Math 632 Notes
Algebraic Geometry 2

Lectures by Karen Smith
Notes by Daniel Hast

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1 Affine schemes

1.1 Motivation and review of varieties

"Classical" setup: What is a variety? A variety X is a set, a topological space, and a ringed space $(X, \mathcal{O}_X)$.

Locally: an *affine* variety. Every point p of a variety X an has open neighborhood U which can be identified with an algebraic set.

What is a scheme? A scheme X is also a set, a topological space, and a ringed space $(X, \mathcal{O}_X)$.

Local picture: *affine scheme.*

We will have a correspondence:

$$\begin{tikzcd}
\left\{\text{affine varieties over } k = \overline{k}\right\} \arrow[r, leftrightarrow] \arrow[d, hook] & \{\text{f.g. reduced } k\text{-algebras}\} \arrow[d, hook] \\
\{\text{affine schemes}\} \arrow[r, leftrightarrow] & \{\text{commutative rings with } 1\}
\end{tikzcd}$$

Recall:

Definition 1.1. An affine algebraic set $V = \mathbb{V}(g_1, \dots, g_r) \subseteq k^n$, where $g_i \in k[x_1, \dots, x_n]$ and $k = \overline{k}$. The coordinate ring of V is

$$k[V] = \{\text{restrictions of polynomials on } k^n \text{ to } V\} = \frac{k[x_1 \dots, x_n]}{\mathbb{I}(V)}.$$

Hilbert's Nullstellensatz: There's a category (anti-)equivalence:

$$\{\text{affine algebraic varieties}\} \longleftrightarrow \{\text{f.g. } k\text{-algebras without nilpotents}\}$$
$$V \to k[V]$$
$$\operatorname{mSpec} R = \{\text{maximal ideals in } R\} \leftarrow R = \frac{k[x_1, \dots, x_n]}{I}$$
$$\left(V \xrightarrow{f} W\right) \longleftrightarrow \left(k[W] \xrightarrow{f^*} k[V]\right)$$

1.2 First attempt at defining an affine scheme

Given a commutative ring R, associate

$$\operatorname{mSpec} R = \left\{\mathfrak{m} \subseteq R \;\middle|\; \mathfrak{m} \text{ is a maximal ideal of } R\right\}.$$

Example 1.2. If $R = \mathbb{Z}$, then

$$\operatorname{mSpec} \mathbb{Z} = \{(2), (3), (5), (7), \dots\}.$$

Fact 1.3 (Hilbert's Nullstellensatz). Given a map of f.g. reduced k-algebras $R \xrightarrow{\varphi} S$, there is an induced map of the corresponding algebraic sets

$$\begin{aligned} \operatorname{mSpec} S &\xrightarrow{f} \operatorname{mSpec} R \\ \mathfrak{m} &\mapsto \varphi^{-1}(\mathfrak{m}). \end{aligned}$$

In particular, $\varphi^{-1}(\mathfrak{m})$ is a maximal ideal of R.

Question: If $R \xrightarrow{\varphi} S$ is a map of rings (i.e., commutative rings with unit), is there an induced map

$$\begin{aligned} \operatorname{mSpec} S &\to \operatorname{mSpec} R \\ \mathfrak{m} &\mapsto \varphi^{-1}(\mathfrak{m}). \end{aligned}$$

Answer: No, not in general!

Example 1.4. Consider the map $\mathbb{Z} \overset{\varphi}{\hookrightarrow} \mathbb{Q}$. We have $\operatorname{mSpec} \mathbb{Q} = (0)$ and

$$\varphi^{-1}(0) = (0) \subseteq \mathbb{Z}.$$

This is not maximal in $\mathbb{Z}$. However, it is still prime.

1.3 Affine schemes

Lemma 1.5. *If $R \xrightarrow{\varphi} S$ is a ring map and $\mathfrak{p} \subseteq S$ is a prime ideal, then $\varphi^{-1}(\mathfrak{p}) \subseteq R$ is prime.*

Proof. We have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \arrow[d, two heads] \\ R/\varphi^{-1}(\mathfrak{p}) \arrow[r, hook] & S/\mathfrak{p}. \end{tikzcd}$$

A subring of an integral domain is an integral domain, and the result follows. □

Definition 1.6. An *affine scheme* (as a set) is $\operatorname{Spec} R$, where R is a ring and

$$\operatorname{Spec} R = \left\{ \mathfrak{p} \mid \mathfrak{p} \subseteq R \text{ is prime in } R \right\}.$$

Example 1.7. For $\operatorname{Spec} \mathbb{Z}$, we have the maximal ideals $(2), (3), (5), (7), \ldots$ and the ideal (0), which we picture as geometrically "containing" all the other points in the spectrum.

Example 1.8. For $\operatorname{Spec} k[x]$, where $k = \overline{k}$, there are two kinds of points: maximal ideals $(x - \lambda)$ and the zero ideal (0). Maximal ideals correspond to a point on the affine line, and the zero ideal is a "fuzzy" point covering the whole line.

Example 1.9. Consider $k[x, y]$ with $k = \overline{k}$. The maximal ideals are those of the form $(x - \alpha_1, y - \alpha_2)$, corresponding to $(\alpha_1, \alpha_2) \in \mathbb{A}^2_k$.

There are also prime ideals of the type (f), where $f \in k[x, y]$ is irreducible. These are irreducible plane curves, which we now think of as points in the spectrum. Finally, there is the ideal (0), corresponding to the whole affine plane.

1.4 The Zariski topology

Definition 1.10. Fix a ring R. The *Zariski topology* on $\operatorname{Spec} R$ has closed sets

$$\mathbb{V}(I) = \left\{\mathfrak{p} \in \operatorname{Spec} R \;\middle|\; \mathfrak{p} \supseteq I\right\}.$$

Remark 1.11. This really forms a topology:

- $\varnothing = \mathbb{V}(R)$
- $\operatorname{Spec} R = \mathbb{V}(0)$
- closed under arbitrary intersection:

$$\bigcap_{\lambda \in \Lambda} \mathbb{V}(I_\lambda) = \mathbb{V}\Big(\sum_{\lambda\in\Lambda} I_\lambda\Big).$$

- closed under finite unions:

$$\mathbb{V}(I_1) \cup \cdots \cup \mathbb{V}(I_r) = \mathbb{V}(I_1 \cap \cdots \cap I_r).$$

For any $\mathfrak{p} \in \operatorname{Spec} R$, what is the *closure* of $\mathfrak{p}$ in the Zariski topology? We have

$$\overline{\mathfrak{p}} = \mathbb{V}(\mathfrak{p}),$$

so $\mathfrak{p} \in \operatorname{Spec} R$ is closed $\iff$ $\mathfrak{p}$ is a maximal ideal.

In other words, $\operatorname{mSpec} R$ is the subset of all closed points of $\operatorname{Spec} R$.

Example 1.12. The closed sets of $\operatorname{Spec} \mathbb{Z}$ are of the form

$$\mathbb{V}(n) = \mathbb{V}(p_1^{a_1} \cdots p_r^{a_r}) = \{(p_1), (p_r), \ldots, (p_r)\},$$

where $p_1, \ldots, p_r$ are prime and $n = p_1^{a_1} \cdots p_r^{a_r}$. Note that any finite set not including (0) is closed.

The zero ideal (0) is not closed; its closure is *all* of $\operatorname{Spec}\mathbb{Z}$, i.e., $\{(0)\}$ is dense.

Example 1.13. If $f \in k[x, y]$ is irreducible, the closure of (f) in $\operatorname{Spec} k[x, y]$ consists of all the points on the affine plane curve defined by $f(x, y) = 0$, plus the point (f) itself.

1.5 The ringed space structure

Caution 1.14. An affine scheme is a set with the structure of a topological space, *plus* a ringed space structure!

Example 1.15. The affine schemes $\operatorname{Spec}\mathbb{R}$ and $\operatorname{Spec}\mathbb{F}_p$ are homeomorphic as topological spaces (since they are both the 1-point space), but they are *not* the same scheme.

Another example: $\operatorname{Spec}\mathbb{R}[t]/(t^2)$ is also a 1-point scheme, but in some sense, it is even more different than the other two.

Proposition 1.16. *If $R \xrightarrow{\varphi} S$ is a map of rings, then the induced map*

$$\begin{aligned} \operatorname{Spec} S &\xrightarrow{f} \operatorname{Spec} R \\ \mathfrak{p} &\mapsto \varphi^{-1}(\mathfrak{p}) \end{aligned}$$

is continuous in the Zariski topology.

Proof sketch. Need to show: If $W \subseteq \operatorname{Spec} R$ is closed, then $f^{-1}(W)$ is closed in $\operatorname{Spec} S$, i.e.,

$$f^{-1}(\mathbb{V}(I)) = \mathbb{V}(IS),$$

where $IS =$ ideal in S generated by $\varphi(I)$. $\square$

Example 1.17. Consider the surjection $R \overset{\varphi}{\twoheadrightarrow} R/I$. We have

$$\begin{tikzcd} \operatorname{Spec}(R/I) \arrow[r, leftrightarrow, "\text{homeomorphism}"] \arrow[rr, bend right, "f"'] & \mathbb{V}(I) \subseteq & \operatorname{Spec} R \end{tikzcd}$$

A surjective homomorphism of rings corresponds to a closed embedding of schemes.

Caution 1.18. There can be many different subscheme structures on a closed set of $\operatorname{Spec} R$.

Example 1.19.

$$\operatorname{Spec} k[x] \twoheadrightarrow \operatorname{Spec} \frac{k[x]}{(x^2)} \twoheadrightarrow \frac{k[x]}{(x)}$$

Example 1.20 (Localization). Let R be a ring and $U \subseteq R$ a multiplicative system in R. We have a natural map

$$R \to R[U^{-1}] = \left\{ \frac{r}{u} \;\middle|\; r \in R,\ u \in U \right\},$$

and hence an induced map

$$\operatorname{Spec} R \leftarrow \operatorname{Spec} R[U^{-1}],$$

corresponding to the subset of primes in R disjoint from U.

Special case: $U = \langle 1, f, f^2, f^3, \ldots \rangle$. Then

$$R[U^{-1}] = R\left[\frac{1}{f}\right],$$

inducing

$$\begin{tikzcd} & \operatorname{Spec} R \arrow[d, phantom, "\cup\!\!\mid"] & \operatorname{Spec} R\left[\frac{1}{f}\right] \arrow[l] \arrow[dl] \\ \operatorname{Spec} R - \mathbb{V}(f) \arrow[r, phantom, "="] & D(f) & \end{tikzcd}$$

where $D(f)$ is an open subset. Subsets of this form form a *basis* for the Zariski topology in $\operatorname{Spec} R$.

Definition 1.21. Let $\mathfrak{p} \in \operatorname{Spec} R$. The *residue field* of $\mathfrak{p}$, denoted $k(\mathfrak{p})$, is

$$R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = (R/\mathfrak{p})_{(\bar{0})} = \operatorname{Frac}(R/\mathfrak{p}).$$

The map $R \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ induces a map of schemes

$$\operatorname{Spec} \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}} \to \operatorname{Spec} R,$$

and we have a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/\mathfrak{p} \arrow[d] \\ R_{\mathfrak{p}} \arrow[r] & R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \end{tikzcd}$$

which induces a diagram of schemes

$$\begin{tikzcd} \operatorname{Spec} R & \operatorname{Spec}(R/\mathfrak{p}) \arrow[l, hook'] \\ \operatorname{Spec} R_{\mathfrak{p}} \arrow[u] & \operatorname{Spec} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \arrow[l] \arrow[u, hook] \end{tikzcd}$$

We think of a "point in the scheme" as corresponding to its residue field

$$\operatorname{Spec}(R/\mathfrak{p})_{(\bar{0})} = \operatorname{Spec} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = \text{``}\mathfrak{p}\text{''}.$$

The scheme $\operatorname{Spec} R_{\mathfrak{p}}$ corresponds to the primes of R contained in $\mathfrak{p}$.

1.6 The ringed space structure, continued

Let R be a ring. We have the set

$$\operatorname{Spec} R = \{\mathfrak{p} \text{ prime ideal in } R\},$$

and the topological space with closed sets

$$\mathbb{V}(I) = \left\{\mathfrak{p} \in \operatorname{Spec} R \;\middle|\; \mathfrak{p} \supseteq I\right\}.$$

A map of rings $R \xrightarrow{\varphi} S$ induces a *continuous* map

$$\begin{aligned} \operatorname{Spec} S &\to \operatorname{Spec} R \\ Q &\mapsto \varphi^{-1}(Q). \end{aligned}$$

Today's goal: Explain how to get a *ringed space* structure on $\operatorname{Spec} R$.

Proposition 1.22. *The topological space* $\operatorname{Spec} R$ *has a basis of (open) sets of the form*

$$D(f) = \operatorname{Spec}(R) \setminus \mathbb{V}(f).$$

Proof. Take $U \subseteq \operatorname{Spec} R$. Then $U = \operatorname{Spec}(R) \setminus \mathbb{V}(I)$ for some ideal I. Consider

$$\widetilde{U} = \bigcup_{f \in I} D(f).$$

We will show that $U = \widetilde{U}$.

If $\mathfrak{p} \in \widetilde{U}$, then $\mathfrak{p} \in D(f) \iff f \notin \mathfrak{p} \iff \mathfrak{p} \in U = \operatorname{Spec} R \setminus \mathbb{V}(I)$ [otherwise $\mathfrak{p} \supseteq I$, but $f \in I,\ f \notin \mathfrak{p}$].

Conversely: $\mathfrak{p} \in U$ means $\mathfrak{p} \not\supseteq I$, so $\exists f \in I$ but not $\mathfrak{p}$. So $\mathfrak{p} \in D(f)$, and hence $\mathfrak{p} \in \widetilde{U}$. $\square$

Proposition 1.23. *The localization map* $R \mapsto R\left[\frac{1}{f}\right]$, $r \mapsto \frac{r}{1}$ *induces*

$$\begin{tikzcd}
\operatorname{Spec} R\left[\frac{1}{f}\right] \arrow[r, hook] \arrow[dr, "\simeq"'] & \operatorname{Spec} R \arrow[d, phantom, "\cup|"] \\
& D(f)
\end{tikzcd}$$

Goal 1.24. We will put a *ringed space* structure on $\operatorname{Spec} R = X$ for each $U \subseteq X$, yielding a ring $\mathscr{O}_X(U)$ satisfying

$$\mathscr{O}_X(X) = X,$$
$$\mathscr{O}_X(D(f)) = R\left[\frac{1}{f}\right],$$

and the "restriction" maps on the rings will be

$$D(f) \subseteq X$$
$$\mathscr{O}_X(X) \xrightarrow{\text{restriction}} \mathscr{O}_X(D(f))$$
$$R \xrightarrow{\text{localization}} R\left[\frac{1}{f}\right].$$

2 Sheaves

2.1 Sheaves

Fix a topological space X and a category C (rings, abelian groups, modules).

Definition 2.1. A *presheaf* on X with values in C is a contravariant functor

$$\{\text{open sets of } X \text{ with inclusions}\} \xrightarrow{\mathscr{F}} C.$$

That is, for each open $U \subseteq X$, we have a ring $\mathscr{F}(U)$, and for each inclusion of open sets $U \subset U'$, we have a ring homomorphism

$$\mathscr{F}(U') \xrightarrow{\rho_{U',U}} \mathscr{F}(U)$$
$$s \mapsto \rho_{U',U}(s) = s\big|_U$$

respecting composition, i.e.,

- $U_1 \subseteq U_2 \subseteq U_3$ induces $\mathscr{F}(U_3) \to \mathscr{F}(U_2) \to \mathscr{F}(U_1)$, which is the same as the map $\mathscr{F}(U_3) \to \mathscr{F}(U_1)$ induced by $U_1 \subseteq U_3$.
- $U \subseteq U \implies \mathscr{F}(U) \xrightarrow{\mathrm{id}_U} \mathscr{F}(U)$.
- $\mathscr{F}(\varnothing) =$ the trivial ring.

Definition 2.2. A *sheaf* is a presheaf $\mathscr{F}$ with the following additional property (sheaf axiom): If $U \subseteq X$ is an open set with an open cover $U = \bigcup_{i \in I} U_i$ and $s_i \in \mathscr{F}(U_i)$ such that

$$s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j} \qquad \forall i, j \in I,$$

then there exists a unique $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i \ \forall i$.

Exercise 2.3. Check that this is equivalent to the definition in Hartshorne (when C is a category where we have a zero).

Example 2.4 (A presheaf which is not a sheaf). Let X be a reducible topological space, and let $\mathscr{F}$ be the presheaf of *constant* $\mathbb{R}$-valued functions:

$$\mathscr{F}(U) = \left\{\text{constant function } U \to \{\lambda\} \;\middle|\; \lambda \in \mathbb{R}\right\}.$$

Since X is reducible, there exist open sets $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \varnothing$; write $U = U_1 \cap U_2$.

Take $u_1 \in U_1$ and $u_2 \in U_2$ to be constant functions with different values. Then this does not agree with the restriction of any single constant function on U.

There is a natural way to fix, or "sheafify" $\mathscr{F}$: take *locally* constant functions on U.

2.2 Stalk of a (pre)sheaf

For $\mathscr{F}$ a presheaf on X and a point $p \in X$, the "stalk" of $\mathscr{F}$ at p is a ring $\mathscr{F}_p$.

Example 2.5 (Main example). If p is a point on a variety X, then $(\mathcal{O}_X)_p = \mathcal{O}_{X,p}$ consists of functions regular at p.

Recall: A directed set I is a partially ordered set such that $\forall i, j \in I$, there exists $k \in I$ such that $i \le k,\ j \le k$.

Example 2.6. If X is a topological space and I is the set of open sets of X, define $U_1 \le U_2$ iff $U_1 \subseteq U_2$.

Say A_i are objects in a category (e.g., rings) indexed by I, some directed set such that if $i \le j$, then there is a map $\varphi_{ij} : A_i \to A_j$, and these maps are functorial.

Definition 2.7. The *direct limit* of $\{A_i\}_{i \in I}$ is

$$A = \varinjlim A_i = \Big(\coprod_{i \in I} A_i\Big)/\sim,$$

where for all $a_i \in A_i$ and $a_j \in A_j$, we have $a_i \sim a_j \iff \exists k$ such that

$$\varphi_{ik}(a_i) = \varphi_{jk}(a_j).$$

This has a ring structure: if $[a],[b] \in \varinjlim_{i\in I} A_i$ with a represented by $a_i \in A_i$ and b represented by $b_j \in A_j$, map both a, b to A_k with $i, j \leq k$, and define the ring operations in that A_k.

Example 2.8. Let the indexing set I be $\mathbb{N}$ with the order $n \leq m \iff n \mid m$. Associate to $n \in \mathbb{N}$ the ring $\mathbb{Z}\left[\frac{1}{n}\right]$.

If $m \leq n = mq$, then we have a map

$$\begin{aligned} \mathbb{Z}\left[\tfrac{1}{m}\right] &\to \mathbb{Z}\left[\tfrac{1}{n}\right] \\ \frac{a}{m^t} &\mapsto \frac{aq^t}{(mq)^t} = \frac{aq^t}{n^t}. \end{aligned}$$

This system of maps is also clearly functorial, i.e., the composition $\mathbb{Z}\left[\frac{1}{2}\right] \to \mathbb{Z}\left[\frac{1}{4}\right] \to \mathbb{Z}\left[\frac{1}{12}\right]$ is the same as the composition $\mathbb{Z}\left[\frac{1}{2}\right] \to \mathbb{Z}\left[\frac{1}{6}\right] \to \mathbb{Z}\left[\frac{1}{12}\right]$.

The direct limit is

$$\varinjlim_{n\in\mathbb{N}} \mathbb{Z}\left[\tfrac{1}{n}\right] = \mathbb{Q}.$$

Definition 2.9. Let $\mathscr{F}$ be a presheaf of rings on X, and let $p \in X$ be a point. The *stalk* of $\mathscr{F}$ at p is

$$\mathscr{F}_p = \varinjlim_{p\in U\subseteq X} \mathscr{F}(U).$$

The indexing set is

$$I = \left\{ U \subseteq X \;\middle|\; U \text{ is open and } p \in U \right\}$$

with the ordering $U_2 \subseteq U_1 \iff U_2 \geq U_1$.

Example 2.10. If $\mathscr{F} = \mathcal{O}_X$ on a variety X, then this *is* the stalk of the structure sheaf at p.

An element $[s] \in \mathscr{F}_p$ is represented by some $s \in \mathscr{F}(U)$, where we think of U as "arbitrarily small."

Example 2.11. The stalk at $0 \in \mathbb{C}$ of the sheaf of analytic functions on $\mathbb{C}$ is the ring of convergent power series at 0.

2.3 Direct and inverse limits

Say $\{A_i\}_{i\in I}$ is a collection of objects in a category, indexed by a poset I, and whenever $i \leq j$ in I, there is a map $A_i \to A_j$ (respectively, $A_i \leftarrow A_j$) such that the diagram commutes (for all $i \leq j \leq k$):

$$\begin{tikzcd} A_i \arrow[r] \arrow[rr, bend right] & A_j \arrow[r] & A_k \end{tikzcd}$$

(Respectively, with the arrows in the opposite direction.)

Assuming I is a *directed* poset:

Definition 2.12. The *direct limit* (also *injective limit*, *inductive limit*, *colimit*) of the direct limit system $\{A_i\}_{i\in I}$, if it exists, is an object A, denoted $\varinjlim A_i$, to which *all* A_i map functorially, which is universal with respect to this property: If there exists an object B to which all A_i map functorially, then there exists a unique map $A \to B$ which makes the diagram commute.

Construction of $\varinjlim A_i$ in abelian groups (or rings):

$$\varinjlim A_i = \coprod A_i / \sim,$$

where $A_i \ni a_i \sim a_j \in A_j \iff \exists k \geq i, j$ such that $\varphi_{ik}(a_i) = \varphi_{jk}(a_j)$.

Remark 2.13. Important idea: Direct limits generalize *union*.

Exercise 2.14. If all A_i are subobjects of some fixed $\widetilde{A}$ and all morphisms are inclusions, then $\varinjlim A_i = \bigcup_{i \in I} A_i$.

Definition 2.15. If $\mathscr{F}$ is a presheaf on X and $p \in X$, then the stalk of $\mathscr{F}$ at p is

$$\mathscr{F}_p = \varinjlim_{p \in U} \mathscr{F}(U).$$

(Here, the limit system is given by restriction.)

Note 2.16. By definition, for all neighborhoods U of p, there exists a unique map

$$\begin{aligned} \mathscr{F}(U) &\to \mathscr{F}_p \\ s &\mapsto s_p. \end{aligned}$$

Terminology: s_p is the *germ* of s at p.

Each $t \in \mathscr{F}_p$ is an equivalence class of sections $t_i \in \mathscr{F}(U_i)$, where $t_i \sim t_j$ means $\exists V \subseteq U_1 \cap U_2$ such that $t_i|_V = t_j|_V$. So, we can represent t by any $t_i \in \mathscr{F}(U_i)$.

Definition 2.17. The *inverse limit* (also *projective limit*, *indirect limit*, *limit*) of the inverse limit system $\{A_i\}_{i \in I}$, if it exists, is an object A, denoted $\varprojlim A_i$, from which *all* A_i map functorially, $A \to A_i$, and A is universal with respect to this property: If there exists an object B from which all A_i map functorially, then there exists a unique map $B \to A$ which makes the digram commute.

Equivalently, an inverse limit is a direct limit in the opposite category.

Construction of $\varprojlim A_i$ in abelian groups (or rings, etc.):

$$\varprojlim A_i = \left\{ (a_i)_{i \in I} \;\middle|\; i \leq j \implies a_i \leftarrow\!\shortmid a_j \right\} \subseteq \prod_{i \in I} A_i.$$

Exercise 2.18. If A_i are all subobjects of some fixed object and the maps $A_i \leftarrow A_j$ are inclusions $A_j \hookrightarrow A_i$, then $\varprojlim A_i = \bigcap_{i \in I} A_i$.

Example 2.19. Consider the inverse system

$$\tfrac{k[x,y]}{(x,y)} \twoheadleftarrow \tfrac{k[x,y]}{(x,y)^2} \twoheadleftarrow \tfrac{k[x,y]}{(x,y)^3} \twoheadleftarrow \tfrac{k[x,y]}{(x,y)^4} \twoheadleftarrow \dots$$

The inverse limit is the ring of formal power series

$$\varprojlim \frac{k[x,y]}{(x,y)^i} = k[\![x,y]\!]\,.$$

2.4 Sheaves on a fixed space

The sheaves on a fixed space X with values in a category C is a *category.*

Definition 2.20. A *morphism of sheaves* of abelian groups on X

$$\mathscr{F} \xrightarrow{\varphi} \mathscr{G}$$

is, for all open $U \subseteq X$, a morphism

$$\mathscr{F}(U) \xrightarrow{\varphi(U)} \mathscr{G}(U)$$

of abelian groups, compatible with restriction maps: for each inclusion $V \subseteq U$ of open sets, the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho"] & \mathscr{G}(U) \arrow[d, "\rho"] \\
\mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V)
\end{tikzcd}$$

Example 2.21. Let $X = \mathbb{C} \setminus \{0\}$, and let $\mathscr{A}$ = sheaf of analytic functions on X. Consider the map

$$\mathscr{A} \xrightarrow{\exp} \mathscr{A}$$
$$f(z) \mapsto \exp(f) = e^{f(z)}.$$

This is a morphism of sheaves of abelian groups, one additive and one multiplicative.

Example 2.22. Let $X = \mathbb{R}$. The map

$$C^\infty \xrightarrow{\frac{d}{dx}} C^\infty$$
$$C^\infty(U) \to C^\infty(U)$$
$$f \mapsto \frac{d}{dx} f$$

is a morphism of sheaves of $\mathbb{R}$-vector spaces.

Definition 2.23. Let X be a topological space, $p \in X$, and k a fixed abelian group. The *skyscraper sheaf* at p with values (sections) k is

$$\underline{k}^{(p)}(U) = \begin{cases} 0 & \text{if } p \notin U \\ k & \text{if } p \in U. \end{cases}$$

with restriction maps $\underline{k}^{(p)}(U) \to \underline{k}^{(p)}(V)$ given by the zero map if $p \notin V$ and the identity if $p \in V$.

Example 2.24 (A morphism of sheaves of R-algebras). Let $p \in X$ be a fixed point. We have the morphism

$$C_X^0 \xrightarrow{\text{eval at } p} \underline{\mathbb{R}}^{(p)}$$
$$C_X^0(U) \to \underline{\mathbb{R}}^{(p)}(U)$$
$$f \mapsto f(p).$$

Proposition 2.25. *If $\mathscr{F} \to \mathscr{G}$ is a morphism of (pre-)sheaves of abelian groups X, then for all $p \in X$, there is an induced map of stalks*

$$\varinjlim_{p \in U} \mathscr{F}(U) = \mathscr{F}_p \to \mathscr{G}_p = \varinjlim_{p \in U} \mathscr{G}(U).$$

Proof. By abstract nonsense: For each open neighborhood U of p, we have an induced map

$$\mathscr{F}(U) \to \mathscr{G}(U) \to \varinjlim_{p \in U} \mathscr{G}(U) = \mathscr{F}_p \to \mathscr{G}_p.$$

By the universal property of direct limits, we have a unique compatible map

$$\mathscr{F}_p = \varinjlim_{p \in U} \mathscr{F}(U) \to \mathscr{G}_p.$$ □

Concretely: Take a germ $s_p \in \mathscr{F}_p$. Represent it by some $s_i \in \mathscr{F}(U_i)$ with $p \in U_i$. We have a diagram

$$\begin{tikzcd} \mathscr{F}_p \arrow[r, dashed, "\exists!"] & \mathscr{G}_p \\ \mathscr{F}(U_i) \arrow[u] \arrow[r, "\varphi(U_i)"] & \mathscr{G}(U_i) \arrow[u] \end{tikzcd}$$

sending $s_i \mapsto \varphi(s_i) \mapsto \varphi(s_p)$, and $s_p \mapsto \varphi(s_p)$ is the well-defined map.

Definition 2.26. A morphism of *sheaves* (not presheaves) $\mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ is *injective* (resp. *surjective*) if for all $p \in X$, the induced map $\mathscr{F}_p \to \mathscr{G}_p$ is injective (resp. surjective).

Note 2.27. Here, we do not define injectivity and surjectivity for presheaves.

Caution 2.28. The above is *false* for presheaves. If $\mathscr{F} \to \mathscr{G}$ is a map of presheaves, and $\mathscr{F}_p \to \mathscr{G}_p$ is injective (or surjective) for all p, then there is an induced map of associated shaves which is injective (or surjective):

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d] & \mathscr{G} \arrow[d] \\ \mathscr{F}^+ \arrow[r, "\varphi^+"] & \mathscr{G}^+ \end{tikzcd}$$

2.5 Morphisms of sheaves, continued

Remark 2.29. Say $\mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ is a morphism of sheaves.

(1) Is it true that φ is injective iff for all $U \subseteq X$, the map $\mathscr{F}(U) \xrightarrow{\varphi(U)} \mathscr{G}(U)$ is injective?

(2) Is it true that φ is surjective iff for all $U \subseteq X$, the map $\mathscr{F}(U) \xrightarrow{\varphi(U)} \mathscr{G}(U)$ is surjective?

It turns out that the first is true, and the second is not.

Proposition 2.30. *If $\mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ is an injective map of sheaves, then $\forall$ open $U \subseteq X$, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*

Proof. Take $U \subseteq X$ open. Consider

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{F}_p \arrow[r] & \mathscr{G}_p \end{tikzcd} \qquad\qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\ s_p \arrow[r, mapsto] & 0 \end{tikzcd}$$

For all $p \in U$, the image $s_p \in \mathscr{F}_p$ is 0. Hence, there exists a neighborhood $U'_p \subseteq U$ of p such that $s|_{U'_p} = 0$.

Now $\left\{U'_p\right\}$ cover U and $s|_{U'_p} = 0$, so $s = 0$ on U (by the sheaf axiom). $\square$

Caution 2.31. Proposition 2.30 is false for surjectivity. There are surjective *sheaf* maps $\mathscr{F} \overset{\varphi}{\twoheadrightarrow} \mathscr{G}$ and open sets $U \subseteq X$ for which $\mathscr{F}(U) \xrightarrow{\varphi(U)} \mathscr{G}(U)$ is not surjective.

Example 2.32. Let $X = \mathbb{C} \setminus \{0\}$, and let $\mathscr{A}$ be the sheaf of analytic functions. We have a map

$$\begin{aligned} \mathscr{A} &\to \mathscr{A}^* \\ f &\mapsto \exp(f) \end{aligned}$$

which is *locally* surjective but not globally.

Example 2.33. Let X be a connected Hausdorff space with at least two distinct points $p, q \in X$. Let $\underline{\mathbb{R}}_X$ be the sheaf of locally constant functions, and let $\mathscr{G}$ be defined by

$$\mathscr{G}(U) = \begin{cases} 0 & \text{if } p, q \notin U, \\ \mathbb{R} \oplus \mathbb{R} & \text{if } p, q \in U, \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

Define a map

$$\begin{aligned} \underline{\mathbb{R}}_X &\xrightarrow{\varphi} \mathscr{G} \\ \underline{\mathbb{R}}_X(U) &\xrightarrow{\varphi(U)} \mathscr{G}(U) \\ f &\mapsto \begin{cases} 0 & \text{if } p, q \notin U, \\ f(p) & \text{if } p \in U,\ q \notin U, \\ (f(p), f(q)) & \text{if } p, q \in U. \end{cases} \end{aligned}$$

This is a surjective map of sheaves, but if U is a connected open set such that $p, q \in U$, then $\varphi(U)$ is not surjective.

2.6 Sheafification

Proposition–Definition 2.34. Given any presheaf $\mathscr{F}$ on X, there is an *associated sheaf* $\mathscr{F}^+$ together with a presheaf map

$$\mathscr{F} \to \mathscr{F}^+$$

which is an isomorphism on stalks. Furthermore, $\mathscr{F}^+$ has the following universal property: for all sheaves $\mathscr{G}$ and morphisms $\mathscr{F} \to \mathscr{G}$, we have a diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \mathscr{F}^+ \arrow[d, dashed, "\exists!"] \\ & \mathscr{G} \end{tikzcd}$$

Proof. To construct $\mathscr{F}^+$, for all open $U \subseteq X$, define

$$\mathscr{F}^+(U) = \left\{ U \xrightarrow{s} \coprod_{p \in U} \mathscr{F}_p \;\middle|\; \begin{array}{l} \bullet\ s(p) \in \mathscr{F}_p \\ \bullet\ \forall q \in U, \text{ there exists a neighborhood } V \subseteq U \text{ of } q \\ \quad \text{and a section } t \in \mathscr{F}(U) \text{ such that } s(x) = t_x \text{ for all} \\ \quad x \in V. \end{array} \right\},$$

and the morphism

$$\begin{aligned} \mathscr{F}(U) &\to \mathscr{F}^+(U) \\ s &\mapsto \begin{bmatrix} U \to \coprod_{p \in U} \mathscr{F}_p \\ q \mapsto s_q \end{bmatrix}. \end{aligned}$$

The verification of the rest is straightforward. □

2.7 Kernel, image, and cokernel of sheaves

Definition 2.35. Let $\mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ be a morphism of sheaves of abelian groups on X. There are naturally arising presheaves:

- presheaf kernel $U \mapsto \ker(\varphi(U))$
- presheaf image $U \mapsto \operatorname{im}(\varphi(U))$
- presheaf cokernel $U \mapsto \operatorname{coker}(\varphi(U))$

By definition, the *kernel* $\ker \varphi$, the *image* $\operatorname{im} \varphi$, and *cokernel* $\operatorname{coker} \varphi$ of φ are the associated sheaves.

Exercise 2.36. The presheaf kernel $U \mapsto \ker(\varphi(U))$ is already a sheaf, so the presheaf kernel of φ is naturally isomorphic to $\ker \varphi$.

2.8 Pushforward/pullback of sheaves

Say $X \xrightarrow{f} Y$ is a continuous map of topological spaces. If $\mathscr{F}$ is a sheaf on X, then there's an easy way to get a sheaf on Y:

Definition 2.37. The *direct image sheaf*, denoted $f_*\mathscr{F}$, assigns to $U \subseteq Y$

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U)).$$

This is a sheaf!

Say $\mathscr{G}$ is a sheaf on Y. There is a natural sheaf on X defined as follows:

Definition 2.38. The *inverse image sheaf*, denoted $f^{-1}\mathscr{G}$ [*not* $f^*\mathscr{G}$] is a sheaf on X defined as follows: For open $U \subseteq X$,

$$f^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V).$$

Note that $f^{-1}\mathscr{G}(U) = \mathscr{G}(f(U))$ if $f(U)$ is open.

3 Ringed spaces and schemes

3.1 Morphisms of ringed spaces

Definition 3.1. A *ringed space* $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on it.

Definition 3.2. A *morphism of ringed spaces*

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y)$$

is a continuous map $X \xrightarrow{f} Y$ together with a map of sheaves of rings

$$\mathcal{O}_Y \to f_*\mathcal{O}_X.$$

Example 3.3. If $X \xrightarrow{f} Y$ is a regular map of varieties, then there is a naturally induced morphism of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{f, f^\#} (Y, \mathcal{O}_Y)$$
$$X \xrightarrow{f} Y$$
$$\mathcal{O}_Y \xrightarrow{f^\#} f_*\mathcal{O}_X$$
$$\mathcal{O}_Y(U) \xrightarrow{f^*} \mathcal{O}_X(f^{-1}(U))$$
$$h \mapsto h \circ f.$$

3.2 The ringed space structure of the spectrum

Let R be a ring. We have the affine scheme $\operatorname{Spec} R = X$, and we want to define $\mathcal{O}_X$ (a sheaf of rings) on X.

Approach: A sheaf is determined by its values on a basis.

Example 3.4. If X is an affine variety, $k[X]$ its coordinate ring, and $k(X)$ its function field, recall:

$$\mathcal{O}_X(U) = \left\{ \varphi \in k(X) \;\middle|\; \begin{array}{c} \forall p \in U, \exists \text{ a representation } \varphi = \frac{f}{g} \text{ where} \\ f, g \in k[X] \text{ such that } g(p) \neq 0 \end{array} \right\},$$

i.e., the $\varphi \in k(X)$ such that $\varphi \in k[X]\left[\frac{1}{g}\right]$ where $p \in D(g)$.

Lemma 3.5. *If X is an affine variety and $U \subseteq X$ is open, then*

$$\mathcal{O}_X(U) = \bigcap_{D(g) \subseteq U} \mathcal{O}_X(D(g)) = \varprojlim_{D(g) \subseteq U} \mathcal{O}_X(D(G)).$$

Proof. Let $D(g) \subseteq U$. Then $\mathcal{O}_X(U) \subseteq \mathcal{O}_X(D(g))$, so $\mathcal{O}_X(U) \subseteq \bigcap_{D(g) \subseteq U} \mathcal{O}_X(D(G))$.

Conversely: Say $\varphi \in \bigcap_{D(g) \subseteq U} \mathcal{O}_X(D(g))$. Then $\varphi \in \mathcal{O}_X(U)$ since $\forall p \in U$, there exists a basic open neighborhood $D(g)$ of p: $p \in D(g) \subseteq U$. We have

$$\varphi \in \mathcal{O}_X(D(g)) = k[X]\left[\frac{1}{g}\right],$$

i.e., φ has a representation $\frac{f}{g^n}$ for some $g^n, f \in k[X]$ and $g^N(p) \neq 0$. $\square$

Remark 3.6. In general, given values of a sheaf $\mathscr{F}$ on a basis $\{U_\lambda\}_{\lambda \in \Lambda}$ (together with the restriction maps $\mathscr{F}(U_\lambda) \to \mathscr{F}(U_\mu)$ whenever $U_\mu \subseteq U_\lambda$), we can reconstruct the sheaf at any open set U as follows: sheafify the presheaf

$$\mathscr{F}(U) = \varprojlim_{U_\lambda \subseteq U} \mathscr{F}(U_\lambda).$$

3.3 Construction of the spectrum

Exercise 3.7. If $\mathscr{F}$ is a sheaf (of abelian groups) on X, and $\{U_\lambda\}_{\lambda \in \Lambda}$ is a basis for the topology on X, and we know $\mathscr{F}(U_\lambda)$ and $\mathscr{F}(U_\lambda) \xrightarrow{\rho} \mathscr{F}(U_{\lambda'})$ (for all U_λ in basis), then $\mathscr{F}$ is uniquely determined by

$$\mathscr{F}(U) = \varprojlim_{U_\lambda \subseteq U} \mathscr{F}(U_\lambda)$$

and all restriction maps $\mathscr{F}(U)|\mathscr{F}(U')$ are likewise uniquely determined.

Let $X = \operatorname{Spec} R$. A basis for the topology is

$$D(g) = X - \mathbb{V}(g) = \left\{ P \in \operatorname{Spec} R \;\middle|\; g \notin P \right\}.$$

Assign $\mathcal{O}_X(D(g)) = R\left[\frac{1}{g}\right]$. If $D(g) \subseteq D(h)$, there's a natural ring map (localization)

$$\mathcal{O}_X(D(h)) = R\left[\frac{1}{h}\right] \xrightarrow{\rho} R\left[\frac{1}{g}\right] = \mathcal{O}_X(D(g)).$$

This is because:

Lemma 3.8. *If $D(g) \subseteq D(h)$, then there exists $f \in R$ and $n \in \mathbb{N}$ such that $g^n = hf$.*

Proof. $D(g) \subseteq D(h) \iff \mathbb{V}(g) \supseteq \mathbb{V}(h) \implies g \in \operatorname{Rad}(H) \implies \exists n$ such that $g^n \in (h)$. $\square$

Now: Define

$$\mathcal{O}_X(U) = \varprojlim_{D(g) \subseteq U} R\left[\frac{1}{g}\right].$$

Easy to check:

Exercise 3.9. If $V \subseteq U$, then there exists a uniquely induced map

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V).$$

Harder to check (done in Shafarevich) that this satisfies the sheaf axioms.

Definition 3.10. The ringed spaced structure on $\operatorname{Spec} R$ is given by

$$\mathcal{O}_X(U) = \varprojlim_{D(g) \subseteq U} R\left[\frac{1}{g}\right].$$

Proposition 3.11.
- $\mathcal{O}_X(D(g)) = R\left[\frac{1}{g}\right]$.
- $\mathcal{O}_{X,P} = \varinjlim_{D(g) \ni P} R\left[\frac{1}{g}\right] = \varinjlim_{g \notin P} R\left[\frac{1}{g}\right] = R_P$.
- $\mathcal{O}_X(X) = \mathcal{O}_X(D(1)) = R\left[\frac{1}{1}\right] = R$.

Example 3.12. Special case: If R is a domain, then all $\mathcal{O}_X(D(g)) = R\left[\frac{1}{g}\right]$ are all subrings of the fraction field of R, and all restriction maps

$$R\left[\frac{1}{h}\right] \to R\left[\frac{1}{g}\right]$$

are inclusions. So

$$\mathcal{O}_X(U) = \bigcap_{D(g) \subseteq U} R\left[\frac{1}{g}\right] = \left\{\varphi \in \operatorname{Frac}(R) \;\middle|\; \exists P \in U\ \exists g \notin P,\ f \in R \text{ such that } \varphi = \tfrac{f}{g}\right\}.$$

Definition 3.13 (Alternate definition, Hartshorne)**.** Let $X = \operatorname{Spec} R$, $U \subseteq X$. Define

$$\mathcal{O}_X(U) = \left\{ U \xrightarrow{\varphi} \coprod_{P \in U} R_P \;\middle|\; \begin{array}{c} \varphi(P) \in R_P, \text{ and } \forall p \in U, \exists \text{ neighborhood } P \in V \subseteq U \\ \text{such that } \exists r, g \in R \text{ such that } \forall Q \in V, \text{ we have} \\ \varphi(Q) = \frac{r}{g},\ g \notin Q \end{array} \right\}.$$

- Easy to see that this is a sheaf.
- $\mathcal{O}_{X,P} = R_P$.
- Harder to see (Hartshorne?): $\mathcal{O}_X(X) = R$ and $\mathcal{O}_X(D(g)) = R\left[\frac{1}{g}\right]$.

3.4 Locally ringed spaces

Definition 3.14. A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if for all $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a *local ring.*

Note 3.15. $(\operatorname{Spec} R, \widetilde{R}) = (X, \mathcal{O}_X)$ is locally ringed.

Recall: A map of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map $X \xrightarrow{f} Y$ together with a map of sheaves of rings

$$\mathcal{O}_Y \to f_*\mathcal{O}_X$$

on Y. (For all $U \subseteq Y$, we have $\mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$.)

Note 3.16. If $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is a map of ringed spaces and $P \in X$, then there is an induced map on stalks

$$\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}.$$

Indeed: if we have $\mathcal{O}_Y \to f_*\mathcal{O}_X$, then we have

$$\mathcal{O}_{Y,f(P)} \to (f_*\mathcal{O}_X)_{f(P)} = \varinjlim_{f(P) \in U} \mathcal{O}_X(f^{-1}(U)) \to \varinjlim_{V \ni P \text{ open}} \mathcal{O}_X(V),$$

where the last map is uniquely given by the universal property of the direct limit over all open $V \ni P$.

Definition 3.17. A *morphism of locally ringed spaces*

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a map of ringed spaces such that for all $P \in X$, the induced map on stalks

$$\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

is a *local map* of local rings. [A *local map* of local rings $(R, \mathfrak{m}) \xrightarrow{\varphi} (S, \mathfrak{n})$ is a ring map such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.]

Proposition 3.18. *If $R \xrightarrow{\varphi} S$ is a map of rings, then there is an induced map*

$$(\operatorname{Spec} S, \widetilde{S}) \xrightarrow{(f, f^\#)} (\operatorname{Spec} R, \widetilde{R})$$

of locally ringed spaces.

Proof sketch. Let f be the map

$$\begin{aligned} \operatorname{Spec} S &\xrightarrow{{}^a\varphi} \operatorname{Spec} R \\ \mathfrak{p} &\mapsto \varphi^{-1}(\mathfrak{p}). \end{aligned}$$

And, let $f^\# : \widetilde{R} \to f_*\widetilde{S}$ be defined on $D(g) \subseteq \operatorname{Spec} R$ by

$$\begin{aligned}\widetilde{R}(D(g)) = R\left[\frac{1}{g}\right] &\xrightarrow{\varphi} S\left[\frac{1}{\varphi(g)}\right] = f_*\widetilde{S}(D(s)) = \widetilde{S}\left(f^{-1}(D(g))\right) = D(\varphi(g)) \\ \frac{r}{g^t} &\mapsto \frac{\varphi(r)}{(\varphi(g))^t}.\end{aligned}$$

□

Exercise 3.19. Check this is a map of *locally* ringed spaces.

3.5 Schemes

Definition 3.20. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is *isomorphic* to $(\operatorname{Spec} R, \widetilde{R})$ as a locally ringed space, for some ring R.

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that for all $P \in X$, there exists a neighborhood $U \subseteq X$ of P such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Exercise 3.21. For any sheaf $\mathscr{F}$ on a topological space X and any open set $U \subseteq X$, there is a sheaf $\mathscr{F}|_U$ defined by $\mathscr{F}|_U(V) = \mathscr{F}(V)$.

Example 3.22. $X = \operatorname{Spec} R$, where $R = k[V]$ for $V \subseteq \mathbb{A}^n_k$ an affine algebraic variety (e.g., $V = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2_{\mathbb{C}}$).

By the Nullstellensatz, there is a natural embedding

$$\begin{gathered} V \to X = \operatorname{Spec} R \\ (\lambda_1, \dots, \lambda_n) = P \mapsto \mathfrak{m}_P = (x - \lambda_1, \dots, x - \lambda_n) \end{gathered}$$

onto the closed points in X. We have the sheaf of regular functions

$$\mathcal{O}_V(D(g)) = R\left[\frac{1}{g}\right].$$

In category-theoretic language, there is a fully faithful embedding of the category of varieties into the category of schemes.

Example 3.23. Let $(V, \mathfrak{m})$ be a discrete valuation ring. (Examples: $\widehat{\mathbb{Z}_{(p)}}, \mathbb{C}[[t]], k[x]_{(x)}$.) The only primes are $(0) \subseteq \mathfrak{m} = (t)$:

$$X = \operatorname{Spec} V = \{\mathfrak{m}, \nu\} = \overline{\{\nu\}}.$$

Open sets: $\varnothing, \{\nu, \mathfrak{m}\}, \{\nu\} = (X - \mathbb{V}(\mathfrak{m})) = D(t)$. We have

$$\widetilde{V}(\varnothing) = 0, \qquad \widetilde{V}(X) = V, \qquad \widetilde{V}(\nu) = V\left[\frac{1}{t}\right] = \operatorname{Frac}(V).$$

Stalks:

$$\widetilde{V}_{\mathfrak{m}} = V, \qquad \widetilde{V}_\nu = \operatorname{Frac}(V).$$

This can be drawn as a single point, plus a "fuzzy" point, e.g., the scheme of the "marked point" $0 \in \mathbb{A}^1$.

3.6 Equivalence of affine schemes and commutative rings

There is a contravariant functor

$$\mathbf{CRing} \to \{\text{affine schemes}\} \subseteq \{\text{schemes}\} \subseteq \{\text{locally ringed spaces}\}$$

sending each ring map $R \xrightarrow{\varphi} S$ to

$$\begin{gathered} \operatorname{Spec} S \xrightarrow{F = {}^a\varphi} \operatorname{Spec} R \\ \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}). \end{gathered}$$

We have a sheaf map on $\operatorname{Spec} R$:

$$\widetilde{R} \xrightarrow{f^\#} f_*\widetilde{S}$$

$$\widetilde{R}(D(g)) = R\left[\frac{1}{g}\right] \xrightarrow{\varphi} \widetilde{S}\left(f^{-1}(D(g))\right) = \widetilde{S}\left(D(\varphi(g))\right) = S\left[\frac{1}{\varphi(s)}\right].$$

This functor defines an (anti)equivalence of categories of commutative rings **CRing** and affine schemes **Aff**.

Proposition 3.24. *Say* $(\operatorname{Spec} B, \widetilde{B}) \xrightarrow{(f, f^\#)} (\operatorname{Spec} A, \widetilde{A})$ *is a morphism of locally ringed spaces. Then* $\left(f, f^\#\right)$ *is induced by the map of rings* $A \xrightarrow{\varphi} B$.

Proof. The map $\widetilde{A} \to f_*\widetilde{B}$ is a map of sheaves of rings on $\operatorname{Spec} A$. Evaluate at $\operatorname{Spec} A$:

$$\widetilde{A}(\operatorname{Spec} A) \to f_*\widetilde{B}(\operatorname{Spec} B) = \widetilde{B}\left(f^{-1}(\operatorname{Spec} A)\right) = \widetilde{B}(\operatorname{Spec} B) = B.$$

Need to show: The ring map $A \xrightarrow{\varphi} B$ induces the map $f : \operatorname{Spec} B \to \operatorname{Spec} A$. In other words, for all $P \in \operatorname{Spec} B$, we need $f(P) = \varphi^{-1}(P)$.

Note: We have a map of *locally ringed spaces* for all $P \in \operatorname{Spec} B$, the induced map on stalks

$$\begin{tikzcd} \widetilde{A}_{f(P)} \arrow[r] \arrow[rr, bend right] & (f_*\widetilde{B})_{f(p)} \arrow[r] & \widetilde{B}_P \end{tikzcd}$$

is a *local* map of local rings. We have a diagram

$$\begin{tikzcd} \widetilde{A}(\operatorname{Spec} A) \arrow[r, equal] & A \arrow[r, "\varphi"] \arrow[d] & B \arrow[r, equal] \arrow[d] & \widetilde{B}(\operatorname{Spec} B) \\ \varinjlim_{D(g) \ni P} \widetilde{A}(D(g)) \arrow[r, equal] \arrow[dr] & A_{f(P)} \arrow[r] \arrow[d, dashed, bend right, "\exists!"'] & B_P & \\ & A_{\varphi^{-1}(P)} \arrow[u, dashed, bend right, "\exists!"'] \arrow[ur] & & \end{tikzcd}$$

where the map from $A_{f(P)}$ to $A_{\varphi^{-1}(P)}$ is by the universal property of the direct limit, and the map in the opposite direction from the universal property of localization. These maps are inverse, giving an isomorphism.[1]

Now $\varphi^{-1}(P)$ and $f(P)$ are two prime ideals in A which have the same localization, hence $\varphi^{-1}(P) = f(P)$. $\square$

[1]All of this is in Hartshorne.

Remark 3.25. An affine scheme is "essentially the same as" a ring. A map of affine schemes is essentially the same as a ring map. In other words, $(X, \mathcal{O}_X)$ is determined by $\mathcal{O}_X(X)$.

There is an especially nice category, "quasi-coherent sheaves" of $\mathcal{O}_X$-modules, which are determined by modules over $\mathcal{O}_X(X)$.

4 The Proj construction

The "Proj" construction is a way to construct a (usually non-affine) scheme from a *graded* ring.

4.1 Graded rings

Definition 4.1. An $\mathbb{N}$-*graded ring* S (or a G-graded ring, where G is any semigroup) is a ring S together with a decomposition

$$S = \bigoplus_{n \in \mathbb{N}} S_n = S_0 \oplus S_1 \oplus S_2 \oplus \dots$$

such that $S_i \cdot S_j \subseteq S_{i+j}$.

Example 4.2 (Main example). $S = k[x, y]$, where $S_n =$ homogeneous polynomials of degree n.

Definition 4.3. An ideal $I \subseteq S$ is *homogeneous* if for all $f = \sum_i f_i \in I$ (where each $f_i \in S_i$), each homogeneous component f_i is in I.

Equivalently: I can be generated by homogeneously elements.

Example 4.4. The ideal

$$S_+ = S_1 \oplus S_2 \oplus S_3 \oplus \dots$$

is called the *irrelevant ideal*.

4.2 The set Proj

Fix an $\mathbb{N}$-graded ring S. Define the set

$$\operatorname{Proj} S = \left\{ \mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p} \text{ homogeneous, } \mathfrak{p} \not\supseteq S_+ \right\} \subseteq \operatorname{Spec} S \setminus \mathbb{V}(S_+).$$

Example 4.5. If $S = k[x, y]$ with $k = \overline{k}$, then

$$\operatorname{Proj} S \subseteq \operatorname{Spec} S \setminus \mathbb{V}(x, y) = \mathbb{A}^2 \setminus \{0\}.$$

Some ideals in $\operatorname{Proj} S$: (x), the generic point (0), etc.

We have $0 \neq \mathfrak{p} \in \operatorname{Proj} S$ iff $\mathfrak{p} \subseteq k[x, y]$ is homogeneous, height 1, generated by an irreducible polynomial. These correspond to irreducible subvarieties of $\mathbb{P}^1$, which correspond to "cone shaped" subvarieties of $\mathbb{A}^2$. In other words,

$$\operatorname{Proj} S = \left\{ \mathfrak{p} = (bx - ay) \mid [a : b] \in \mathbb{P}^1_k \right\} \cup \{(0)\}.$$

Example 4.6. The scheme

$$\operatorname{Proj} k[x,y,z] \subseteq \operatorname{Spec} S \setminus \mathbb{V}(x,y,z)$$

has three types of points, each corresponding to an irreducible subvariety of the classical $\mathbb{P}^2_k$:

(1) generic point (0)

(2) height 1 ideals, which correspond to $\mathfrak{p} = (f)$ irreducible, homogeneous.

(3) height 2 ideals, corresponding to points $[a : b : c]$ in $\mathbb{P}^2$, i.e., lines through $(0,0,0)$ in k^3:

$$\mathfrak{p} = (cx - az, cy - bz, bx - ay).$$

4.3 The Zariski topology on Proj

As a topological space, $\operatorname{Proj} S$ has the subspace topology induced from $\operatorname{Spec} S$.

In other words, the closed sets of $\operatorname{Proj} S$ are

$$\mathbb{V}(I) = \left\{ \mathfrak{p} \in \operatorname{Proj} S \;\middle|\; \mathfrak{p} \supseteq I \right\} \subseteq \operatorname{Proj} S,$$

where I is homogeneous.

The following open sets form a basis for the topology:

$$D_+(f) = \left\{ \mathfrak{p} \in \operatorname{Proj} S \;\middle|\; f \notin \mathfrak{p} \right\} \subseteq \operatorname{Proj} S.$$

Exercise 4.7. The height 2 ideals in Example 4.6 are *closed* in $\operatorname{Proj} S$.

4.4 Proj as a ringed space

Fix an $\mathbb{N}$-graded ring S. We now define the sheaf of rings $\mathcal{O}_X = \widetilde{S}$ on $X = \operatorname{Proj} S$.

On basic open sets $D_+(f)$, it is the ring

$$\mathcal{O}_X(D_+(f)) \stackrel{\text{def}}{=} \left[S\left[\frac{1}{f}\right] \right]_0 = \left\{ \frac{s}{f^t} \;\middle|\; \deg s = t \cdot d \right\}.$$

Example 4.8. If $X = \operatorname{Proj} k[x,y]$, we compute

$$\mathcal{O}_X(D_+(x)) = \left[k\left[x, y, \frac{1}{x}\right] \right]_0 = k\left[\frac{y}{x}\right].$$

Lemma 4.9. *If $D_+(f) \supseteq D_+(h)$ with f, h homogeneous, then without loss of generality, $h = gf$.*

Now we can define the restriction maps to be the following localization:

$$\begin{aligned} &\mathcal{O}_X(D_+(f)) \to \mathcal{O}_X(D_+(gf)) \\ &\left[S\left[\frac{1}{f}\right] \right]_0 \to \left[S\left[\frac{1}{gf}\right] \right]_0 = \left[S\left[\frac{1}{f}\right]\left[\frac{1}{g}\right] \right]_0. \end{aligned}$$

Note 4.10. If f is homogeneous of degree d, then

$$\left[S\left[\frac{1}{f}\right]\right]_0 = \bigcup_{t\in\mathbb{N}} \frac{S_{dt}}{f^t} \to \left[S\left[\frac{1}{gf}\right]\right]_0 = \left[S\left[\frac{1}{f}\right]\left[\frac{1}{g}\right]\right]_0 = \left[S\left[\frac{1}{f}\right]\right]_0\left[\left(\frac{g^d}{f^{\deg g}}\right)^{-1}\right].$$

(The last equality is an exercise.)

For any open $U \subseteq \operatorname{Proj} S$, define

$$\mathcal{O}_X(U) = \varprojlim_{D_+(f)\subseteq U} \mathcal{O}_X(D_+(f)) = \varprojlim_{D_+(f)\subseteq U} \left[S\left[\frac{1}{f}\right]\right]_0.$$

Exercise 4.11. Check that this is a sheaf.

4.5 Proj is locally ringed

We have now defined the ringed space $(\operatorname{Proj} S, \widetilde{S})$. Is it *locally* ringed?

Compute $\mathcal{O}_{X,\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Proj} S$:

$$\begin{aligned}
\mathcal{O}_{X,\mathfrak{p}} &= \varinjlim_{\mathfrak{p}\in D_+(f)} \mathcal{O}_X(D_+(f)) = \varinjlim_{\substack{f\notin\mathfrak{p} \\ \text{homog.}}} \left[S\left[\frac{1}{f}\right]\right]_0 \\
&= \left\{\frac{s}{f} \;\middle|\; s, f \text{ homogeneous of same degree, } f \notin \mathfrak{p}\right\} \\
&= \left[SU^{-1}\right]_0,
\end{aligned}$$

where U is the multiplicative system of homogeneous elements *not* in $\mathfrak{p}$.

Claim 4.12. $[SU^{-1}]_0$ *is a local ring whose maximal ideal is*

$$\left[\mathfrak{p}SU^{-1}\right]_0 = \left\{\frac{s}{f} \;\middle|\; s, f \textit{ homogeneous of same degree, } f \notin \mathfrak{p},\ s \in \mathfrak{p}\right\}.$$

To show that $[\mathfrak{p}SU^{-1}]_0$ is maximal, let $\frac{r}{f} \in [SU^{-1}]_0 \setminus [\mathfrak{p}SU^{-1}]_0$ be arbitrary. Then $r \notin \mathfrak{p}$, so $\frac{f}{r} \in [SU^{-1}]_0$.

Example 4.13. Consider $\mathfrak{p} = (x, y) \in \operatorname{Proj} k[x, y, z] = X$ with the usual grading. Then, for example,

$$\mathcal{O}_X(D_+(z)) = \left[k\left[x, y, z, \tfrac{1}{z}\right]\right]_0 = k\left[\frac{x}{z}, \frac{y}{z}\right],$$

so

$$\mathcal{O}_{X,\mathfrak{p}} = \varinjlim_{f\notin\mathfrak{p}} \left[S\left[\frac{1}{f}\right]\right]_0 = k\left[\frac{x}{z}, \frac{y}{z}\right] U^{-1} = k\left[\frac{x}{z}, \frac{y}{z}\right]_{\left(\frac{x}{z}, \frac{y}{z}\right)}.$$

Note that $[k[x, y, z]U^{-1}]$, without taking the degree 0 component, is *not* local.

4.6 Proj is a scheme

So far: $(\operatorname{Proj} S, \widetilde{S})$ is locally ringed.

Claim 4.14. $(\operatorname{Proj} S, \widetilde{S})$ *is a* scheme, *meaning that it has a cover by open sets* $\{U_\lambda\}$ *such that*

$$(U_\lambda, \mathcal{O}_X\big|_{U_\lambda}) \xrightarrow{\simeq} (\operatorname{Spec} A_\lambda, \widetilde{A_\lambda})$$

as locally ringed spaces.

Note 4.15. Since

$$\operatorname{Proj} S = \bigcup_{f \in S_+ \text{homog.}} D_+(f),$$

it suffices to check that each $D_+(f)$ is affine.

Proposition 4.16. *For all homogeneous* $f \in S_+$*, the basic open set* $U = D_+(f) \subseteq \operatorname{Proj} S$ *is an affine scheme. Namely,*

$$(U, \mathcal{O}_X\big|_U) \xrightarrow{\simeq} (\operatorname{Spec} A, \widetilde{A}),$$

where

$$A = \mathcal{O}_X(U) = \left[S\left[\frac{1}{f}\right]\right]_0.$$

Example 4.17. If $S = k[x, y, z]$, then

$$\operatorname{Proj} S = D_+(x) \cup D_+(y) \cup D_+(z),$$

so, as we will show,

$$D_+(x) \cong \operatorname{Spec} \underbrace{k\left[\frac{y}{x}, \frac{z}{x}\right]}_{\widetilde{S}(D_+(x))}.$$

To prove Proposition 4.16:

(1) Find φ homeomorphism

$$U = D_+(f) \xrightarrow{\varphi} \operatorname{Spec} A.$$

(2) Check that $\widetilde{A} \to \varphi_*(\widetilde{S})|_U$ is an isomorphism of sheaves.

(3) Check that φ induces an isomorphism of locally ringed spaces.

To find φ: We have

$$S \to S\left[\frac{1}{f}\right] \supseteq A = \left[S\left[f^{-1}\right]\right]_0.$$

Observe that if $\mathfrak{p} \in \operatorname{Proj} S$ with $f \notin \mathfrak{p}$, then $\mathfrak{p}S[f^{-1}]$ is a prime ideal, so

$$\mathfrak{p}S\left[f^{-1}\right] \cap A = \left[\mathfrak{p}S\left[f^{-1}\right]\right]_0$$

is prime in A. So define

$$\begin{aligned} \operatorname{Proj} S \supseteq D_+(f) &\xrightarrow{\varphi} \operatorname{Spec} A \\ \mathfrak{p} &\mapsto \varphi(\mathfrak{p}) = \left[\mathfrak{p}S\left[f^{-1}\right]\right]_0 \\ D_+(f) \cap \mathbb{V}(I) &\longleftrightarrow \mathbb{V}\left(\left[IS\left[f^{-1}\right]\right]_0\right). \end{aligned}$$

Exercise 4.18. Check that this is a homeomorphism which induces an isomorphism of locally ringed spaces.

4.7 Examples of Proj

Example 4.19 (Projective n-space over A). Define *projective n-space* to be

$$\mathbb{P}^n_A \overset{\text{def}}{=} (X = \operatorname{Proj} A[x_0, \ldots, x_n], \mathcal{O}_X)$$

with the standard grading. This has a cover by affine schemes $\{D_+(x_i)\}_{i=0}^n$, where

$$D_+(x_i) = \operatorname{Spec}\left[A[x_0, \ldots, x_n]\left[x_i^{-1}\right]\right]_0 = \operatorname{Spec} A\left[\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right],$$

a polynomial ring in n variables.

Example 4.20 (Weighted projective space over A). Let $S = A[x_0, \ldots, x_n]$, and say the degree of x_i is d_i. Denote the scheme

$$\mathbb{P}^n_A(d_0, \ldots, d_n) \overset{\text{def}}{=} \operatorname{Proj} A[x_0, \ldots, x_n].$$

For example, consider $\operatorname{Proj} k[x, y, z]$ with $\deg x = 2$ and $\deg y = \deg z = 1$:

$$D_+(x) = \operatorname{Spec}\left[k\left[x, y, z, \frac{1}{x}\right]\right]_0 = \operatorname{Spec} k\left[\frac{y^2}{x}, \frac{yz}{x}, \frac{z^2}{x}\right] \cong \operatorname{Spec}\frac{k[S, T, V]}{(SV - T^2)}.$$

By contrast, with $\operatorname{Proj} k[x, y, z]$ with the grading,

$$D_+(x) = \operatorname{Spec} k\left[\frac{y}{x}, \frac{z}{x}\right].$$

Remark 4.21. We have a natural map

$$\begin{array}{ccccccc} \operatorname{Proj} A[x_0, \ldots, x_n] & \supseteq & D_+(x_i) & = & \operatorname{Spec} A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] & & A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \\ \downarrow & & & & \downarrow & & \uparrow \\ \operatorname{Spec} A & & & & \operatorname{Spec} A & & A \end{array}$$

which expresses $\operatorname{Proj} A[x_0, \ldots, x_n]$ as a scheme of finite type over A.

Example 4.22. Consider

$$S = \frac{k[x, y, z]}{(x^2 + y^2 - z^2)}$$

with the standard grading. We have

$$\begin{array}{ccc} \operatorname{Proj} S & \subseteq & \operatorname{Spec} S \setminus \mathbb{V}(x, y, z) \\ \downarrow & & \\ \operatorname{Spec} k & & \end{array}$$

The closed points in $\operatorname{Proj} S$ are in bijective correspondence with classical points on the projective variety $\mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2_k$.

There is also a generic point (0). This is the only non-closed point; the dimension of $\operatorname{Proj} S$ is 1.

On $D_+(z)$:

$$D_+(z) = \operatorname{Spec}\left[S\left[\frac{1}{z}\right]\right]_0 = \operatorname{Spec}\frac{k\left[\frac{x}{z}, \frac{y}{z}\right]}{\left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right)}.$$

We call this ring a *conic*, even if k is replaced with some arbitrary ring.

Example 4.23. There is the conic

$$\operatorname{Proj}\frac{\mathbb{Z}[x,y,z]}{(x^2+y^2-z^2)},$$

called a "conic in $\mathbb{P}^2_{\mathbb{Z}}$." We then have

$$D_+(z) = \operatorname{Spec}\frac{\mathbb{Z}\left[\frac{x}{z}, \frac{y}{z}\right]}{\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1}.$$

This has dimension 2.

4.8 Example from the homework

Let A be a k-algebra $k \to A$; i.e., $\operatorname{Spec} A$ is a k-scheme $\operatorname{Spec} A \to \operatorname{Spec} k$.

Say

$$\operatorname{Spec}\frac{k[\varepsilon]}{\varepsilon^2} \to \operatorname{Spec} A$$

is a map of k-schemes. Then we have a commutative diagram of k-algebra maps

$$\begin{tikzcd} A \arrow[r] \arrow[dr, two heads, "\eta"'] & \frac{k[\varepsilon]}{\varepsilon^2} \arrow[d, two heads, "\text{kill } \varepsilon"] \\ & k, \end{tikzcd}$$

where the kernel of η is $\mathfrak{m}_P \in \operatorname{Spec} A$. We then have:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[d, hook] \\ \operatorname{Spec}\frac{k[\varepsilon]}{\varepsilon^2} \arrow[d] \\ \operatorname{Spec} A \end{tikzcd}$$

We get that a k-scheme map $\operatorname{Spec}\frac{k[\varepsilon]}{\varepsilon^2} \to \operatorname{Spec} A$ is giving us a k-rational point $\mathfrak{m}_P \in \operatorname{Spec} A$ together with a tangent vector to $\mathfrak{m}_P$.

4.9 Étale neighborhoods

What about the scheme $\operatorname{Spec} A/\mathfrak{m}_P^2$? This is essentially the point $\mathfrak{m}_P$, together with its cotangent space.

If $A = k[x,y]$ and $\mathfrak{m}_P = (x,y)$, then we have

$$\operatorname{Spec} k \hookrightarrow \operatorname{Spec} \frac{k[x,y]}{(x,y)^2} \hookrightarrow \operatorname{Spec} k[x,y]$$

$$k \twoheadleftarrow \frac{k[x,y]}{(x,y)^2} \twoheadleftarrow k[x,y]$$

sending $f = \lambda_0 + \lambda_1 x + \lambda_2 y + \dots$ to $\lambda_0 + \lambda_1 x + \lambda_2 y + \mathfrak{m}^2$, and then to $\lambda_0 + \mathfrak{m}$.

Likewise, $k[x,y]/(\mathfrak{m}^3)$ preserves the degree ≤ 2 terms of f, and similarly for higher degree.

Taking limits, we have

$$\operatorname{Spec} k[[x,y]] = \varinjlim \left(\operatorname{Spec} k \hookrightarrow \operatorname{Spec} \frac{k[x,y]}{\mathfrak{m}} \hookrightarrow \operatorname{Spec} \frac{k[x,y]}{\mathfrak{m}^2} \hookrightarrow \operatorname{Spec} \frac{k[x,y]}{\mathfrak{m}^3} \hookrightarrow \dots \right)$$
$$k[[x,y]] = \varprojlim \left(k \twoheadleftarrow \frac{k[x,y]}{\mathfrak{m}} \twoheadleftarrow \frac{k[x,y]}{\mathfrak{m}^2} \twoheadleftarrow \frac{k[x,y]}{\mathfrak{m}^3} \twoheadleftarrow \dots \right).$$

Now consider $\mathbb{V}(y^2 - x^2 - x^3) \subseteq \operatorname{Spec} k[x,y] = \mathbb{A}^2$. The polynomial $y^2 - x^2 - x^3$ does not factor in $k[x,y]$; however, pulling back to $\operatorname{Spec} k[[x,y]]$,

$$y^2 - x^2 - x^3 = \left(y - x\sqrt{1+x}\right)\left(y + x\sqrt{1+x}\right)$$

in $k[[x,y]]$, where $\sqrt{1+x}$ is given by its Taylor series expansion. (By Hensel's lemma, z^2+1+x has a solution in $k[[x,y]]$.) This can be thought of as an "étale neighborhood" which is smaller than the Zariski neighborhoods.

5 Constructions on schemes

5.1 Products in the category of S-schemes

Let S be any scheme. [Classical example: $\operatorname{Spec} k$, where $k = \overline{k}$.]

Given two S-schemes $X \to S$, $Y \to S$, is there a product? That is, a scheme "$X \times_S Y$" together with "projections" (maps of S-schemes)

$$X \times_S Y \xrightarrow{\pi_1} X$$
$$X \times_S Y \xrightarrow{\pi_2} Y$$

satisfying the universal property:

$$\begin{tikzcd}
 & W \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & X \times_S Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X \arrow[dr] & & Y \arrow[dl] \\
 & S &
\end{tikzcd}$$

If there exists an S-scheme W fitting in the above diagram, then there is a unique S-scheme map $W \dashrightarrow X \times_S Y$ making the diagram commute.

Theorem 5.1. *Products exist in the category of S-schemes (unique up to unique isomorphism).*

Proof sketch. Do the affine case, and glue together. Affine case: $S = \operatorname{Spec} A$, and we have maps $A \to R$, $A \to T$ (i.e., S-schemes $\operatorname{Spec} R \to \operatorname{Spec} A$, $\operatorname{Spec} T \to \operatorname{Spec} A$).

We need an A-scheme Z together with

$$Z \xrightarrow{\pi_1} \operatorname{Spec} R$$
$$Z \xrightarrow{\pi_2} \operatorname{Spec} T$$

making the product diagram commute.

Let $Z = \operatorname{Spec}(R \otimes_A T)$. Then

$$\begin{tikzcd}
 & B & \\
 & R \otimes_A T \arrow[u, dashed, "\exists!"] & \\
T \arrow[ur] \arrow[uur, bend left] & & T \arrow[ul] \arrow[uul, bend right] \\
 & A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The details of the proof are all worked out in Hartshorne. $\square$

5.2 Examples of products

Example 5.2. Over $S = \operatorname{Spec} k$, let

$$X = \operatorname{Spec} k[x_1, x_2] = \mathbb{A}_k^2,$$
$$Y = \operatorname{Spec} k[y_1, y_2] = \mathbb{A}_k^2.$$

Then

$$X \times Y = \operatorname{Spec}\left(k[x_1,x_2]\otimes_k k[y_1,y_2]\right) = \operatorname{Spec} k[x_1,x_2,y_1,y_2] = \mathbb{A}^4_k.$$

Example 5.3. Over $S = \operatorname{Spec}\mathbb{R}$, let $X = \operatorname{Spec}\mathbb{C}$. Then

$$\mathbb{A}^2_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} = \operatorname{Spec}\left(\mathbb{R}[x,y]\otimes_{\mathbb{R}}\mathbb{C}\right) = \operatorname{Spec}\mathbb{C}[x,y].$$

Example 5.4. Over $\mathbb{R}$, consider

$$\operatorname{Spec}\frac{\mathbb{R}[x,y]}{(x^2+y^2)} \to \operatorname{Spec}\mathbb{R}.$$

Base change: make it a scheme over $\mathbb{C}$:

$$\operatorname{Spec}\left(\frac{\mathbb{R}[x,y]}{(x^2+y^2)}\right) \times_{\mathbb{R}} \operatorname{Spec}\mathbb{C} = \operatorname{Spec}\frac{\mathbb{C}[x,y]}{(x^2+y^2)}.$$

Not irreducible.

Caution 5.5. The product $X \times_S Y$ can be non-irreducible, non-reduced, etc., even if X, Y, S are all irreducible.

5.3 Products of S-schemes, continued

Global picture:

$$\begin{tikzcd}
& X \times_S Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

Local picture:

$$\begin{tikzcd}
& \operatorname{Spec} R\otimes_A T \arrow[dl] \arrow[dr] & \\
\operatorname{Spec} R \arrow[dr] & & \operatorname{Spec} T \arrow[dl] \\
& \operatorname{Spec} A &
\end{tikzcd}$$

Example 5.6. Let $X = \operatorname{Proj} k[x,y,z]$ (with the standard grading), $Y = \operatorname{Spec} k[t]$: both k-schemes. We have

$$X = D_+(x)\cup D_+(y)\cup D_+(z) = \operatorname{Spec} k\left[\frac{y}{x},\frac{z}{x}\right] \cup \operatorname{Spec} k\left[\frac{x}{y},\frac{z}{y}\right] \cup \operatorname{Spec} k\left[\frac{x}{z},\frac{y}{z}\right].$$

So

$$\begin{aligned}
X \times_k Y &= (D_+(x)\times_k \operatorname{Spec} k[t]) \cup (D_+(y)\times_k \operatorname{Spec} k[t]) \cup (D_+(z)\times_k \operatorname{Spec} k[t]) \\
&= \operatorname{Spec} k\left[t,\frac{y}{x},\frac{y}{z}\right] \cup \operatorname{Spec} k\left[t,\frac{x}{y},\frac{z}{y}\right] \cup \operatorname{Spec} k\left[t,\frac{x}{z},\frac{y}{z}\right] \\
&= \operatorname{Proj} k[t][x,y,z].
\end{aligned}$$

To summarize:

$$\begin{aligned} X \times_k Y &= \operatorname{Proj} k[t][x,y,z] = \mathbb{P}^2_{k[t]} \\ \mathbb{P}^2_k \times \mathbb{A}^1 &= \mathbb{P}^2_{k[t]} = \mathbb{P}^2_{\mathbb{A}^1}. \end{aligned}$$

We have the family

$$\begin{aligned} \mathbb{P}^2_{k[t]} &\xrightarrow{\pi_2} \operatorname{Spec} k[t] = \mathbb{A}^1_k \\ \mathbb{P}^2 \times \mathbb{A}^1 &\to \operatorname{Spec} k[t] \\ \mathbb{P}^2 \times \{\lambda\} &\mapsto \lambda. \end{aligned}$$

5.4 Base change

Given a family $X \to B$ of schemes ("given a B-scheme") parametrized by B, we can "change base" for any $B' \to B$ by considering the new family

$$X \times_B B' \xrightarrow{\pi_2} B'.$$

This is called "*base change* to B'."

Example 5.7. Given $\mathbb{P}^2_k \to \operatorname{Spec} k$, we get a new family

$$\mathbb{P}^2_{k[t]} = \mathbb{P}^2_k \times_k \operatorname{Spec} k[t] \to \operatorname{Spec} k[t].$$

Example 5.8 (Main applications). Consider

$$\operatorname{Proj} \frac{\mathbb{Z}[x,y,z]}{(x^3+y^3+z^3)} \to \operatorname{Spec} \mathbb{Z}.$$

The map $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Z}$ lets us change base:

$$\operatorname{Proj} \frac{\overline{\mathbb{Q}}[x,y,z]}{(x^3+y^3+z^3)} = \operatorname{Proj} \frac{\mathbb{Z}[x,y,z]}{(x^3+y^3+z^3)} \times_{\mathbb{Z}} \overline{\mathbb{Q}} \xrightarrow{\pi_2} \operatorname{Spec} \overline{\mathbb{Q}}.$$

5.5 Fibers

First, we motivate the definition with an example.

Example 5.9. Consider the following product over $\operatorname{Spec} k[t]$:

$$\begin{tikzcd} \left(\operatorname{Spec} \frac{k[t]}{(t-\lambda)}\right) \otimes_{k[t]} \left(\operatorname{Spec} \frac{k[t][x,y]}{(xy-t)}\right) \arrow[r] \arrow[d] & \operatorname{Spec} \frac{k[t][x,y]}{(xy-t)} \arrow[d] \\ \operatorname{Spec} \frac{k[t]}{(t-\lambda)} \arrow[r] & \operatorname{Spec} k[t] \end{tikzcd}$$

We have

$$\left(\operatorname{Spec}\frac{k[t]}{(t-\lambda)}\right)\otimes_{k[t]}\left(\operatorname{Spec}\frac{k[t][x,y]}{(xy-t)}\right)=\operatorname{Spec}\left(\frac{k[t]}{(t-\lambda)}\otimes_{k[t]}\frac{k[t][x,y]}{(xy-t)}\right)$$
$$=\operatorname{Spec}\frac{\frac{k[t]}{(t-\lambda)}[x,y]}{(xy-t)}$$
$$=\operatorname{Spec}\frac{k[x,y]}{(xy-\lambda)}$$
$$=\text{fiber over } \lambda\in\mathbb{A}^1.$$

In more detail, we have the map

$$X=\operatorname{Spec}\frac{k[t,x,y]}{(ty-x^2)}\to\operatorname{Spec}k[t]=S.$$

A point $(t-\lambda)=P\in\operatorname{Spec}k[t]$ corresponds to

$$Y=\operatorname{Spec}k(P)=\frac{k[t]}{(t-\lambda)}\hookrightarrow\operatorname{Spec}k[t].$$

We have

$$\begin{tikzcd}
\operatorname{Spec}\frac{k[x,y]}{(\lambda y-x^2)} \arrow[r] \arrow[d] & \operatorname{Spec}\frac{k[t,x,y]}{(ty-x^2)} \arrow[d] \\
\operatorname{Spec}\frac{k[t]}{(t-\lambda)} \arrow[r, hook] & \operatorname{Spec}k[t].
\end{tikzcd}$$

This is the (classical) fiber over λ.

Definition 5.10. If $X\xrightarrow{f}B$ is a morphism of schemes and $p\in B$, then the (scheme theoretic) *fiber* is $X\times_B\operatorname{Spec}k(p)$. [Here, if $p\in U=\operatorname{Spec}R\subseteq B$, then $k(p)=R_p/pR_p\leftarrow R$.]

Example 5.11. Continuing from Example 5.9, we compute the fiber over the origin in $\operatorname{Spec}k[t]$:

$$f^{-1}(\text{origin})=\operatorname{Spec}k(\text{origin})\times_{\operatorname{Spec}k[t]}\operatorname{Spec}\frac{k[t,x,y]}{(ty-x^2)}$$
$$=\operatorname{Spec}\left(\frac{k[t]}{(t)}\otimes_{k[t]}\frac{k[t,x,y]}{(ty-x^2)}\right)=\operatorname{Spec}\frac{k[x,y]}{(x^2)}.$$

Now, the fiber of the generic point:

$$\begin{tikzcd}
\operatorname{Spec}\frac{k(t)[x,y]}{(ty-x^2)} \arrow[r, hook] \arrow[d] & \operatorname{Spec}\frac{k[t,x,y]}{(ty-x^2)} \arrow[d, "f"] \\
\operatorname{Spec}k(t) \arrow[r, hook] & \operatorname{Spec}k[t]
\end{tikzcd}$$

So

$$f^{-1}(\text{generic point})=\operatorname{Spec}\frac{k(t)[x,y]}{(ty-x^2)}.$$

Example 5.12. Given a morphism of affine schemes $\operatorname{Spec} S \xrightarrow{f} \operatorname{Spec} R$ and a point $\mathfrak{p} \in \operatorname{Spec} R$, we have

$$f^{-1}(\mathfrak{p}) = \operatorname{Spec}\left(\frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}} \otimes_R S\right).$$

Exercise: Check that there is a homeomorphism

$$\operatorname{Spec}\left(\frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}} \otimes_R S\right) \cong f^{-1}(\mathfrak{p}) \subseteq \operatorname{Spec} S.$$

In other words, $f^{-1}(\mathfrak{p})$ is also the set-theoretic fiber.

Example 5.13. Consider the family

$$\operatorname{Proj} \frac{\mathbb{Z}[x,y,z]}{(x^3+y^3+z^3)} \xrightarrow{f} \operatorname{Spec} \mathbb{Z}.$$

If p is prime, then

$$f^{-1}(p) = \operatorname{Proj} \frac{(\mathbb{Z}/p\mathbb{Z})[x,y,z]}{(x^3+y^3+z^3)},$$

and

$$f^{-1}(\text{generic point}) = \operatorname{Proj} \frac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3)}.$$

This is *reduction to characteristic* p. So if a property holds for a typical $\mathbb{Z}/p\mathbb{Z}$, then it holds for the generic fiber $\mathbb{Q}$. In this case, $p = 3$ is the "non-typical" case.

6 Quasi-coherent sheaves

Most important class: *coherent* sheaves (of modules) on *schemes.*

6.1 Sheaves of modules

Definition 6.1. Fix a ringed space $(X, \mathcal{O}_X)$. An $\mathcal{O}_X$-module (or a *sheaf of modules* on X) $\mathscr{F}$ is a sheaf of abelian groups on X such that for all open $U \subseteq X$, there is an action of $\mathcal{O}_X(U)$ on $\mathscr{F}(U)$ making $\mathscr{F}(U)$ into an $\mathcal{O}_X(U)$-module, compatibly with restriction: for all open $U' \subseteq U$, and for all $r \in \mathcal{O}_X(U)$ and $m \in \mathscr{F}(U)$,

$$(rm)\big|_{U'} = (r\big|_{U'})m\big|_{U'} \in \mathscr{F}(u').$$

Note 6.2. For any $P \in X$, the stalk $\mathscr{F}_P$ is an $\mathcal{O}_{X,P}$-module.

Example 6.3 (Trivial examples). A ringed space $\mathcal{O}_X$ is itself a $\mathcal{O}_X$-module. Also, there is the free $\mathcal{O}_X$-module of rank n:

$$\underbrace{\mathcal{O}_X \oplus \ldots \oplus \mathcal{O}_X}_{n \text{ copies}}$$

Example 6.4 (Vector bundles). Let X be a smooth manifold. Then (X, C_X^∞) is a ringed space given by

$$C_X^\infty(U) = \left\{ U \xrightarrow{\varphi} \mathbb{R} \mid \varphi \text{ smooth} \right\}.$$

Say

$$V \xrightarrow{\pi} X$$

is a rank n vector bundle over X. Let $\mathcal{V}$ be the sheaf of smooth sections of V over X:

$$\mathcal{V}(U) = \left\{ U \xrightarrow{s} V \mid s \text{ smooth}, \ \pi \circ s = \mathrm{id}_U \right\}.$$

Observe that $\mathcal{V}$ is a sheaf of abelian groups:

$$(U \xrightarrow{s_1} V) + (U \xrightarrow{s_2} V) = \big(U \to V, \ x \mapsto s_1(x) + s_2(x)\big),$$

where $s_1(x) + s_2(x) \in V_x = \pi^{-1}(x)$ in V; this is an $\mathbb{R}$-vector space.

Moreover, $\mathcal{V}$ is a C^∞-module: for any open $U \subseteq X$, $\mathcal{V}(U)$ is a $C^\infty(U)$-module, given for all $s \in \mathcal{V}(U)$ and $f \in C^\infty(U)$ by

$$\begin{aligned} fs : U &\to V \\ x &\mapsto f(x)s(x). \end{aligned}$$

Example 6.5 (The trivial vector bundle). Consider $V \times \mathbb{R}^n \supseteq U \times \mathbb{R}^n$ as a vector bundle over $V \supseteq U$. Then

$$[C_X^\infty(U)]^{\oplus n} = \mathcal{V}(U) \cong C_X^\infty(U) \oplus \ldots \oplus C_X^\infty(U).$$

Easy to check: $\mathcal{V} \cong C_X^{\oplus n}$.

The stalks of $\mathcal{V}$ are, for $x \in X$,

$$\mathcal{V}_x = \varinjlim_{U \ni x} \mathcal{V}(U) = \left(C_{X,x}^\infty\right)^{\oplus n}.$$

Given open $U' \subseteq X$, we have

$$\begin{tikzcd} V \arrow[d, "\pi"'] & \supseteq & \pi^{-1}(U') \arrow[r, "\varphi", "\simeq"'] \arrow[d] & U^1 \times \mathbb{R}^n \arrow[dl] \\ X & \ni & U' & \end{tikzcd}$$

We have

$$\mathcal{V}\big|_{U'} \cong (C_{U'}^\infty)^{\oplus n}.$$

6.2 Quasi-coherent sheaves

Idea: "globalization" of the idea of an R-module. Rings are to schemes are modules are to quasi-coherent sheaves.

Example 6.6. Fix a ring A and an A-module M. The sheaf associated to M on $\operatorname{Spec} A$ is

$$\widetilde{M}\big(D(g)\big) = M\big[g^{-1}\big] = M \otimes_A A\big[g^{-1}\big],$$

which is obviously an $A[g^{-1}] = \mathcal{O}_X\big(D(g)\big)$-module.

It is easy to check that $\widetilde{M}$ is a presheaf of modules on $\operatorname{Spec} A$:

$$\widetilde{M}(U) = \varprojlim_{D(g)\subseteq U} \widetilde{M}\big(D(g)\big) = \varprojlim_{D(g)\subseteq U} M\big[g^{-1}\big].$$

Easy to see the stalks

$$\widetilde{M}_{\mathfrak{p}} = \varinjlim_{\mathfrak{p}\in D(g)} \widetilde{M}\big(D(g)\big) = \varinjlim_{g\notin\mathfrak{p}} M\big[g^{-1}\big] = M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}.$$

Proposition 6.7. *In fact, $\widetilde{M}$ as we defined it is a* sheaf. *[so $\widetilde{M}(\operatorname{Spec} A) = M$]*

Proof. The sheafification of $\widetilde{M}$ on U is

$$\widetilde{M}(U) = \left\{ U \to \coprod_{P\in U} M_P \;\middle|\; \begin{array}{c} s(P) \in M_P, \text{ and } \forall P \in U, \exists \text{ neighborhood } D(g) \text{ and} \\ m \in M\big[g^{-1}\big] \text{ such that } \forall Q \in D(g),\ s(Q) = \text{germ of } m \\ \text{in } M_Q \end{array} \right\}.$$

See Hartshorne II.5 for the details. □

Definition 6.8. A *quasi-coherent sheaf* on a scheme $(X, \mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X$-modules M such that X has an open cover by affines $U_i = \operatorname{Spec} A_i$ where $M|_{U_i} = \widetilde{M_i}$ for some A_i-module M_i.

If X is Noetherian, we say M is *coherent* if the M_i can be taken to be finitely generated A_i-modules.[2]

Proposition 6.9. *If $\mathscr{F}$ is a quasi-coherent sheaf on an affine scheme* $\operatorname{Spec} A$*, then $\mathscr{F} = \widetilde{M}$ for some A-module M (i.e., $M = \mathscr{F}(\operatorname{Spec} A)$).*

Proof. Prove the following lemma: If $\mathscr{F}$ is a quasi-coherent sheaf on any scheme $(X, \mathcal{O}_X)$ and $\operatorname{Spec} A = U \subseteq X$ is any open *affine* subset, then $\mathscr{F}|_U = \widetilde{\mathscr{F}(U)}$. (See Hartshorne.) □

Fact 6.10. If $M \to N$ is an A-module homomorphism, then there is a naturally induced homomorphism of $\mathcal{O}_X$-modules $\widetilde{M} \to \widetilde{N}$ on $X = \operatorname{Spec} A$.

There is a functor

$$A\text{-}\mathbf{Mod} \to \{\text{sheaves of abelian groups}\}$$

which induces an equivalence of categories

$$A\text{-}\mathbf{Mod} \xrightarrow{\simeq} \{\text{quasi-coherent sheaves on } \operatorname{Spec} A\}.$$

[2]If we do not require X to be Noetherian, then the correct criterion is that the M_i be *coherent* A_i-modules. A *coherent module* is a finitely generated module whose finitely generated submodules are finitely presented.

6.3 Examples of quasi-coherent sheaves

Fix a scheme X. Some quasi-coherent sheaves on X:

- $\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_X$-module.
- $\mathcal{O}_X^{\oplus n}$ is the free quasi-coherent $\mathcal{O}_X$-module of rank n.
- If $V \xrightarrow{\pi} X$ is an algebraic vector space of varieties, then the sheaf of *regular* sections $\mathcal{V}$ is a quasi-coherent sheaf on X. The stalks are

  $$\mathcal{V}_P = \mathcal{O}_{X,P}^{\oplus \operatorname{rank} V},$$

  and the vector bundle is given by

  $$\begin{tikzcd} U \times \mathbb{A}_k^n \arrow[drr] & \pi^{-1}(U) \arrow[l, "\simeq"'] & \subseteq & V \arrow[d, "\pi"] \\ & & U \arrow[ul, "s"] & \subseteq & X, \end{tikzcd}$$

  where for $p \in U$,

  $$p \mapsto \big(p, s_1(p), \ldots, s_n(p)\big)$$

  with $s_i \in \mathcal{O}_X(U)$.
- Say $X = \operatorname{Spec} A$, and let M be any locally free (projective), rank 1 module[3] which is *not* free. Then $\widetilde{M}$ is a quasi-coherent sheaf on X, not free, but the stalks are

  $$\widetilde{M}_P = M_P \cong A_P.$$

Example 6.11. Consider $A = \mathbb{Z}\big[\sqrt{-5}\big]$ and $I = \big(2, 1 + \sqrt{-5}\big) \subseteq A$. Then I is height 1, and is *not* free as an A-module. However, it is locally free: since A is a Dedekind domain, $I_P \subseteq A_P$ is principal, so we can write $I_P = (f \cdot A_P)$, and

$$\begin{aligned} A_P &\xrightarrow{\simeq} f \cdot A_P \\ 1 &\mapsto f \end{aligned}$$

is clearly an isomorphism.

Example 6.12. Let $Y \overset{i}{\subseteq} X$ be a closed subscheme. Then we have the map of sheaves of rings on X

$$\mathcal{O}_X \twoheadrightarrow i_* \mathcal{O}_Y.$$

So the *kernel*

$$\mathscr{I} = \mathscr{I}_Y \subseteq \mathcal{O}_X$$

[3] If M is an A-module, where A is a domain, then the *rank* of M is the dimension of $M \otimes_A K$ as a K-vector space, where $K = \operatorname{Frac}(A)$ is the fraction field.

is a sheaf of ideals in X (i.e., $\mathcal{O}_X$-modules). It is *quasi-coherent*: For any affine $U \subseteq X$, we have $Y \cup U \subseteq X \cap U = U$, and the exact sequence

$$0 \to \mathscr{I}_Y(U) \subseteq \mathcal{O}_X(U) \twoheadrightarrow \mathcal{O}_Y(Y \cap U).$$

On U, need

$$\mathscr{I}\big|_U = \widetilde{\mathscr{I}(U)} = \widetilde{I}.$$

If $U = \operatorname{Spec} A$, $Y \cap U = \operatorname{Spec}(A/I)$ under the closed embedding $Y \cap U \hookrightarrow U$ induced by $A \twoheadrightarrow A/I$.

We need to show that for any basic open $\operatorname{Spec} A[g^{-1}] = D(g) \subseteq U$. Need: $\mathscr{I}\big(D(g)\big) = I[g^{-1}]_{A[g^{-1}]}$. The following are exact sequences:

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

$$0 \longrightarrow I[g^{-1}] \longrightarrow A[g^{-1}] \longrightarrow (A/I)[g^{-1}] \longrightarrow 0$$

Proposition 6.13. *The category of quasi-coherent sheaves on a scheme X is closed under taking direct sums, kernels, cokernels, direct limits, and inverse limits.*

6.4 Equivalence of modules and q.c. sheaves

Fix a ring A, and let $X = \operatorname{Spec} A$. There is an equivalence of categories

$$\begin{aligned} \{A\text{-modules}\} &\longleftrightarrow \{\text{quasi-coherent sheaves on } \operatorname{Spec} A\} \subseteq \{\text{sheaves of modules on } \operatorname{Spec} A\} \\ M &\mapsto \widetilde{M} \\ A &\mapsto \widetilde{A} = \mathcal{O}_X \\ \mathscr{F}(\operatorname{Spec} A) &\mapsfrom \mathscr{F} \end{aligned}$$

Given an element $f \in M$, the "value of f at $\mathfrak{p} \in \operatorname{Spec} A$" is the image of f in

$$\left(A \to \frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}}\right) \otimes_A M$$
$$M \to \frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}} = \text{fiber of } \widetilde{M} \text{ over } \mathfrak{p}.$$

Operations on A-modules (e.g., $M \otimes_A N$, where M, N are A-modules) induce operations of sheaves of A-modules (e.g., $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} = (\widetilde{M \otimes_A N})$), which induce operations on quasi-coherent sheaves on *arbitrary* schemes: if $\mathscr{F}, \mathscr{G}$ are quasi-coherent sheaves on X, define $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ by locally on affines U setting

$$(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})(U) = \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U).$$

Exercise 6.14. If $\mathscr{F}, \mathscr{G}$ are quasi-coherent, then $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is quasi-coherent.

Caution 6.15. It is *not* true for *all* open U that $(\mathscr{F} \otimes \mathscr{G})(U) = \mathscr{F}(U) \otimes \mathscr{G}(U)$.

6.5 Key functors on quasi-coherent sheaves

Given A-modules M, N, the hom-set $\operatorname{Hom}_A(M,N)$ induces a sheaf

$$\mathscr{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) = \widetilde{\operatorname{Hom}_A(M,N)}.$$

Say we have a map of schemes

$$X \xrightarrow{f} Y, \qquad\qquad \mathcal{O}_Y \to f_*\mathcal{O}_X.$$

If $\mathscr{F}$ is quasi-coherent on X, then $f_*\mathscr{F}$ is quasi-coherent on Y.

Local picture: $A \xrightarrow{f^\#} B$ is a ring map. If M is a B-module, then M is an A-module by restriction of scalars, and the quasi-coherent sheaf $\widetilde{M}$ on $\operatorname{Spec} B$ is sent to $f_*\widetilde{M} = {}_A\widetilde{M}$, a quasi-coherent sheaf on $\operatorname{Spec} A$.

In other words, f_* is "restriction of scalars" to Y.

Caution 6.16. If $\mathscr{F}$ is *coherent*, then $f_*\mathscr{F}$ is quasi-coherent, but in general *not* coherent. For example, consider

$$X = \operatorname{Spec} k[t] \xrightarrow{f} \operatorname{Spec} k$$
$$\widetilde{M} = \widetilde{k[t]} = \mathcal{O}_X \mapsto f_*\mathcal{O}_X = {}_k\big(\widetilde{k[t]}\big),$$

but $k[t]$ is *not* a finitely generated k-module.

Next, if $\mathscr{F}$ is a quasi-coherent sheaf on Y, then $f^{-1}(\mathscr{F})$ is a sheaf of modules over $f^{-1}\mathcal{O}_X$: for any open $U \subseteq X$,

$$f^{-1}(\mathscr{F})(U) = \text{“}\mathscr{F}\big(f(U)\big)\text{”} = \varinjlim_{V \supseteq f(U)} \mathcal{O}_Y(V),$$

so $f^{-1}\mathscr{F}(U)$ is a module over $\varinjlim_{V\supseteq f(U)} \mathcal{O}_Y(V) = f^{-1}\mathcal{O}_Y(U)$.

Definition 6.17. For $\mathscr{F}$ any sheaf of $\mathcal{O}_Y$-modules on a ringed space $(Y, \mathcal{O}_Y)$ and any morphism $(X, \mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y, \mathcal{O}_Y)$, define

$$f^*\mathscr{F} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{F}.$$

In practice, consider the local picture: given a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, $A \to B$ and an A-module M, we have

$$f^*\widetilde{M} = \widetilde{(B \otimes_A M)}.$$

6.6 Example on Proj

Consider $S = k[x,y]$ with the standard $\mathbb{N}$-grading, and let M be the Z-graded S-module given by $M = S$ as an abelian group, and as an S-module with the grading shifted by d:

$$M_n = S_{n+d}.$$

Notation: $M = S(d)$. This is the same S-module structure as the trivial module S, but with a different grading.

Define a sheaf on $\operatorname{Proj} S$ as follows:

$$D_+(f) \mapsto \widetilde{M}\big(D_+(f)\big) = \big[M[f^{-1}]\big]_0.$$

This is a $[S[f^{-1}]]_0$-module: if $\deg s = t \cdot \deg f$ and $\deg m = e \cdot \deg f$, then

$$\left(\frac{s}{f^t} \cdot \frac{m}{f^e}\right) = \frac{sm}{f^{t+\ell}},$$

and all of the above are degree 0.

Remark 6.18. For *any* $\mathbb{Z}$-graded S-module M over *any* $\mathbb{N}$-graded ring S,

$$\widetilde{M}(U) = \varprojlim_{D_+(f) \subseteq U} \big[M[f^{-1}]\big]_0$$

is a module over

$$\mathcal{O}_X(U) = \varprojlim_{D_+(f) \subseteq U} \big[S[f^{-1}]\big]_0.$$

We denote this sheaf $\widetilde{M}$.

Exercise 6.19. This is a quasi-coherent sheaf on $\operatorname{Proj} S$. On an affine set $D_+(f)$,

$$\widetilde{M}\big|_{D_+(f)} = \widetilde{M[f^{-1}]_0},$$

where $D_+(f) = \operatorname{Spec}[S[f^{-1}]]_0$.

Returning to our example with Proj, let us compute $\widetilde{M} = \widetilde{S(1)}$ on $\operatorname{Proj} S = \operatorname{Proj} k[x,y] = \mathbb{P}^1_k$:

$$\begin{aligned}
\mathbb{P}^1_k = D_+(x) \cup D_+(y) &= \operatorname{Spec} k\left[\frac{y}{x}\right] \cup \operatorname{Spec} k\left[\frac{x}{y}\right] \\
\widetilde{M}\big(D_+(x)\big) = \left[M\left[\frac{1}{x}\right]\right]_0 &= \left\{ \frac{m}{x^t} \;\middle|\; m \in M_t = [S(1)]_t = S_{t+1} \right\} \\
&= \left\{ \frac{x^{t+1}}{x^t}, \frac{x^t y}{x^t}, \dots, \frac{xy^t}{x^t}, \frac{y^{t+1}}{x^t} \right\} = x \cdot \left[S\left[\frac{1}{x}\right]\right]_0 \\
&= x \cdot k\left[\frac{y}{x}\right] = x \cdot \mathcal{O}_X\big(D_+(x)\big),
\end{aligned}$$

which is a free $\mathcal{O}_X\big(D_+(x)\big)$-module of rank 1.

On $D_+(y)$:

$$\widetilde{M}\big(D_+(y)\big) = y \cdot \mathcal{O}_X\big(D_+(y)\big) = y \cdot k\left[\frac{x}{y}\right] = \left[k\left[x, y, \frac{1}{y}\right]\right]_1 = \left[M\left[\frac{1}{y}\right]\right]_0.$$

So this is a locally free $\mathcal{O}_X$-module of rank 1.

Exercise 6.20. If $M = S(d)$, then

$$\widetilde{M}\big(D_+(x)\big) = \left[M\left[\frac{1}{x}\right]\right]_0 = \left[S\left[\frac{1}{x}\right]\right]_d = x^d \cdot \left[S\left[\frac{1}{x}\right]\right]_0,$$

so $\widetilde{S(d)}$ is locally free of rank 1 for *all* $d \in \mathbb{Z}$.

6.7 Twists of the structure sheaf on Proj

Definition 6.21. If S is an $\mathbb{N}$-graded ring, the quasi-coherent sheaves $\widetilde{S(d)}$ are called "*twists*" of the structure sheaf.

Proposition 6.22. *If S is a domain, finitely generated over S_0 by elements of degree 1, then $\widetilde{S(d)}$ is locally free of rank 1.*

Proof. We have

$$\operatorname{Proj} S = D_+(x_0) \cup \cdots \cup D_+(x_n),$$

where $x_0, \ldots, x_n$ are degree 1 generators for S as an S_0-algebra. Then

$$\widetilde{S(d)}\big|_{D_+(x_i)} = x_i^d \cdot \mathcal{O}_X\big(D_+(x_i)\big). \qquad \square$$

Claim 6.23. $\widetilde{S(d)} \not\cong \widetilde{S(d')}$ *is general.*

For example, consider $S = k[x,y]$ with the standard grading, $\operatorname{Proj} S = \mathbb{P}^1_k$. Let us compute $\varphi \in \widetilde{S(d)}(\mathbb{P}^1)$:

$$\varphi\big|_{D_+(x)} = x^d \cdot f\left(\frac{y}{x}\right) \in \widetilde{S(d)}\big(D_+(x)\big) = x^d \cdot k\left[\frac{y}{x}\right]$$

and

$$\varphi\big|_{D_+(y)} = y^d \cdot g\left(\frac{x}{y}\right),$$

so

$$x^d \cdot f\left(\frac{y}{x}\right) = y^d \cdot g\left(\frac{x}{y}\right).$$

Clearly,

$$S_d = \{\text{homogeneous polynomials of degree } d \text{ in } x, y\} \subseteq \widetilde{S(d)}(\mathbb{P}^1).$$

Indeed,

$$x^a y^{d-1} = x^d \cdot \left(\frac{y^{d-a}x^a}{x^d}\right) = y^d \cdot \left(\frac{x^a y^{d-a}}{y^d}\right).$$

It's not too hard to show that the above is actually equality:

$$\widetilde{S(d)}(\mathbb{P}^1) = S_d.$$

So the S_d are vector spaces of different dimension, and hence are not isomorphic.

In $p = [0:1] \in \mathbb{P}^1$, we have $\mathscr{I}_p \subseteq \mathcal{O}_{\mathbb{P}^1}$:

$$\mathscr{I}_p\big|_{D_+(x)} = \mathcal{O}_{\mathbb{P}^1},$$
$$\mathscr{I}_p\big|_{D_+(y)} = \left(\frac{y}{x}\right) \subseteq k\left[\frac{y}{x}\right].$$

If $f \in S_d$ is a homogeneous polynomial of degree d, then

$$M = f \cdot S \subseteq S,$$
$$[M]_d = [f \cdot S]_d \to S_0.$$

We have an isomorphism of graded S-modules

$$S(-d) \xrightarrow{\simeq} f \cdot S$$
$$1 \mapsto f,$$

where $1 \in S(-d)$ is a generator for the S-module $S(-d)$. So

$$\widetilde{S(-d)} = \widetilde{(f \cdot S)}$$

is the ideal sheaf of the closed subscheme of $\operatorname{Proj} S$ defined by

$$S \twoheadrightarrow S/(f)$$
$$\operatorname{Proj} S \hookleftarrow \operatorname{Proj} S/(f).$$

7 Separated and proper morphisms

Guest lectures by David Speyer.

7.1 Notation and motivation

- If X is a scheme over $\operatorname{Spec} k$ (denoted X/k), then "X is [adjective]" means "$X \to \operatorname{Spec} k$ is [adjective]".
- Separated "means" Hausdorff.
- Proper "means" compact.

Motivation: For $X/\mathbb{C}$ of finite type, there is a topological space X^{an}. The point set of X^{an} is

$$X(\mathbb{C}) = \operatorname{Hom}(\operatorname{Spec} \mathbb{C}, X) = \left\{ x \in X \;\middle|\; k(x) = \mathbb{C} \right\}.$$

Then X is separated $\iff$ X^{an} is Hausdorff, and X is proper $\iff$ X^{an} is compact.

7.2 Separated morphisms

Motivation: Let X be a topological space. Let Δ be the diagonal in $X \times X$. Then the following are equivalent:

- X is Hausdorff.
- For all $x, y \in X$ with $x \neq y$, there exist open $U, V \subseteq X$ with $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

- For all $(x, y) \in X \times X$ with $(x, y) \notin \Delta$, there exists an open W with $(x, y) \in W$ and $W \cap \Delta = \varnothing$.
- $(X \times X) \setminus \Delta$ is open in $X \times X$.
- Δ is closed in $X \times X$.

Definition 7.1. A scheme X over S is *separated* if Δ is closed in $X \times_S X$, or equivalently, if $\Delta \hookrightarrow X \times_S X$ is a closed embedding.

Note 7.2. If we have morphisms $X \to B \to C$, then using the universal property, we have

$$\Delta \to X \times_B X \hookrightarrow X \times_C X.$$

Check in Hartshorne if this is true: separated over C implies separated over B.

Example 7.3 (The line with two origins). Here is the standard example of a nonseparated scheme: Take two copies of $\mathbb{A}^1$. Inside each, we have $\mathbb{A}^1 \setminus \{0\}$. Glue these open subsets by identity, but don't glue the origins $0, 0'$. Call this space X.

Now consider the product $X \times X$. This consists of the affine plane, but with two copies of each axis and four copies of the origin. The diagonal contains two of the four origin points, namely $(0, 0)$ and $(0', 0')$, but $(0, 0')$ and $(0', 0)$ are also in the closure of the diagonal. Therefore, X is not separated.

Example 7.4 (An orbit space). The punctured plane $\mathbb{A}^2 \setminus \{(0, 0)\}$ has an action of

$$\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$$

by

$$t : (x, y) \mapsto \left(tx, t^{-1}y\right).$$

Write the coordinates on $\mathbb{A}^2$ by (x, y). Consider affine open subsets

$$\begin{aligned} U_1 &= \{x \neq 0\} = \operatorname{Spec} k[x, x^{-1}, y], \\ U_2 &= \{y \neq 0\} = \operatorname{Spec} k[x, y, y^{-1}]. \end{aligned}$$

Then

$$\begin{aligned} U_1/\mathbb{G}_m &= \operatorname{Spec} k[xy] \cong \mathbb{A}^1, \\ U_2/\mathbb{G}_m &= \operatorname{Spec} k[xy] \cong \mathbb{A}^1. \end{aligned}$$

The projection map $U_1 \to U_1/\mathbb{G}_m$ is

$$(x, y) \mapsto xy.$$

So $\left(\mathbb{A}^2 \setminus \{(0, 0)\}\right)/\mathbb{G}_m$ is $\mathbb{A}^1$ glued to $\mathbb{A}^1$ along

$$(U_1 \cap U_2)/\mathbb{G}_m = \mathbb{A}^1 \setminus \{(0, 0)\}.$$

Remark 7.5. Gluing can created nonseparatedness!

However, open and closed subschemes of separated schemes are separated. Since $\mathbb{A}^n$ and $\mathbb{P}^n$ are separated, anything affine, projective, or quasiprojective is separated.

7.3 Properties of separated schemes

Theorem 7.6. *If X is separated, Z is reduced, $f, g : Z \to X$ are two morphisms, and $U \subseteq Z$ is a dense open subset such that $f|_U = g|_U$, then $f = g$ on Z.*

Proof. Consider the map

$$\begin{aligned} h &: Z \to X \times X \\ h(z) &= \big(f(z), g(z)\big). \end{aligned}$$

This is the map given by the diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[d, dashed, "h"] \arrow[dr, "f"] & \\ X & X \times X \arrow[l] \arrow[r] & X \end{tikzcd}$$

Since X is separated, $\Delta \subset X \times X$ is closed, so $h^{-1}(\Delta)$ is closed in Z. Since $h^{-1}(\Delta)$ contains U, the closed subscheme $h^{-1}(\Delta)$ is supported on all of Z. But Z is reduced, so $h^{-1}(\Delta) = Z$, and so $f = g$. □

Caution 7.7. Here is why we assumed Z is reduced. Consider

$$Z = \operatorname{Spec} \frac{k[x, y]}{(y^2, xy)}.$$

Note that $Z^{\text{red}} = \operatorname{Spec} k[x]$. We will find two morphisms that agree on Z^{red}, but not on Z.

Consider the maps

$$\begin{aligned} f &: Z \to \mathbb{A}^2 \\ \frac{k[x, y]}{(y^2, xy)} &\leftarrow k[x, y] \\ x &\mapsfrom x \\ y &\mapsfrom y \end{aligned}$$

and

$$\begin{aligned} g &: Z \to \mathbb{A}^2 \\ \frac{k[x, y]}{(y^2, xy)} &\leftarrow k[x, y] \\ x &\mapsfrom x \\ 0 &\mapsfrom y. \end{aligned}$$

Inside Z, we have

$$U = \operatorname{Spec} \frac{k[x, y, x^{-1}]}{(xy, y^2)} = \operatorname{Spec} \frac{k[x, y, x^{-1}]}{(y)} = \operatorname{Spec} k[x, x^{-1}].$$

This sort of situation can occur anywhere on the nonreduced locus of the scheme Z.

Fact 7.8 (Key fact). If Z is reduced, and W is a closed subscheme supported on Z, then $Z = W$. For any scheme Z, the subscheme Z^{red} is supported on all of Z.

Example 7.9. Consider the maps

$$\begin{aligned} f : \mathbb{A}^1 &\to \mathbb{A}^2 \setminus \{0\} \\ u &\mapsto (u, 1) \end{aligned}$$

and

$$\begin{aligned} g : \mathbb{A}^1 &\to \mathbb{A}^2 \setminus \{0\} \\ u &\mapsto (1, u). \end{aligned}$$

Sending these to the quotient space

$$\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^2 \setminus \{0\} \to \left(\mathbb{A}^2 \setminus \{0\}\right)/\mathbb{G}_m$$
$$\mathbb{A}^1 \xrightarrow{g} \mathbb{A}^2 \setminus \{0\} \to \left(\mathbb{A}^2 \setminus \{0\}\right)/\mathbb{G}_m,$$

we see that $(1, u)$ and $(u, 1)$ are in the same orbit when $u \neq 0$, but in different orbits (the y-axis and the x-axis) when $u = 0$.

Theorem 7.10. *If X is a separated scheme, and U and V are open affine subsets in X, then $U \cap V$ is affine.*

Proof. See Hartshorne. □

Example 7.11 (Nonseparated counterexample). Glue together two copies of $\mathbb{A}^2$ except at the origin. Then the intersection of the two affine planes is $\mathbb{A}^2 \setminus \{0\}$.

7.4 Proper morphisms

Definition 7.12. A map $f : X \to Y$ is *closed* if, for any closed $K \subseteq X$, the set $f(K)$ is closed in Y.

Example 7.13. The inclusion map $A^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$ is not closed.

Example 7.14. The map

$$\operatorname{Spec} k[x, x^{-1}] \sqcup \operatorname{Spec} k \to \operatorname{Spec} k[x]$$

given by "filling in" the discrete point into the hole is not closed.

Example 7.15. The map

$$\begin{aligned} \mathbb{A}^2 &\to \mathbb{A}^1 \\ (x, y) &\mapsto x \end{aligned}$$

is not closed. Indeed, the hyperbola $xy = 1$ is closed in $\mathbb{A}^2$, but its image is $\mathbb{A}^1 \setminus \{0\}$.

Definition 7.16. A scheme X over S is called *proper* if $X \to S$ is separated, of finite type, and *universally closed*: for every $B \to S$, the projection $X \times B \to B$ is closed.

So the previous example shows that $\mathbb{A}^1$ is not proper.

Remark 7.17. Using the same definition, a topological space X is proper $\iff$ compact.[4] Let's see that proper $\implies$ sequentially compact.

Let $x_1, x_2, \ldots$ be a sequence in X. Let $B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\right\} \subset \mathbb{R}$. Set

$$S = \left\{\left(x_i, \tfrac{1}{i}\right)\right\} \subseteq X \times B.$$

Then the projection of S to B, namely $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is not closed, so S must not be closed. Thus, a point of $\overline{S} \setminus S$ must be the form $(x, 0)$, where x is an accumulation point of $\{x_i\}$.

Proper says: For any B/k, U dense in B, V in $X \times B$ projecting onto U, and any $u \in \overline{U}$, there is some $v \in \overline{V}$ over u.

7.5 Proper morphisms, continued

A morphism $X \to Y$ is proper if it is of finite type, separated, and for all $B \to Y$, the map $X \times_Y B \to B$ is closed.

This means that: "If you have a 'path' in Y which approaches a limit u in B, and you lift that to a 'path' in $X \times_Y B$, then that path upstairs accumulates at some v above u."

Example 7.18. The map

$$\begin{aligned} \mathbb{A}^1 &\to \mathbb{A}^1 \\ t &\mapsto t^2 \end{aligned}$$

is proper, even though $\mathbb{A}^1$ itself is not proper.

Returning to the general case, let us formulate this property of proper maps more precisely: For any $g : B \to Y$, U dense in B, V in $X \times_Y B$ projecting onto U, if $u \in \overline{g(U)} \subseteq Y$, then there is a point $v \in \overline{V}$ above u.

A map $f : X \to Y$ of topological spaces obeys this condition (for all $B \to Y$, the map $X \times_Y B \to B$ is closed) $\iff$ for any $K \subseteq Y$ with K compact, $f^{-1}(K)$ is also compact.

7.6 Facts about proper morphisms

Proposition 7.19. *The projective space $\mathbb{P}^n_k$ is proper, i.e., for any $K \subseteq \mathbb{P}^n \times B$ with K closed, the projection of K onto B is closed.*

If $f_t(x, y)$ and $g_t(x, y)$ are some homogeneous polynomials in x, y, then letting t vary in $\mathbb{A}^1$, the equations

$$f_t(x, y) = g_t(x, y) = 0$$

define a closed subscheme of $\mathbb{P}^1 \times \mathbb{A}^1$. The set of t for which there is a common root of $f_t(x, y)$ and $g_t(x, y)$ is closed.

Similarly, over any base scheme S:

[4] A full proof can be found at `http://ncatlab.org/toddtrimble/published/Characterizations+of+compactness`.

Proposition 7.20. *$\mathbb{P}^n_S \to S$ is proper.*

Proposition 7.21. *If X/k is proper, so is any closed subscheme of X, and so is any surjective image of X.*

Fact 7.22. Proper maps have the following useful properties:

- Proper maps are closed.
- If X is proper and $f : X \to Y$ is a morphism, then $f(X)$ is closed in Y.
- If $X \to Y$ and $Y \to Z$ are proper, then the composition $X \to Z$ is proper.
- $X \to Y$ is proper and affine[5] $\iff$ $X \to Y$ is finite.

7.7 Valuation rings

Definition 7.23. Let R be a domain, and let $K = \operatorname{Frac} R$. We say R is a *valuation ring* if, for all $u \in K^\times$, either u or u^{-1} is in R. That is, for any $a, b \in R$ with $a, b \neq 0$, either $a \mid b$ or $b \mid a$.

Example 7.24. The ring $R = k[[t]]$ is a valuation ring: If

$$\begin{aligned} a &= a_i t^i + a_{i+1} t^{i+1} + \dots, \\ b &= b_j t^j + \dots, \end{aligned}$$

then $\frac{b}{a} \in k[[t]]$ if $i \leq j$, and $\frac{a}{b} \in k[[t]]$ if $i \geq j$.

Example 7.25. Here are a few more valuation rings:

$$\begin{aligned} k[t]_{(t)} &= \left\{ \frac{f}{g} \;\middle|\; f, g \in k[t],\ t \nmid g(t) \right\} \\ \mathbb{Z}_p &= \varprojlim \mathbb{Z}/p^n\mathbb{Z} \\ \mathbb{Z}_{(p)} &= \left\{ \frac{a}{b} \in \mathbb{Q} \mid p \nmid b \right\}. \end{aligned}$$

Example 7.26. Valuation rings are not necessarily discrete. Here is a non-discrete valuation ring:

$$\bigcup_{n=1}^{\infty} k\left[\left[t^{1/n}\right]\right].$$

Given a valuation ring, we can define a valuation. Let

$$\begin{aligned} A &= K^\times / R^\times, \\ A_+ &= (R \setminus \{0\}) / R^\times \subseteq A. \end{aligned}$$

For example, if $R = k[[t]]$, then $A = \mathbb{Z}$ and $A_+ = \mathbb{Z}_{\geq 0}$; and if $R = \bigcup_{n=1}^{\infty} k\left[\left[t^{1/n}\right]\right]$, then $A = \mathbb{Q}$ and $A_+ = \mathbb{Q}_{\geq 0}$.

Then A is an ordered abelian group. Moreover,

[5] A morphism $f : X \to Y$ is *affine* provided that for any affine open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is also affine.

- Every $a \in A$ is either in A_+ or in $-A_+$.
- $A_+ \cap -A_+ = \{0\}$.
- A_+ is closed under addition.

Define the map

$$v : K^\times \to A = K^\times/R^\times.$$

Then

- $v(xy) = v(x) + v(y)$,
- $v(x+y) \geq \min\bigl(v(x), v(y)\bigr)$,
- $R = v^{-1}(A_+)$.

We can carry out this process in reverse:

Definition 7.27. Let k be a field. A *valuation* is a map

$$v : k^\times \to A$$

for an ordered abelian group A, such that

- $v(xy) = v(x) + v(y)$,
- $v(x+y) \geq \min\bigl(v(x), v(y)\bigr)$.

The corresponding valuation ring is $v^{-1}(A_+)$.

Example 7.28. Take the map

$$v : k(x,y)^\times \to \mathbb{Q} + \mathbb{Q}\sqrt{2} \subseteq \mathbb{R}$$

defined by $v(x) = 1$, $v(y) = \sqrt{2}$, and $v(k^\times) = 0$.

7.8 Spectra of valuation rings

Let v be a valuation. Then $R = v^{-1}(A_{\geq 0})$ is a ring, and $\mathfrak{m} = v^{-1}(A_{>0})$ is a maximal ideal. Indeed:

Proposition 7.29. *$R/\mathfrak{m}$ is a field.*

Proof. If $\bar{u} \in (R/\mathfrak{m}) - \{0\}$, lift to $u \in R - \mathfrak{m}$. Then $v(u) = 0$. So $u^{-1} \in R$, and the class of u^{-1} in $R/\mathfrak{m}$ is an inverse to $\bar{u}$. □

So $\mathfrak{m}$ is a closed point of $\operatorname{Spec} R$, and (0) is another point of $\operatorname{Spec} R$.

Example 7.30. Let $A = \mathbb{Z}^2$ with the lexicographic ordering. We have the valuation

$$\begin{aligned} v : k(x,y)^* &\to \mathbb{Z}^2 \\ x &\mapsto (1,0) \\ y &\mapsto (0,1) \\ k^\times &\mapsto (0,0). \end{aligned}$$

Then the prime ideals of the associated valuation ring are (0), $v^{-1}(A_{(>0,*)}$, and $v^{-1}(A_{>(0,0)}$.

7.9 The valuative criterion

Theorem 7.31. *A scheme X/k is separated (resp. proper) iff the following criterion holds: For every valuation ring (R, v) which is a k-algebra, we have $v(k) = 0$; and for every map*

$$f : \operatorname{Spec} \operatorname{Frac} R \to X,$$

there is at most (resp. at least[6]) one way to extend f to a map $\operatorname{Spec} R \to X$.

Proof. See Hartshorne §2.4. □

7.10 Projective space is proper

We now use the valuative criterion to prove that $\mathbb{P}^n$ is proper. To check this, let R be a valuation ring with fraction field K and valuation $v : K^\times \to A$.

Let $f : \operatorname{Spec} K \to \mathbb{P}^n$ be a morphism, and let $(x_0 : x_1 : \cdots : x_n) \in K^{n+1} \setminus \{0\}$ be arbitrary. Let $v_i = v(x_i)$ or ∞ if $x_i = 0$. Without loss of generality, $v_0 \leq v_1, v_2, \ldots, v_n$. So

$$P := \left(1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \cdots : \frac{x_n}{x_0}\right)$$

represents the same map $\operatorname{Spec} K \to \mathbb{P}^n$. But $v(x_i/x_0) \geq 0$, so $\frac{x_i}{x_0} \in R$. Thus P gives a map $\operatorname{Spec} R \to \mathbb{P}^n$.

Example 7.32. Consider two copies of $\mathbb{A}^2$, with $\mathbb{A}^1 \times (\mathbb{A}^1 - \{0\})$ glued to $\mathbb{A}^1 \times (\mathbb{A}^1 - \{0\})$ by gluing (x, y) to $(x + y^{-1}, y)$.

To have a map $\operatorname{Spec} \operatorname{Frac} R \to X$, we must have $(x, y) \in R^2$ with $v(y) > 0$ and $x + y^{-1} \in R$. If $v(u) < v(w)$, then $v(u + w) = v(u)$; also, if $v(x) \gg 0$ and $v(y) > 0 \implies v(y^{-1}) < 0$, then $v(x + y^{-1}) < 0$. So no such $(x, y) \in R^2$ exists, hence this scheme is separated.

8 Quasi-coherent sheaves, continued

...

9 Divisors on schemes

9.1 Assumptions on schemes

Fix a scheme X.

Assumption (*): X Noetherian, separated, integral, regular in codimension 1.

Definition 9.1. A scheme is *regular in codimension 1* if for all codimension 1 integral subscheme $Y \subseteq X$, the stalk $\mathcal{O}_{X,y}$ (local ring, dimension 1) of the generic point y of Y is regular.

[6] And therefore exactly one, since proper morphisms are defined to be separated.

Remark 9.2. Look at the subset

$$W = \left\{P \in X \,\middle|\, \mathcal{O}_{X,P} \text{ is regular}\right\} \subseteq X.$$

Non-obvious fact: W is open.[7]

A scheme is regular in codimension 1 $\iff$ the closed set $X - W$ has codimension ≥ 2.

Remark 9.3. If $X = \operatorname{Spec} A$, then "regular in codimension 1" means that $A_\mathfrak{p}$ is regular for all height 1 prime ideals $\mathfrak{p}$ in A.

We will introduce two types of divisors under assumption (*):

Cartier divisors or "locally principal" divisors $\subseteq$ Weil divisors.

9.2 Weil divisors

Definition 9.4 (Weil divisors).

- Assume (*) is satisfied for X. A *prime divisor* is an integral codimension 1 closed subscheme of X.
- A *(Weil) divisor* is a formal $\mathbb{Z}$-linear combination of prime divisors

$$D = \sum_{i=1}^{t} a_i Y_i,$$

  where $Y_i \subseteq X$ are prime divisors and $a_i \in \mathbb{Z}$.
- $\operatorname{Div}(X)$ = free abelian group generated by prime divisors.
- If all $a_i \geq 0$, then say D is *effective*.

Example 9.5. The subscheme

$$\operatorname{Spec} \frac{k[x,y]}{(x^2)} \subseteq \operatorname{Spec} k[x,y]$$

corresponds to the divisor

$$2 \cdot \operatorname{Spec} \frac{k[x,y]}{(x)}.$$

9.3 Aside: Normal rings

Let A be a Noetherian domain, let $K = \operatorname{Frac}(A)$ be its fraction field, and let $\mathfrak{p} \subseteq A$ be a height 1 prime. Then

$$A \subseteq A_\mathfrak{p} \subseteq K,$$

and

$$A \hookrightarrow \bigcap_{\mathfrak{p} \text{ ht } 1} A_\mathfrak{p} \overset{\text{thm}}{=} \text{normalization of } A.$$

If A is normal, then $A_\mathfrak{p}$ is normal for all $\mathfrak{p}$ height 1, so $A_\mathfrak{p}$ is regular.

[7] This was an open question in general for a number of years. It was proven by Serre.

9.4 The valuation associated to a prime divisor

Under assumption (*), if $Y \subseteq X$ is a prime divisor, let $\xi \in X$ and $y \in Y$ denote the generic points. Then $\mathcal{O}_{X,y}$ is a DVR, so we get a valuation of $K =$ "function field of X", the stalk of $\mathcal{O}_X$ at the generic point of X.

We have an inclusion $\mathcal{O}_{X,y} \subseteq K$. Indeed, restricting to an affine patch

$$\varnothing \neq Y \cap U \hookrightarrow U = \operatorname{Spec} A,$$

then $Y \cap U$ corresponds to a height 1 prime $\mathfrak{p}$ in A. Then

$$\begin{tikzcd} \mathcal{O}_{X,y} \arrow[r, hook] \arrow[d, equal] & \mathcal{O}_{X,\xi} \\ A_{\mathfrak{p}} \arrow[r, hook] & A_{(0)} = K. \end{tikzcd}$$

This gives the "valuation of Y", denoted v_Y:

$$\begin{aligned} v_Y : K^* &\to \mathbb{Z} \\ f &\mapsto v_Y(f) = \text{"order of } f \text{ in } \mathcal{O}_{X,y}\text{"} \end{aligned}$$

Example 9.6. Here's an example that isn't from 631:

$$Y = \operatorname{Spec} \mathbb{Z}/(7) \subseteq \operatorname{Spec} \mathbb{Z}.$$

Let

$$f = \frac{17}{49} \in \mathbb{Q} = K.$$

Then

$$v_Y\left(\frac{17}{49}\right) = v_Y(17) - v_Y(49) = 0 - 2 = -2.$$

Note 9.7. Because X is separated, the valuation v_Y uniquely determines Y. That is, use the valuative criterion for separatedness:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \mathcal{O}_{X,y} \arrow[r] \arrow[ur] & \operatorname{Spec} \mathbb{Z}, \end{tikzcd}$$

where the map $\operatorname{Spec} \mathcal{O}_{X,y} \to X$ sends the closed point of $\operatorname{Spec} \mathcal{O}_{X,y}$ to the generic point of $Y \subseteq X$. By the valuative criterion, this is the unique such map.

Lemma 9.8. *For all $f \in K^*$, there are at most finitely many prime divisors Y such that $v_Y(f) \neq 0$.*

Proof. Choose affine $U \subseteq X$. Write $f = \frac{h}{g}$. Then

$$v_Y(f) = v_Y(h) - v_Y(g).$$

Without loss of generality, we can assume $f \in A$ such that $U = \operatorname{Spec} A \subseteq X$ is an affine chart.

Which $Y \subseteq \operatorname{Spec} A$ can be such that $v_Y(f) \neq 0$? Observe that

$$v_Y(f) \neq 0 \iff f \in \mathfrak{p}_Y = \text{ideal of } Y.$$

By the following commutative algebra fact, we are done. $\square$

Fact 9.9 (Commutative algebra). If A is a Noetherian domain and $f \neq 0$, then there are finitely many primes of height 1 ("minimal primes") containing f.

9.5 The divisor class group

Proposition–Definition 9.10. Fix X satisfying (*). Let K = function field of X (stalk at the generic point of X). There is a group homomorphism

$$\begin{aligned} K^* &\to \operatorname{Div}(X) \\ f &\mapsto \operatorname{div} f \stackrel{\text{def}}{=} \sum_{\substack{Y \subseteq X \\ \text{prime}}} v_Y(f) Y. \end{aligned}$$

Its image $P(X)$ is the subgroup of *principal divisors*. The quotient group

$$\operatorname{Cl}(X) = \operatorname{Div}(X)/P(X)$$

is called the *divisor class group*.

Example 9.11. $\operatorname{Cl}(\operatorname{Spec} k[x,y]) = 0$ because every height 1 prime $\mathfrak{p}$ is principal, so if $\mathfrak{p} = (f) \subseteq k[x,y]$ is prime, height 1, then

$$\operatorname{div} f = \mathfrak{p} \in \operatorname{Div}(\operatorname{Spec} k[x,y]).$$

Indeed, $v_{\mathfrak{p}}(f) = 1$, and $v_{\mathfrak{q}}(f) = 0$ for all $\mathfrak{q} \neq \mathfrak{p}$.

Theorem 9.12 (see Hartshorne)**.** *$\operatorname{Spec} A$ has trivial class group $\iff$ A is a UFD.*

Proposition 9.13. *There is a natural map*

$$\begin{aligned} \operatorname{Div}\left(\operatorname{Proj} k[x_0, \dots, x_n]\right) = \operatorname{Div}(\mathbb{P}^n_k) &\xrightarrow{\deg} \mathbb{Z} \\ D = \sum_{i=1}^{t} n_i Y_i &\mapsto \sum_{i=1}^{t} n_i d_i, \end{aligned}$$

where Y_i corresponds to $\mathfrak{p}_i = (F_i)$ with F_i homogeneous of degree d_i. The kernel of this map is $P(\mathbb{P}^N_k)$, so

$$\operatorname{Cl}(\mathbb{P}^n) \cong \mathbb{Z}.$$

Proof. See Hartshorne. $\square$

9.6 Cartier divisors

If we assume (*), then we can think of Cartier divisors as special kinds of Weil divisors. However, Cartier divisors can be defined on *arbitrary* schemes.

Here, we will only define Cartier divisors on integral schemes; Hartshorne defines them in full generality using the total quotient ring.

Definition 9.14. Fix an integral scheme X. Let K = function field of X, and let $\mathscr{K}$ be the constant sheaf on X of K. A *Cartier divisor* is a global section φ of the sheaf $\mathscr{K}^*/\mathcal{O}_X^*$.

More concretely: φ is data $\{U_\lambda, f_\lambda\}$, where $\bigcup_{\lambda\in\Lambda} U_\lambda = X$ is an open cover of X, and $f_\lambda \in \mathscr{K}^*(U_\lambda) = K^*$, such that each f_λ and f_μ agree on $U_\lambda \cap U_\mu$, i.e.,

$$f_\lambda f_\mu^{-1} \in \mathcal{O}_X^*(U_\lambda \cap U_\mu).$$

[If we do *not* assume X integral, instead of K, use the sheaf of "total quotient rings" $\mathscr{K}$, the sheaf associated to the presheaf which assigns to $U \subseteq X$ the ring

$$\mathscr{K}(U) = \mathcal{O}_X(U)\left[\{\text{non-zerodivisors}\}^{-1}\right],$$

which agrees with this definition when X is integral.]

Remark 9.15. Since $\mathscr{K}^*/\mathcal{O}_X^*$ is a sheaf of abelian groups, Cartier divisors form a group.

Proposition 9.16. *Assume X satisfies (*). There is a natural map of groups*

$$\begin{aligned} \{\textit{Cartier divisors on } X\} &\to \operatorname{Div}(X) \\ \varphi = \{(U_\lambda, f_\lambda)\}_{\lambda\in\Lambda} &\mapsto \text{“}\operatorname{div}\varphi\text{”}, \end{aligned}$$

where $\operatorname{div}\varphi$ *is the unique divisor D on X such that*

$$D\big|_{U_\lambda} = \operatorname{div}_{U_\lambda}(f_\lambda) = \sum_{\substack{Y\subseteq X \text{ prime} \\ Y\cap U_\lambda \neq \varnothing}} v_Y(f_\lambda)\cdot Y.$$

9.7 Summary of Weil divisors

Recall assumption (*): X is a Noetherian integral separated scheme, regular in codimension 1 [always holds when X is normal].

Example 9.17. Here is a scheme which satisfies (*), but is not normal:

$$X = \operatorname{Spec} k[s^4, s^3t, t^3s, t^4].$$

Indeed,

$$s^2t^2 = \frac{(s^3t)^2}{s^4}$$

is in the normalization, but not in the ring.

Let us now briefly review Weil divisors. Let K = function field of X. Consider a Weil divisor $D = \sum n_i Y_i$, where $n_i \in \mathbb{Z}$ and each $Y_i \subseteq X$ is a prime divisor (i.e., a codimension 1, integral, closed subscheme).

In an affine patch $U = \operatorname{Spec} A \subseteq X$,

$$D\big|_U = \sum n_i (Y_i \cap U).$$

Each nonempty $Y_i \cap U \subseteq U$ corresponds to a prime $\mathfrak{p}_i \subseteq A$ of height 1, the "generic point of Y_i". This induces a DVR

$$\mathcal{O}_{X,Y_i} = A_{\mathfrak{p}_i},$$

which induces a valuation v_{Y_i} on K.

Each $f \in K^* = K - \{0\}$ determines a *principal* divisor, the "divisor of zeros and poles":

$$\operatorname{div}_X f = \sum_{\substack{Y \subseteq X \\ \text{prime}}} v_Y(f) \cdot Y \in P(X) \subseteq \operatorname{Div}(X).$$

There is a group homomorphism

$$\begin{aligned} K^* &\xrightarrow{\text{div}} \operatorname{Div}(X) \\ f &\mapsto \operatorname{div}_X f. \end{aligned}$$

The cokernel is called the *divisor class group* $\operatorname{Cl}(X)$.

For any $f \in \mathcal{O}_X(U)$,

$$\operatorname{div}_U f = \sum_{\substack{Y \subseteq X \\ \text{prime}}} v_Y(f) \cdot Y \geq 0.$$

Indeed, if $f \in \mathcal{O}_X(U)$, then $f \in \mathcal{O}_{X,Y}$, so $v_Y(f) \geq 0$.

Caution 9.18. The converse is false; contrary to our initial intuition, there are effective principal divisors $\operatorname{div}_U f$ such that $f \notin \mathcal{O}_X(U)$. For example, if X is the scheme from Example 9.17, then $\operatorname{div}_X(s^2t^2) \geq 0$, but $s^2t^2 \notin \mathcal{O}_X(X)$.

Proposition 9.19. *If X is normal, then for all $f \in K^*$ and for all open $U \subseteq X$,*

$$\operatorname{div}_U f \geq 0 \iff f \in \mathcal{O}_X(U).$$

Proof. Reduce to the case where $U = \operatorname{Spec} A$ is affine. If $\operatorname{div}_U f \geq 0$, then ... □

9.8 An explicit example

Consider

$$\begin{aligned} X &= \mathbb{P}^3_k = \operatorname{Proj} k[x_0, x_1, x_2, x_3] \supseteq U_0 = \operatorname{Spec} k\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right], \\ K &= k\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right), \\ f &= \frac{x_1^2 x_0 - x_2^3}{x_0^3} = \left(\frac{x_1}{x_0}\right)^2 - \left(\frac{x_2}{x_0}\right)^3. \end{aligned}$$

Note: Most prime divisors in $\mathbb{P}^3$ have generic point in U_0. In fact, *only* $H_0 = \mathbb{V}(x_0) \subseteq \mathbb{P}^3$ does not.

Let's compute the associated principal divisor:

$$\operatorname{div}_{\mathbb{P}^3}(f) = \sum_{\substack{Y \subseteq \mathbb{P}^3 \\ \text{prime}}} v_Y(f) \cdot Y$$

$$\operatorname{div}_{U_0}(f) = \sum_{\substack{\mathfrak{p} \subseteq k\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right] \\ \text{ht 1 prime}}} v_{\mathfrak{p}}\left(\left(\frac{x_1}{x_0}\right)^2 - \left(\frac{x_2}{x_0}\right)^3\right) \cdot \mathfrak{p} = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(t_1^2 - t_2^3)\mathfrak{p} = S = \mathbb{V}(t_1^2 - t_2^3) \subseteq U_0.$$

To see what happens at H_0, we need to choose an affine chart containing the generic point of H_0:

$$U_1 = \operatorname{Spec} k\left[\frac{x_0}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}\right] = \operatorname{Spec} k[x_{0/1}, x_{2/1}, x_{3/1}],$$
$$f = \frac{(x_1^2 x_0 - x_2^3)/x_1^3}{(x_0/x_1)^3} = \frac{x_{0/1} - x_{2/1}^3}{x_{0/1}^3}.$$

We just need to look at the valuation v_{H_0} of the valuation ring

$$\mathcal{O}_{X,H_0} = k[x_{0/1}, x_{2/1}, x_{3/1}]_{(x_{0/1})}.$$

This is given by

$$v_{H_0}(f) = v_{H_0}(x_{0/1} - x_{2/1}^3) - v_{H_0}(x_{0/1}^3) = -3.$$

Thus,

$$\operatorname{div}_{\mathbb{P}^3}(f) = \sum v_Y(f) \cdot Y = \mathbb{V}(x_0 x_1^2 - x_2^3) - 3H_0.$$

9.9 Summary of Cartier divisors

Recall, on a scheme satisfying (*):

Definition 9.20. A *Cartier divisor* is a Weil divisor which is *locally principal*, i.e., writing

$$D = \sum_{Y_i \text{ prime}} n_i Y_i \in \operatorname{Div} X,$$

there exists an open cover $\{U_\lambda\}$ of X and $f_\lambda \in K^*$ such that $D|_{U_\lambda} = \operatorname{div}_{U_\lambda}(f_\lambda)$.

Equivalently: A Cartier divisor is a global section of $K^*/\mathcal{O}_X^*$. [Advantage: This makes sense even if X does not satisfy (*).]

Example 9.21. On $\mathbb{P}^3$, let

$$D = S + 5H_0 = \mathbb{V}(x_1^2 x_0 - x_2^3) + 5 \cdot \mathbb{V}(x_1).$$

Take the standard cover U_0, U_1, U_2, U_3. Then

$$\begin{aligned} D \cap U_0 &= \operatorname{div}_{U_0}\left(x_{1/0}^2 - x_{2/0}^3\right) = 1 \cdot S, \\ D \cap U_1 &= \operatorname{div}_{U_1}\left(\left(x_{0/1} - x_{2/1}^3\right) \cdot x_{0/1}^5\right), \\ D \cap U_2 &= \operatorname{div}_{U_2}\left(\frac{\left(x_0 x_1^2 - x_2^3\right)\left(x_0^5\right)}{x_2^8}\right), \end{aligned}$$

etc. So D is locally principal!

The above situation occurs in more generality:

Definition 9.22. We say that X is *locally factorial* provided that $\mathcal{O}_{X,P}$ is a UFD for all $P \in X$.

Theorem 9.23. *If X is locally factorial, then* every *Weil divisor is Cartier.*

9.10 Sheaf associated to a divisor

Assume X is *normal*, not just (*). Let K be the function field of X. For $D \in \operatorname{Div} X$, we define a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(D)$ which is a subsheaf of K:

$$\mathcal{O}_X(D)(U) = \left\{ f \in K^* \;\middle|\; \operatorname{div}_U f + D\big|_U \geq 0 \right\} \cup \{0\} \subseteq K.$$

If $U \subseteq U'$ is an open inclusion, then restriction is given by

$$\begin{aligned} \mathcal{O}_X(D)(U') &\hookrightarrow \mathcal{O}_X(D)(U) \\ f &\mapsto f. \end{aligned}$$

Hence, $\mathcal{O}_X(D)$ is a presheaf.

Easy to check:

- $\mathcal{O}_X(D)$ is a sheaf.
- $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$-module: for any $f, g \in \mathcal{O}_X(D)(U)$, we have $f + g \in \mathcal{O}_X(D)(U)$.

  Exercise 9.24. $v_Y(f+g) \geq \min\{v_Y(f) + v_Y(g)\}$.

  Also, we define

  $$\operatorname{div}(rf) = \operatorname{div} r + \operatorname{div} f,$$

  which is still effective.

Also easy to check:

- If $D = 0$, then $\mathcal{O}_X(D) = \mathcal{O}_X$ (uses normality).[8]
- If $U = X - \operatorname{Supp} D$, then $\mathcal{O}_X(D)|_U = \mathcal{O}_X|_U$. This is a "rank 1 subsheaf of K."

Proposition 9.25. *If D is Cartier, then $\mathcal{O}_X(D)$ is locally free of rank* 1 *(i.e.,* invertible*).*

[8] Hartshorne uses the notation $\mathscr{L}(D)$; this is somewhat outdated.

Proof. If D is Cartier, then there is an open cover $\{U_\lambda, f_\lambda\}$ such that $D|_{U_\lambda} = \operatorname{div}_{U_\lambda} f_\lambda$. For all λ,

$$\begin{aligned}\mathcal{O}_X(D)(U_\lambda) &= \left\{g \in K^* \;\middle|\; \operatorname{div}_{U_\lambda} g + D\big|_{U_\lambda} \geq 0\right\} \cup \{0\} \\ &= \left\{g \in K^* \;\middle|\; \operatorname{div}_{U_\lambda} g + \operatorname{div}_{U_\lambda} f_\lambda \geq 0\right\} \cup \{0\}.\end{aligned}$$

We have $\operatorname{div}_{U_\lambda} g + \operatorname{div}_{U_\lambda} f_\lambda = \operatorname{div}_{U_\lambda}(g f_\lambda) \geq 0 \iff g f_\lambda \in \mathcal{O}_X(U_\lambda) \iff g \in \mathcal{O}_X(U_\lambda) \cdot f_\lambda^{-1} \subseteq K$.

Thus, $\mathcal{O}_X(D)$ is free on U_λ, generated by f_λ^{-1}. □

Proposition 9.26. *Let X be a normal scheme satisfying (*). (Actually, arbitrary X is fine, too.)*

(1) There is a one-to-one correspondence

$$\begin{aligned}\operatorname{CDiv}(X) \longleftrightarrow \{\textit{invertible subsheaves of } K\} &\overset{X \text{ integral}}{\longleftrightarrow} \{\textit{invertible sheaves on } X\} \\ D \mapsto \mathcal{O}_X(D).&\end{aligned}$$

(2) Given two Cartier divisors D_1, D_2,

$$\mathcal{O}_X(D_1 - D_2) \cong \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} [\mathcal{O}_X(D_2)]^{-1}.$$

(3) $D_1 \sim D_2 \iff \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$.

Proof sketch. (1) Fix an invertible subsheaf $\mathscr{L}$ of K.[9] Take an open cover $\{U_\lambda\}$ such that $\mathscr{L}|_{U_\lambda} \subseteq K$ is free of rank 1 on U_λ, generated by f_λ^{-1} via the map

$$\begin{aligned}\mathcal{O}_X\big|_{U_\lambda} &\cong \mathscr{L}\big|_{U_\lambda} \\ 1 &\mapsto f_\lambda^{-1}.\end{aligned}$$

Let $D = \{U_\lambda, f_\lambda\}$. It is easy to check that $\mathcal{O}_X(D) = \mathscr{L}$.

(2) A commutative algebra fact: $\mathcal{O}(-D) = [\mathcal{O}_X(D)]^{-1}$. The local picture to show this: Let A be a domain with fraction field K. Let $M = Af$ be a rank 1 free A-submodule of K. Then

$$M^* = \operatorname{Hom}_A(M, A) = \operatorname{Hom}_A(A \cdot f, A) = \frac{1}{f} \cdot A.$$

(3) It is equivalent to show $D = \operatorname{div} f \iff \mathcal{O}_X(D) \cong \mathcal{O}_X$.

Say we have

$$\begin{aligned}\mathcal{O}_X &\xrightarrow{\simeq} \mathcal{O}_X(D) \subseteq K \\ 1 &\mapsto f^{-1}.\end{aligned}$$

[9] If X is integral with generic point η, then we have

$$\begin{aligned}\mathcal{O}_X &\hookrightarrow \mathcal{O}_{X,\eta} = K \\ \mathscr{L} = \mathcal{O}_X \otimes \mathscr{L} &\hookrightarrow \mathscr{L} \otimes K = K,\end{aligned}$$

so any invertible sheaf is a subsheaf of K.

Check that $D = \operatorname{div} f$.

Conversely, if $D = \operatorname{div} f$, then check that there is a map

$$\begin{aligned} \mathcal{O}_X &\xrightarrow{\simeq} \mathcal{O}_X(D) \subseteq K \\ 1 &\mapsto f^{-1} \end{aligned}$$

which is an isomorphism. □

Remark 9.27. From what we have just shown, $\operatorname{Pic} X := \operatorname{CDiv}(X)/P(X)$ is isomorphic to the group of isomorphism classes of invertible sheaves (under $\otimes$).

9.11 Summary of the correspondence

Let X be a Noetherian integral separated scheme, and let K be its function field.

Last time, we defined a map

$$\begin{aligned} \operatorname{WDiv} X &\to \{\text{coherent } \mathcal{O}_X\text{-modules}\} \subseteq K \\ D &\mapsto \mathcal{O}_X(D) \end{aligned}$$

which restricts to an isomorphism

$$\begin{aligned} \operatorname{CDiv}(X) &\xrightarrow{\simeq} \{\text{invertible sheaves}\} \subseteq K \\ D\big|_U = \operatorname{div}_U f &\mapsto \mathcal{O}(D)(U) = f^{-1} \cdot \mathcal{O}_X(U) \end{aligned}$$

given on principal divisors by

$$\begin{aligned} P(X) &\xrightarrow{\simeq} \{\text{invertible sheaves} \cong \mathcal{O}_X\} \\ D = \operatorname{div} f &\mapsto \frac{1}{f}\mathcal{O}_X \cong \mathcal{O}_X. \end{aligned}$$

These are homomorphisms with respect to addition of divisors and the tensor operation on coherent $\mathcal{O}_X$-modules.

Aside 9.28 (not in Hartshorne). The image of $\operatorname{WDiv} X$ under the above map is the set of *reflexive* subsheaves of K. For any $\mathcal{O}_X$-module $\mathscr{F}$, there is a natural map $\mathscr{F} \to \mathscr{F}^{**}$. We say that $\mathscr{F}$ is *reflexive* if this is an isomorphism.

Corollary 9.29. *By the above correspondence,*

$$\operatorname{Pic} X \overset{def}{=} (\{\textit{invertible sheaves}\}/\cong) \cong (\operatorname{CDiv}(X)/\equiv).$$

Proof. The natural group homomorphism

$$\begin{aligned} \operatorname{CDiv}(X) &\to \operatorname{Pic} X \\ D &\mapsto [\mathcal{O}_X(D)] \end{aligned}$$

is surjective, and its kernel is $P(X)$. (Recall: On an integral scheme, *every* invertible sheaf is isomorphic to a subsheaf of K.) □

9.12 Examples of sheaves associated to divisors

First, an important general example:

Example 9.30. Say $Y \subseteq X$ is a prime divisor on X. Then $\mathscr{I}_Y \subseteq \mathcal{O}_X$, and we have a sheaf $\mathcal{O}_X(-Y)$ which is given on U by

$$\mathcal{O}_X(-Y)(U) = \left\{ f \in K^* \;\middle|\; \operatorname{div}_U f - Y\big|_U \geq 0 \right\}.$$

Fact 9.31. $\mathcal{O}_X(-Y) = \mathscr{I}_Y \subseteq X$.

More generally: If $D = \sum_{i=1}^{t} a_i Y_i$ is an effective divisor (i.e., each $a_i > 0$), then

$$\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$$

is an ideal sheaf defining a closed subscheme of X.[10]

Let us compute a more explicit example.

Example 9.32. Consider

$$\begin{aligned} X &= \mathbb{P}^2 = \operatorname{Proj} \overbrace{k[x_0, x_1, x_2]}^{S} \\ D &= C + 3H_0 = \mathbb{V}(x_0x_1^2 - x_2^3) + 3 \cdot \mathbb{V}(x_0) \\ F &= \left(x_0x_1^2 - x_2^3\right)\left(x_0^3\right). \end{aligned}$$

Then $(F) \subseteq S$, inducing an inclusion

$$\mathscr{I}_D = (\widetilde{F}) \subseteq \widetilde{S} = \mathcal{O}_{\mathbb{P}^2},$$

and

$$(\widetilde{F})(U_1) = \left[FS\left[\frac{1}{x_1}\right]\right]_0 = \left(\frac{F}{x_1^6}\right) k\left[\frac{x_0}{x_1}, \frac{x_2}{x_1}\right],$$

where

$$\frac{F}{x_1^6} = \left(x_{0/1} - x_{2/1}^3\right) x_{0/1}^3 = \left(s - t^3\right) s^3.$$

We have $FS \to S(-6)$, the free S-module generated 1, which has degree 6. Then

$$\begin{gathered} (FS) \cong S(-6), \\ \mathcal{O}_{\mathbb{P}^2}(-D) = \mathscr{I}_D = (\widetilde{FS}) \cong \widetilde{S(-6)} = \mathcal{O}_X(-6). \end{gathered}$$

Recall: $\operatorname{Pic}(\mathbb{P}^2) = \mathbb{Z}$. So since D is degree 6 in $\mathbb{P}^2$,

$$\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^2}(6).$$

[10] As a set, this subscheme corresponds to the union of the components of D, i.e.,

$$\operatorname{Supp} D = Y_1 \cup \cdots \cup Y_t.$$

Example 9.33. Continuing from the previous example:

$$\mathbb{P}^2 = \operatorname{Proj} k[x_0, x_1, x_2] \supseteq U_1 = \operatorname{Spec} k[x_{0/1}, x_{2/1}],$$

$$\mathcal{O}_{\mathbb{P}^2}(6)(U_1) = \widetilde{S(6)}(U_1) = \left[S(6)\left[\frac{1}{x_1}\right]\right]_0 = \left[S\left[\frac{1}{x_1}\right]\right]_6 = x_1^6 \cdot \left[S\left[\frac{1}{x_1}\right]\right]_0 = x_1^6 k[x_{0/1}, x_{2/1}].$$

On U_i, it is generated by $x_i^6 \cdot \mathcal{O}_X(U_i)$. The transition functions are:

$$\begin{aligned} &\mathcal{O}_{\mathbb{P}^2}(6)(U_i)\big|_{U_i \cap U_1} \to \mathcal{O}_{\mathbb{P}^2}(6)(U_1)\big|_{U_i \cap U_1} \\ &\qquad x_i^6 \mapsto x_1^6, \qquad\qquad \text{“multiplication by } \left(\frac{x_1}{x_i}\right)^6 \in \mathcal{O}_X(U_i \cap U_x)\text{”} \end{aligned}$$

If we do the same thing with the sheaf $\mathcal{O}_{\mathbb{P}^2}(D)$ (from Example 9.32), then we get the same transition functions. As a Cartier divisor, D is given locally on U_i by

$$D\big|_{U_i} = \operatorname{div}_{U_i}\left(\frac{F}{x_i^6}\right).$$

So

$$\mathcal{O}_X(D)(U_i) = \left(\frac{x_i^6}{F}\right) \cdot \mathcal{O}_X(U_i).$$

On $U_i \cap U_1$,

$$\frac{x_i^6}{F} \cdot a = \frac{x_1^6}{F} \cdot \frac{x_i^6}{x_1^6} \cdot a,$$

meaning that we have the same transition functions.

10 Maps to projective space

We are interested in maps from A-schemes to

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n] = \operatorname{Proj} S.$$

10.1 Initial remarks

Recall: $\mathbb{P}^n_A$ has an invertible sheaf

$$\mathcal{O}(1) = \widetilde{S(1)}$$

which is *globally* generated by sections $x_0, \ldots, x_n$.

Given any morphism $X \xrightarrow{\varphi} \mathbb{P}^n_A$ of A-schemes, the sheaf $\mathscr{L} = \varphi^*\mathcal{O}(1)$ is an invertible sheaf on X, globally generated by $s_i = \varphi^*(x_i)$.

Here is the picture on an affine chart:

$$\begin{aligned} X &\xrightarrow{\varphi} \mathbb{P}^n_A \\ \varphi^{-1}(U_0) &\to \operatorname{Spec} A\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right] = U_0 \end{aligned}$$

(Note that $\mathcal{O}(1)$ is generated by x_0.)

$$\mathcal{O}_X\big(\varphi^{-1}(U_0)\big) \leftarrow A\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right].$$

This is free, rank 1, generated by $s_0 = x_0 \otimes 1$:

$$\mathcal{O}(1)(U_0) \otimes_{A\left[\frac{x}{x_0}\right]} \mathcal{O}_X\big(\varphi^{-1}(U_0)\big).$$

10.2 Invertible sheaves and $\mathbb{P}^n$

Theorem 10.1. *Let X be a scheme over A.*

(1) If $\varphi : X \to \mathbb{P}^n_A$ is a morphism of A-schemes, then $\mathscr{L} = \varphi^*\mathcal{O}(1)$ is an invertible sheaf on X which is globally generated by

$$\varphi^*(x_i) = 1 \otimes x_i \in \varphi^*\mathcal{O}(1).$$

(2) Conversely, if $\mathscr{L}$ is an invertible sheaf on X, and $s_0, \dots, s_n$ are a set of global generators for $\mathscr{L}$, then there is a unique morphism of A-schemes $\varphi : X \to \mathbb{P}^n_A$ such that $\varphi^*\mathcal{O}(1) = \mathscr{L}$ and $\varphi^*(x_I) = s_i$.

Remark 10.2. The map in (2) can be intuitively thought of as

$$\begin{aligned} X &\to \mathbb{P}^n \\ x &\mapsto [s_0(x) : \cdots : s_n(x)] \qquad\qquad \frac{s_i}{s_j} \in \mathcal{O}_X(U_j). \end{aligned}$$

Proof of part (2). Given $\mathscr{L}$ and $s_0, \dots, s_n \in \mathscr{L}(X)$, let

$$X_i = \left\{x \in X \;\middle|\; s_i \text{ generates } \mathscr{L} \text{ at } x\right\},$$

(i.e., the image of s_i in $\mathscr{L}_x$ generates $\mathscr{L}_x$ as an $\mathcal{O}_{X,x}$-module)

$$= \left\{x \in X \;\middle|\; s_i \notin \mathfrak{m}_x\mathscr{L}_x, \text{ where } \mathfrak{m}_x \subseteq \mathcal{O}_{X,x} \text{ is the maximal ideal}\right\}$$

by Nakayama's lemma. Easy to check: $X_i \subseteq X$ is open (Hartshorne, II, Lemma 5.14).

Claim 10.3 (Main point of proof)**.** *On X_i, we can think of s_j/s_i as an element of $\mathcal{O}_X(X_i)$.*

Here is why: on X_i,

$$\mathscr{L}(X_i) = \mathcal{O}_X(X_i) \cdot s_i,$$

and we can restrict the global generator $s_j \in \mathscr{L}(X)$ to $\mathscr{L}(X_i)$, so that

$$s_j = r \cdot s_i \implies \frac{s_j}{s_i} = r \in \mathcal{O}_X(X_i).$$

Plan: Trying to define a map

$$X \xrightarrow{\varphi} \mathbb{P}^n_A.$$

We'll give maps

$$X_i \xrightarrow{\varphi_i} U_i = \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$$

which agrees on $X_i \cap X_j$. Giving φ_i is equivalent to giving an A-algebra map

$$\begin{aligned} A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] &\to \mathcal{O}_X(X_i) \\ \frac{x_j}{x_i} &\mapsto \frac{s_j}{s_i} \in \mathcal{O}_X(X_i). \end{aligned}$$

To check that these morphisms glue up to a morphism $X \to \mathbb{P}^n_A$, observe that

$$\frac{x_j}{x_k} = \frac{x_j/x_i}{x_k/x_i} \mapsto \frac{s_j/s_i}{s_k/s_i} = \frac{s_j}{s_k}. \qquad \square$$

Remark 10.4 (Important point). The sections s_i cannot be "evaluated at P" so that $s_i(P) \in k$. But their ratios s_j/s_i *are* regular functions on X_i.

10.3 Some examples

Consider $\mathbb{P}^1_A = \operatorname{Proj} A[x, y]$, and let $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1_A}(d)$ for some $d > 0$. Consider the global sections $s_i = x^{d-i}y^i$ for $i = 0, \dots, d$. We get an A-morphism

$$\begin{gathered} \mathbb{P}^1_A \xrightarrow{\nu_d} \mathbb{P}^d_A \\ \text{“}[x : y] \mapsto \left[x^d : x^{d-1}y : \cdots : xy^{d-1} : y^d\right]\text{”} \\ X_i = \left\{P \in X \;\middle|\; s_i \text{ generates } \mathscr{L} \text{ at } P\right\} \to \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \frac{x_d}{x_i}\right] = U_i \\ \mathcal{O}_X(X_i) \leftarrow A\left[\frac{x_0}{x_i}, \dots, \frac{x_d}{x_i}\right] \\ \left(\frac{y}{x}\right)^j = \frac{x^{d-j}y^i}{x^d} = \frac{s_j}{s_i} \leftarrow \frac{x_j}{x_0} \end{gathered}$$

This is the d-th Veronese map.

What if we use a different set of global generators of the same size that differ linearly from the s_i? Then we get the same map, up to a linear change of coordinates.

The global sections x^d, y^d also globally generate $\mathcal{O}(d)$. This gives a map

$$\begin{gathered} \mathbb{P}^1_A \to \mathbb{P}^1_A \\ \text{“}[x : y] \mapsto \left[x^d : y^d\right]\text{”} \end{gathered}$$

which is given by the d-th Veronese map ν_d, followed by a projection to $\mathbb{P}^1_A$.

10.4 Automorphisms of projective space

Theorem 10.5. *Let k be any field. The automorphism group of $\mathbb{P}^n_k$ (as a k-scheme) is* $\operatorname{PGL}(n, k) = \operatorname{GL}(n+1, k)/k^*$.

Proof. We have a natural homomorphism

$$\begin{aligned}\mathrm{GL}(n+1,k) &\to \operatorname{Aut}\mathbb{P}^n_k\\ g &\mapsto g\end{aligned}$$

whose kernel consists of all "scalar multiplication" linear transformations, i.e., k^*. Thus, we have an injection

$$\mathrm{PGL}(n,k) \hookrightarrow \operatorname{Aut}(\mathbb{P}^n_k).$$

Say we have a k-automorphism

$$\begin{aligned}\mathbb{P}^n_k &\xrightarrow{\varphi} \mathbb{P}^n_k,\\ \varphi^*\mathcal{O}(1) &= \mathscr{L},\\ \varphi^*(x_i) &= s_i.\end{aligned}$$

Recall that $\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ via the isomorphism $\mathcal{O}(d) \longleftrightarrow d$.

Because φ is an automorphism, the induced map

$$\begin{aligned}\operatorname{Pic}\mathbb{P}^n_k &\to \operatorname{Pic}\mathbb{P}^n_k\\ \mathscr{L} &\mapsto \varphi^*\mathscr{L}\end{aligned}$$

is an automorphism of groups. Since $\mathcal{O}(-1)$ has no global sections,

$$\varphi^*\mathcal{O}(1) = \mathcal{O}(1).$$

Given an automorphism $\varphi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ corresponding to $\mathscr{L} = \mathcal{O}(1) = \varphi^*\mathcal{O}(1)$ and $s_i = \varphi^*(x_i) \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$. So we can write $s_i = a_{i0}x_i + \cdots + a_{in}x_n$, whence

$$[s_0 : \cdots : s_n] = A \cdot [x_0 : \cdots : x_n],$$

where A is a matrix, and we are done. □

10.5 Connection with linear systems

Fix X, an invertible sheaf $\mathscr{L}$, and a nonzero global section $s \in \mathscr{L}(X)$. There is a corresponding Cartier divisor, called "the divisor of zeros of s".

Definition 10.6. The *divisor of zeros* $(s)_0$ is the Cartier divisor defined as follows. Fix a trivialization of $\mathscr{L}$:

$$\begin{aligned}g_\lambda \cdot \mathcal{O}_X\big|_{U_\lambda} = \mathscr{L}\big|_{U_\lambda} &\xrightarrow[\varphi_\lambda]{\simeq} \mathcal{O}_X\big|_{U_\lambda}\\ g_\lambda &\leftarrow\!\shortmid 1\\ s &\mapsto \varphi_\lambda(s) = r_\lambda.\end{aligned}$$

So $s|_{U_\lambda} = g_\lambda \cdot r_\lambda \in \mathscr{L}(U_\lambda)$, where $r_\lambda \in \mathcal{O}_X(U_\lambda)$. Define $(s)_0$ on U_λ as $\operatorname{div}(r_\lambda)$.

This is a well-defined divisor on X since on $U_\lambda \cap U_{\lambda'}$,

$$r_\lambda = s_{\lambda\lambda'} r_{\lambda'} \in \mathcal{O}^*_X(U_\lambda \cap U_{\lambda'}).$$

Example 10.7. On $\mathbb{P}^1_k$, let $\mathscr{L} = \mathcal{O}(d)$ and $s = xy^{d-1}$. Write

$$\begin{aligned} H_0 &= \mathbb{V}(x) \subseteq \mathbb{P}^1_k, \\ H_1 &= \mathbb{V}(y) \supseteq \mathbb{P}^1_K. \end{aligned}$$

The corresponding divisor is

$$(s)_0 = H_0 + (d-1)H_1.$$

On $U_0 = \operatorname{Spec} k[y/x]$,

$$\begin{aligned} \mathcal{O}(d)\big|_{U_0} &= x^d \cdot k\left[\frac{y}{x}\right] \\ s &= xy^{d-1} = x^d \left(\frac{xy^{d-1}}{x^d}\right) \\ r_0 &= \left(\frac{y}{x}\right)^{d-1}, \end{aligned}$$

so $(s)_0|_{U_0} = \operatorname{div}_{U_0} r_0$.

Example 10.8. Let $\mathscr{L} = \mathcal{O}(d)$ on $\mathbb{P}^n_k$. Then

$$\begin{aligned} [k[x_0, \ldots, x_n]]_d = \{\text{global sections of } \mathcal{O}(d)\} &\xrightarrow{\text{"divisor of zeros"}} \{\text{effective divisors}\} \\ F_d &\mapsto \mathbb{V}(F_d) \subseteq \mathbb{P}^n_k. \end{aligned}$$

This gives the complete linear system of all effective divisors in $\mathbb{P}^n$ of degree d.

Proposition 10.9. *If $s \in \mathscr{L}(X)$ is a nonzero global section of an invertible sheaf $\mathscr{L}$ of X, let D be its divisor of zeros. Then there is an isomorphism*

$$\mathcal{O}_X(D) \xrightarrow[\simeq]{\text{"multiplication by } s\text{"}} \mathscr{L}.$$

Proof. Take U such that

$$\begin{aligned} g \cdot \mathcal{O}_X\big|_U = \mathscr{L}\big|_U &\xrightarrow{\simeq} \mathcal{O}_X\big|_U \\ s = r\cdot &\longleftrightarrow r. \end{aligned}$$

We have $D|_U = \operatorname{div}_U r$. Then

$$\mathcal{O}_X(D)(U) = \left\{ f \in K^* \;\middle|\; \operatorname{div}_U f + D \geq 0 \right\} = \frac{1}{r} \cdot \mathcal{O}_X\big|_U.$$

Consider the map

$$\begin{aligned} \mathcal{O}_X(D) &\xrightarrow{\simeq} \mathscr{L} \\ \mathcal{O}_X(D)\big|_U &\to \mathscr{L}\big|_U \\ \frac{1}{r} \cdot \mathcal{O}_X\big|_U &\xrightarrow{\text{"mult. by } s\text{"}} g \cdot \mathcal{O}_X(U) \\ \frac{f}{r} &\mapsto \frac{sf}{r} = g \cdot f. \end{aligned}$$

This glues on the different patches to give the desired isomorphism. ◻

Example 10.10. Again, consider $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1_k}(d)$. Then $x_0 x_1^{d-1} = F_d \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(d))$ corresponds to

$$D = H_0 + (d-1)H_1 = \mathbb{V}(x_0) + (d-1)\mathbb{V}(x_1) = \mathbb{V}(F_d).$$

By Proposition 10.9, $\mathscr{L} \cong \mathcal{O}(D)$. Note that D is an effective divisor.

Example 10.11 (The hyperplane bundle). On $\mathbb{P}^n$, consider a global section $L = \sum_{i=0} a_i x_i$ of $\mathcal{O}_{\mathbb{P}^n_k}(1)$ corresponding to a divisor H.

The full vector space of global sections of $\mathcal{O}(1)$ is in bijection with the full set of hyperplanes in $\mathbb{P}^n_k$ (so $\mathcal{O}(1)$ is the "*hyperplane bundle*").

Remark 10.12. A bad abuse of notation that you might see sometimes: "$\mathcal{O}(1) = \mathcal{O}(H) = \mathcal{O}(C - H_1)$". Don't do this; it's confusing!

Remark 10.13 (Connection to 631). Fix a divisor D. Then

$$|D| = \left\{ D' \in \operatorname{Div} X \;\middle|\; D' \geq 0,\ D' = D + \operatorname{div} f \right\} = \left\{ f \in K^* \;\middle|\; \operatorname{div} f + D \geq 0 \right\} = \mathcal{O}_X(D).$$

10.6 Example: An elliptic curve

Let k be any field. Consider an elliptic curve

$$E = \mathbb{V}\left(\frac{y^2 z - x^3 - xz - z^3}{z^3}\right) \subseteq \mathbb{A}^2 = \operatorname{Spec} k\left[\frac{x}{z}, \frac{y}{z}\right] \subseteq \operatorname{Proj} k[x, y, z].$$

Taking the projective closure, this looks like

$$E = \operatorname{Proj} \frac{k[x, y, z]}{(y^2 z - x^3 - xz - z^3)} \hookrightarrow \mathbb{P}^2.$$

Observe that y, z globally generate $\mathscr{L} = \varphi^* \mathcal{O}(1)$ on E. The associated map is

$$\begin{aligned} E &\to \mathbb{P}^1 \\ [x : y : z] &\mapsto [y : z] \\ \left[\frac{x}{z} : \frac{y}{z} : 1\right] &\mapsto \left[\frac{y}{z} : 1\right]. \end{aligned}$$

10.7 Divisors and projective morphisms

10.7.1 Summary of invertible sheaves and projective morphisms

Fix an A-scheme X. Then

$$\begin{aligned} \{A\text{-morphisms } X \to \mathbb{P}^n_A\} &\longleftrightarrow \left\{ \begin{array}{c} \text{invertible sheaves on } X\text{, plus a set of} \\ n+1 \text{ global sections which generate the} \\ \text{sheaf} \end{array} \right\} \\ \left[X \xrightarrow{\varphi} \mathbb{P}^n_A\right] &\mapsto \mathscr{L} = \varphi^* \mathcal{O}_{\mathbb{P}^n_A}(1),\ s_i = \varphi^*(x_i),\ i = 0, \ldots, n \\ \left[x \mapsto [s_0(x) : \cdots : s_n(x)]\right] &\leftarrow [\mathscr{L},\ s_0, \ldots, s_n \in \mathscr{L}(X)]. \end{aligned}$$

If $A = k$, $\mathscr{L} = \mathcal{O}_X(D) \subseteq K$, and X integral, then each $s_i \in K$. Then $s_i(x)$ makes sense as an element of k for each k-point $x \in X$.

10.7.2 Alternate, classical perspective

Let us now translate this into the language of linear systems of divisors. Let $A = k$, and assume X is normal.

Recall that each $s \in \mathscr{L}(X)$ has an associated *effective* divisor D, the "divisor of zeros of s", denoted

$$D = (s)_0 = \{s = 0\} \subseteq X.$$

We have $\mathcal{O}_X(D) \cong \mathscr{L}$. [If $\mathscr{L} = \mathcal{O}_X(D')$, then $s \in \mathscr{L}(X) = \{f \in K^* \mid \operatorname{div} f + D' \geq 0\}$.]

Given two different global sections s_1 and s_2 of $\mathscr{L}(X)$, the corresponding divisors of zeros D_1 and D_2 are *linearly equivalent.*

Observe that $s \in \mathscr{L}(X)$ generates $\mathscr{L}$ and $P \in X \iff s$ generates $\mathscr{L}_P \iff s \notin \mathfrak{m}_P \mathscr{L}_P \iff s$ does not vanish at $P \iff P \notin \operatorname{Supp} D$. We have

$$X_s = \left\{P \in X \;\middle|\; s \text{ generates } \mathscr{L}\right\} = X - \operatorname{Supp} D.$$

Global sections $s_0, \ldots, s_n \in \mathscr{L}(X)$ *fail* to generate at $P \iff P \in \bigcap_{i=0}^n \operatorname{Supp} D_i$, where $D_i = (s_i)_0$. So $s_0, \ldots, s_n$ generate $\mathscr{L} \iff \bigcap_{i=0}^n \operatorname{Supp} D_i = \varnothing$.

Here is how complete linear systems fit into the picture:

$$\{k\text{-vector space } \mathscr{L}(X)\} \longleftrightarrow \left\{\text{complete linear system } |D| = \left\{D' = (s)_0 \;\middle|\; s \in \mathscr{L}(X)\right\}\right\}.$$

For any representative D in the complete linear system, $\mathscr{L} \cong \mathcal{O}_X(D)$. Linear systems correspond to vector subspaces:

$$\{\text{subvector space } V \subseteq \mathscr{L}(X)\} \longleftrightarrow \left\{\text{linear system } \mathfrak{D} = \left\{D = (s)_0 \;\middle|\; s \in V \setminus \{0\}\right\}\right\}.$$

There is also a correspondence between *base loci*:

$$\left\{P \in X \;\middle|\; \begin{array}{c}\text{the elements of } V \\ \text{fail to generated } \mathscr{L} \\ \text{at } P\end{array}\right\} \longleftrightarrow \operatorname{Bs}(D) \stackrel{\text{def}}{=} \bigcap_{D \in \mathfrak{D}} \operatorname{Supp} D \subseteq X.$$

Definition 10.14. The *base locus* of V is the set of points $P \in X$ such that the elements of V fail to generate $\mathscr{L}$ at P.

The *base locus* of a linear system D is

$$\operatorname{Bs}(D) \stackrel{\text{def}}{=} \bigcap_{D \in \mathfrak{D}} \operatorname{Supp} D \subseteq X.$$

Fix a basis $s_0, \ldots, s_n \in V \subseteq \mathscr{L}(X)$. These generate $\mathscr{L}$ on the open set $X - \operatorname{Bs}(D)$. The linear system $\mathfrak{D}$ igves a map

$$\begin{aligned} X - \operatorname{Bs}(D) &\to \mathbb{P}^n_A \\ x &\mapsto [s_0(x) : \cdots : s_n(x)], \end{aligned}$$

which extends to a *rational* map $X \to \mathbb{P}^n_A$.

Remark 10.15. The sheaf $\mathscr{L}$ is globally generated $\iff |D|$ is a base-point-free linear system.

Also, $\mathscr{L}$ is very ample $\iff \exists s_0, \ldots, s_n \in \mathscr{L}(X)$ globally generate and define an immersion in $\mathbb{P}^n_A$.

Remark 10.16.

- $\mathscr{L}$ is very ample over $k \iff |D|$ defines an embedding.
- $\mathscr{L}$ is ample over $A \iff \mathscr{L}^n$ is very ample for some $n > 0$.

10.8 Example: A blowup of projective space

Example 10.17. Let X be the blowup of $\mathbb{P}^2 = \operatorname{Proj} k[x,y,z]$ at $[0:0:1]$. Then

$$X = \{(p,\ell) \mid p \in \ell\} = \left\{ [x:y:z],[s:t] \;\middle|\; \operatorname{rank}\begin{pmatrix} x & y \\ s & t \end{pmatrix} = 1 \right\} = \mathbb{V}(xt - sy) \subseteq \overset{[x:y:z]}{\mathbb{P}^2} \times \overset{[s:t]}{\mathbb{P}^1},$$

where $\mathbb{P}^1 =$ lines in $\mathbb{P}^2$ through $[0:0:1]$. We have

$$\begin{aligned} \operatorname{Pic}\mathbb{P}^2 &= \mathbb{Z}\cdot H, \\ \operatorname{Pic}X &= \mathbb{Z}\,(\pi^*H) \oplus \mathbb{Z}E, \end{aligned}$$

where E is the exceptional divisor $\pi^{-1}([0:0:1])$. The blowup map $X \xrightarrow{\pi} \mathbb{P}^2$ is given by $\mathscr{L} = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and π^*x, π^*y, π^*z. Since π is well-defined but not an embedding, $\mathscr{L}$ is globally generated by π^*x, π^*y, π^*z, but not very ample.

Write $L_1 := \mathbb{V}(x) = (x)_0$, $L_2 := \mathbb{V}(y) = (y)_0$, $L_\infty := \mathbb{V}(z) = (z)_0$ for the divisors of zeros in $\mathbb{P}^2$. Then in X,

$$\begin{aligned} (\pi^*x)_0 &= \widetilde{L}_1 + E, \\ (\pi^*y)_0 &= \widetilde{L}_2 + E, \\ (\pi^*z)_0 &= L_\infty. \end{aligned}$$

The corresponding linear system on X is $|\pi^*H| =$ divisors on X satisfying either

- birational transforms of lines in $\mathbb{P}^2$ not through $[0:0:1]$,
- $E + L$, where L is the birational transform of a line through $[0:0:1]$.

Example 10.18. Let's look at the other projection now:

$$\mathbb{P}^2 \times \mathbb{P}^1 \supseteq \mathbb{V}(xt - ys) = X \xrightarrow{\nu} \mathbb{P}^1_k = \operatorname{Proj} k[s,t].$$

This collapses to the central line E. So this is essentially the tautological bundle. It is given by

$$\mathscr{M} = \nu^*\mathcal{O}_{\mathbb{P}^1}(1)$$

and s,t. The corresponding vector space is

$$V = \left\{ bs + at \;\middle|\; a,b \in k \right\} \subseteq \mathscr{M}(X).$$

The corresponding system of divisors are

$$\{bs + at = 0\} = \{[a:b],[a:b:z]\},$$

which is the line in X corresponding to the line through $[0:0:1]$ in $\mathbb{P}^2$ determining the point $[a:b] \in \mathbb{P}^1$.

For each of the divisors D above, $D \sim -E$.

Example 10.19. Consider

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend right, "\varphi"'] & \mathbb{P}^2 \times \mathbb{P}^1 \arrow[r, hook, "\sigma"] & \mathbb{P}^5. \end{tikzcd}$$

Write $\varphi^*\mathcal{O}(1) = \mathscr{N}$. The global sections sx, sy, sz, tx, ty, tz generate the linear system $|L_\infty - E|$. Note that the image actually lands in $\mathbb{P}^4 = \mathbb{V}(sy - tx)$.

11 Cohomology of sheaves

11.1 Big picture

We now turn to the cohomology of sheaves of abelian groups on schemes.

Fix a scheme X. We're interested in the functor

$$\begin{aligned} \{\text{sheaves of abelian groups on } X\} &\xrightarrow{\Gamma} \{\text{abelian groups}\} \\ \mathscr{F} &\mapsto \Gamma(X, \mathscr{F}) = \text{global sections of } \mathscr{F}. \end{aligned}$$

This is covariant and left exact, i.e., given an exact sequence of sheaves

$$0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0,$$

we have an exact sequence

$$0 \to \Gamma(X, \mathscr{A}) \to \Gamma(X, \mathscr{B}) \to \Gamma(X, \mathscr{C}).$$

The propose of sheaf cohomology is to construct a collection of (additive) functors: for each $i = 0, 1, 2, \ldots,$

$$\{\text{sheaves of abelian groups on } X\} \xrightarrow{H^i} \{\text{abelian groups}\}$$

such that

(1) $H^0(\mathscr{F}) \overset{\text{def}}{=} H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$;

(2) If $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ is a short exact sequence of sheaves, then we get a long exact sequence of cohomology

$$0 \to H^0(\mathscr{A}) \to H^0(\mathscr{B}) \to H^0(\mathscr{C}) \to H^1(\mathscr{A}) \to H^1(\mathscr{B}) \to H^1(\mathscr{C}) \to H^2(\mathscr{A}) \to \ldots.$$

Remark 11.1 (Right derived functors). In general, given *any* left exact covariant functor from one abelian category to another, we can always construct "right derived functors" (provided the source category "has enough injectives", which is always true for sheaves). [An *injective object* in a category is an object I such that $\mathrm{Hom}(-, I)$ is exact.]

If the original functor is $\mathscr{A} \xrightarrow{F} \mathscr{B}$, we'll get $\forall i \in \mathbb{Z}_{\geq 0}$ a functor

$$\mathscr{A} \xrightarrow{R^i F} \mathscr{B}$$

such that

- $R^0 F = F$;
- For any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathscr{A}$, we get a long exact sequence

$$0 \to R^0 F M_1 \to R^0 F M_2 \to R^0 F M_3 \to R^1 F M_1 \to R^1 F M_2 \to \ldots;$$

- If I is injective, then $R^i I = 0$ for all $i > 0$.

Example 11.2 (Ext). Fix a commutative ring R and an R-module M. Then we have a functor

$$\begin{aligned} R\text{-}\mathbf{Mod} &\xrightarrow{\operatorname{Hom}(M,-)} R\text{-}\mathbf{Mod} \\ A &\mapsto \operatorname{Hom}(M, A). \end{aligned}$$

This is covariant and left exact. Given

$$0 \to A \to B \to C \to 0,$$

we obtain the Ext long exact sequence

$$0 \to \operatorname{Hom}(M, A) \to \operatorname{Hom}(M, B) \to \operatorname{Hom}(M, C) \to \operatorname{Ext}^1(M, A) \to \operatorname{Ext}^1(M, B) \to \dots.$$

The right derived functors are called $\operatorname{Ext}^i(M, -)$.

Example 11.3 (Tor). We also have a functor

$$\begin{aligned} R\text{-}\mathbf{Mod} &\xrightarrow{-\otimes M} R\text{-}\mathbf{Mod} \\ A &\mapsto A \otimes_R M \end{aligned}$$

which is covariant and *right*-exact. Since we have enough *projectives*, there are *left* derived functors Tor^i such that, given a short exact sequence

$$0 \to A \to B \to C \to 0,$$

there is the Tor long exact sequence

$$\begin{aligned} \dots \to \operatorname{Tor}^2(M, B) \to \operatorname{Tor}^2(M, C) \to \operatorname{Tor}^1(M, A) \to \operatorname{Tor}^1(M, B) \\ \to \operatorname{Tor}^1(M, C) \to M \otimes A \to M \otimes B \to M \otimes C \to 0. \end{aligned}$$

11.2 Motivation: global generation

Let X be a projective scheme over k, and let $\mathscr{L}$ be an invertible sheaf. Consider

$$\begin{aligned} X &\dashrightarrow \mathbb{P}\big(\Gamma(X, \mathscr{L})\big) \\ x &\mapsto [s_0(x) : \dots : s_n(x)]. \end{aligned}$$

In order to use this, we need to know what is $\dim_k(\Gamma(X, \mathscr{L}))$. In other words, given $P \in X$, when is $\mathscr{L}$ globally generated at P?

Fix $P \in X$. Suppose $P \overset{i}{\hookrightarrow} X$ is a k-point. Then we have an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathfrak{m}_P \hookrightarrow \mathcal{O}_X \to i_* \mathcal{O}_P = \frac{\mathcal{O}_X}{\mathfrak{m}_P \mathcal{O}_X} \to 0.$$

Tensor with $\mathscr{L}$. Locally free $\implies$ flat, so we get an exact sequence

$$\begin{aligned} 0 \to \mathfrak{m}_P \otimes_{\mathcal{O}_X} \mathscr{L} \to \mathscr{L} &\xrightarrow{\text{“eval at } P\text{”}} \frac{\mathscr{L}}{\mathfrak{m}_P \mathscr{L}} \to 0 \\ s &\mapsto s \pmod{\mathfrak{m}_P \mathscr{L}} \end{aligned}$$

Now, $\mathscr{L}$ is globally generated at $P \iff$ the sequence

$$\begin{aligned} 0 \to \Gamma(X, \mathfrak{m}_P \otimes \mathscr{L}) \to \Gamma(X, \mathscr{L}) \to \Gamma(X, \mathscr{L}/\mathfrak{m}_P\mathscr{L}) \to 0 \\ s \mapsto s(P) \end{aligned}$$

is *still* exact.

In general, cohomology gives us a long exact sequence:

$$\begin{aligned} 0 \to \Gamma(X, \mathfrak{m}_P \otimes \mathscr{L}) \to \Gamma(X, \mathscr{L}) \to \Gamma(X, \mathscr{L}/\mathfrak{m}_P\mathscr{L}) \\ \to H^1(X, \mathfrak{m}_P \otimes \mathscr{L}) \to H^1(X, \mathscr{L}) \to \dots. \end{aligned}$$

Often, we prove that $\mathscr{L}$ is globally generated at P by showing $H^1(X, \mathfrak{m}_P \otimes \mathscr{L}) = 0$.

11.3 Motivation: invariants of schemes

We can use cohomology to define new *invariants* of schemes.

Example 11.4. If X is a smooth projective curve over k, then its (arithmetic) *genus* is $\dim_k H^1(X, \mathcal{O}_X)$.

Let $C \overset{i}{\subseteq} \mathbb{P}^2$ be a smooth curve. How can we compute the genus of C? By definition,

$$g = \dim_k H^1(C, \mathcal{O}_C).$$

There is an exact sequence of $\mathcal{O}_{\mathbb{P}^2}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-C) = \mathscr{I}_C \to \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_C \to 0.$$

We can also write this has

$$0 \to \mathcal{O}(-d) \hookrightarrow \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0.$$

This is because

$$\begin{aligned} \Gamma(\mathbb{P}^2, i_*\mathcal{O}_C) = \Gamma(C, \mathcal{O}_C), \\ i_*\mathcal{O}_C(\mathbb{P}^2) = \mathcal{O}_C(C). \end{aligned}$$

There's a corresponding long exact sequence:

$$\begin{aligned} 0 \to \Gamma\big(\mathbb{P}^2, \mathcal{O}(-d)\big) \to \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to \Gamma(C, \mathcal{O}_C) \\ \to H^1\big(\mathbb{P}^2, \mathcal{O}(-d)\big) \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to H^1(C, \mathcal{O}_C) \\ \to H^2\big(\mathbb{P}^2, \mathcal{O}(-d)\big) \to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to \dots. \end{aligned}$$

Theoretically, if we know *all* $H^i(\mathbb{P}^n, \mathcal{O}(d))$ for all i, n, d, then we could compute $H^1(C, \mathcal{O}_C)$.

In fact, in this case:

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0,$$

so

$$H^1(C, \mathcal{O}_C) \cong H^2\big(\mathbb{P}^2, \mathcal{O}(-d)\big).$$

Also, by Serre duality (or using some commutative algebra), $H^2(\mathbb{P}^2, \mathcal{O}(-d))$ is dual to $\big[k[x,y,z]\big]_{d-3}$.

Thus, the genus of C is

$$\begin{aligned} g &= \dim_k H^1(C, \mathcal{O}_C) = \dim_k H^2\big(\mathbb{P}^2, \mathcal{O}(-d)\big) = \dim_k \big[k[x,y,z]\big]_{d-3} \\ &= \binom{d-3+2}{2} = \frac{(d-1)(d-2)}{2}. \end{aligned}$$

11.4 Abelian categories and injective objects

An *abelian category* is a category where "exact sequences make sense": kernels exist, cokernels exist, can add objects and morphisms, etc.

Example 11.5. Here are some abelian categories:

- Abelian groups
- Vector spaces over a fixed field k.
- Modules over a fixed ring R.
- Sheaves of abelian groups on a fixed topological space X.
- Sheaves of modules on a fixed ringed space $(X, \mathcal{O}_X)$.
- Quasi-coherent sheaves on a fixed scheme X.
- Coherent sheaves on a fixed scheme X.
- Finitely-generated modules over a ring R.
- Finitely-generated abelian groups.

Some things that aren't abelian categories:

- Topological spaces
- Manifolds
- Complex manifolds
- Varieties
- Rings (assuming you're sensible and require rings to have a multiplicative identity)
- Schemes

Definition 11.6. A object (in an abelian category) I is *injective* provided that $\operatorname{Hom}(-, I)$ is exact.

Equivalently, given $A \hookrightarrow B$ and $A \to I$, we have a lifting

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & B \arrow[d, dashed, "\exists"] \\ & I. \end{tikzcd}$$

Example 11.7. In the category of abelian groups, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, and $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ are injective objects.

Lemma 11.8. *If $I \hookrightarrow M$, where I is injective, then this splits, so $M \cong I \oplus N$ for some N.*

Definition 11.9. An abelian category has *enough injectives* if every object embeds into an object object.

Example 11.10. Of the abelian categories we listed in Example 11.5, the following have enough injectives:

- Abelian groups
- Vector spaces over a fixed field k.
- Modules over a fixed ring R.
- Sheaves of abelian groups on a fixed topological space X.
- Sheaves of modules on a fixed ringed space $(X, \mathcal{O}_X)$.
- Quasi-coherent sheaves on a fixed scheme X.

However, these do *not* have enough injectives:

- Coherent sheaves on a fixed scheme X.
- Finitely-generated modules over a ring R.
- Finitely-generated abelian groups.

Note 11.11. If we have enough injectives, then every object has an injective resolution.

We can construct an injective resolution

$$0 \to \mathscr{F} \to I^0 \to I^1 \to I^2 \to \dots,$$

which is exact by diagram chasing.

Aside 11.12 (The language of derived categories). The *derived category* is formed from chain complexes with a notion of isomorphism. We can embed an object $\mathscr{F}$ in the derived category via

$$0 \to \mathscr{F} \to 0 \to 0 \to \dots,$$

and think of $\mathscr{F}$ as "(quasi-)isomorphic in the derived category" to

$$0 \to I^0 \to I^1 \to I^2 \to \dots$$

because it has isomorphic cohomology.

11.5 Grothendieck's derived functors

(1) Start with a functor [left exact, covariant] Γ from one abelian category [with enough injectives] to another.

$$\{\text{Sheaves of abelian groups on } X\} \xrightarrow{\Gamma(X,-)} \{\text{Abelian groups}\}$$
$$\mathscr{F} \mapsto \Gamma(X, \mathscr{F}) = \mathscr{F}(X).$$

(2) Fix $\mathscr{F}$ in the source category.

(3) To compute the derived functor $R^i\Gamma$ of $\mathscr{F}$, take an injective resolution of $\mathscr{F}$:

$$0 \to \mathscr{F} \to I^0 \to I^1 \to I^2 \to \dots,$$

also denoted

$$0 \to \mathscr{F} \to I^\bullet.$$

(In practice, this is the impossible part.)

(4) Apply the functor Γ to $I^\bullet$ to get a sequence of objects in the target:

$$0 \to \Gamma(I^0) \to \Gamma(I^1) \to \Gamma(I^2) \to \dots.$$

(5) Define

$$R^i\Gamma(\mathscr{F}) = \frac{\ker\left(\Gamma(I^i) \to \Gamma(I^{i+1})\right)}{\operatorname{im}\left(\Gamma(I^{i-1}) \to \Gamma(I^i)\right)}.$$

Definition 11.13 (sheaf cohomology)**.** The cohomology of a sheaf $\mathscr{F}$ is

$$H^i(X, \mathscr{F}) \stackrel{\text{def}}{=} R^i\Gamma(X, \mathscr{F}).$$

Proposition 11.14 (easy to check)**.** *(0) This is independent of the choice of injective resolution.*

(1) $R^0\Gamma(\mathscr{F}) = \Gamma(\mathscr{F})$.

(2) Given a short exact sequence $0 \to A \to B \to C \to 0$*, there is a long exact sequence*

$$0 \to R^0\Gamma(A) \to R^0\Gamma(B) \to R^0\Gamma(C) \to R^1\Gamma(A) \to R^1\Gamma(B) \to \dots.$$

(3) If I *is injective, then* $R^i\Gamma(I) = 0$ *for all* $i > 0$*.*

(Use diagram chasing, the snake lemma, and Lemma 11.8.)

11.6 Acyclic sheaves

Definition 11.15. A sheaf $\mathscr{F}$ is "*acyclic* for Γ" if $R^i\Gamma(\mathscr{F}) = 0$ for all $i > 0$.

Example 11.16 (Main example of acyclic sheaves for Γ). A sheaf $\mathscr{F}$ is *flasque* if, for all nonempty open inclusions $U \subseteq V$, the restriction map $\mathscr{F}(V) \xrightarrow{\rho} \mathscr{F}(U)$ is surjective. Flasque sheaves are acyclic for Γ.

Example 11.17. If X is an integral scheme, then the constant sheaves K and K^* are acyclic for Γ.

Example 11.18 (Another important example). Let X be a smooth manifold. Any sheaf "with partitions of unity" is acyclic for Γ.[11] For instance, C_X^∞ is acyclic for Γ.

Proposition 11.19 (Hartshorne III.1.2A)**.** *In computing $H^i(X, \mathscr{F}) = R^i\Gamma(\mathscr{F})$, instead of resolving $\mathscr{F}$ by* injectives, *we can resolve $\mathscr{F}$ by* acyclic *(for Γ) sheaves.*

Example 11.20 (de Rham cohomology). Let X be a smooth (compact) manifold. We have the *de Rham complex*

$$0 \to \underline{\mathbb{R}} \to C_X^\infty \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \dots,$$

which is exact (as *sheaves*, not globally) by the Poincaré lemma. Note that Ω_X^1 is locally free of rank $= \dim X$ over C_X^∞. Moreover, C_X^∞ and Ω_X^i are *acyclic sheaves* (for Γ).

Thus, we can compute $H^i(X, \underline{\mathbb{R}})$ using the de Rham resolution

$$0 \to C_X^\infty(X) \xrightarrow{d} \Omega_X^1(X) \xrightarrow{d} \Omega_X^2(X) \xrightarrow{d}\to \dots,$$

which is usually called the "de Rham complex" for X. By definition, the *de Rham cohomology* of X is

$$H^i_{\mathrm{DR}}(X) = i\text{-th cohomology of the de Rham complex} = H^i(X, \underline{\mathbb{R}}).$$

Remark 11.21. There is a complex analogue of de Rham cohomology, known as *Dolbeault cohomology*.

Example 11.22 (Another cool application). Let X be an integral scheme, K its function field. We have a short exact sequence of sheaves

$$0 \to \mathcal{O}_X^* \hookrightarrow K^* \to K^*/\mathcal{O}_X^* \to 0,$$

which induces a long exact sequence of cohomology

$$\begin{array}{c} 0 \to \Gamma(X, \mathcal{O}_X^*) \to \Gamma(X, K^*) \xrightarrow{d} \Gamma(X, K^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, K^*) = 0. \\ f \mapsto \mathrm{div}(f) \end{array}$$

The cokernel of d is

$$\mathrm{coker}(d) = \frac{\mathrm{CDiv}(X)}{P(X)} = \mathrm{Pic}(X).$$

Thus, $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

[11] This is known as a *fine* sheaf.

11.7 Cohomology between categories

Let X be a scheme, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. We can think of $\mathscr{F}$ in three different categories:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{quasi-}\\ \text{coherent}\\ \text{sheaves on a}\\ \text{scheme } X\end{array}\right\} \arrow[r, "\text{“forget”}"] \arrow[dr, "\Gamma_{\text{scheme}}"'] & \left\{\begin{array}{c}\mathcal{O}_X\text{-modules}\\ \text{on a ringed}\\ \text{space}\\ (X, \mathcal{O}_X)\end{array}\right\} \arrow[r, "\text{“forget”}"] \arrow[d, "\Gamma_{\text{ring}}"] & \left\{\begin{array}{c}\text{sheaves of abelian}\\ \text{groups on a}\\ \text{topological space } X\end{array}\right\} \arrow[dl, "\Gamma_{\text{top}}"] \\
& \mathbf{Ab} &
\end{tikzcd}$$

However, these categories do not have the same injectives! For example, consider $\operatorname{Spec} R$, where $(R, \mathfrak{m})$ is a local ring: this is *not* an injective $\mathbb{Z}$-module.

Theorem 11.23. *Injective objects in the category of quasi-coherent sheaves are* flasque *(hence* acyclic*) in the category of sheaves of abelian groups.*

11.8 Vanishing in some special cases

Theorem 11.24 (Grothendieck's vanishing theorem)**.** *Let X be a Noetherian topological space, and let $\mathscr{F}$ be a sheaf of abelian groups on X. Then*

$$H^p(X, \mathscr{F}) = 0 \qquad \forall p > \dim X.$$

Theorem 11.25. *If X is an affine scheme and $\mathscr{F}$ is quasi-coherent, then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. Let $X = \operatorname{Spec} A$. Then $\mathscr{F} = \widetilde{M}$ for some A-module M. Consider a resolution of M by injective A-modules

$$0 \to M \to I^0 \to I^1 \to I^2 \to \ldots.$$

By the equivalence of categories, this yields an exact sequence of quasi-coherent sheaves

$$0 \to \widetilde{M} \to \widetilde{I^0} \to \widetilde{I^1} \to \widetilde{I^2} \to \ldots,$$

and since $\operatorname{Hom}_X(-, \widetilde{I}) = \operatorname{Hom}_A(-, I)$, this is an injective resolution. Taking global sections yields again

$$0 \to M \to I^0 \to I^1 \to I^2 \to \ldots,$$

which is exact. Thus, the p-th cohomology is zero for $p > 0$. □

Theorem 11.26 (Serre)**.** *Let X be a Noetherian separated scheme. The following are equivalent:*

(1) X is affine.

(2) $H^p(X, \mathscr{F}) = 0$ for all $p > 0$ and all quasi-coherent sheaves $\mathscr{F}$.

(3) $H^1(X, \mathscr{I}) = 0$ for all quasi-coherent ideal sheaves $\mathscr{I} \subseteq \mathcal{O}_X$.

12 Čech cohomology

12.1 Serre's approach to cohomology

Let X be a topological space, and let $\mathscr{F}$ be a sheaf of abelian groups on X. Fix an open cover $\mathcal{U} = \{U_i\}_{i\in I}$.

Definition 12.1. The *Čech cohomology* $\check{H}^p(\mathcal{U}, \mathscr{F})$ of $\mathscr{F}$ with respect to $\mathcal{U}$ is the p-th cohomology of the Čech complex for $\mathscr{F}$ w.r.t. $\mathcal{U}$:

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d, equal] & C^1(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d, equal] & C^2(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d, equal] & \dots \\
0 \arrow[r] & \prod_{i\in I} \mathscr{F}(U_i) \arrow[r] & \prod_{i<j} \mathscr{F}(U_i\cap U_j) \arrow[r] & \prod_{i<j<k} \mathscr{F}(U_i\cap U_j\cap U_k) \arrow[r] & \dots
\end{tikzcd}$$

The maps are given by (for instance)

$$\begin{aligned}
\prod_{i\in I} \mathscr{F}(U_i) &\to \prod_{i<j} \mathscr{F}(U_i\cap U_j) \\
(s_i)_{i\in I} &\mapsto \left(s_j\big|_{U_i\cap U_j} - s_i\big|_{U_i\cap U_j}\right)_{i<j}.
\end{aligned}$$

We have $\check{H}^0(\mathcal{U}, \mathscr{F}) = \mathscr{F}(X)$ by the map $s \mapsto (s|_{U_i})_{i\in I}$.

Theorem 12.2 (Serre)**.** *If X is a Noetherian separated scheme, and $\mathscr{F}$ is a quasi-coherent sheaf, then if $\mathcal{U}$ is an* affine *cover, then $\check{H}^p(\mathcal{U}, \mathscr{F})$ is the same for all $\mathcal{U}$, and isomorphic to $H^p(X, \mathscr{F})$.*

Idea: For any cover $\mathcal{U}$, there's always a map

$$\check{H}^p(\mathcal{U}, \mathscr{F}) \to H^p(X, \mathscr{F}).$$

It is an isomorphism if X is a Noetherian separated scheme, $\mathcal{U}$ is affine, and $\mathscr{F}$ is quasi-coherent.

12.2 Twisting on the projective line

As an example of how to compute Čech cohomology, consider

$$\begin{aligned}
X &= \mathbb{P}^1_A = \operatorname{Proj} A[x, y], \\
\mathscr{F} &= \mathcal{O}_X(d), \\
\mathcal{U} &= U_0 \cup U_1, \\
U_0 &= \operatorname{Spec} A\left[\frac{y}{x}\right] = D_+(x), \\
U_1 &= \operatorname{Spec} A\left[\frac{x}{y}\right] = D_+(y).
\end{aligned}$$

Compute $\check{H}^\bullet(\mathcal{U}, \mathcal{O}_X(d))$:

$$0 \to \mathcal{O}_X(d)(U_0) \oplus \mathcal{O}_X(d)(U_1) \to \mathcal{O}_X(d)(U_0 \cap U_1) \to 0.$$

This is the d-graded piece of

$$\begin{aligned} 0 \to \big[A[x,y][x^{-1}]\big] \oplus \big[A[x,y,y^{-1}]\big] &\xrightarrow{\partial} A\left[x, y, \frac{1}{xy}\right] \to 0 \\ \left(\frac{f}{x^t}, \frac{g}{y^t}\right) &\mapsto \frac{g}{y^t} - \frac{f}{x^t} = \frac{x^t g - y^t f}{(xy)^t}. \end{aligned}$$

So

$$\begin{aligned} \check{H}^1\big(\mathcal{U}, \mathcal{O}(d)\big) &= \text{cokernel of } \partial \text{ in degree } d \\ &= \left[\frac{h}{(xy)^t} \,\middle|\, \text{where } \forall t,\ h \in \big[A[x,y]\big]_{2t+d}\right] / (\operatorname{im} \partial). \end{aligned}$$

If $d \geq -1$, then for any $h = \sum_{i,j} a_{ij} x^i y^j \in \big[A[x,y]\big]_{2t+d}$, we cannot have a monomial in the sum with $i \leq t-1$ and $j \leq t-1$, so $h \in (x^t, y^t)$, and so we can write

$$h = -g x^t + f y^t$$

for some g, f. Hence,

$$H^1\big(\mathbb{P}^1, \mathcal{O}(d)\big) = 0 \qquad \forall d \geq -1.$$

However, in the case $d = -2$,

$$\left[\frac{(xy)^{t-1}}{(xy)^t}\right] = \left[\frac{1}{xy}\right]$$

is a nonzero cohomology class.

12.3 The Čech complex

Definition 12.3. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups on X, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X.

The *Čech complex* of $\mathscr{F}$ w.r.t. $\mathcal{U}$ is

$$0 \to C^0(\mathcal{U}, \mathscr{F}) \to C^1(\mathcal{U}, \mathscr{F}) \to C^2(\mathcal{U}, \mathscr{F}) \to \dots,$$

where

$$C^p(\mathcal{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \xrightarrow{\partial^p} \prod_{j_0 < \cdots < j_{p+1}} \mathscr{F}(U_{j_0} \cap \cdots \cap U_{j_{p+1}}) = C^{p+1}(\mathcal{U}, \mathscr{F})$$

$$(s_{i_0,\dots,i_p})_{i_0 < \cdots < i_p} \mapsto \sum_{k=0}^{p+1} (-1)^k (s_{j_0,\dots,\hat{j}_k,\dots,j_{p+1}})_{j_0 < \cdots < j_{p+1}}.$$

Exercise 12.4 (Easy exercise). This is really a complex, i.e., $\partial^{p+1} \circ \partial^p = 0$.

Definition 12.5. The *Čech cohomology* of $\mathscr{F}$ w.r.t. $\mathcal{U}$, denoted $\check{H}^p(\mathcal{U}, \mathscr{F})$, is the p-th cohomology of $\check{H}^\bullet(\mathcal{U}, \mathscr{F})$.

Remark 12.6 (Easy). For *any* cover $\mathcal{U}$,

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \mathscr{F}(X).$$

However, the Čech cohomology $\check{H}^p(\mathcal{U}, \mathscr{F})$ $\forall p \geq 1$ definitely depends on the cover $\mathcal{U}$ in general.

Theorem 12.7. *There is a natural map*

$$\check{H}^p(\mathcal{U}, \mathscr{F}) \to H^p(X, \mathscr{F}) \qquad \forall p$$

which is an isomorphism when X is a Noetherian separated scheme, $\mathcal{U}$ is an affine cover, and $\mathscr{F}$ is quasi-coherent.

Proof sketch. Consider a sheafified version of the Čech complex

$$0 \to \prod_{i \in I} \mathscr{F}\big|_{U_i} \to \prod_{i<j} \mathscr{F}\big|_{U_i \cap U_j} \to \dots.$$

This is a *resolution* of $\mathscr{F}$ by sheaves of abelian groups. Embed this into injectives to get a map of complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \prod_{i\in I} \mathscr{F}|_{U_i} \arrow[r] \arrow[d] & \prod_{i<j} \mathscr{F}|_{U_i \cap U_j} \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{I}^0 \arrow[r] & \mathscr{I}^1 \arrow[r] & \dots
\end{tikzcd}$$

When $\mathcal{U}$ is affine and $\mathscr{F}$ is quasi-coherent, then the sheaf Čech cohomology is a resolution of $\mathscr{F}$ by *acyclic* objects, so we can use it to compute cohomology. □

Aside 12.8. We say that a cover $\mathcal{U}'$ is a *refinement* of a cover $\mathcal{U}$ if for all $U' \in \mathcal{U}'$, there exists $U \in \mathcal{U}$ such that $U' \subseteq U$.

In this situation, there is an induced map of corresponding Čech complexes

$$\check{C}^\bullet(\mathcal{U}, \mathscr{F}) \to \check{C}^\bullet(\mathcal{U}', \mathscr{F}),$$

which induces a map

$$\check{H}^p(\mathcal{U}, \mathscr{F}) \to \check{H}^p(\mathcal{U}', \mathscr{F}).$$

This forms a direct limit system, and we get the limit

$$\varinjlim_{\mathcal{U} \text{ open cover}} \check{H}^p(\mathcal{U}, \mathscr{F}) = \check{H}^p(X, \mathscr{F}) \to H^p(X, \mathscr{F}).$$

12.4 Cohomology of projective space

Let A be a Noetherian ring. We will compute

$$H^1\big(\mathbb{P}^1_A \mathcal{O}(D)\big) = \check{H}^1\big(\mathcal{U}, \mathcal{O}(d)\big)$$

using the cover

$$\begin{aligned} \mathcal{U} &= U \cup V, \\ U &= D_+(y) = \operatorname{Spec} A\left[\frac{x}{y}\right], \\ V &= D_+(x) = \operatorname{Spec} A\left[\frac{y}{x}\right], \\ U \cap V &= D_+(xy) = \operatorname{Spec}\left[A\left[x, y, \frac{1}{xy}\right]\right]_0 = \operatorname{Spec} A\left[\frac{x}{y}, \frac{y}{x}\right]. \end{aligned}$$

The Čech complex is

$$0 \to \mathcal{O}(d)\big(D_+(y)\big) \times \mathcal{O}(d)\big(D_+(x)\big) \to \mathcal{O}(d)(U \cap V) \to 0,$$

which is the d-th graded piece of

$$0 \to A[x,y]\left[\frac{1}{y}\right] \times A[x,y]\left[\frac{1}{x}\right] \to A\left[x, y, \frac{1}{xy}\right] \to 0.$$

The middle map is defined by

$$\begin{aligned} \big(0, x^a y^b\big) &\mapsto x^a y^b, \\ \big(x^i y^j, 0\big) &\mapsto -x^i y^j. \end{aligned}$$

The cokernel $H^1(\mathbb{P}^1, \mathcal{O}(d))$ is the free A-module spanned by $\{x^i y^j\}$ for $i+j = d,\ i < 0,\ j < 0$. Thus:

- $H^1(\mathbb{P}^1, \mathcal{O}(d)) = 0$ for all $d \geq -1$.
- $H^1(\mathbb{P}^1, \mathcal{O}(-2)) = \left[\frac{1}{xy}\right] A \cong A$.
- There is a perfect pairing[12]

  $$\big[A[x,y]\big]_{-d-2} \times H^1\big(\mathbb{P}^1, \mathcal{O}(d)\big) \to H^1\big(\mathbb{P}^1, \mathcal{O}(-2)\big) \cong A$$

  $$\big(x^i y^j, \big[x^a, y^b\big]\big) \mapsto \big[x^{a+i} y^{b+j}\big] = \begin{cases} \left[\frac{1}{xy}\right] & \text{iff } i = -a-1 \text{ and } j = -b-1, \\ 0 & \text{otherwise.} \end{cases}$$

  In other words,

  $$H^1\big(\mathbb{P}^1, \mathcal{O}(d)\big) \cong \left[(A[x,y])_{-d-2}\right]^\vee.$$

[12] Recall: If V, W are free A-modules and $\langle : \rangle\ V \times W \to A$ is a bilinear map, then we say $\langle \cdot, \cdot \rangle$ is a *perfect pairing* if the maps

$$\begin{aligned} V &\to W^\vee = \operatorname{Hom}_A(W, A) \qquad & W &\to V^\vee \\ v &\mapsto (w \mapsto \langle v, w \rangle) \qquad & w &\mapsto (v \mapsto \langle v, w \rangle) \end{aligned}$$

are isomorphisms of A-modules. That is, if $V \times W \to A$ is a perfect pairing, then $V^\vee \cong W$ and $W^\vee \cong V$.

Theorem 12.9. *$H^1\big(\mathbb{P}^1, \mathcal{O}(-m)\big)$ is dual to $\big[A[x,y]\big]_{m-2}$. Therefore, $H^1\big(\mathbb{P}^1, \mathcal{O}(-m)\big)$ is a free A-module of rank $m-1$.*

Remark 12.10. This is a case of Serre duality.

Here is the generalization to higher-dimensional projective space:

Theorem 12.11. *For all integers $n \geq 1$, the cohomology of $\mathcal{O}(d)$ on $\mathbb{P}^n_A$ is as follows:*

- *$H^i\big(\mathbb{P}^n, \mathcal{O}(d)\big) = 0$ for $0 < i < n$ or $i > n$ and $\forall d$.*
- *There is the natural map $\big[A[x_0, \ldots, x_n]\big]_d \xrightarrow{\simeq} H^0\big(\mathbb{P}^n, \mathcal{O}(d)\big)$.*
- *$H^n\big(\mathbb{P}^n, \mathcal{O}(-n-1)\big) \cong A$.*
- *There is a perfect pairing*
$$\big[A[x_0, \ldots, x_n]\big]_d \times H^n\big(\mathbb{P}^n, \mathcal{O}(-d-n-1)\big) \to H^n\big(\mathbb{P}^n, \mathcal{O}(-n-1)\big) \cong A.$$

Proof sketch. Look at $\check{C}^\bullet(\mathcal{U}, \mathcal{O}(m))$, where $\mathcal{U} = D_+(x_0) \cup \cdots \cup D_+(x_n)$ is the standard cover. Then we have

$$\ldots \to \prod_{i=0}^{n} A[x_0, \ldots, x_n]\left[\frac{1}{x_0 \cdots \hat{x}_i \cdots x_n}\right] \xrightarrow{\partial} A[x_0, \ldots, x_n]\left[\frac{1}{x_0 \cdots x_n}\right] \to 0.$$

A basis is $\big(x_0^{i_0} \cdot \ldots \cdot x_n^{i_n}\big)_{\sum i_k = m}$. The image of ∂ is the free A-module spanned by $x_0^{i_0} \cdot \ldots \cdot x_n^{i_n}$, where at least one $i_k \geq 0$. Thus, the cokernel is the free A-module spanned by $x_0^{i_0} \cdot \ldots \cdot x_n^{i_n}$ where *all* $i_k < 0$ and $\sum_k i_k = m$.

Note that the critical value is $m = -n-1$, where

$$H^n\big(\mathbb{P}^n, \mathcal{O}(-n-1)\big) = A\left[\frac{1}{x_0 \ldots x_n}\right]. \qquad \square$$

12.5 Serre duality

Over a field k, consider the sheaf $\Omega_{\mathbb{P}^n/k}$ on $\mathbb{P}^n_k$. This is a locally free sheaf of rank n; on $U_i = \operatorname{Spec} k[x_{0/i}, \ldots, x_{n/i}]$, it is the free $\mathcal{O}_{U_i}$-module spanned by $dx_{0/i}, \ldots, d_{x_{n/i}}$.

Define the *canonical sheaf*

$$\omega_{\mathbb{P}^n_k} := \bigwedge^n \Omega_{\mathbb{P}^n/k}.$$

This is locally free of rank 1 (invertible) on $\mathbb{P}^n$.

Exercise 12.12. $\omega_{\mathbb{P}^n_k} \cong \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$.

Theorem 12.13 (Serre duality)**.** *Let X be a smooth projective variety over k of dimension n, let $\mathscr{L}$ be an invertible sheaf, and define*

$$\omega_X := \bigwedge^n \Omega_{X/k}.$$

Then $H^n(X, \omega_X) \cong k$, and for all i, there is a perfect pairing

$$H^i(X, \mathscr{L}) \times H^{n-i}(X, \mathscr{L}^{-1} \otimes \omega_X) \to H^n(X, \omega_X) \cong k.$$

So $H^i(X, \mathscr{L})$ is dual to $H^{n-i}(X, \mathscr{L}^{-1} \otimes \omega_X)$ over k.

Remark 12.14 (Special case)*.* Let X be a smooth projective curve over k. By Serre duality and the definition of genus,

$$\operatorname{genus} X = \dim_k H^1(X, \mathcal{O}_X) = \dim_X H^0(X, \omega_X).$$

Remark 12.15 (Local cohomology)*.* Let $S = k[x_0, \dots, x_n]/I$, let $X = \operatorname{Proj} S$, let $d = \dim X$, let $\mathcal{U}$ be a cover by $d+1$ open affines $\{D_+(f_i)\}_{i=0,\dots,d}$, let M be an S-module, and let $\mathscr{F} = \widetilde{M}$.

12.6 Cohomology of projective schemes

...

Theorem 12.16. *Let X be a projective scheme over a Noetherian ring A. For any coherent sheaf $\mathscr{F}$ on X, $\mathscr{F}(X)$ is a finitely-generated A-module.*

Note 12.17*.* This is wildly false without the projective assumption: if $X = \operatorname{Spec} k[x] = \mathbb{A}^1_k$, then $\mathcal{O}_X(X) = k[x]$ is not a finitely-generated k-module.

More generally:

Theorem 12.18. *Let X be projective over a Noetherian ring A, let $\mathscr{F}$ be coherent, and let $\mathscr{L}$ be a very ample line bundle on X. Then*

(1) For all i, $H^i(X, \mathscr{F})$ is finitely-generated over A.

(2) There exists N_0 such that for all $n \geq N_0$ and all $i > 0$,

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0.$$

Note 12.19 (Some current research)*.* In (2), the N_0 that "works" depends on $\mathscr{F}$ and X. There are two different research directions:

(1) Fix X, and try to find N_0 that works for *all* $\mathscr{F}$ in some sense "positive" (for all ample invertible sheaves $\mathscr{F}$):

$$H^n(X, \mathscr{L}^n) = 0 \qquad \forall n \geq N_0.$$

This uses "characteristic p techniques".

(2) Fix a distinguished $\mathscr{F}$ (usually $\mathscr{F} = \omega_X$, and assume X is smooth). Try to find N_0 that works for all $\mathscr{L}$ very ample:

$$H^i(X, \omega_X \otimes \mathscr{L}^n) = 0.$$

Theorem 12.20 (Smith)*. If X is smooth and $\mathscr{L}$ is very ample, then $H^i(X, \mathscr{L}^n \otimes \omega_X) = 0$ for all $n > \dim X$.*

Proof of Theorem 12.18, part (1). First, we reduce to the case $X = \mathbb{P}^n_A$. Consider $X \overset{i}{\hookrightarrow} \mathbb{P}^n_A$, $\mathscr{L} = i^*\mathcal{O}_{\mathbb{P}^n_A}(1)$. We claim that

$$H^i(X, \mathscr{F}) = H^i(\mathbb{P}^n_A, i_*\mathscr{F}).$$

Indeed, let $\mathcal{U} = \{U_i\}$ be the standard affine cover of $\mathbb{P}^n$. Then $\mathcal{U} \cap X = \{U_i \cap X\}$ is an affine cover of X, and

$$H^i(X, \mathscr{F}) = \text{cohomology of } \check{C}(\mathcal{U} \cap X, \mathscr{F}) : 0 \to \prod_{i=0}^{n} \mathscr{F}(U_i \cap X) \to \dots$$

$$H^i(\mathbb{P}^n_A, i_*\mathscr{F}) = \text{cohomology of } \check{C}(\mathcal{U}, i_*\mathscr{F}) : 0 \to \prod_{i=0}^{n} i_*\mathscr{F}(U_i) \to \dots,$$

and these are *exactly* the same complex.

So, without loss of generality, $X = \mathbb{P}^n_A$, and $\mathscr{F} = \widetilde{M}$ for some finitely-generated graded $S = A[x_0, \dots, x_n]$-module M.

Say M is generated over S by $m_1, \dots, m_t$, where $\deg m_i = d_i$. We map onto M by the degree-preserving map of graded S-modules

$$0 \to N \to S(-d_1) \oplus \dots \oplus S(-d_t) \twoheadrightarrow M \to 0$$
$$e_i = (0, \dots, 1, \dots, 0) \mapsto m_i.$$

This induces by the $\widetilde{\cdot}$ functor

$$0 \to \mathscr{K} = \widetilde{N} \to \mathcal{O}_{\mathbb{P}^n}(-d_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(-d_t) \twoheadrightarrow \mathscr{F} \to 0.$$

We get a long exact sequence of cohomology

$$H^i(\mathbb{P}^n, \mathscr{K}) \to H^i\Big(\mathbb{P}^n, \bigoplus_{i=1}^{t} \mathcal{O}(-d_i)\Big) \to H^i(\mathbb{P}^N_A, \mathscr{F}) \to H^{i+1}(\mathbb{P}^n_A, \mathscr{K}) \to \dots$$

The cohomology module

$$H^i\Big(\mathbb{P}^n, \bigoplus_{i=1}^{t} \mathcal{O}(-d_i)\Big) = \bigoplus_{i=1}^{t} H^i\big(\mathbb{P}^n, \mathcal{O}(-d_i)\big)$$

is finitely-generated over A by explicit computation. For $i = n$, this becomes

$$\bigoplus_{i=1}^{t} H^n\big(\mathbb{P}^n_A, \mathcal{O}(-d_i)\big) \twoheadrightarrow H^n(\mathbb{P}^n, \mathscr{F}) \to 0.$$

The homomorphic image of a finitely-generated A-module is also finitely-generated, hence $H^n(\mathbb{P}^n, \mathscr{F})$ is finitely-generated over A.

Now use reverse induction on i: Assume that for all $\mathscr{F}$ coherent on $\mathbb{P}^n_A$, the cohomology module $H^{i+1}(\mathbb{P}^n_A, \mathscr{F})$ is finitely-generated over A. Then we have

$$\bigoplus_{i=1}^{t} H^i\big(\mathbb{P}^n, \mathcal{O}(-d_i)\big) \xrightarrow{d} H^i(\mathbb{P}^n, \mathscr{F}) \xrightarrow{d'} H^{i+1}(\mathbb{P}^n, \mathscr{K}).$$

The modules on the left and the right are finitely-generated. Breaking this up into a short exact sequence, we obtain

$$0 \to \operatorname{im} d \to H^i(\mathbb{P}^n, \mathscr{F}) \to \operatorname{im} d' \to 0.$$

Since A is Noetherian, $\operatorname{im} d$ and $\operatorname{im} d'$ are finitely-generated, hence $H^i(\mathbb{P}^n, \mathscr{F})$ is as well. $\square$

Proof of Theorem 12.18, part (2). Again, take $X \overset{i}{\hookrightarrow} \mathbb{P}^n_A$ and $\mathscr{L} = i^*\mathcal{O}_{\mathbb{P}^n_A}(1)$. Then

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = H^i\big(\mathbb{P}^n, i_*(\mathscr{F} \otimes \mathscr{L}^n)\big).$$

By the projection formula,

$$i_*(\mathscr{F} \otimes \mathscr{L}^n) = i_*\big(\mathscr{F} \otimes (i^*\mathcal{O}(1))^n\big) = i_*\big(\mathscr{F} \otimes i^*\mathcal{O}(n)\big) = i_*\mathscr{F} \otimes \mathcal{O}(n).$$

So

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = H^i\big(\mathbb{P}^n, i_*(\mathscr{F} \otimes \mathscr{L}^n)\big) = H^i\big(\mathbb{P}^n, (i_*\mathscr{F}) \otimes \mathcal{O}_{\mathbb{P}^n}(n)\big).$$

Since $\mathcal{O}_{\mathbb{P}^n}(n)$ is locally free and hence flat, we have a short exact sequence

$$0 \to \mathscr{K}(n) \to \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbb{P}^n}(-d_i + n) \to \mathscr{F}(n) \to 0.$$

We proceed similarly to the proof of part (1); the details are left as an exercise. $\square$

13 Curves

13.1 Main setting

Definition 13.1. By a *curve*, we mean a projective, integral, smooth scheme X of dimension 1 over a field k. (If $k = \mathbb{C}$, these are (compact) Riemann surfaces.)

Questions:

- Classify curves up to isomorphism.
- Study maps between them.
- Study covers of $\mathbb{P}^1$ by curves: $X \twoheadrightarrow \mathbb{P}^1$.

To answer these, we need to understand invertible sheaves $\mathscr{L}$ on X and $H^0(X, \mathscr{L})$.

13.2 The Riemann–Roch theorem

Fix a basis $s_0, \ldots, s_n$ for $H^0(X, \mathscr{L})$. This defines a map

$$\begin{aligned} X &\to \mathbb{P}\big(H^0(X,\mathscr{L})\big) = \mathbb{P}^n \\ x &\mapsto [s_0(x) : \cdots : s_n(x)]\,. \end{aligned}$$

Recall: $\mathscr{L} \cong \mathcal{O}_X(D)$ for some divisor $D = \sum_{i=1}^r n_i P_i$ (where P_i are points) on X.

Remark 13.2. In general, it can be hard to compute $h^0(X,\mathscr{L}) := \dim H^0(X,\mathscr{L})$. But it is easier to compute

$$\chi(X,\mathscr{L}) = h^0(X,\mathscr{L}) - h^1(X,\mathscr{L}) + h^2(X,\mathscr{L}) - h^3(X,\mathscr{L}) + \cdots = \sum_{i=0}^{\dim X} (-1)^i \dim H^i(X,\mathscr{L}).$$

Formulas for $\chi(\mathscr{L})$ can be given in terms of invariants of X and $\mathscr{L}$ (called *Riemann–Roch formulas*).

Remark 13.3. For an invertible sheaf $\mathscr{L}$ on a *curve*, the *degree of* $\mathscr{L}$ is defined as $\sum_i n_i$, where $D = \sum_i n_i P_i$ such that $\mathscr{L} \cong \mathcal{O}(D)$.

Theorem 13.4 (Riemann–Roch for curves)**.** *Let X be a curve of genus g, and let $\mathscr{L}$ be a "line bundle" (invertible sheaf) on X. Then*

$$\chi(X,\mathscr{L}) = \deg(\mathscr{L}) + 1 - g.$$

Note 13.5. Using Serre duality,

$$\chi(X,\mathscr{L}) \overset{\text{def}}{=} h^0(X,\mathscr{L}) - h^1(X,\mathscr{L}) = h^0(X,\mathscr{L}) - h^0(X,\omega_X \otimes \mathscr{L}^{-1}).$$

So, in dimension 1, we can rewrite the Riemann–Roch theorem as

$$h^0(X,\mathscr{L}) = \deg \mathscr{L} + 1 - g + h^0(X,\omega_X \otimes \mathscr{L}^{-1}).$$

Proof of Riemann–Roch. We can view the term $1-g$ in terms of the trivial line bundle:

$$\chi(\mathcal{O}_X) = h^0(X,\mathcal{O}_X) - h^1(X,\mathcal{O}_X) = 1 - g.$$

So the theorem just states that $\chi(\mathscr{L}) = \chi(\mathcal{O}_X) + \deg \mathscr{L}$.

We will use induction on $\deg \mathscr{L}$. In the case of the inductive step where $\mathscr{L}$ has degree $d > 0$, write $\mathscr{L} = \mathcal{O}(D)$, where $D = \sum_{i=1}^t n_i P_i$. Take one point P in the support. Then

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to k(P) \to 0.$$

Tensor with $\mathscr{L}$ to get

$$0 \to \mathcal{O}_X(D-P) \to \mathcal{O}_X(D) \to \mathscr{L} \otimes k(P) = k(P) = k \to 0,$$

which induces a cohomology exact sequence

$$\begin{aligned} 0 &\to H^0\big(X,\mathcal{O}_X(D-P)\big) \to H^0\big(X,\mathcal{O}_X(D)\big) \to k \\ &\to H^1\big(X,\mathcal{O}_X(D-P)\big) \to H^1\big(X,\mathcal{O}_X(D)\big) \to 0. \end{aligned}$$

Hence, the alternating sum of the dimensions is zero:

$$\begin{aligned}\chi(\mathcal{O}_X(D)) &= h^0(X,\mathcal{O}(D)) - h^1(X,\mathcal{O}(D))\\ &= h^0(X,\mathcal{O}(D-P_1)) - h^1(X,\mathcal{O}(D-P_1)) + 1\\ &= \chi(\mathcal{O}_X(D-P_1)) + 1\\ &= \chi(\mathcal{O}_X) + \deg(D-P_1) + 1 = \chi(\mathcal{O}_X) + \deg D.\end{aligned}$$

Aside 13.6 (General fact). If $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ is a short exact sequence of coherent sheaves on a projective variety Z, then

$$\chi(Z,\mathscr{B}) = \chi(Z,\mathscr{A}) + \chi(Z,\mathscr{C}).$$

Returning to the proof, for any D and any P,

$$\begin{aligned}\chi(\mathcal{O}_X(D)) &= \chi(\mathcal{O}_X(D-P)) + 1,\\ \chi(\mathcal{O}_X(D)) &= \chi(\mathcal{O}_X(D+P)) - 1.\end{aligned}$$

To prove Riemann–Roch, now write

$$D = \sum_{i=1}^{t} n_i P_i - \sum_{i=1}^{s} m_i Q_i, \qquad\qquad n_i, m_i > 0,\ P_i, Q_i \in X.$$

Hence

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \sum_{i=1}^{t} n_i - \sum_{i=1}^{s} m_i = \chi(\mathcal{O}_X) + \deg D.$$

Recall that

$$\chi(\mathcal{O}_X) = \dim H^0(X,\mathcal{O}_X) - \dim H^1(X,\mathcal{O}_X) = 1 - g.$$

So

$$\chi(D) = \chi(\mathcal{O}_X) + \deg D = 1 - g + \deg D. \qquad \square$$

13.3 Remark on arbitrary fields

Let us make sense of the Riemann–Roch theorem over fields that are not necessarily algebraically closed.

The only place the assumption $k = \overline{k}$ is used is to say $k = k(P)$. If $k \neq \overline{k}$, we still have a finite extension

$$k \hookrightarrow k(P) = \frac{k(X \cap U)}{\mathfrak{m}_P}.$$

Hence

$$\deg k(P) = \dim_k k(P),$$

and the same argument goes through, except that

$$\chi(\mathcal{O}_X(D)) = \mathcal{O}_X(D-P) + \dim_k k(P).$$

So the statement of Riemann–Roch is the same over arbitrary fields, once we use the following revised definition:

Definition 13.7. For a divisor $D = \sum_{i=1}^t n_i P_i$, let

$$\deg(D) \overset{\text{def}}{=} \sum_{i=1}^{t} n_i \deg_k k(P_i).$$

An alternative approach is to "base change" to $\overline{k}$:

$$\begin{tikzcd} \overline{X} = X \times_k \overline{k} \arrow[r] \arrow[d] & \operatorname{Spec} \overline{k} \arrow[d] \\ X \arrow[r] & \operatorname{Spec} k. \end{tikzcd}$$

A divisor $D = \sum_i n_i P_i$ with $P_i \subseteq X$ induces an inclusion

$$P_i \times_k \overline{k} \subseteq X \times_k \overline{k}.$$

Hence, from the exact sequence

$$0 \to \mathcal{O}_X(-P) \hookrightarrow \mathcal{O}_X \to \frac{\mathcal{O}_X}{\mathcal{O}_X(-P)} \to 0,$$

we can tensor with $\overline{k}$ to obtain an exact sequence

$$0 \to \mathcal{O}_{\overline{X}}(-P) \hookrightarrow \mathcal{O}_{\overline{X}} \to \frac{\mathcal{O}_{\overline{X}}}{\mathcal{O}_{\overline{X}}(-P)} \to 0.$$

Example 13.8. If $X = \mathbb{P}^1_{\mathbb{R}} = \operatorname{Proj} \mathbb{R}[x, y]$, then

$$\overline{X} = X \times_{\mathbb{R}} \mathbb{C} = \operatorname{Proj} \mathbb{C}[x, y] = \mathbb{P}^1_{\mathbb{C}}.$$

Consider the divisor

$$D = P = \left(t^2 + 1\right) \subseteq \mathbb{P}^1_{\mathbb{R}}.$$

We have $k(P) = \mathbb{C}$, so $\deg_{\mathbb{R}} k(P) = 2$. Viewed in $\mathbb{P}^1_{\mathbb{C}}$,

$$D \times_{\mathbb{R}} \mathbb{C} = P_1 + P_2,$$

where $P_1 = [i : 1]$ and $P_2 = [-i : 1]$.

Returning to the general case,

$$\chi(\mathcal{O}_X(D)) = 1 - g + \deg(D \times_k \overline{k}).$$

Writing

$$D = \sum_i n_i P_i,$$

$$D \times_k \overline{k} = \sum_i n_i \sum_j m_{ij} Q_{ij},$$

we have

$$\deg D = \deg D \times_k \overline{k} = \sum_{n-I} \deg_k k(P_i).$$

Letting v be the projection $X \times_k \overline{k} \to X$, for any coherent sheaf $\mathscr{F}$ on X, the cohomology is

$$H^p(X, \mathscr{F}) \otimes_k \overline{k} = H^p(\overline{X}, v^* \mathscr{F}) = H^p(X \times_k \overline{k}, \mathscr{F} \otimes_k \overline{k}).$$

13.4 Divisors of degree zero

Assume $k = \overline{k}$. What can we say about divisors of degree 0 on a curve X?

Proposition 13.9. *If* $f \in k(X)$, *then* $\deg(\operatorname{div} f) = 0$.

Proof. Consider the rational map

$$\begin{aligned} X &\dashrightarrow k = \mathbb{A}^1_k \subseteq \mathbb{P}^1_k \\ x &\mapsto f(x), \end{aligned}$$

which extends to a map

$$\begin{aligned} X &\xrightarrow{\varphi} \mathbb{P}^1_k \\ x &\mapsto [f(x) : 1]. \end{aligned}$$

Recall: If $X \twoheadrightarrow Y$ is a finite map of projective varieties, then fibers of all points have the same cardinality (counting multiplicities).

Hence the divisor of zeros and poles of f is given by

$$\operatorname{div} f = \text{"zeros of } f\text{"} - \text{"poles of } f\text{"} = \varphi^{-1}([0:1]) - \varphi^{-1}([1:0]) = \sum_i n_i P_i - \sum_i m_i Q_i,$$

and so $\deg(\operatorname{div} f) = 0$. $\square$

13.5 Degree zero divisors

Let $\operatorname{Div}^0(X)$ be the subgroup of degree zero divisors on X. Then we have a short exact sequence

$$0 \to \operatorname{Div}^0(X) \to \operatorname{Div}(X) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

As we just showed, the group $P(X)$ of principal divisors is contained in $\operatorname{Div}^0(X)$, so this induces

$$0 \to \frac{\operatorname{Div}^0(X)}{P(X)} \to \frac{\operatorname{Div}(X)}{P(X)} \xrightarrow{\deg} \mathbb{Z} \to 0,$$

denoted

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

The subgroup $\operatorname{Pic}^0(X)$ turns out to have the structure of a *variety* over k. This is a smooth projective (abelian) variety, called the *Jacobian variety* of X. Its dimension is $g(X)$.

In the case of an elliptic curve (i.e., $g(x) = 1$),

$$\operatorname{Pic}^0(X) \cong X.$$

This is the usual group structure on an elliptic curve.

Remark 13.10 (Higher dimension)*.* Let X be a smooth projective variety over a field $k = \overline{k}$. In higher dimension, "degree" makes no sense. However, we still have a subgroup

$$\operatorname{Pic}^0(X) \subseteq \operatorname{Pic}(X) = \frac{\operatorname{Div}(X)}{P(X)} \twoheadrightarrow \operatorname{NS}(X) \cong \mathbb{Z}^r \to 0,$$

the group of *numerically trivial divisors*. It turns out that $\operatorname{Pic}^0(X)$ is again an abelian variety, called the *Picard variety* of X.

The cokernel $\operatorname{NS}(X)$ is a finitely-generated, torsion-free abelian group, the *Néron–Severi group* of X.

Returning to divisors of degree zero on a curve:

Lemma 13.11. *(a) If* $h^0(X, D) \neq 0$*, then* $\deg D \geq 0$*.*

(b) If $h^0(X, D) \neq 0$ *and* $\deg D = 0$*, then* $D \sim \operatorname{div}(f) \sim 0$ *is principal.*

(c) If $\mathscr{L}$ *is a* non-trivial *invertible sheaf of degree* 0*, then* $H^0(X, \mathscr{L}) = 0$*.*

Proof. (a) Observe that

$$h^0(X, D) = \dim_k H^0(X, \mathcal{O}(D)) = \dim_k \left\{ f \in k(X)^* \;\middle|\; \operatorname{div} f + D \geq 0 \right\} \cup \{0\}.$$

If $f \in H^0(X, D)$ is nonzero, then $\operatorname{div} f + D \geq 0$, so

$$\deg D = \deg(\operatorname{div} f + D) \geq 0.$$

Alternatively, write $\mathscr{L} \cong \mathcal{O}_X(D)$. Then for all $s \in H^0(X, \mathscr{L})$, we can look at $(s)_0$, the divisor of zeros, which is automatically effective.

(b) If $\deg D = 0$ and $f \in k(X)^*$, then $\operatorname{div} f + D \geq 0$ is degree zero, so $\operatorname{div} f + D = 0$. Hence

$$D = -\operatorname{div}(f) = \operatorname{div}(1/f),$$

so D is principal. □

13.6 Divisors of positive degree

Divisors of negative degree have no global sections! So, to understand maps from a curve to $\mathbb{P}^n$ ("to do geometry for curves"), we should focus on divisors of *positive* degree. [In higher dimension, we also want to understand "positive" divisors. A major question is what "positive" should mean in the higher-dimensional context.]

Example 13.12. Consider a smooth, degree-d plane curve

$$X = \operatorname{Proj} S = \operatorname{Proj} \frac{k[x, y, z]}{(F_d)} = \mathbb{V}(F_d) \subseteq \mathbb{P}^2_k = \operatorname{Proj} k[x, y, z].$$

Write $\mathscr{L} = i^*\mathcal{O}(1)$, and let $s = ax + by + cz \in H^0(X, \mathscr{L})$, where $a, b, c \in k$. If $H = \mathbb{V}(ax + by + cz)$, then

$$(s)_0 = \text{"divisor of zeros of } ax + by + cz\text{"} = H \cap X = \mathbb{V}(s, F_d) = \sum_{i=1}^{d} P_i,$$

where the P_i are points (not necessarily distinct) on X.

We have

$$\mathscr{L}^n = \mathcal{O}_X\big(n \cdot (H \cap X)\big),$$

so

$$\deg(\mathscr{L}^n) = n \cdot d.$$

By Riemann–Roch for $\mathscr{L}^n$ on X,

$$\chi(\mathscr{L}^n) = 1 - g + \deg \mathscr{L}^n,$$

so

$$\dim H^0(X, \mathscr{L}^n) = 1 - g + nd + \dim H^1(X, \mathscr{L}^n).$$

By Serre vanishing, $\dim H^1(X, \mathscr{L}^n) = 0$ for sufficiently large n. We have

$$\Gamma_*(X, \mathscr{L}) = \bigoplus_{n \in \mathbb{N}} H^0(X, \mathscr{L}^n) \hookrightarrow \bigoplus_{n \in \mathbb{Z}} S_n = S,$$

with equality in large degree, and we have

$$\dim(S_n) = d \cdot n + 1 - g.$$

Note that $\dim(S_n)$ is the Hilbert function of n evaluated at n, and $d \cdot n + 1 - g$ is a polynomial of degree 1 in n. Thus this is the Hilbert polynomial for S.

13.7 Base-point-free and very ample linear systems

Question: Given a curve X and a divisor D, how can we tell if

$$|D| = \big\{ D' \bigm| D' \geq 0,\ D' \sim D \big\}$$

is base-point-free or very ample?

The following are equivalent:

- $|D|$ is base-point-free.
- For all $P \in X$, there exists $D' \in |D|$ such that $P \notin D'$.
- For all $P \in X$, $\mathcal{O}_X(D)$ has a global section $s \in \Gamma(X, \mathcal{O}_X(D))$ such that $s(P) \neq 0$.
- $\mathcal{O}_X(D)$ is globally generated.

For very ample, look at a basis $s_0, \ldots, s_n \in H^0(X, \mathcal{O}_X(D))$. The map

$$\begin{aligned} X &\hookrightarrow \mathbb{P}^n \\ x &\mapsto [s_0(x) : \cdots : s_n(x)] \end{aligned}$$

is a closed embedding. (In this case, members of $|D|$ are hyperplane sections of $X \subseteq \mathbb{P}^n$.)

Example 13.13. Consider $X = \mathbb{P}^1 = \operatorname{Proj} k[x,y]$ and

$$\mathscr{D} = \operatorname{span}\left\{x^4, x^3y, x^2y^2, xy^3\right\} \subseteq H^0\big(X, \mathcal{O}(4)\big).$$

The associated map is

$$\begin{aligned} X &\overset{\varphi}{\dashrightarrow} \mathbb{P}^3 \\ [x:y] &\mapsto \left[x^4 : x^3y : x^2y^2 : xy^3\right], \end{aligned}$$

defined everywhere expect at $[0:1]$, which is a base point of $\mathscr{D}$. This map extends to the Veronese embedding

$$\begin{aligned} X &\xrightarrow{\nu_3} \mathbb{P}^3 \\ [x:y] &\mapsto \left[x^3 : x^2y : xy^2 : y^3\right], \end{aligned}$$

which corresponds to $|\mathcal{O}(3)|$:

$$\mathscr{L} = \nu_3^*\mathcal{O}(1) = \mathcal{O}(3),$$

which is globally generated by the pullbacks x^3, x^2y, xy^2, y^3 of x_0, x_1, x_2, x_3.

Remark 13.14. When $|D|$ is base-point-free,

$$\begin{aligned} X &\to \mathbb{P}^n = \mathbb{P}\big(H^0(X, \mathcal{O}_X(D))\big) \\ x &\mapsto [s_0(x) : \cdots : s_n(x)], \end{aligned}$$

and the members of $|D|$ are the pullbacks of hyperplane sections.

Proposition 13.15. *Let $|D|$ be a linear system of divisors.*

- *$|D|$ is base-point-free $\iff$ for all $P \in X$,*

$$\dim |D - P| = |D| - 1.$$

- *$|D|$ is very ample $\iff$ for all $P, Q \in X$ (including $P = Q$),*

$$\dim |D - P - Q| = \dim |D| - 2.$$

Remark 13.16. We have a bijection

$$\begin{aligned} \mathbb{P}\big(H^0(X, \mathcal{O}_X(D))\big) &\to |D| \\ f &\mapsto \text{divisor of zeros of } (\operatorname{div} f + D), \end{aligned}$$

so we can think of $|D|$ as a projective space.

Proof of Proposition 13.15. Take any $P \in X$. We have an exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X &\xrightarrow{\text{eval at } P} k(P) = k \to 0 \\ f &\mapsto f(P). \end{aligned}$$

Tensoring with $\mathcal{O}_X(D)$, we obtain an exact sequence

$$0 \to \mathcal{O}_X(D-P) \to \mathcal{O}_X(D) \to k(P) \to 0,$$

which yields a long exact sequence of cohomology

$$0 \to H^0(D-P) \to H^0(D) \to k \to \dots.$$

So

$$\dim H^0\big(X, \mathcal{O}_X(D)\big) = \begin{cases} \dim H^0\big(X, \mathcal{O}_X(D-P)\big) + 1 & \text{iff “eval at } P\text{” is surjective,} \\ \dim H^0\big(X, \mathcal{O}_X(D-P)\big) & \text{iff “eval at } P\text{” is } 0. \end{cases}$$

Note that $f \mapsto f(P)$ is zero $\iff$ $\mathcal{O}_X(D)$ is *not* globally generated at P $\iff$ P is a zero of every section of $\mathcal{O}_X(D)$ $\iff$ P is a base point of $|D|$. This proves the first part of Proposition 13.15.

For the “very ample” part, first observe that if $|D|$ is very ample, then $|D|$ is base-point-free, so for all $P \in X$,

$$\dim |D-P| = \dim |D| - 1.$$

Hence, for all $Q \in X$,

$$\dim |D-P-Q| = \begin{cases} \dim |D-P| - 1 = \dim |D| - 2 & \text{iff } Q \text{ is } \textit{not} \text{ a base point of } |D-P|, \\ \dim |D-P| & \text{iff } Q \text{ is a base point of } |D-P|. \end{cases}$$

Observe that

$$H^0\big(X, \mathcal{O}(D-Q-P)\big) \subseteq H^0\big(X, \mathcal{O}(D-P)\big) \subsetneqq H^0\big(X, \mathcal{O}_X(D)\big).$$

If there exists $Q \neq P$ such that $\dim |D-P-Q| = \dim |D-P|$, i.e., Q is a base point of $|D-P|$, i.e., for all $s = \sum_{i=0}^n a_i s_i \in H^0(X, D-P)$, we have $s(Q) = 0$.

Find a hyperplane $H = \sum_i a_i x_i \subseteq \mathbb{P}^n$ which passes through P and not Q. Then

$$\varphi^* H = \text{divisor on } X \text{ of zeros of } \varphi^*\left(\sum a_i x_i\right).$$

But we have

$$\varphi^*\left(\sum a_i x_i\right) = \sum a_i \varphi^* x_i = \sum a_i s_i = s \in H^0\big(X, \mathcal{O}_X(D)\big),$$

so $s(P) = 0 \implies s \in H^0(D-P)$. But $s(Q) \neq 0$.

To summarize: we have $P, Q \in X \subseteq \mathbb{P}^n$. Find H such that $P \in H$ and $Q \notin H$. ...

Now for the case $P = Q$. We have

$$|D-2P| \subseteq |D-P| \subsetneqq |D|,$$

viewed as hyperplane sections, and we want to show that the first inclusion is proper. If $|D-2P| = |D-P|$, then every $\ell = \sum_i a_i x_i$ vanishing at $P \in X$ vanishes to order 2.

Choose an affine chart so that P is the origin in $X \cap \mathbb{A}^n \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$. Then we have

$$k[t_1, \ldots, t_n] \twoheadrightarrow \frac{k[t_1, \ldots, t_n]}{I}$$
$$\frac{(t_1, \ldots, t_n)}{(t_1, \ldots, t_n)^2} = \frac{\mathfrak{m}_P}{\mathfrak{m}_P^2} \twoheadrightarrow \frac{\mathfrak{m}_P}{\mathfrak{m}_P^2}.$$

Find a section of $\mathcal{O}(1)$ on $\mathbb{P}^n$ whose local defining equation in a neighborhood of P is a *generator* of $\mathfrak{m}_P$ (meaning that the equation is *not* in $\mathfrak{m}_P^2$). In other words, $s \in H^0(X, D-P)$, but $s \notin H^0(X, D-2P)$.

(See Hartshorne for the other direction of the proof.) $\square$

Corollary 13.17.
- *If* $\deg D \geq 2g$*, then* $|D|$ *is base-point-free.*
- *If* $\deg D \geq 2g+1$*, then* $|D|$ *is very ample.*

Remark 13.18 (Some classical language). If there exists $D' \in |D|$ such that $P \in \operatorname{Supp} D'$ but $Q \notin \operatorname{Supp} D'$, then we say $|D|$ "separates points P and Q". We say that $\mathscr{L} = \mathcal{O}_X(D)$ "separates" P and Q provided that there exists $s \in H^0(X, \mathscr{L})$ such that $s(P) = 0$ but $s(Q) \neq 0$. In either case, the map

$$X \xrightarrow{\varphi} \mathbb{P}^n$$

is such that $\varphi(P) \neq \varphi(Q)$. This is the case if and only if

$$\dim |D - P - Q| = \dim |D| - 2$$

for all $P \neq Q$.

We also say that $|D|$ "separates tangent vectors at P" provided that

$$|D - 2P| \subsetneq |D - P|,$$

or equivalently, $X \xrightarrow{\varphi} \mathbb{P}^n$ induces an injective map of vector spaces

$$T_P X \xrightarrow{d_P \varphi} T_P \mathbb{P}^n,$$

i.e., φ is an embedding at P. If $|D|$ separates *all* points, then $\varphi_{|D|}$ is injective.

Recall from last time: if D has degree $\geq 2g-1$, then

$$h^1(D) = h^0(K_X - D) = 0.$$

Proof of Corollary 13.17. Suppose $\deg D \geq 2g$. To show $|D|$ is base-point-free, we need to show that for all $P \in X$,

$$\dim |D - P| = \dim |D| - 1.$$

Compute using Riemann–Roch:

$$h^0(D-P) = 1 - g + \deg(D-P) + h^1(D-P).$$

Since $\deg(D-P) = \deg D - 1 \geq 2g-1$, we have $h^1(D-P) = 0$, so

$$h^0(D-P) = 1 - g + \deg(D-P) = 1 - g + \deg D - 1 = \deg D - g.$$

The proof of the very ample part is similar. $\square$

13.8 Classification of curves

Let us classify curves by genus:

- $g(X) = 0 \iff X \cong \mathbb{P}^1$.
- $g(X) = 1 \iff X = \mathbb{V}(F_3) \hookrightarrow \mathbb{P}^2$ via the linear system $\varphi_{|3P_0|}$. We have

$$\begin{tikzcd} X \arrow[r, hook, "\varphi_{|3P_0|}"] \arrow[dr, "\varphi_{|2P_0|}", "2:1"'] & \mathbb{P}^2 \arrow[d, dashed] \\ & \mathbb{P}^1 \end{tikzcd} \qquad \begin{tikzcd} {[x:y:1]} \arrow[d, mapsto] \\ {[x:1]} \end{tikzcd}$$

The equation F_3 is given by

$$F_3(x,y,z) = F_3\left(\frac{x}{z}, \frac{y}{z}, 1\right) = f(x,y) = y^2 - x(x-1)(x-\lambda).$$

By the *Hurwitz formula*, there are 4 ramification points $0, 1, \infty, \lambda$.

For $g(X) \geq 2$, consider the canonical divisor K_X. We have

$$\deg K_X = 2g - 2.$$

Claim 13.19. *$|K_X|$ has no base points when $g \geq 2$.*

(We will not prove this claim here; it does not follow from Corollary 13.17.)

Note that

$$\dim(K_X) = \dim H^0(X_1, \omega_X) - 1 = \dim H^0\big(X, \mathcal{O}_X(K_X)\big) - 1 = g - 1.$$

We get a map

$$X \xrightarrow{\varphi_{|K_X|}} \mathbb{P}^{g-1}.$$

In the case $g = 2$, we get a finite cover of $\mathbb{P}^1$; the degree of the cover is

$$\deg K_X = 2g - 2 = 2$$

because members of $|K_X|$ are $\varphi^*(P)$. Thus, *every* genus 2 curve is a 2-to-1 cover of $\mathbb{P}^1$, ramified at 6 points $0, 1, \infty, a, b, c$ by the Hurwitz formula. So we can parametrize genus 2 curves by a family

$$\mathcal{M}_2 \subseteq \frac{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}{S_6}.$$

So, to summarize:

- $g(X) = 2$: $X \overset{2:1}{\twoheadrightarrow} \mathbb{P}^1$ ramified at 6 points (3 degrees of freedom).
- $g(X) \geq 3$: $|K_X|$ is either very ample (yielding $\varphi_{|K_X|} : X \hookrightarrow \mathbb{P}^{g-1}$) or it gives a map $X \twoheadrightarrow \mathbb{P}^1$.

As a special case, consider X of genus 3, not hyperelliptic. Then $|K_X|$ is very ample, so we have an embedding

$$X = \mathbb{V}(F_4) \hookrightarrow \mathbb{P}^2.$$

Members of $|K_X|$ are hyperplane sections, and

$$\deg K_X = 2g - 2 = 6 - 2 = 4.$$

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