{ \frac{x \cdot x^2}{F_3} : \frac{x \cdot xy}{F_3} : \frac{x \cdot xz}{F_3} : \frac{x \cdot y^2}{F_3} : \frac{x \cdot yz}{F_3} : \frac{x \cdot z^2}{F_3} \right\}.$$

What is the map $\varphi_{\mathcal{W}_L}$ corresponding to $\mathcal{W}_L$? It is

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^5 \\ [x:y:z] &\mapsto \left[\frac{x^3}{F_3} : \frac{x^2y}{F_3} : \cdots : \frac{xz^2}{F_3}\right] = \left[x^2 : xy : \cdots : z^2\right]. \end{aligned}$$

Note that $\mathcal{W}_L$ gives the same map as $|C_2|$.

Note 19.27. Let $X \dashrightarrow \varphi\mathbb{P}^n$ and $D \in \mathrm{Div}(\mathbb{P}^n)$. What is $\varphi^* D$? We have

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi"] & \mathbb{P}^n \\ U \arrow[u, phantom, "\cup|"] \arrow[ur, "\varphi_U"'] & \end{tikzcd}$$

and $X \setminus U$ has codimension ≥ 2. Then

$$\varphi^* D \stackrel{\text{def}}{=} \overline{\varphi_U^* D},$$

the unique divisor D' on X such that $D'|_U = (\varphi_n^* D)$.

Example 19.28. In general, the Veronese map $\mathbb{P}^n \xrightarrow{\nu_d} \mathbb{P}^{\binom{n+d}{d}-1}$ corresponds to the complete linear system $|dH|$ on $\mathbb{P}^n$.

Definition 19.29. A divisor D is *very ample* if the map $\varphi_{|D|} : X \dashrightarrow \mathbb{P}^n$ corresponding to the complete linear system $|D|$ is an embedding.

A divisor D is *ample* if $\exists m \in \mathbb{N}$ such that mD is very ample.

Example 19.30. Consider the projection

$$\mathbb{P}^3 \overset{\varphi}{\dashrightarrow} \mathbb{P}^2$$
$$[x : y : z : w] \mapsto [x : y : z]$$

from $p = [0 : 0 : 0 : 1]$. Let $H = \mathbb{V}(ax + by + cz) \in |H|$. Then hyperplanes H correspond to hyperplanes on $\mathbb{P}^3$ which contain p, i.e.,

$$|H_p| = \text{linear system on } \mathbb{P}^3 \text{ of hyperplanes through } p.$$

This is fixed component free, since the base locus is $\{p\}$, the locus of indeterminacy of φ.

Example 19.31. Let $\widetilde{\mathbb{P}}^2 \xrightarrow{\pi} \mathbb{P}^2$ be the blowup at a point $p \in \mathbb{P}^2$.

This corresponds to the linear system $\pi^* |L|$ (where $|L|$ is the complete linear system of lines on $\mathbb{P}^2$), which includes "lines" L which don't meet the exceptional divisor E.

This is base point free, but not very ample.

20 Differential forms

20.1 Sections

Recall from the homework: The *tautological bundle* is

$$T = \{(x, \ell) \mid x \in \ell\} \subseteq k^{n+1} \times \mathbb{P}^n$$

with the projection map $T \xrightarrow{\pi} \mathbb{P}^n$. The fiber

$$\pi^{-1}(\ell) = \{(x, \ell) \mid x \in \ell\}$$

is the set of points in the line which is ℓ.

A *section* is a morphism $\mathbb{P}^n \xrightarrow{s} T$ such that $\pi \circ s = \mathrm{id}|_{\mathbb{P}^n}$. A section of the tautological bundle is given by a choice of representative of each line, i.e., for all $\ell \in \mathbb{P}^n$, $s(\ell) \in \pi^{-1}(\ell)$.

We can add two sections $s_1, s_2 : \mathbb{P}^n \to T$ by adding outputs:

$$\begin{aligned} s_1 + s_2 : \mathbb{P}^n &\to T \\ \ell &\mapsto s_1(\ell) + s_2(\ell). \end{aligned}$$

We can also multiply a section $s : \mathbb{P}^n \to T$ by any function $f : \mathbb{P}^n \to k$:

$$\begin{aligned} fs : \mathbb{P}^n &\to T \\ fs(\ell) &= f(\ell)s(\ell) \in \pi^{-1}(\ell). \end{aligned}$$

20.2 Differential forms

Definition 20.1. A *differential form* ψ on X is an assignment associating to each $x \in X$ some $\psi(x) \in (T_x X)^*$.

Put differently, a differential form is a section of the cotangent bundle of X.

Example 20.2. If f is a regular function on X, then df is a differential form:

$$df(x) = d_x f = \sum_{i=1}^{n} \left.\frac{\partial f_i}{\partial x_i}\right|_x \big(x - x_i(x)\big)\big|_{T_x X \subseteq T_x \mathbb{A}^n}.$$

We can add two differential forms:

$$(\psi_1 + \psi_2)(x) = \psi_1(x) + \psi_2(x).$$

Can also multiply ψ by any k-valued function φ:

$$(\varphi\psi)(x) = \varphi(x) \cdot \psi(x).$$

In other words, the set of all differential forms $\Psi[x]$ on X forms a module over $\mathfrak{F}(x)$, the ring of all functions on X.

Example 20.3. Consider $\mathbb{A}^n$ with coordinates $x_1, \ldots, x_n$. The cotangent space at x is spanned by $d_x x_1, \ldots, d_x x_n$.

Example 20.4. In $\mathbb{R}^2$, $\sin x\, dy + \cos x\, dx \in \Psi[x]$ is a differential form.

20.3 Regular differential forms

Definition 20.5. A differential form ψ on X is *regular* if $\forall x \in X$, there is an open neighborhood $U \ni x$ such that $\psi|_U$ agrees with $\sum_{i=1}^{t} g_i df_i$, where $f_i, g_i \in \mathcal{O}_X(U)$.

In other words, viewing ψ as a section of the cotangent bundle of X, the section map is regular.

Example 20.6. The differential form

$$\psi = 2x\,d(xy) = 2x\,(x\,dy + y\,dx) = 2x^2\,dy + 2xy\,dx$$

is a regular differential form in $\mathbb{A}^2$.

Notation 20.7. For $U \subseteq X$ open, let $\Omega_X(U)$ be the set of *regular* differential forms on the variety U.

Note 20.8. $\Omega_X(U)$ is a module over $\mathcal{O}_X(U)$. In fact, Ω_X is a *sheaf* of $\mathcal{O}_X$-modules.

Example 20.9. On $\mathbb{A}^n$, Ω_X is the free $\mathcal{O}_X$-module generated by $dx_1, \ldots, dx_n$.

Theorem 20.10. *If X is smooth, then Ω_X is a locally free $\mathcal{O}_X$-module of rank* $\dim X$.

Proof sketch. Take $x \in X$, and take local parameters $x_1, \ldots, x_n$ at x. Show that $dx_1, \ldots, dx_n$ are a free basis for Ω_X in some neighborhood of x. (Use Nakayama's lemma.) $\square$

Proposition 20.11. *Let $V \subseteq \mathbb{A}^n$ be an affine variety with ideal $\mathbb{I}(V) = (g_1, \ldots, g_t) \subseteq k[\mathbb{A}^n]$. Then $\Omega_V(V)$ is the $\mathcal{O}_V(V)$-module*

$$\frac{k[V]\,dx_1|_V + \cdots + k[V]\,dx_n|_V}{k[V]\textit{-submodule generated by } (dg_1, \ldots, dg_t)}.$$

Note that if g vanishes on V, then $dg = 0$ on V.

Example 20.12. Let $V = \mathbb{V}(t - s^2) \subseteq \mathbb{A}^2$. Then

$$\Omega_V = \frac{k[V]\,dt + k[V]\,ds}{(dt - 2s\,ds)}.$$

This is free, since $dt = 2s\,ds$ in Ω_V, so the generator dt is redundant, and $\Omega_V = k[V]\,ds$.

Example 20.13. Consider $\mathbb{P}^1$ with homogeneous coordinates x, y, and with $t = \frac{x}{y}$, $s = \frac{y}{x}$. Say ψ is a global regular differential form on $\mathbb{P}^1$. Then

$$\psi\big|_{U_y} \in \Omega_{\mathbb{P}^1}(U_y) = k[t]\,dt$$
$$\psi\big|_{U_x} \in \Omega_{\mathbb{P}^1}(U_x) = k[s]\,ds.$$

If we have $p(t)\,dt \in k[t]\,dt$ and $q(s)\,ds \in k[t]\,dt$, then

$$p(t)\,dt = q(1/t)\,d(1/t)$$

on $U_x \cap U_y$. Then

$$p(t)\,dt = -q(1/t)\frac{dt}{t^2},$$

so

$$t^2 p(t) = -q(1/t)$$

in $k[t, t^{-1}]$. Thus $p = q = 0$, i.e., there are no nontrivial global regular differential forms on $\mathbb{P}^1$.

However, on $X = \mathbb{V}(x^3 + y^3 + z^3) \subseteq \mathbb{P}^2$, there is a 1-dimensional k-vector space of global differential forms. And, on $X = \mathbb{V}(x^4 + y^4 + z^4) \subseteq \mathbb{P}^2$, the space $\Omega_X(X)$ is 3-dimensional over k.

Definition 20.14. If X is a smooth projective curve, then the *genus* of X is the dimension of $\Omega_X(X)$ as a k-vector space.

20.4 Rational differential forms and canonical divisors

A rational differential form on X is intuitively $f_1 dg_1 + \cdots + f_r dg_r$, where f_i and g_i are *rational* functions on X. Formally:

Definition 20.15. A *rational differential form* on X is an equivalence class of pairs (U, φ) where $U \subseteq X$ is open and $\varphi \in \Omega_X(U)$. [As with rational functions, $(U, \varphi) \sim (U', \varphi')$ means $\varphi|_{U \cap U'} = \varphi'|_{U \cap U'}$.]

We can define the divisor of a rational differential form.

Definition 20.16. If ω is a rational differential form on a smooth curve X, then $\operatorname{div}(\omega) \in \operatorname{Div}(X)$ is called a *canonical divisor*.

The canonical divisors form a linear equivalence class on X, denoted K_X. Also,

$$\dim \mathscr{L}(K_X) = \operatorname{genus}(X).$$

Example 20.17. On $\mathbb{P}^1$, the canonical divisor $K_{\mathbb{P}^1}$ is the class of degree -2 divisors.

20.5 Canonical divisors, continued

Let X be smooth (or, X normal, and work on $X_{\text{sm}} \subseteq X$; since $\operatorname{codim}(X \setminus X_{\text{sm}}) \geq 2$, we won't miss any divisors).

Consider the sheaf Ω_X of regular differential forms on X. [In U, $\Omega_X(U)$ is the set of differential forms φ on U such that $\forall x \in U$, there exists an open neighborhood where φ agrees with $\sum f_i dg_i$, where f_i, g_i are *regular* functions.]

The sheaf Ω_X is a *locally* free $\mathcal{O}_X$-module of rank $d = \dim X$.

Fact 20.18. The set of rational differential forms[10] forms a vector space over $k(X)$.

Definition 20.19. A *separating transcendence basis* for $k(X)$ over k is a set of algebraically independent elements $\{u_i\}$ over which $k(X)$ is *separable* algebraic [i.e., $k(u_1, \ldots, u_n) \hookrightarrow k(X)$ is separable algebraic].

Example 20.20. Consider $X = \mathbb{P}^2$. Then

$$k\left(\frac{x}{y}, \frac{z}{y}\right) \xrightarrow{\simeq} k(\mathbb{P}^2),$$

so $\frac{x}{y}, \frac{z}{y}$ is a separating transcendence basis. In characteristic $\neq 2, 3$,

$$k\left(\left(\frac{x}{y}\right)^2, \left(\frac{z}{y}\right)^3\right) \hookrightarrow k\left(\frac{x}{y}, \frac{z}{y}\right)$$

is also a separating transcendence basis.

Theorem 20.21. *If $u_1, \ldots, u_n$ is a separating transcendence base for $k(X)$, then $du_1, \ldots, du_n$ is a basis for the space of rational differential forms on X over $k(X)$.*

[10] Shafarevich denotes this $\Theta(X)$.

Proof sketch. We have $k(u_1, \ldots, u_n) \hookrightarrow k(X)$. Given $\sum f_i dg_i$ with $f_i, g_i \in k(X)$, it suffices for each $g = g_i \in k(X)$ that we can write

$$dg = r_1 du_1 + \cdots + r_n du_n$$

for $r_i \in k(X)$.

Then g satisfies a minimal polynomial

$$g^m + a_1 g^{m-1} + \cdots + a_m = 0$$

with $a_i \in k(u_1, \ldots, u_n)$. Apply "$d$":

$$mg^{m-1}dg + g^m da_1 + a_1 \cdot (m-1)g^{m-2}dg + \cdots + da_m = 0. \tag{*}$$

Solve for dg_i:

$$(\text{rational function})\, dg \in k(X)\text{-span of } du_1, \ldots, du_n.$$

(Check the coefficient on dg is not zero if (*) is separable.) So $dg \in k(X)$-span of $du_1, \ldots, du_n$. □

20.6 The canonical bundle on X

For each $p \in \mathbb{N}$, look at the sheaf $\bigwedge^p \Omega_X$ of p-differentiable forms on X, which assigns to open $U \subseteq X$ the set of all regular p-forms: $\forall x \in U$, $\varphi(x) : \bigwedge^p T_x X \to k$. Locally these look like $\sum f_i dg_{i_1} \wedge \cdots \wedge dg_{i_p}$.

Rational p-forms are defined analogously.

Corollary 20.22. *The set of rational p-forms on X is a $k(X)$-vector space of dimension $\binom{n}{p}$.*

Proof. If $u_1, \ldots, u_n$ is a separating transcendence basis, then $\left\{du_{i_1} \wedge \cdots \wedge du_{i_p}\right\}$ is a basis for rational p-forms over $k(X)$. □

Definition 20.23. The *canonical sheaf* (or *dualizing sheaf*) of X (where X is smooth, $\dim X = n$) is

$$\omega_X = \bigwedge^n \Omega_X.$$

Note 20.24. (1) ω_X is locally free of rank 1.

(2) The set of *rational canonical* (n-)forms is a vector space of dimension 1 over $k(X)$.

Example 20.25. On $\mathbb{P}^2$, let $s = \frac{x}{y}$ and $t = \frac{z}{y}$, and consider

$$fd\left(\frac{x}{z}\right) \wedge d\left(\frac{y}{z}\right).$$

We have

$$\begin{aligned} d\left(\frac{x}{z}\right) \wedge d\left(\frac{y}{z}\right) &= d\left(\frac{s}{t}\right) \wedge d\left(\frac{1}{t}\right) \\ &= \left(\frac{t\,ds - s\,dt}{t^2}\right) \wedge \frac{(-dt)}{t^2} \\ &= \frac{-t\,ds \wedge dt}{t^4} = \frac{-ds \wedge dt}{t^3}. \end{aligned}$$

On U_z, there are no zeros or poles. On U_y, we have a pole of order 3 along $t = 0$ (the divisor $\mathbb{V}(z) \subset \mathbb{P}^2$).

So:

$$\operatorname{div}\left(d\left(\frac{x}{z}\right) \wedge d\left(\frac{y}{z}\right)\right) = -3L_\infty,$$

where $L_\infty = \mathbb{V}(z) \subset \mathbb{P}^2$.

Definition 20.26. The divisor of a rational canonical form φ on X is the divisor

$$\operatorname{div}(\varphi) = \sum_{\substack{D \text{ prime} \\ \text{divisor}}} \nu_D(\varphi) D,$$

where $\nu_D(\varphi)$ is computed as follows: Pick any $u_1, \dots, u_n$ parameters for a point $x \in D$. Write

$$\varphi = f \cdot du_1 \wedge \cdots \wedge du_n,$$

where $f \in k(X)$. Then $\nu_D(\varphi) = \nu_D(f)$.

Note 20.27. The divisor $\operatorname{div}(\omega)$ is not necessarily principal.

Proposition 20.28. *For all $f \in k(X)$, ω a rational canonical form,*

$$\operatorname{div}(f\omega) = \operatorname{div}(f) + \operatorname{div}(\omega).$$

In particular, any two rational canonical forms define the same divisor class.

Definition 20.29. The divisor $\operatorname{div}(\omega)$ is called a *canonical divisor*. By Proposition 20.28, they form a class, called the *canonical class* K_X.

Example 20.30. On $\mathbb{P}^2$, $K_{\mathbb{P}^2}$ is the class of divisors of degree -3.

We can use the canonical class (or multiples of it) to *classify* varieties.

If we embed

$$\begin{tikzcd} X \arrow[r, hook, "|dK_X|"] & \mathbb{P}^n \\ Y \arrow[ur, hook, "|dK_Y|"'] & \end{tikzcd}$$

then $X \cong Y \iff$ there is a *projective change of coordinates* taking X to Y.

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