Math 847 Notes
Modular Forms

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Chapter I

Modular forms

I.1 2014-01-22: Introduction to modular forms

I.1.1 Modular forms

Vaguely: modular forms are holomorphic functions $f : \mathfrak{h} \to \mathbb{C}$ (where $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$) that

- satisfy symmetry properties with respect to certain finite index subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, where $\mathrm{GL}_2^+(\mathbb{R}) = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det g > 0\}$ acts on $\mathfrak{h}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

- behave well near $i\infty$.

One symmetry: for all Γ, if $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ for some $h \in \mathbb{Z}$, then $f(z+h) = f(z)$. So we get a Fourier expansion:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n(f) q_h^n,$$

where $q_h = e^{2\pi i z/h}$. As $q_h \to 0$, $z \to i\infty$.

Definition I.1.1 (holomorphic at ∞)**.** We say f is "*holomorphic at* ∞" if

$$f = \sum_{n \geq 0} a_n(f) q_h^n.$$

If also $a_0(f) = 0$, we say f "vanishes at ∞" and say f is *cuspidal*.

Definition I.1.2 (automorphy factor)**.** Recall the action $\mathrm{GL}_2^+(\mathbb{R}) \circlearrowright \mathfrak{h}$. The *automorphy factor* for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $z \in \mathfrak{h}$ is

$$j(\alpha, z) = cz + d.$$

Definition I.1.3 (weight k action)**.** For all $k \in \mathbb{Z}$, define the *weight k action* $\mathrm{GL}_2^+(\mathbb{R}) \circlearrowright \{f : \mathfrak{h} \to \mathbb{C}\}$ by

$$(f|_k\alpha)(z) = \det(\alpha)^{k/2} j(\alpha, z)^{-k} f(\alpha \cdot z).$$

Definition I.1.4 (congruence subgroups)**.** Let $N \in \mathbb{Z}_{\geq 1}$. Define the *principal congruence subgroup* $\Gamma(N) \leq \mathrm{SL}_2(\mathbb{Z})$ by

$$\Gamma(N) = \big\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \bigm| \gamma \equiv I \pmod N\big\} = \ker\big(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\big).$$

This is a finite-index subgroup.

A subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* (of level N) if $\Gamma(N) \leq \Gamma$. For example:

$$\begin{aligned}
\Gamma_0(N) &= \left\{\gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N\right\}, \\
\Gamma_1(N) &= \left\{\gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N\right\}.
\end{aligned}$$

Definition I.1.5 (modular form)**.** Fix a congruence subgroup Γ. A map $f : \mathfrak{h} \to \mathbb{C}$ is a *modular form* of weight k and level Γ if

- f is holomorphic;
- $f|_k\gamma = f$ for all $\gamma \in \Gamma$;
- $(f|_k\alpha)(z)$ is holomorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Denote the space of modular forms of weight k and level Γ by $M_k(\Gamma)$.

Fact I.1.6. $\dim M_k(\Gamma) < \infty$.

Definition I.1.7 (cusp form)**.** If in addition $(f|_k\alpha)$ vanishes at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, then f is a *cusp form*. Denote the space of cusp forms by $S_k(\Gamma)$.

Remark I.1.8. The only weight 0 forms are the constant functions.

We can think of f as a function on $\Gamma\backslash\mathfrak{h}$. The spaces $\Gamma\backslash\mathfrak{h}$ are known as *modular curves*.

I.1.2 Eisenstein series

Let

$$\Gamma_\infty = \left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \bigm| n \in \mathbb{Z}\right\} \leq \mathrm{SL}_2(\mathbb{Z}) = \Gamma.$$

Then $j(\gamma, z) = 1$ for all $\gamma \in \Gamma_\infty$, so $1|_k\gamma = j(\gamma, z)^{-k} \cdot 1 = 1$. Consider the sum

$$\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} 1|_k\gamma.$$

If $k \geq 4$ is even, then this converges absolutely, and we can verify explicitly that this is a modular form. This is an example of an *Eisenstein series*. This is, up to a constant factor, equal to

$$E_k = \frac{3(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Fact I.1.9. $M(\Gamma) = \bigoplus_k M_k(\Gamma)$ is freely generated (as an algebra) by E_4 and E_6.

Example I.1.10. $\Delta(z) = q \prod_{n \geq 1} (1-q^n)^{24}$.

I.2 2014-01-24: Applications of modular forms

Guest lecture by Jordan Ellenberg.

I.2.1 Sums of squares

(Serre, *Course in Arithmetic*, part III.)

Let $r_k(n)$ be the number of representations of n as a sum of k squares:

$$\begin{aligned} r_k(n) &= \#\left\{(a_1, \ldots, a_k) \in \mathbb{Z}^k \;\middle|\; a_1^2 + \cdots + a_k^2 = n\right\} \\ &= \#\left\{a \in \mathbb{Z}^k \;\middle|\; \|a\| = \sqrt{n}\right\}. \end{aligned}$$

What can we say about $r_k(n)$ (e.g., asymptotics as $n \to \infty$ with k fixed)? For example,

$$\sum_{n=1}^{N} r_2(n) = \#\left\{\text{lattice points at distance} \leq \sqrt{N} \text{ from } 0\right\} \sim \text{area of circle} = \pi N.$$

More generally,

$$\sum_{n=1}^{N} r_k(n) = \#\left\{\text{lattice points inside a } k\text{-ball of radius } \sqrt{N}\right\} = \text{constant}_k \cdot N^{k/2}.$$

So, on average, $r_k(n) \sim cN^{\frac{k}{2}-1}$.

Theorem I.2.1 (Lagrange's theorem)**.** $r_4(n) > 0$ *for all* $n \geq 0$.

I.2.2 Theta functions

We can put $r_k(n)$ in a generating function

$$\sum_{n=0}^{\infty} r_k(n) q^n.$$

Define

$$\theta := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \dots.$$

Note that

$$\theta^k = \sum_n r_k(n) q^n.$$

Taking $q = e^{2\pi i\tau}$, we can think of θ as a function on $\mathfrak{h}$, the upper half-plane, satisfying

$$\theta(\tau) = \theta(\tau + 1).$$

This is a "modular form" of weight 1/2. In the definition of modular form, this requires us to say something like

$$f|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = (c\tau + d)^{-1/2} f(\tau).$$

To remove quotes, raise to the 4-th power: θ^4 is a modular form of weight 2 under a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level 4.

Recall:

$$E_4 \sim \sum d_3(n) q^n,$$
$$E_6 \sim \sum d_5(n) q^n,$$

where $d_i(n) = \sum_{k|n} k^i$. More generally, θ^k is a modular form of weight $k/2$ for all even k. We can decompose

$$\theta^k = E_{k/2} + f,$$

where $E_{k/2}$ is an Eisenstein series and f is a cusp form. For $k \geq 5$, the n-th coefficient of $E_{k/2}$ is very close to $n^{\frac{k}{2}-1}$, and the n-th coefficient of f is $o(n^{\frac{k}{2}-1})$.

Remark I.2.2 (For those who know Hecke operators). θ^4 is actually a Hecke eigenform, which is why there's a *formula* for $r_4(n)$.

I.2.3 More general thetas

Let Q be an arbitrary positive-definite quadratic form in k variables over $\mathbb{Z}$. Then

$$\theta_Q = \sum_{n\in\mathbb{Z}^k} q^{Q(n)}$$

is again a modular form for some subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (depending on Q).

More generally still: Let Q be a positive definite symmetric matrix. Then we define

$$\sum_{A\in M_{m\times k}(\mathbb{Z})} q^{A^T Q A}.$$

(Note that $A^T Q A$ is an $m \times m$ positive semidefinite symmetric matrix.) This can be thought of as a formal sum in a power series ring, or as a function on the Siegel upper half plane. This is *still* a *Siegel modular form*, which we also call an *automorphic form* on Sp_{2m}.

This gives you a way of addressing questions like: Is every positive definite 2×2 matrix the sum of k "squares" $A^T A$, where A is a 1×2 matrix?

I.3 2014-01-27: Modular curves

Earlier, we defined congruence subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ and modular forms $M_k(\Gamma)$, which are "functions on $\Gamma\backslash\mathfrak{h}$." We want to think of the *modular curve* $\Gamma\backslash\mathfrak{h}$ as a geometric object, a Riemann surface. (cf. Diamond–Shurman, chapters 2 & 3)

Our goal is to turn $\Gamma\backslash\mathfrak{h}$ into a Riemann surface $Y(\Gamma)$, then compactify to get a compact Riemann surface $X(\Gamma)$.

I.3.1 Example: Fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$

Definition I.3.1. A *fundamental domain* for Γ acting on $\mathfrak{h}$ is a subset $\mathcal{F} \subseteq \mathfrak{h}$ such that:

(0) $\mathcal{F}$ is a connected closed domain in $\mathfrak{h}$.

(1) $\mathfrak{h} = \bigcup_{\gamma\in\Gamma} \gamma \cdot \mathcal{F}$.

(2) $\gamma \cdot \mathcal{F}^\circ \cap \mathcal{F}^\circ = \varnothing$ for all $\gamma \in \Gamma$ with $\gamma \neq \pm I$, where $\mathcal{F}^\circ$ is the interior of $\mathcal{F}$.

Theorem I.3.2. *(i) $\mathcal{F} = \left\{z \in \mathfrak{h} : |\mathrm{Re}(z)| \leq \frac{1}{2},\ |z| \geq 1\right\}$ is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$.*

(ii) If $z_1 \in \mathcal{F}$ such that $z_2 = \gamma \cdot z_1 \in \mathcal{F}$ for some $\gamma \neq \pm I$, then one of the following is true:

(a) $\mathrm{Re}(z_1) = \pm\frac{1}{2}$ and $\gamma \cdot z_1 = z_1 \mp 1$, or

(b) $|z_1| = 1$ and $\gamma \cdot z_1 = -\frac{1}{z_1}$.

Proof. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so $S \cdot z = -\frac{1}{z}$ and $T^{\pm 1} \cdot z = z \pm 1$.

(i) We define an algorithm that takes $z \in \mathfrak{h}$ as input, and outputs $\gamma \in \langle S, T\rangle \subseteq \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot z \in \mathcal{F}$.

Step 1: Apply T or T^{-1} until $|\mathrm{Re}(z)| \leq \frac{1}{2}$.

Step 2: If $|z| \geq 1$, then we are done. Otherwise, apply S. Note that, if $\mathrm{Im}(z) > 1$, then

$$\mathrm{Im}\left(\frac{-1}{z}\right) = \mathrm{Im}\left(\frac{z}{|z|^2}\right) > \mathrm{Im}(z).$$

Step 3: If $|\mathrm{Re}(z)| \leq \frac{1}{2}$, then we are done; otherwise, go to step 1.

This process must eventually terminate. Indeed, note that if $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, then

$$\mathrm{Im}(\alpha \cdot z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Given z, there are only finitely many pairs $(c, d) \in \mathbb{Z}^2$ such that $|cz + d| < 1$. Moreover,

$$T^{\pm 1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

But step 2 strictly increases $\mathrm{Im}(z)$, so it can only occur finitely many times.

(ii) The proof is left as an exercise. □

I.3.2 $\mathrm{SL}_2(\mathbb{R})$-structure of the upper half-plane

Theorem I.3.3. *As $\mathrm{SL}_2(\mathbb{R})$-sets, $\mathfrak{h} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.*

Proof. This follows from the following two facts:

(i) $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathfrak{h}$.

(ii) There exists $z \in \mathfrak{h}$ such that $\mathrm{Stab}(z) = \mathrm{SO}_2(\mathbb{R})$.

These can both be verified by direct computation:

(i) $\frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot i = x + iy$.

(ii) $\frac{ai+b}{ci+d} = i \iff ai + b = -c + di$, so $a = d$ and $b = -c$, whence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$. □

Remark I.3.4.

(i) $\mathrm{SL}_2(\mathbb{R})$ is a locally compact topological group, and $\mathrm{SL}_2(\mathbb{Z})$ is a discrete subgroup.

(ii) The action $\mathrm{SL}_2(\mathbb{R}) \times \mathfrak{h} \to \mathfrak{h}$ is continuous.

(iii) The map $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \xrightarrow{\simeq} \mathfrak{h}$ given by $\alpha\,\mathrm{SO}_2(\mathbb{R}) \mapsto \alpha \cdot i$ is a homeomorphism.

Next time, we will prove that the action is "properly discontinuous".

I.4 2014-01-29: Construction of modular curves

I.4.1 The action is properly discontinuous

Proposition I.4.1. *For all $z_1, z_2 \in \mathfrak{h}$, there are open neighborhoods $U_i \ni z_i$ such that, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, if $\gamma(U_1) \cap U_2 \neq \varnothing$, then $\gamma \cdot z_1 = z_2$. (That is, the action is "properly discontinuous".)*

We prove this proposition in several steps.

Claim I.4.2. *Given any neighborhoods $U_i' \ni z_i$ such that $\overline{U_i'}$ is compact, $\gamma(U_1') \cap U_2' \neq \varnothing$ for only finitely many $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.*

Proof. Let $\mathrm{SL}_2(\mathbb{R}) \xrightarrow{\pi} \mathfrak{h}$ be the map $\alpha \mapsto \alpha \cdot i$, and consider the closed subsets $S_i = \pi^{-1}(\overline{U_i'}) \subseteq \mathrm{SL}_2(\mathbb{R})$. Let $\bigcup_j V_{i,j} \supseteq S_i$ be open covers with $\overline{V_{i,j}}$ compact. Then $\overline{U_i'} \subseteq \bigcup_j \pi(V_{i,j})$. Since $\overline{U_i'}$ is compact, we can take $\overline{U_i'} \subseteq \bigcup_{j=1}^{n_i} \pi(V_{i,j})$. Hence

$$S_i \subseteq \bigcup_{j=1}^{n_i} \pi^{-1}(\pi(V_{i,j})) \subseteq \bigcup_{j=1}^{n} \overline{V_{i,j}}\,\mathrm{SO}_2(\mathbb{R}),$$

so S_i is compact. Observe that

$$\left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma(U_1') \cap U_2' \neq \varnothing\right\} \subseteq \mathrm{SL}_2(\mathbb{Z}) \cap S_2 S_1^{-1}.$$

Since $\mathrm{SL}_2(\mathbb{Z})$ is discrete and $S_2 S_1^{-1}$ is compact, the above set is finite. □

Let $F = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma(U_1') \cap U_2' \neq \varnothing\}$. By Claim I.4.2, F is finite. The following completes the proof of the proposition.

Claim I.4.3. *Let $U_{i,\gamma}$ be disjoint neighborhoods of $\gamma \cdot z_1$ and z_2, respectively. Then for $i = 1, 2$,*

$$U_i = U_i' \cap \bigcap_{\gamma \in F} \gamma^{-1}(U_{i,\gamma})$$

are the neighborhoods stated to exist in Proposition I.4.1.

This is easily verified, completing the proof of the proposition. □

Corollary I.4.4. *If Γ is any congruence subgroup, then $Y(\Gamma) = \Gamma\backslash\mathfrak{h}$ is Hausdorff.*

I.4.2 Make $Y(\Gamma)$ a complex manifold

To give $Y(\Gamma)$ the structure of a complex manifold, we find for all $y \in Y(\Gamma)$ a neighborhood $\tilde{U}_y \subseteq Y(\Gamma)$ and a homeomorphism $\varphi_y : \tilde{U}_y \xrightarrow{\simeq} V_y \overset{\text{open}}{\subseteq} \mathbb{C}$ with holomorphic transition maps.

Proposition I.4.1 says that, if $z \in \mathfrak{h}$ is only fixed by $\Gamma \cap \{\pm I\}$, then $\pi_\Gamma : \mathfrak{h} \to Y(\Gamma)$ is a homeomorphism near z.

Definition I.4.5. For $z \in \mathfrak{h}$, let $\Gamma_z := \mathrm{Stab}_z(\Gamma) = \{\gamma \in \Gamma \mid \gamma \cdot z = z\}$. We call z an *elliptic point* for Γ if $\pm I \cdot \Gamma_z \supsetneq \{\pm I\}$. We also call $\pi_\Gamma(z) \in Y(\Gamma)$ and $\gamma \in \Gamma_z, \gamma \neq \pm I$ *elliptic.*

Example I.4.6. If $z = i$, then

$$\mathrm{SL}_2(\mathbb{Z})_i = \left\{\pm I,\ \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\} = \langle S \rangle .$$

If $z = \rho = e^{i\pi/3} = 1 + e^{2\pi i/3}$, then $\mathrm{SL}_2(\mathbb{Z})_\rho = \langle TS \rangle$, where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Here is a convenient transformation: given $\tau \in \mathfrak{h}$, let

$$F_\tau(z) = \begin{pmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{pmatrix} \cdot z = \frac{z - \tau}{z - \overline{\tau}}.$$

So $F_\tau(\tau) = 0$ and $F_\tau(\overline{\tau}) = \infty$.

[See §2.2 of Diamond–Shurman for details.]

More generally:

Proposition I.4.7. *For all Γ and elliptic τ, Γ_τ is finite cyclic. Hence,*

$$h_\tau = \begin{cases} \#\Gamma_\tau & \text{if } -I \notin \Gamma, \\ \frac{1}{2}\#\Gamma_\tau & \text{if } -I \in \Gamma. \end{cases}$$

I.5 2014-01-31: Elliptic points and compactification

I.5.1 Elliptic points

A few remarks:

- The only elliptic points for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$ are i and $\rho = e^{i\pi/3}$.
- For any congruence subgroup Γ, there are only finitely many elliptic points in $\Gamma\backslash\mathfrak{h}$.
- $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ is called *elliptic* iff $\mathrm{tr}(\alpha)^2 < 4\det(\alpha)$.
- $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is elliptic iff γ is conjugate in $\mathrm{SL}_2(\mathbb{Z})$ to one of the following:

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{\pm 1}.$$

  The following have *no* elliptic elements:

  (i) $\Gamma(N)$ for $N \geq 2$.

  (ii) $\Gamma_1(N)$ for $N \geq 4$.

  (iii) $\Gamma_0(N)$ if there is prime $p \mid N$ such that $p \equiv 11 \pmod{12}$.

I.5.2 Compactification

Earlier, we showed that $Y(\Gamma)$ is a Riemann surface. We would like to work with a *compact* Riemann surface instead; to obtain one, we need to compactify!

Let us move from $\mathfrak{h}$ to the Poincaré disc $\mathscr{D} := \{|z| < 1\}$. This is another model of the hyperbolic plane. We can move from the upper half-plane to the Poincaré disc via the *Cayley transformation*

$$\begin{aligned} C &: \mathfrak{h} \to \mathscr{D}, \\ C(z) &= \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \cdot z = \frac{z-i}{z+i}. \end{aligned}$$

Note that $C(0) = -1$, $C(\infty) = 1$, $C(i) = 0$, $C(1) = -i$, and $C(\rho) = \frac{-i}{2+\sqrt{3}}$. Moreover, the boundary $\partial\mathscr{D} = \{|z| = 1\}$ corresponds to the projective real line $\mathbb{R} \cup \{\infty\} = \mathbb{P}^1(\mathbb{R})$.

The fundamental domain $\mathcal{F}$ corresponds to a hyperbolic triangle in $\mathscr{D}$, with a missing point on $\partial\mathscr{D}$. "Filling in" this point gives a compactification.

More generally, if Γ is any congruence subgroup, then there is a fundamental domain which is a convex polygon in hyperbolic geometry, with finitely many cusps on the boundary to be filled in. The idea is to add points to $\mathfrak{h}$ to get $\mathfrak{h}^*$, then look at $\Gamma\backslash\mathfrak{h}^* =: X(\Gamma)$.

For $X(1) = X(\Gamma(1)) = X(\mathrm{SL}_2(\mathbb{Z}))$, we just need to add a single point ∞ to $Y(\Gamma)$, so we can take $\mathfrak{h}^* = \mathfrak{h} \cup \mathrm{SL}_2(\mathbb{Z}) \cdot \infty$, where we think of $\mathfrak{h}$ as a subset of

$$\mathbb{P}^1(\mathbb{C}) = \big\{[\alpha : \beta] \;\big|\; (\alpha, \beta) \neq (0,0)\big\} / ([\alpha : \beta] \sim [\lambda\alpha : \lambda\beta] \; \forall \lambda \in \mathbb{C}^\times).$$

Note that $[\alpha : \beta] \in \mathbb{P}^1(\mathbb{C})$ corresponds to $\frac{\alpha}{\beta}$ if $\beta \neq 0$, and to $\infty = [1:0]$ if $\beta = 0$. We extend the $\mathrm{SL}_2(\mathbb{Z})$-action by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [\alpha : \beta] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{c\frac{\alpha}{\beta} + b}{c\frac{\alpha}{\beta} + d}.$$

Observe that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \frac{a}{c} \in \mathbb{Q} \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q}).$$

So, let $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$.

Remark I.5.1. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be any discrete subgroup. An element $\gamma \in \Gamma$ is called *parabolic* if $\mathrm{tr}(\gamma)^2 = 4$, so γ has a unique fixed point $x_\gamma \in \mathbb{P}^1(\mathbb{C})$, and $x_\gamma \in \mathbb{P}^1(\mathbb{R})$. Here, take

$$\mathfrak{h}^* = \mathfrak{h} \cup \left\{ x_\gamma \;\middle|\; \gamma \in \Gamma \text{ is parabolic} \right\}.$$

For $\mathrm{SL}_2(\mathbb{Z})$, $\infty = x_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}$.

I.5.3 The topology on $\mathfrak{h}^*$

We want $\Gamma \backslash \mathfrak{h}^*$ to be Hausdorff.

Example I.5.2. $\Gamma_0(11)\backslash\mathfrak{h}$ has two cusps, at ∞ and 0. These are indeed different cusps:

$$\begin{pmatrix} a & b \\ 11c & d \end{pmatrix} \cdot \infty = \begin{pmatrix} a & b \\ 11c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{a}{11c},$$

which is 0 iff $a = 0$, which is impossible because then the determinant of $\begin{pmatrix} a & b \\ 11c & d \end{pmatrix}$ would be $11bc \neq 1$. However, $\frac{a}{11c}$ can get arbitrarily close to 0, so we need to be careful if we want the topology to be Hausdorff!

In general, here is how we define the topology on $\mathfrak{h}^*$: The neighborhoods of ∞ are of the form

$$U_M := \left\{ z \in \mathfrak{h} \;\middle|\; \mathrm{Im}(z) > M \right\} \cup \{\infty\}$$

for all $M > 0$. The neighborhoods of $\frac{a}{c} \in \mathbb{Q}$ are $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot U_M$ for all $M > 0$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. (These neighborhoods look like circles in the upper half-plane, tangent to the real line $\mathbb{R}$ at the point $\frac{a}{c}$.)

I.5.4 The topology of $X(\Gamma)$

Give $X(\Gamma) = \Gamma\backslash\mathfrak{h}^*$ the quotient topology, and let $\pi_\Gamma : \mathfrak{h}^* \to X(\Gamma)$ be the natural projection.

Theorem I.5.3. *$X(\Gamma)$ is Hausdorff, connected, and compact.*

Proof. To prove compactness, consider the fundamental domain

$$\mathcal{F}^* = \left\{ z \in \mathfrak{h} : |z| \geq 1,\ |\mathrm{Re}(z)| \leq \tfrac{1}{2} \right\} \cup \{\infty\} \subseteq \mathfrak{h}^*.$$

This is a closed subset of $\mathbb{P}^1(\mathbb{C})$, hence is compact. (See [Diamond–Shurman] for details.) $\square$

I.5.5 Complex structure at the cusps

Let $s \in \mathbb{P}^1(\mathbb{Q})$. Choose $\gamma_s \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_s \cdot s = \infty$. Consider

$$h_{s,\Gamma} = \#\left(\mathrm{SL}_2(\mathbb{Z})_s / \{\pm I\}\, \Gamma_s\right),$$

the "width of s in Γ". Take U_M (for, say, $M = 2$) to $V = \psi(U_M) \subseteq \mathbb{C}$ via $\psi(z) = e^{2\pi i z/h}$, then descend to $X(\Gamma)$ via π_Γ and proceed similarly to with elliptic points.

Chapter II

Dimensions of spaces of modular forms

II.1 2014-02-03: Dimension formulas

Guest lecture by Nigel Boston.

Now we have the Riemann surfaces $Y(\Gamma)$ and $X(\Gamma)$. Our goals for today:

(1) Understand modular forms as functions on $X(\Gamma)$.

(2) Use this to find the dimensions of spaces of modular forms $M_k(\Gamma)$ and $S_k(\Gamma)$.

II.1.1 Motivational example

If $f \in M_k(\Gamma)$, then $f : \mathfrak{h} \to \mathbb{C}$ satisfies $f(\gamma \cdot z) = j(\gamma, z)^k f(z)$ for all $\gamma \in \Gamma$. Thus, f is well-defined as a function $Y(\Gamma) \to \mathbb{C}$ if and only if $k = 0$.

What do we do if $k \neq 0$? Say $k = 2$, and consider the differential dz on $\mathfrak{h}$. Then for all $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$,

$$d(\alpha \cdot z) = d\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2}\,dz = \frac{ad-bc}{(cz+d)^2}\,dz = \frac{\det\alpha}{(cz+d)^2}\,dz.$$

So if $f \in M_2(\Gamma)$, let $\omega_f(z) = f(z)\,dz$ (a differential on $\mathfrak{h}$). Then

$$\omega_f(\gamma \cdot z) = f(\gamma \cdot z)\,d(\gamma \cdot z) = \frac{(cz+d)^2 f(z)\,dz}{(cz+d)^2} = f(z)\,dz = \omega_f(z),$$

i.e., $\omega_f(z)$ is Γ-invariant. Thus, $\omega_f(z)$ is well-defined on $Y(\Gamma) = \Gamma\backslash\mathfrak{h}$.

The differential dz has a "simple pole" at ∞, but if $f(z) \in S_2(\Gamma)$, then $f(z)$ vanishes at ∞, so $f(z)\,dz$ is a holomorphic differential on $X(\Gamma)$. In fact, we shall see that

$$\Omega_{\mathrm{hol}}(X(\Gamma)) \cong S_2(\Gamma)$$

as $\mathbb{C}$-vector spaces. (Here, Ω_{hol} denotes the space of holomorphic differentials.) As a consequence,

$$\dim_{\mathbb{C}} S_2(\Gamma) = \dim_{\mathbb{C}} \Omega_{\mathrm{hol}}(X(\Gamma)) = \text{genus of } X(\Gamma).$$

Example II.1.1. Recall the fundamental domain for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acting on $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$. Take the three points i, ρ, ∞, together with a fourth point z_0 in the interior; joining these together gives a triangulation of $X(\Gamma)$. The Euler characteristic is thus

$$\chi(X(\Gamma)) = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces}) = 4 - 6 + 4 = 2.$$

But $\chi(X(\Gamma)) = 2 - 2 \cdot \mathrm{genus}(X(\Gamma))$, so $\mathrm{genus}(X(\Gamma)) = 0$.[1]

Thus, $\dim_{\mathbb{C}} S_2(\mathrm{SL}_2(\mathbb{Z})) = 0$, and so there are no nontrivial weight 2 cusp forms for $\mathrm{SL}_2(\mathbb{Z})$.

Example II.1.2. Recall the fundamental domain for $\Gamma = \Gamma_0(11)$ on $\mathfrak{h}^*$. Using Sage, we can compute that the fundamental domain is isomorphic to a torus. So

$$\dim_{\mathbb{C}} S_2(\Gamma_0(11)) = \mathrm{genus}(X_0(11)) = 1.$$

II.1.2 Maps between compact Riemann surfaces and the Riemann–Hurwitz formula

More generally, for all congruence subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, there is a natural quotient map

$$f_\Gamma : X(\Gamma) \to X(1) = X(\mathrm{SL}_2(\mathbb{Z})).$$

The Riemann–Hurwitz formula applied to f_Γ will relate $\mathrm{genus}(X(\Gamma))$ to $\mathrm{genus}(X(1)) = 0$, yielding a formula for $\dim_{\mathbb{C}} S_2(\Gamma)$.

For $k \neq 0, 2$, we will need tensor products of differentials, and then we can use Riemann–Roch to compute the dimensions of $M_k(\Gamma)$ and $S_k(\Gamma)$.

Let $f : X \to Y$ be a holomorphic map of (connected) compact Riemann surfaces.

Fact II.1.3. f is constant or surjective.

Proof sketch. $f(X)$ is the continuous image of a compact, connected set, so $f(X)$ is compact and connected, and hence is closed in Y. If f is nonconstant, then it is open (by the open mapping theorem). So if f is nonconstant, then $f(X)$ is a connected, clopen subset of Y. Since Y is connected, $f(X) = Y$. □

Fact II.1.4. If f is nonconstant, then f has a well-defined *degree* $\deg(f) \geq 1$ such that, for all but finitely many $y \in Y$, $\#f^{-1}(y) = \deg(f)$.

For each $x \in X$, the *ramification degree* of f at x, denoted $e_x \geq 1$, is the multiplicity with which f takes x to 0 in local coordinates; i.e., $f(z) = f(x) + \sum_{i=e_x}^{\infty} c_i(z-x)^i$, where $c_{e_x} \neq 0$. Then for all $y \in Y$,

$$\deg f = \sum_{x \in f^{-1}(y)} e_x.$$

Theorem II.1.5 (Riemann–Hurwitz)**.** *If $f : X \to Y$ is nonconstant, then*

$$2g(X) - 2 = (\deg f)(2g(Y) - 2) + \sum_{x \in X} (e_x - 1).$$

[1] One can also determine that $X(\mathrm{SL}_2(\mathbb{Z}))$ has genus zero in several other ways. For example, the j-function gives an explicit isomorphism with the Riemann sphere.

II.2 2014-02-05: Riemann–Hurwitz formula

Guest lecture by Jordan Ellenberg.

II.2.1 Remarks on Riemann–Hurwitz

Let X, Y be compact Riemann surfaces. Recall the Riemann–Hurwitz formula:

$$2g(X) - 2 = \deg(f) \cdot (2g(Y) - 2) + \sum_{x \in X} (e_x - 1),$$

where e_x is the ramification degree of f at x. (Since $e_x = 1$ for all but finitely many $x \in X$, the above sum is finite.)

Definition II.2.1. $\chi(X) = 2g(X) - 2$ is the *Euler characteristic* of X.

When there's no ramification, $f : X \to Y$ is a covering map, and

$$\chi(X) = \deg(f) \cdot \chi(Y).$$

This is also a property of the Euler characteristic in higher dimension.

Topologically, being unramified means every $y \in Y$ has a neighborhood U_y such that $f^{-1}(U_y)$ is the disjoint union of the right number of discs $\coprod U_{x_i}$.

Example II.2.2. Consider a cover of $\mathbb{P}^1(\mathbb{C})$ given by adjoining a square root $\sqrt{f(x)}$ of a polynomial of degree $2n$. Continuously winding around the zeros of f "transports" between the two branches. This yields a Riemann surface X of genus $n-1$, which has an involution $\gamma : X \to X$ such that $X/\gamma \cong \mathbb{P}^1(\mathbb{C})$. So we have a degree 2 map

$$X \xrightarrow{\pi} X/\gamma = \mathbb{P}^1(\mathbb{C}),$$

ramified exactly at the roots of f. By Riemann–Hurwitz,

$$\begin{aligned} 2g(X) - 2 &= \deg(\pi) \cdot (2g(\mathbb{P}^1(\mathbb{C})) - 2) + \sum_{\text{6 points}} (e_x - 1) \\ &= 2 \cdot (2 \cdot 0 - 2) + \sum_{\text{6 points}} (2 - 1) = -4 + 6, \end{aligned}$$

so $g(X) = 2$.

II.2.2 Prototypical ramified cover

Consider the map

$$\mathfrak{h}^*/\Gamma(p) \to \mathfrak{h}^*/\Gamma(1) \cong \mathbb{P}^1(\mathbb{C}).$$

Notice that $\Gamma(p)$ is a normal subgroup of $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$, and $\Gamma(1)/\Gamma(p) = \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})/\{\pm 1\}$. So

$$G = \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})/\{\pm 1\} \circlearrowright \mathfrak{h}^*/\Gamma(p) = X(p),$$

and $X(p)/G \cong X(1)$. Thus, the degree of the map is $|G| = \frac{1}{2}(p^3 - p)$.

Write $Y = X/G$. For each $y \in Y$, $\pi^{-1}(y)$ is an *orbit* of G on X. A point $x \in X$ is ramified iff $|Gx| < |G|$.[2]

To compute $g(X(p))$, we need to find the ramification points in $X(p) \to X(1)$. These are points $[z] \in \mathfrak{h}^*/\Gamma(p)$ where $g_1[z] = g_2[z]$ for some distinct $g_1, g_2 \in G = \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})/\{\pm 1\}$, i.e., $g_1 z = g_2 z$ for $g_1, g_2 \in \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ which lie in *different cosets* of $\Gamma(p)$.

The ramification comes from stabilizers in the upper half-plane: points which are fixed by nontrivial elements of $\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$. In other words, ramification only occurs above the cusps and the elliptic points.

II.3 2014-02-07: Ramification at elliptic points

II.3.1 The local picture

We have the map $f_\Gamma : X(\Gamma) \to X(1)$, and we want to compute the ramification index e_x of f_Γ for x above i, ρ, ∞.

Say $f_\Gamma(x) = y \in X(1)$. We can find $\tau \in \mathfrak{h}^*$ such that $\pi_\Gamma(\tau) = x$ and $\pi_{\mathrm{SL}_2(\mathbb{Z})}(\tau) = y$, and a neighborhood $U \ni \tau$ and charts φ_x, φ_y near x and y giving a commutative diagram

$$\begin{tikzcd}
& U \arrow[r, "\mathrm{id}"] \arrow[dl, "F_\tau"'] \arrow[d, "\pi_\Gamma"'] & U \arrow[d, "\pi_{\mathrm{SL}_2(\mathbb{Z})}"] \arrow[dr, "F_\tau"] & \\
F_\tau(U) \arrow[dr, "P_{h_x}"'] & \pi_\Gamma(U) \arrow[r, "f_\Gamma"] \arrow[d, "\varphi_x"', "\simeq"] & \pi_{\mathrm{SL}_2(\mathbb{Z})}(U) \arrow[d, "\varphi_y", "\simeq"'] & F_\tau(U) \arrow[dl, "P_{h_y}"] \\
& V_x \arrow[r, dashed, "f_{\mathrm{loc}}"] & V_y &
\end{tikzcd}$$

with f_{loc} defined by the above diagram, and where

$$P_h = \begin{cases} z \mapsto e^{2\pi i z/h} & \text{near a cusp,} \\ z \mapsto z^h & \text{near an elliptic point} \end{cases}$$

and

$$h_x = \begin{cases} \text{width of the cusp } \tau & \text{if } \tau \text{ is a cusp of } \Gamma, \\ \text{period of the elliptic } \tau & \text{if } \tau \text{ is elliptic for } \Gamma, \\ 1 & \text{otherwise,} \end{cases}$$

and similarly with $\mathrm{SL}_2(\mathbb{Z})$ instead of Γ.

II.3.2 Ramification indices

Let $\psi_x = P_{h_x} \circ F_\tau$ and $\psi_y = P_{h_y} \circ F_\tau$. Two cases:

[2]For those who like stacks, this has an interpretation in terms of stacks.

(i) $\tau \in \mathfrak{h}$, i.e., τ is not a cusp. In this case, $P_h(z) = z^h$, so $f_{\mathrm{loc}} \circ \psi_x = \psi_y$. Then $f_{\mathrm{loc}}(z) = z^{h_y/h_x}$. We know $h_x, h_y \in \{1,2,3\}$, and f_{loc} is holomorphic, so $h_y/h_x \in \mathbb{Z}$. Hence, either $h_x = 1$ or $h_x = h_y$. Thus, by definition,

$$e_x = \frac{h_y}{h_x} = \left.\begin{cases} h_y & \text{if } \tau \text{ is elliptic for } \mathrm{SL}_2(\mathbb{Z}) \text{ but not } \Gamma, \\ 1 & \text{otherwise} \end{cases}\right\} = \left[\mathrm{SL}_2(\mathbb{Z})_\tau : \{\pm I\}\,\Gamma_\tau\right].$$

(ii) $\tau \in \mathfrak{h}^* \setminus \mathfrak{h} = \mathbb{P}^1(\mathbb{Q})$, i.e., τ is a cusp. Then $P_h(z) = e^{2\pi i z/h}$, so $f_{\mathrm{loc}}(z) = z^{h_x/h_y}$. Thus,

$$e_x = \frac{h_x}{h_y} = \left[\mathrm{SL}_2(\mathbb{Z})_\tau : \{\pm I\}\,\Gamma_\tau\right].$$

But $y = \infty$, so $h_y = 1$, hence $e_x = h_x$.

So, we have elliptic points

$$y_2 := \mathrm{SL}_2(\mathbb{Z}) \cdot i \in X(1),$$
$$y_3 := \mathrm{SL}_2(\mathbb{Z}) \cdot \rho \in X(1)$$

and the cusp

$$y_\infty := \mathrm{SL}_2(\mathbb{Z}) \cdot \infty \in X(1),$$

with $h_{y_2} = 2$, $h_{y_3} = 3$, and $h_{y_\infty} = 1$.

Every elliptic point of Γ of period 2 is in $f_\Gamma^{-1}(y_2)$, of period 3 in $f_\Gamma^{-1}(y_3)$, and every cusp of Γ is in $f_\Gamma^{-1}(y_\infty)$. Let

$$\nu_2 := \#\text{ of elliptic points of } X(\Gamma) \text{ of period } 2,$$
$$\nu_3 := \#\text{ of elliptic points of } X(\Gamma) \text{ of period } 3,$$
$$\nu_\infty := \#\text{ of cusps of } X(\Gamma).$$

For $h \in \{2,3\}$,

$$\deg(f_\Gamma) = \sum_{x \in f_\Gamma^{-1}(y_h)} e_x = \nu_h \cdot 1 + \left(\# f_\Gamma^{-1}(y_h) - \nu_h\right) \cdot h.$$

Hence,

$$\sum_{x \in f_\Gamma^{-1}(y_h)} (e_x - 1) = (h-1)\left(\# f_\Gamma^{-1}(y_h) - \nu_h\right) = \frac{h-1}{h}\left(\deg(f_\Gamma) - \nu_h\right).$$

Likewise, $\deg(f_\Gamma) = \sum_{x \in f_\Gamma^{-1}(y_\infty)} e_x$ and $\nu_\infty = \# f_\Gamma^{-1}(y_\infty)$, so

$$\sum_{x \in f_\Gamma^{-1}(y_\infty)} (e_x - 1) = \deg(f_\Gamma) - \nu_\infty.$$

II.3.3 Genus of $X(\Gamma)$

Let $\mu := \deg(f_\Gamma) = [\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$.

Theorem II.3.1. *The genus of $X(\Gamma)$ is*

$$g(X(\Gamma)) = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}.$$

Proof. By the Riemann–Hurwitz formula,

$$2g(X(\Gamma)) - 2 = \mu \cdot (2 \cdot 0 - 2) + \sum_{x \in X(\Gamma)} (e_x - 1),$$

so

$$g(X(\Gamma)) = 1 - \mu + \frac{1}{2}\left(\frac{1}{2}(\mu - \nu_2) + \frac{2}{3}(\mu - \nu_3) + \mu - \nu_\infty\right). \qquad \square$$

II.4 2014-02-10: Meromorphic differentials on Riemann surfaces

II.4.1 Meromorphic modular forms

Definition II.4.1. Let $k \in \mathbb{Z}$, and let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. A *meromorphic modular form* of weight k and level Γ is a function $f : \mathfrak{h} \to \mathbb{P}^1(\mathbb{C})$ such that:

(i) f is meromorphic;

(ii) $f|_k\gamma = f$ for all $\gamma \in \Gamma$;

(iii) $f|_k\alpha$ is meromorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Denote the $\mathbb{C}$-vector space of all such forms by $A_k(\Gamma)$. Then

$$S_k(\Gamma) \subseteq M_k(\Gamma) \subseteq A_k(\Gamma).$$

II.4.2 Differential forms

Given a Riemann surface X, the "best" way to think about differential forms on X is to construct the cotangent bundle $p : \Omega_X \to X$; this is a line bundle.

Definition II.4.2. A *differential form* ω is a section of $p : \Omega_X \to X$, i.e., a map $\omega : X \to \Omega_X$ such that $p \circ \omega = \mathrm{id}$.

Concretely: X is covered by open sets U equipped with maps $\varphi : U \xrightarrow{\simeq} V \subseteq \mathbb{C}$. So, to describe differential forms on X, we need to define $\Omega(V)$ for $V \subseteq \mathbb{C}$ an open subset (the local picture), and to see what independence of the chart for differential forms on X implies for the transition maps on overlaps.

Let $V \subseteq \mathbb{C}$ be open.

- dz is in $\Omega(V)$.
- Because V is 1-dimensional, $\Omega(V)$ should be 1-dimensional over

  $$\mathfrak{M}(V) = \{\text{meromorphic functions } f : V \to \mathbb{C}\},$$

  the "ring of functions on V".

Hence, we define

$$\Omega^1(V) = \left\{ f(z)\, dz \;\middle|\; f \in \mathfrak{M}(V) \right\}.$$

For $n \in \mathbb{Z}_{\geq 0}$, define

$$\Omega^{\otimes n}(V) := \Omega^1(V) \otimes_{\mathfrak{M}(V)} \dots \otimes_{\mathfrak{M}(V)} \Omega^1(V) = \left\{ f(z)\,(dz)^n \;\middle|\; f \in \mathfrak{M}(V) \right\}.$$

Then we define the ring

$$\Omega(V) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Omega^{\otimes n}(V),$$

where $(dz)^n \cdot (dz)^m = (dz)^{n+m}$. This is the local picture.

II.4.3 Pullbacks of differential forms

Next, let us study how gluing works. In particular, we need to understand the relation between $\Omega(V_1)$ and $\Omega(V_2)$ given a holomorphic map $\varphi : V_1 \to V_2$.

$n = 0$ $\Omega^{\otimes 0}(V) = \mathfrak{M}(V)$, and a holomorphic map $\varphi : V_1 \to V_2$ induces a "pull-back" map

$$\begin{aligned} \varphi^* : \mathfrak{M}(V_2) &\to \mathfrak{M}(V_2) \\ f &\mapsto f \circ \varphi. \end{aligned}$$

Given $V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3$, we have $(\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^*$.

$n = 1$ If $f(z_2)\, dz_2$ on V_2, then

$$\Omega^1(V_1) \ni \varphi^*(f(z_2)\, dz_2) = f(\varphi(z_1))\, d(\varphi(z_1)) = \big(f(\varphi(z_1))\varphi'(z_1)\big)\, dz_1.$$

General n The pullback map is given by

$$\begin{aligned} \varphi^* : \Omega^{\otimes n}(V_2) &\to \Omega^{\otimes n}(V_1) \\ f(z_1)(dz_2)^n &\mapsto f(\varphi(z_1))\,(\varphi'(z_1))^n\,(dz_1)^n. \end{aligned}$$

Remark II.4.3. (i) If $\varphi : V_1 \hookrightarrow V_2$ is an inclusion, then $\varphi^*(\omega) = \omega|_{V_1}$.

(ii) If $\varphi : V_1 \twoheadrightarrow V_2$ is surjective, then φ^* is injective.

(iii) If $\varphi : V_1 \xrightarrow{\simeq} V_2$ is an isomorphism, then $(\varphi^{-1})^* = (\varphi^*)^{-1}$.

II.4.4 Gluing differential forms

Consider a Riemann surface X with coordinate charts $\varphi_j : U_j \xrightarrow{\simeq} V_j \subseteq \mathbb{C}$ (where $j \in J$). Given two coordinate charts φ_j, φ_k, we have holomorphic transition maps $\varphi_{j,k}$ and $\varphi_{k,j}$.

Definition II.4.4. A *global* meromorphic differential form ω on X of degree $n \geq 0$ is a *compatible* system $(\omega_j)_{j\in J} \in \prod_{j\in J} \Omega^{\otimes n}(V_j)$, i.e.,

$$\varphi_{k,j}^*(\omega_k\big|_{V_{k,j}}) = \omega_j\big|_{V_{j,k}}.$$

II.5 2014-02-12: Meromorphic modular forms and differentials

Recall that

$$\Omega^{\otimes n}(X) = \left\{\omega = (\omega_j)_{j\in J} \text{ on } X \text{ of deg } n \geq 0 \;\middle|\; \omega_j \in \Omega^{\otimes n}(V_j)\right\}.$$

Our goal is to show that, for $k \in \mathbb{Z}_{\geq 0}$ even, Γ a congruence subgroup, there is an isomorphism of $\mathbb{C}$-vector spaces

$$\begin{aligned} \omega : A_k(\Gamma) &\xrightarrow{\simeq} \Omega^{\otimes k/2}(X(\Gamma)) \\ f &\mapsto \omega(f). \end{aligned}$$

The map $\pi_\Gamma : \mathfrak{h}^* \to X(\Gamma)$ induces a map of differentials

$$\pi_\Gamma^* : \Omega^{\otimes n}(X(\Gamma)) \to \Omega^{\otimes n}(\mathfrak{h}).$$

Claim II.5.1. *A collection $(\omega_j)_{j\in J} \in \prod_{j\in J} \Omega^{\otimes n}(V_j)$ is compatible iff the $\psi_j^*(\omega_j)$ to $\mathfrak{h}$ are the restriction of a global meromorphic differential $f(\tau)(d\tau)^{\otimes n} \in \Omega^{\otimes n}(\mathfrak{h})$, where $f \in A_{2n}(\Gamma)$.*

Claim II.5.2. *For all $f \in A_{2n}(\Gamma)$, there exists $(\omega_j)_{j\in J}$ with $\omega_j \in \Omega^{\otimes n}(V_j)$ such that $\psi_j^*(\omega_j) = f(\tau)(d\tau)^n|_{U_j}$.*

Define π_Γ^* given $\omega = (\omega_j)_{j\in J} \in \Omega^{\otimes n}(X(\Gamma))$. Let $U_j' = U_j \cap \mathfrak{h}$, $V_j' = \psi_j(U_j')$, and $\omega_j' = w_j|_{V_j'}$. Define $\tilde{\omega}_j := \psi_j^*(\omega_j)$ on U_j'. These $\tilde{\omega}_j$ are compatible:

$$\begin{tikzcd}
& U_j' \cap U_k' \arrow[dl, "\psi_j"'] \arrow[d, "\pi_r"] \arrow[dr, "\psi_k"] & \\
V_{j,k}' \arrow[rr, bend right, "\varphi_{k,j}"'] & \pi_\Gamma(U_j' \cap U_k') \arrow[l, "\simeq"', "\varphi_j"] \arrow[r, "\simeq", "\varphi_k"'] & V_{k,j}'
\end{tikzcd}$$

$$\tilde{\omega_k}\big|_{U_j'\cap U_k'} = \psi_k^*(\omega_k\big|_{V_{k,j}'}) = \psi_j^*\left(\varphi_{k,j}^*(\omega_k\big|_{V_{k,j}'})\right) = \psi_j^*(\omega_j\big|_{V_{j,k}'}) = \tilde{\omega}_j\big|_{U_j'\cap U_k'}.$$

Hence, the $\tilde{\omega}_j$ give $\pi_\Gamma^*(\omega) := \tilde{\omega} = f(\tau)(d\tau)^{\otimes n} \in \Omega^{\otimes n}(\mathfrak{h})$.

Claim II.5.3. *$f \in A_{2n}(\Gamma)$.*

Proof. For all $\gamma \in \Gamma$, we have $\gamma : \mathfrak{h} \to \mathfrak{h},\ \tau \mapsto \gamma\tau$, so $\gamma^* : \Omega^{\otimes n}(\mathfrak{h}) \to \Omega^{\otimes n}(\mathfrak{h})$. We have

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\gamma"] \arrow[dr, "\pi_\Gamma"'] & \mathfrak{h} \arrow[d, "\pi_\Gamma"] \\ & X(\Gamma), \end{tikzcd}$$

i.e., $\pi_\Gamma \circ \gamma = \pi_\Gamma$, so $\gamma^* \circ \pi_\Gamma^* = \pi_\Gamma^*$. Thus,

$$\begin{aligned} \pi_\Gamma^*(\omega) &= \gamma^*\big(\pi_\Gamma^*(\omega)\big) = \gamma^*\big(f(\tau)(d\tau)^n\big) = f(\gamma\cdot\tau)\big((\gamma\cdot\tau)'\big)^n(d\tau)^n \\ &= f(\gamma\cdot\tau)j(\gamma,\tau)^{-2n}(d\tau)^n = f(\tau)(d\tau)^n, \end{aligned}$$

i.e., $(f|_{2n}\gamma) = f$. It remains to check $f|_{2n}\alpha$ is meromorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. □

Claim II.5.4. *Let $f \in A_{2n}(\Gamma)$, $\psi_j : U_j \cap V_j$, and $U_j' = U_j \cap \mathfrak{h}$. Then there is $\omega_j \in \Omega^{\otimes n}(V_j)$ such that $\psi_j^*(\omega_j|_{V_j'}) = f(\tau)(d\tau)^n|_{U_j'}$.*

Proof. Consider $\delta_j = \begin{pmatrix} 1 & -\tau_j \\ 1 & -\overline{\tau_j} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. Write $\delta_j \cdot \tau = f_{\tau_j}(\tau)$. Since F_{τ_j} is a bijection, there exists

$$\lambda_j := \left(F_{\tau_j}^{-1}\right)^* \left(f(\tau)(d\tau)^n\big|_{U_j'}\right)$$

such that $f(\tau)(d\tau)^n|_{U_j'} = F_{\tau_j}^*(\lambda_j)$. Let $\alpha = \delta_j^{-1}$. Then

$$\lambda_j = \alpha^*\big(f(\tau)(d\tau)^n\big) = f(\alpha\cdot z)\frac{\det(\alpha)^n}{j(\alpha,z)^{2n}}(dz)^n = (f|_{2n}\alpha)(z)(dz)^n.$$

So for any $\gamma \in \Gamma$,

$$\left(\alpha^{-1}\gamma\alpha\right)^*\lambda_j = (f|_{2n}\alpha\alpha^{-1}\gamma\alpha)(z)(dz)^n = (f|_{2n}\alpha)(z)(dz)^n = \lambda_j.$$

Hence, λ_j is invariant under $\delta_j\Gamma\delta_j^{-1}$. □

II.6 2014-02-14: Divisors and Riemann–Roch

II.6.1 Meromorphic modular forms and differentials

Continuing from last time: λ_j is $\delta_j\Gamma\delta_j^{-1}$-invariant. If τ_j is not a cusp, then $\delta_j : \tau_j \mapsto 0$, and

$$h_j = \#\left\{\pm I\right\}(\delta_j\Gamma\delta_j^{-1})_0/\left\{\pm I\right\}$$

is cyclic generated by $r_{h_j} : z \mapsto \mu_{h_j}z$, where $\mu_{h_j} = e^{2\pi i/h_j}$. We have

$$(f|_{2n}\alpha)(z)(dz)^n = \lambda_j = r_{h_j}^*(\lambda_j) = (f|_{2n}\alpha)(\mu_{h_j}z)(d(\mu_{h_j}z))^n = (f|_{2n}\alpha)(\mu_{h_j}z)\mu_{h_j}^n(dz)^n,$$

so there exists $g_j \in \mathfrak{M}(V_j')$ such that

$$(\mu_{h_j}z)^n(f|_{2n}\alpha)(\mu_{h_j}z) = z^n(f|_{2n}\alpha)(z) = g_j(z^{h_j}) = g_j(q),$$

where $q = z^{h_j}$. Define

$$\omega_j' := \frac{g_j(q)}{(h_j q)^n}(dq)^n.$$

Hence

$$\psi_j^*(\omega_j') = F_{\tau_j}^*\big(\rho_j^*(\omega_j')\big).$$

We want to show that $\rho_j^*(\omega_j') = \lambda_j$. Indeed,

$$\rho_j^*(\omega_j') = \frac{g_j(z^{h_j})}{(h_j z^{h_j})^n}(h_j z^{h_j-1})^n (dz)^n = \frac{g_j(z^{h_j})}{z^n}(dz)^n = (f|_{2n}\alpha)(z)dz = \lambda_j.$$

At a cusp, $\delta_j : \tau_j \mapsto \infty$, h_j is the width, and $\rho_j(z) = e^{2\pi i z/h_j}$. We now instead have

$$\omega_j' = \frac{g_j(q)}{(2\pi i q/h_j)^n}(dq)^n,$$

and the rest proceeds similarly.

This completes the proof of the isomorphism

$$\omega : A_k(\Gamma) \xrightarrow{\simeq} \Omega^{\otimes k/2}(X(\Gamma))$$

for k even.

II.6.2 Divisors on Riemann surfaces

Let X be a compact Riemann surface.

Definition II.6.1. The *divisor group* on X, denoted $\mathrm{Div}(X)$, is the free abelian group on the set of points in X. In other words, a *divisor* D on X is a finite formal sum $D = \sum_{x\in X} n_x x$, where $n_x \in \mathbb{Z}$ and all but finitely many n_x are zero.

- Write $D \geq D'$ if $n_x \geq n_x'$ for all $x \in X$.
- There is a natural homomorphism

$$\begin{aligned} \deg : \mathrm{Div}(X) &\to \mathbb{Z}, \\ \sum_{x\in X} n_x x &\mapsto \sum_{x\in X} n_x. \end{aligned}$$

Example II.6.2 (Principal divisors). For any $f \in \mathfrak{M}(X)$ and $x \in X$, define the *order of vanishing* $\mathrm{ord}_x(f)$ by locally writing $f_x(z) = \sum_{n\in\mathbb{Z}} a_n (z-x)^n$, and setting

$$\mathrm{ord}_x(f) = \begin{cases} \min\{n \in \mathbb{Z} \mid a_n \neq 0\} & \text{if } f \neq 0, \\ \infty & \text{if } f = 0. \end{cases}$$

For $f \neq 0$, define the *principal divisor*

$$\mathrm{div}(f) := \sum_{x\in X} \mathrm{ord}_x(f) x.$$

This gives a homomorphism

$$\mathfrak{M}(X)^\times \to \mathrm{Div}(X).$$

Note II.6.3.
- By a general theorem of complex analysis, $\deg(\operatorname{div}(f)) = 0$.
- $\operatorname{ord}_x(f_1 + f_2) \geq \min(\operatorname{ord}_x f_1, \operatorname{ord}_x f_2)$.

Definition II.6.4. Let D be a divisor. The *linear space* of D is

$$L(D) := \left\{ f \in \mathfrak{M}(X) \;\middle|\; \operatorname{div}(f) + D \geq 0 \right\} \cup \{0\}.$$

Example II.6.5.
(1) If $D = -2 \cdot 0 + 3 \cdot 1$, then $L(D)$ consists of functions with a zero of order at least 2 at 0, and a pole of order at most 3 at 1.
(2) If $g \in \mathfrak{M}(X)^\times$, then $L(\operatorname{div}(g)) = \{f \mid fg \text{ is holomorphic}\}$.

Fact II.6.6.
- $L(D)$ is a $\mathbb{C}$-vector space.
- $\ell(D) := \dim_{\mathbb{C}} L(D) < \infty$.

If $\omega \in \Omega^{\otimes n}(X)$ with $\omega \neq 0$, and $x \in U \xrightarrow[\simeq]{\varphi} V \subseteq \mathbb{C}$ is a local chart, set $\omega = \varphi^*(\omega_x)$ with $\omega_x = f_x(q)(dq)^n$. Define

$$\operatorname{ord}_x(\omega) := \operatorname{ord}_{q=\varphi(x)} f_x$$

and

$$\operatorname{div}(\omega) = \sum_{x \in X} \operatorname{ord}_x(\omega) x.$$

Then

$$\operatorname{div}(\omega_1\omega_2) = \operatorname{div}(\omega_1) + \operatorname{div}(\omega_2).$$

Definition II.6.7. If $\lambda \in \Omega^1(X)$ with $\lambda \neq 0$, then $\operatorname{div}(\lambda)$ is called a *canonical divisor*.

II.6.3 The Riemann–Roch theorem

Theorem II.6.8 (Riemann–Roch). *For all $D \in \operatorname{Div}(X)$,*

$$\ell(D) = \deg(D) - g(X) + 1 + \ell(\operatorname{div}(\lambda) - D),$$

where $\operatorname{div}(\lambda)$ is a canonical divisor and $g(X)$ is the genus of X.

II.7 2014-02-16: Riemann–Roch

II.7.1 Consequences of Riemann–Roch

Let $\lambda \in \Omega^1(X)$ be nonzero and $D \in \operatorname{Div}(X)$ be arbitrary.

Theorem II.7.1 (Riemann–Roch). $\ell(D) = \deg(D) - g(X) + 1 + \ell(\operatorname{div}(\lambda) - D)$.

Corollary II.7.2. *(a)* $\ell(\operatorname{div}(\lambda)) = g$.

(b) $\deg(\operatorname{div}(\lambda)) = 2g - 2$.

(c) If $\deg(D) < 0$, *then* $\ell(D) = 0$.

(d) If $\deg(D) > 2g-2$, then $\ell(D) = deg(D) - g + 1$. (This is the "simple form" of Riemann–Roch.)

Proof. (a) If $D = 0$, then $\ell(D) = 1$. Plug in $D = 0$.

(b) Plug in $D = \operatorname{div}(\lambda)$.

(c) If $\ell(D) > 0$, then there exists f such that $\operatorname{div}(f) \geq -D$. So $0 = \deg(\operatorname{div}(f)) \geq -\deg(D)$.

(d) In this case, $\deg(\operatorname{div}(\lambda) - D) < 0$, so the result follows from (c). $\square$

II.7.2 Application to meromorphic modular forms

Let $k \geq 2$ be even. Here is our plan:

(1) Show there exists nonzero $f_0 \in A_k(\Gamma)$.

(2) Show $A_k(\Gamma) = \mathfrak{M}(X(\Gamma)) \cdot f_0$.

(3) For any $g \in A_k(\Gamma)$, we will make sense of $\operatorname{div}(g) \in \operatorname{Div}_{\mathbb{Q}}(X(\Gamma)) := \operatorname{Div}(X(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

(4) $M_k(\Gamma) = L(\operatorname{div}(f_0))$, and similarly for $S_k(\Gamma)$.

(5) Relate $\operatorname{ord}_x(\omega(f))$ to $\operatorname{ord}_\tau(f)$.

For any $f \in A_k(\Gamma)$ and $\gamma \in \Gamma$,

$$(f|_k\gamma)(z) = (cz+d)^k f(z).$$

Note that $(cz+d)^k \notin \{0, \infty\}$ for all $z \in \mathfrak{h}$, so $\operatorname{ord}_\tau f = \operatorname{ord}_{\gamma\cdot\tau} f$.

Let $\tau \in \mathfrak{h}$ and $q_0 = \pi_\Gamma(\tau) \in X(\Gamma)$. Then

$$f(z) = \sum_{n \geq \operatorname{ord}_\tau f} a_n (z-\tau)^n.$$

In local coordinates, $(q - q_0)$ is $(z-\tau)^{h_\tau}$. So

$$f(q) = \sum_{n \geq \operatorname{ord}_\tau f} a_n \left((z-\tau)^{h_\tau}\right)^{n/h_\tau} = \sum_{n \geq \operatorname{ord}_\tau f} a_n (q-q_0)^{n/h_\tau}.$$

Hence, we can define

$$\operatorname{ord}_{q_0}(f) := \frac{\operatorname{ord}_\tau f}{h_\tau} \in \tfrac{1}{2}\mathbb{Z} \cup \tfrac{1}{3}\mathbb{Z}.$$

Say $x = \pi_\Gamma(s)$, where $s \in \mathbb{P}^1(\mathbb{Q})$. Let h_x be the width of x. If $s = \infty$, then

$$\{\pm I\}\,\Gamma_\infty = \{\pm I\} \left\langle \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \right\rangle,$$

but since $-I$ might not be in Γ, this only implies Γ_∞ is one of the following:

$$\{\pm I\} \left\langle \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \right\rangle, \quad \left\langle -\begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \right\rangle.$$

If $x = \pi_\Gamma(s)$ with $s \in \mathbb{P}^1(\mathbb{Q})$, choose $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha \cdot \infty = s$. If

$$(\alpha^{-1}\Gamma\alpha)_\infty = \left\langle -\begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \right\rangle,$$

then x is called an *irregular cusp*, and then $\begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \notin \alpha^{-1}\Gamma\alpha$. So if k is odd, then

$$\left((f|_k\alpha)|_k - \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix} \right)(z) = j\left(-\begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix}, z \right)^k (f|_k\alpha)(z) = (-1)^k (f|_k\alpha)(z).$$

so h_x is not the period. Thus,

$$\operatorname{ord}_x f := \begin{cases} \operatorname{ord}_s f & \text{if } x \text{ is regular or } k \text{ is even,} \\ \frac{1}{2}\operatorname{ord}_s f & \text{if } x \text{ is irregular and } k \text{ is odd.} \end{cases}$$

II.7.3 Order of differentials

Let $\omega \in \Omega^{\otimes n}(X(\Gamma))$, so $\omega = \omega(f)$ for some $f \in A_{2n}(\Gamma)$.

If x is not a cusp, let $g(q) = z^{h_x} f(z)$, where $q = z^{h_x}$. Then

$$\operatorname{ord}_x \omega = \operatorname{ord}_0 \frac{g(q)}{(h_x q)^{k/2}} = \operatorname{ord}_0 g(q) - \frac{k}{2} = \operatorname{ord}_x(f) + \frac{k}{2} \cdot \frac{1}{h_x} - \frac{k}{2}.$$

In particular, if $h_x = 1$, then $\operatorname{ord}_x(\omega) = \operatorname{ord}_x(f)$.

If x is a cusp, let $g(q) = f(e^{2\pi i q/h_x})$. Then

$$\operatorname{ord}_x \omega = \operatorname{ord}_0 \frac{g(q)}{(2\pi i q/h_x)^{k/2}} = \operatorname{ord}_x f - \frac{k}{2}.$$

We formally define

$$\operatorname{div}(d_\tau) := -\sum_i \frac{1}{2} x_{2,i} - \sum_i \frac{2}{3} x_{3,i} - \sum_i x_{\infty,i} \in \operatorname{Div}_{\mathbb{Q}}(X),$$

where $x_{2,i}, x_{3,i}, x_{\infty,i}$ range over the order 2 elliptic points, order 3 elliptic points, and cusps, respectively.

II.8 2014-02-19: Computing the dimensions

II.8.1 Computing the dimension of $M_k(\Gamma)$

Assume[3] there exists $f_0 \neq 0$ in $A_k(\Gamma)$ for $k \geq 0$.

Proposition II.8.1. *For $k \geq 0$, $A_k(\Gamma) = \mathfrak{M}(X(\Gamma)) \cdot f_0$.*

[3] For now; we'll prove this later.

Proof. Let $f \in A_k(\Gamma)$. Then $f/f_0 \in A_0(\Gamma)$ is Γ-invariant, so descends to $X(\Gamma)$. □

So

$$\begin{aligned} M_k(\Gamma) &= \left\{ f f_0 \;\middle|\; f \in \mathfrak{M}(X(\Gamma)),\ f f_0 = 0 \text{ or } \operatorname{div}(f f_0) \geq 0 \right\} \\ &\cong \left\{ f \in \mathfrak{M}(X(\Gamma)) \;\middle|\; f = 0 \text{ or } \operatorname{div}(f) + \operatorname{div}(f_0) \geq 0 \right\} \\ &= L(\operatorname{div}(f_0)). \end{aligned}$$

The problem is, $\operatorname{div}(f_0)$ is usually not integral. To fix this, for $D = \sum n_x x \in \operatorname{Div}_{\mathbb{Q}}(X(\Gamma))$, define $\lfloor D \rfloor := \sum \lfloor n_x \rfloor x \in \operatorname{Div}(X(\Gamma))$. Since $\operatorname{div}(f)$ is integral for $f \in M(X(\Gamma))$,

$$M_k(\Gamma) \cong L(\lfloor \operatorname{div} f_0 \rfloor).$$

For $k \geq 0$ even, $\omega(f_0) = f_0(\tau)(d\tau)^{k/2}$, so

$$\operatorname{div}(\omega(f_0)) = \operatorname{div}(f_0) + \frac{k}{2} \operatorname{div}(d\tau)$$

is integral. Hence,

$$\lfloor \operatorname{div} f_0 \rfloor = \operatorname{div}(\omega(f_0)) + \sum_i \left\lfloor \frac{k}{4} \right\rfloor x_{2,i} + \sum_i \left\lfloor \frac{k}{3} \right\rfloor x_{3,i} + \sum_i \frac{k}{2} x_{\infty,i}.$$

We need to know $\deg(\operatorname{div}(\omega(f_0)))^i$.

If $\lambda \in \Omega^1(X(\Gamma))$ (so $\lambda^{k/2} \in \Omega^{\otimes k/2}(X(\Gamma))$), then

$$\deg(\operatorname{div}(\lambda^{k/2})) = \frac{k}{2} \deg(\operatorname{div}(\lambda)) = \frac{k}{2}(2g-2) = k(g-1).$$

For all $\omega \in \Omega^{k/2}(X(\Gamma)) \in \mathfrak{M}(X(\Gamma)) \cdot \lambda^{k/2}$, writing $\omega = f \cdot \lambda^{k/2}$, we have

$$\operatorname{div}(\omega) = \operatorname{div}(f) + \operatorname{div}(\lambda^{k/2}),$$

so

$$\deg(\operatorname{div}(\omega)) = \deg(\operatorname{div}(f)) + \deg(\operatorname{div}(\lambda^{k/2})) = \deg(\operatorname{div}(\lambda^{k/2})).$$

Hence,

$$\begin{aligned} \deg(\lfloor \operatorname{div} f_0 \rfloor) &= k(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \nu_2 + \left\lfloor \frac{k}{3} \right\rfloor \nu_3 + \frac{k}{2} \nu_\infty \\ &\geq \frac{k}{2}(2g-2) + \left(\frac{k-2}{4} \right) \nu_2 + \left(\frac{k-2}{3} \right) \nu_3 + \frac{k}{2} \nu_\infty \\ &= 2g - 2 + \frac{k-2}{2} \underbrace{\left(2g - 2 + \frac{\nu_2}{2} + \frac{2\nu_3}{3} + \nu_\infty \right)}_{= \frac{\mu}{6} > 0 \text{ by the genus formula}} + \nu_\infty \\ &\geq 2g - 2 + \nu_\infty \qquad (\text{if } k \geq 2) \\ &> 2g - 2 \end{aligned}$$

because there are always cusps. Therefore, we can use the simple version of Riemann–Roch: if $k \geq 2$ is even, then

$$\dim_{\mathbb{C}} M_k(\Gamma) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \nu_2 + \left\lfloor \frac{k}{3} \right\rfloor \nu_3 + \frac{k}{2} \nu_\infty.$$

II.8.2 Computing the dimension of $S_k(\Gamma)$

Similarly:

$$S_k(\Gamma) \cong L\Big(\lfloor \operatorname{div} f_0 \rfloor - \sum_i x_{\infty,i}\Big).$$

If $k = 2$, then

$$\lfloor \operatorname{div} f_0 \rfloor = \operatorname{div}(\omega(f_0)) + 0 + 0 + \sum_i x_{\infty,i},$$

so $S_2(\Gamma) \cong L(\operatorname{div}(\omega(f_0)))$, so $\dim_{\mathbb{C}} S_2(\Gamma) = g(X(\Gamma))$.

If $k \geq 4$, then $\frac{k-2}{2} > 0$, so

$$2g - 2 + \frac{k-2}{2}\left(\frac{\mu}{6}\right) > 2g - 2.$$

By the simple version of Riemann–Roch, for $k \geq 4$,

$$\dim_{\mathbb{C}} S_k(\Gamma) = \dim_{\mathbb{C}} M_k(\Gamma) - \nu_\infty.$$

II.8.3 Negative weight modular forms

If $k = 0$, then

$$M_0(\Gamma) = \{f \text{ holomorphic}\} = \text{constants} = \mathbb{C}.$$

So $\dim M_0(\Gamma) = 1$ and $\dim S_0(\Gamma) = 0$.

Suppose $k < 0$. Recall $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$, given by

$$\Delta = q \prod_{n \geq 0} (1 - q^n)^{24}.$$

So $\Delta(\tau) \neq 0$ for all $\tau \in \mathfrak{h}$. Let $f \in M_k(\Gamma)$. Then $f^{12}\Delta^{-k} \in S_0(\Gamma) = 0$, so $f = 0$. Thus,

$$M_k(\Gamma) = S_k(\Gamma) = 0$$

for $k < 0$.

II.8.4 Example: modular forms for $\mathrm{SL}_2(\mathbb{Z})$

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z}))$, and let $k \geq 4$ be even. Then

$$\begin{aligned} \dim_{\mathbb{C}} S_k(\mathrm{SL}_2(\mathbb{Z})) &= \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor - 1 & \text{if } k \equiv 2 \pmod{12}, \\ \left\lfloor \frac{k}{12} \right\rfloor & \text{otherwise,} \end{cases} \\ \dim_{\mathbb{C}} M_k(\mathrm{SL}_2(\mathbb{Z})) &= \dim_{\mathbb{C}} S_k(\mathrm{SL}_2(\mathbb{Z})) + 1. \end{aligned}$$

II.8.5 Odd weight modular forms

Let $k > 0$ be odd. If $-I \in \Gamma$, then $j(-I, z)^k = -1$, so

$$f(\tau) = (f|_k - I)(\tau) = -f(\tau),$$

and there are no nontrivial modular forms of weight k.

Assume $-I \notin \Gamma$. Given $f \in A_k(\Gamma)$, look at $\omega(f^2)$.

Theorem II.8.2. *Suppose $k \geq 3$ is odd and $-I \notin \Gamma$. Then*

$$\begin{aligned}\dim_{\mathbb{C}} M_k(\Gamma) &= (k-1)(g-1) + \left\lfloor \frac{k}{3} \right\rfloor \nu_3 + \frac{k}{2}\nu_\infty^{\mathrm{reg}} + \frac{k-1}{2}\nu_\infty^{\mathrm{irr}}, \\ \dim_{\mathbb{C}} S_k(\Gamma) &= \dim M_k(\Gamma) - \nu_\infty^{\mathrm{reg}}.\end{aligned}$$

Now consider $k = 1$:

- If $\nu_\infty^{\mathrm{reg}} > 2g - 2$, then

$$\begin{aligned}\dim_{\mathbb{C}} M_1(\Gamma) &= \frac{\nu_\infty^{\mathrm{reg}}}{2}, \\ \dim_{\mathbb{C}} S_1(\Gamma) &= 0.\end{aligned}$$

- Otherwise:

$$\begin{aligned}\dim_{\mathbb{C}} M_1(\Gamma) &\geq \frac{\nu_\infty^{\mathrm{reg}}}{2}, \\ \dim_{\mathbb{C}} S_1(\Gamma) &= \dim_{\mathbb{C}} M_1(\Gamma) - \frac{\nu_\infty^{\mathrm{reg}}}{2}.\end{aligned}$$

Chapter III

Eisenstein series

III.1 2014-02-21 [missing]

III.2 2014-02-24

III.2.1 Cusps of $\Gamma(N)$

Recall: the cusps of $\Gamma(N)$ are given by

$$\begin{pmatrix} a \\ c \end{pmatrix} \equiv \begin{pmatrix} a' \\ c' \end{pmatrix} \pmod N$$

with $\gcd(a, c) = 1$, so the cusps of $\Gamma(N)$ are represented by $\frac{a}{c}$ with $\gcd(a, c) = 1$, i.e., by elements of $(\mathbb{Z}/N)^2$ of order exactly N.

Remark III.2.1.

$$\nu_\infty(\Gamma(N))) = \begin{cases} \frac{N^2}{2} \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right) & \text{if } N > 2, \\ 3 & \text{if } N = 2. \end{cases}$$

III.2.2 Cocycle relations

We have the following cocycle relation for $j(\gamma, z)$:

Lemma III.2.2. *For all $\alpha, \beta \in \mathrm{GL}_2^+(\mathbb{R})$ and $z \in \mathfrak{h}$,*

$$j(\alpha\beta, z) = j(\alpha, \beta z) j(\beta, z).$$

Let x be a cusp of Γ, and let $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma \cdot x = \infty$. Then:

Lemma III.2.3. *Let $k \geq 0$. Then*

$$j(\sigma\gamma\sigma^{-1}, z)^k = 1 \qquad \forall \gamma \in \Gamma_x$$

iff k is even or $-I \notin \Gamma$ or x is regular.

Definition III.2.4. Let $k \geq 3$, and take $\Gamma = \Gamma(N)$. Let $x_i \in \mathfrak{h}^*$ be cusps of $\Gamma(N)$, and let $\sigma_i \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_i \cdot x_i = \infty$. (If k is odd, assume the x_i are regular.) Suppose $x_0 = \infty$ and $\sigma_0 = I$. Define:

$$g_i(z) = g_i(z; k, N) \stackrel{\text{def}}{=} \sum_{\gamma \in \Gamma_{x_i} \backslash \Gamma} j(\sigma_i, z)^{-k}|_k \gamma.$$

Our goal is to show that $\{g_i(z)\}$ is a basis of the Eisenstein space $\mathcal{E}(\Gamma(N))$ for all $k \geq 3$, i.e.:

(i) $g_i \in M_k(\Gamma(N))$.

(ii) The g_i are linearly independent.

Note III.2.5. $-I \in \Gamma(N)$ iff $N \in \{1, 2\}$.

Lemma III.2.6. *There is a bijection*

$$\Gamma_\infty \backslash \Gamma \longleftrightarrow \left\{ (c, d) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} \gcd(c, d) = 1, \\ (c, d) \equiv (0, 1) \pmod{N}, \\ \textit{if } N \in \{1, 2\}, \ \textit{then } d > 0 \end{array} \right\}$$

Note that

$$\pm \begin{pmatrix} 1 & Ne \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + cNe & b + dNe \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$$

with $a'd - b'c = 1$ and $xd - yc = 1$. So $\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and $N \mid a - a'$, $N \mid b - b'$.

Let $C_N = 1$ if $N \geq 3$, and $C_N = \frac{1}{2}$ if $N = 1, 2$. So

$$g_0(z) = C_n \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d) \equiv (0,1) \quad (N) \\ \gcd(c,d) = 1}} \frac{1}{(cz + d)^k}.$$

For $v = (c, d) \mod N$ of order N in $(\mathbb{Z}/N)^2$, let

$$G_k^v(z) := C_N \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ \gcd(c,d) = 1 \\ (c,d) \equiv v \quad (N)}} \frac{1}{(cz + d)^k}.$$

Let

$$\tilde{E}_k(z) := \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d) \neq 0}} \frac{1}{(cz + d)^k}.$$

Proposition III.2.7. *For all $k \geq 3$, $\tilde{E}_k(z)$ converges absolutely and uniformly on compact subsets of $\mathfrak{h}$.*

Proof. Let $z \in \mathfrak{h}$, and denote $L_z := \{cz + d \mid c, d \in \mathbb{Z}\} \subseteq \mathbb{C}$. For $n \in \mathbb{Z}_{>0}$, let M_n be the boundary of the parallelogram whose vertices are $\pm nz \pm n$. Then $\bigcup_{n \geq 1}(L \cap M_n) = L \setminus \{0\}$, and $\#(L \cap M_n) = 8n$.

Let $r(z)$ be the distance from the origin to M_1; this is a continuous function of z. If $\omega \in L \cap M_n$, then $|\omega| \geq r(z) \cdot n$. Thus,

$$\sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq 0}} \frac{1}{|cz+d|^k} = \sum_{n=1}^{\infty} \sum_{\omega \in L \cap M_n} |\omega|^{-k} \leq \sum_{n=1}^{\infty} 8n \cdot \frac{1}{(r(z)n)^k} = 8r(z)^{-k}\zeta(k-1). \qquad \square$$

Lemma III.2.8. *(i) Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *such that* $(c, d) \equiv v \pmod N$. *Then*

$$G_k^v(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma\sigma} j(\gamma, z)^{-k}.$$

(ii) For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$\left(G_k^v|_k\gamma\right) = G_k^{v\gamma}.$$

Proof. (i) If $(c', d') \equiv (0, 1) \mod N$, then

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ c & d \end{pmatrix} \pmod N.$$

(ii) We have

$$\begin{aligned}\left(G_k^v|_k\gamma\right)(z) &= C_N j(\gamma, z)^{-k} \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma\sigma} j(\gamma', \gamma \cdot z)^{-k} \\ &= C_N \sum_{\gamma \in \Gamma_\infty \backslash \Gamma\sigma} j(\gamma'\gamma, z)^{-k} \\ &= C_N \sum_{\gamma'' \in \Gamma_\infty \backslash \Gamma\sigma\gamma} j(\gamma'', z)^{-k} = G_k^{v\gamma}(z). \qquad \square\end{aligned}$$

III.3 2014-02-26: Eisenstein series

Let $\bar{\nu} \in (\mathbb{Z}/N\mathbb{Z})^2$. Define

$$G_k^{\bar{\nu}}(z) = C_N \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ (c,d)\equiv\bar{\nu} \quad (N)}} \frac{1}{(cz+d)^k},$$

where $C_N = \frac{1}{2}$ if $N = 1, 2$, and $C = 1$ otherwise. This set of functions (with $\bar{\nu}$ varying and k fixed) is permuted by $\mathrm{SL}_2(\mathbb{Z})$, whence $G_k^{(0,1)}(z)$ is preserved by $\mathrm{Stab}_{(0,1)} \mathrm{SL}_2(\mathbb{Z}) = \Gamma_1(N)$.

How does $G_k^{\bar{\nu}}(z)$ behave as we move towards the boundary of $\mathfrak{h}^*$, e.g., what happens to G_k as $z \to i\infty$? (Note that if we wanted to describe behavior as $z \to p/q = \gamma\infty$, we can just study $G_k^{\bar{\nu}}(\gamma z) = c_\gamma G_k^{\bar{\nu}'}(z)$ as $z \to \infty$.)

When $k > 2$, the sum $\sum (cz+d)^{-k}$ is uniformly convergent. So

$$\lim_{z\to i\infty} C_N \sum \frac{1}{(cz+d)^k} = C_N \sum \lim_{z\to i\infty} \frac{1}{(cz+d)^k}.$$

Note that $\lim_{z\to i\infty}(cz+d)^{-k} = 0$ whenever $c \neq 0$. Since $\gcd(c,d) = 1$, if $c = 0$, then $d = \pm 1$. Thus,

$$C_N \sum \lim_{z\to i\infty} \frac{1}{(cz+d)^k} = C_N \sum_{\substack{d\in\{\pm1\}\\(0,d)\equiv\bar{\nu}}} \frac{1}{d^k} = \begin{cases} 1 & \text{if } \bar{\nu} = (0,1) \text{ and } N \geq 3,\\ (-1)^k & \text{if } \bar{\nu} = (0,-1) \text{ and } N \geq 3,\\ 1 & \text{if } \bar{\nu} = (0,1) \text{ and } N \in \{1,2\} \text{ and } k \text{ even},\\ 0 & \text{otherwise.}\end{cases}$$

To sum up, $\lim_{z\to i\infty} G_k^{\bar{\nu}}(z)$ is one of $0, 1, (-1)^k$.

We can use this computation to show these forms are a basis of the space of Eisenstein series. We have an exact sequence

$$\begin{aligned} 0 \to S_k(\Gamma(N)) \to M_k(\Gamma(N)) &\xrightarrow{E} \mathbb{C}^{\{\text{cusps of } \Gamma(N)\}}\\ f &\mapsto \big(s \mapsto \lim_{z\to s} f(z)\big).\end{aligned}$$

What we have just done is computed $E(G_k^{(0,1)})$.

Consider

$$\begin{aligned}\tilde{E}_k^{\bar{\nu}}(z) &= \sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)\\(c,d)\equiv\bar{\nu}\quad(N)}} \frac{1}{(cz+d)^k}\\ &= \sum_{\substack{n=1\\(n,N)=1}}^{\infty} \sum_{\substack{(c,d)\\ \gcd=n}} \frac{1}{(cz+d)^k}\\ &= \sum_{\substack{n=1\\(n,N)=1}}^{\infty} \frac{1}{n^k} \sum_{\substack{(c',d')\\ \gcd=n}} \frac{1}{(c'z+d')^k}\end{aligned}$$

(where $c = nc'$, $d = nd'$, and $\gcd(c',d') = 1$)

$$\begin{aligned} &= \sum_n \frac{1}{n^k} G_k^{n^{-1}\bar{\nu}}(z)\\ &= \sum_{x\in(\mathbb{Z}/N\mathbb{Z})^\times} \Big(\sum_{n\equiv x\quad(N)} \frac{1}{n^k}\Big) G_k^{x^{-1}\bar{\nu}}(z)\\ &= \sum_{x\in(\mathbb{Z}/N\mathbb{Z})^\times} \zeta_x(k) G_k^{x^{-1}\bar{\nu}}(z).\end{aligned}$$

III.4 2014-02-28: The q-expansion

Recall $\tilde{E}_k^{\bar{v}}$ from last time: Fix $(c_v, d_v) \in \mathbb{Z}^2$ such that $\gcd(c_v, d_v) = 1$ and $(c_v, d_v) \equiv \bar{v} \pmod{N}$. Then

$$\tilde{E}_k^{\bar{v}} = \sum \frac{1}{(cz+d)^k} = \sum_{c\equiv c_v} \sum_{d\in\mathbb{Z}} \frac{1}{(cz+d_v+Nd)^k} = \frac{1}{N^k} \sum_{c\equiv c_v} \sum_{d\in\mathbb{Z}} \frac{1}{\left(\frac{cz+d_v}{N}+d\right)^k}.$$

The $c = 0$ term gives a constant term

$$\text{constant term} = \begin{cases} 0 & \text{if } c_v \not\equiv 0 \pmod{N}, \\ \sum_{\substack{d\equiv d_v \\ d\neq 0}} \frac{1}{d^k} & \text{if } c_v \equiv 0. \end{cases}$$

If $c > 0$, then $\tau := \frac{cz+d_v}{N} \in \mathfrak{h}$.

Lemma III.4.1. *We have*

$$\sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^k} = \Omega_k \sum_{m=1}^{\infty} m^{k-1} q^m,$$

where $\tau \in \mathfrak{h}$, $k \geq 2$, $q = e^{2\pi i\tau}$, and $\Omega_k := \frac{(-2\pi i)^k}{(k-1)!}$.

Proof. Use Poisson summation: let $\mathfrak{h} : \mathbb{R} \to \mathbb{C}$ such that

- $\int_{-\infty}^{\infty} |h(x)|\, dx < \infty$, and
- $\sum_{d\in\mathbb{Z}} h(x+d)$ converges absolutely and uniformly on compact subsets, and is C^∞.

Then

$$\sum_{d\in\mathbb{Z}} h(x+d) = \sum_{m\in\mathbb{Z}} \widehat{h}(m) e^{2\pi i m x},$$

where $\widehat{h}$ is the Fourier transform

$$\widehat{h}(x) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i x t}\, dt.$$

Let $\tau := x + iy$ for fixed $y > 0$. Applying Poisson summation to $h(x) = \frac{1}{\tau^k} = \frac{1}{(x+iy)^k}$ for $k \geq 2$, it suffices to show that

$$\widehat{h}(m) = \begin{cases} 0 & \text{if } m \leq 0, \\ \Omega_k m^{k-1} & \text{if } m > 0. \end{cases}$$

For $m = 0$, observe that

$$\widehat{h}(0) = \int_{-\infty}^{\infty} h(t)\, dt = \int_{-\infty}^{\infty} \frac{1}{(x+iy)^k} = \frac{1}{(1-k)(x+iy)^{k-1}} \bigg|_{-\infty}^{\infty} = 0.$$

If $m > 0$, then replacing t with $t - iy$ by a change of variables,

$$\widehat{h}(m) = \int_{-\infty}^{\infty} \frac{1}{(t+iy)^k} e^{-2\pi imt}\, dt = e^{2\pi my} \int_{iy-\infty}^{iy+\infty} \frac{1}{t^k} e^{-2\pi imt}\, dt.$$

Let $f_m(z) = z^{-k} e^{-2\pi imz}$. This is meromorphic with a pole at $z = 0$. We can check that

$$-2\pi i \operatorname{Res}_{z=0} f_m(z) = \Omega_k m^{k-1},$$

and contour integration shows this is indeed the value of the above integral.

If $m < 0$, then

$$\begin{aligned}
\widehat{h}(m) &= \int_{-\infty}^{\infty} \frac{1}{(t+iy)^k} e^{-2\pi imt}\, dt \\
&= \int_{\infty}^{-\infty} \frac{1}{(-t+iy)^k} e^{-2\pi i|m|t}\, dt \\
&= (-1)^{k+1} \int_{-\infty}^{\infty} \frac{1}{(t-iy)^k} e^{-2\pi i|m|t}\, dt.
\end{aligned}$$

We can now do a contour integral containing no poles, so by the residue theorem, the integral is zero. □

Continuing with the Eisenstein series from before,

$$\tilde{E}_k^{\bar{v}}(z) = \frac{1}{N^k} \sum_{c \equiv c_v} \sum_{d \in \mathbb{Z}} \frac{1}{\left(\frac{cz+d_v}{N} + d\right)^k}.$$

If $c < 0$, $\tau = \frac{cz+d_v}{N}$, $\mu_N = e^{2\pi i/N}$, and $q_N = e^{2\pi i\tau/N}$, then

$$e^{2\pi i\tau m} = e^{2\pi imcz/N} e^{2\pi id_v m/N} = q_N^{cm} \mu_N^{d_v m},$$

so the $c > 0$ part is equal to

$$\frac{\Omega_k}{N^k} \sum_{\substack{c \equiv c_v \\ c>0}} \sum_{m=1}^{\infty} m^{k-1} \mu_N^{d_v m} q_N^{cm} = \frac{\Omega_k}{N^k} \sum_{n=1}^{\infty} \sum_{\substack{m \mid n \\ \frac{n}{m} \equiv c_v \\ m>0}} m^{k-1} \mu_N^{d_v m} q_N^{n}.$$

If $c < 0$, $\tau = -\left(\frac{cz+d_v}{N}\right) \in \mathfrak{h}$, then the $c < 0$ part is

$$(-1)^k \frac{\Omega_k}{N^k} \sum_{\substack{c \equiv c_v \\ c<0}} \sum_{m=1}^{\infty} m^{k-1} \mu_N^{-d_v m} q_N^{-cm} = \frac{\Omega_k}{N^k} \sum_{n=1}^{\infty} \sum_{\substack{m \mid n \\ \frac{n}{m} \equiv c_v \\ m<0}} -m^{k-1} \mu_N^{d_v m} q_N^{n}.$$

This computation yields the following:

Theorem III.4.2. *For $k \geq 3$ and $\bar{v} \in (\mathbb{Z}/N)^2$ of order N,*

$$\tilde{E}_k^{\bar{v}}(z) = \delta(c_{\bar{v}})\zeta_{\bar{d}_v}^{\pm}(k) + \frac{\Omega_k}{N^k}\sum_{n=1}^{\infty}\sigma_{k-1}^{\bar{v}}(n)q_N^n,$$

where

$$\begin{aligned}\delta(c_{\bar{v}}) &= \begin{cases}1 & \textit{if } c_{\bar{v}} \equiv 0 \pmod{N},\\ 0 & \textit{otherwise},\end{cases}\\ \zeta_{\bar{d}_v}^{\pm}(k) &= \sum_{\substack{d\in\mathbb{Z}\\ d\equiv d_v}},\\ \sigma_{k-1}^{\bar{v}}(n) &= \sum_{\substack{m\mid n\\ \frac{n}{m}\equiv c_v}} \operatorname{sgn}(m)m^{k-1}\mu_N^{d_v m}.\end{aligned}$$

Remark III.4.3. If Γ is a congruence subgroup such that $\Gamma(N) \leq \Gamma$, let

$$\tilde{E}_{k,\Gamma}^{\bar{v}} := \sum_{\gamma\in\Gamma(N)\backslash\Gamma} \tilde{E}_k^{\bar{v}}|_k\gamma \in M_k(\Gamma).$$

Example III.4.4. For $N = 1$ and $v = (0,1)$, we have

$$\zeta_{\bar{d}_v}^{\pm}(k) = \sum_{d\neq 0}\frac{1}{d^k} = \begin{cases}0 & \text{if } k \text{ is odd},\\ 2\zeta(k) & \text{if } k \text{ is even},\end{cases}$$

and

$$\sigma_{k-1}^{\bar{v}}(n) = \sum_{m\mid n}\operatorname{sgn}(m)m^{k-1} = \begin{cases}0 & \text{if } k \text{ is odd},\\ 2\sigma_{k-1}(n) & \text{if } k \text{ is even}.\end{cases}$$

Also, $\sigma_{k-1}(n) = \sum_{\substack{d\mid n\\ d>0}} d^{k-1}$, so we get $\tilde{E}_k(z) = 0$ if k is odd, and

$$\tilde{E}_k(z) = 2\zeta(k) + 2\Omega_k\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

if k is even.

III.5 2014-03-03

Let

$$E_k(z) = \frac{1}{2\Omega_k}\tilde{E}_k(z) = \begin{cases}\dfrac{\zeta(1-k)}{2} + \displaystyle\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, & \text{k even},\\ 0, & k \text{ odd}.\end{cases}$$

Another useful normalization:

$$G_k(z) = \frac{1}{2}\left(\sum_{m=1}^{\infty}\frac{\mu(m)}{m^k}\right)\tilde{E}_k(z).$$

Fact III.5.1. $\displaystyle\sum_{m=1}^{\infty} \frac{\mu(m)}{m^k} = \frac{1}{\zeta(k)}$, so if k is even, then

$$G_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Define:

$$\Delta(z) := \frac{1}{1728}\left(G_4^3 - G_6^2\right).$$

Then $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.

Consider the j-invariant:

$$j(z) = \frac{G_4^3}{\Delta} = \frac{1}{q} + 744 + 196884q + \cdots \in A_0(X(1)) = \mathfrak{M}(X(1)) \cdot j.$$

Theorem III.5.2. *$j'(z) \in A_2(\mathrm{SL}_2(\mathbb{Z}))$, so there exists $f_0 \neq 0$ in $A_k(\mathrm{SL}_2(\mathbb{Z}))$ for all $k \geq 0$ even (namely, $f_0 = (j')^{k/2}$).*

Proof. For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, consider the map $\mathfrak{h} \xrightarrow{\gamma} \mathfrak{h} \xrightarrow{j} \mathbb{C}$. We have

$$j(z) = (j \circ \gamma)(z) = j'(\gamma z) \cdot \gamma'(z) = j'(\gamma z) \frac{1}{j(\gamma, z)^2} = (j'|_2 \gamma)(z).$$

This is holomorphic on $\mathfrak{h}$ and meromorphic at ∞, so $j' \in A_2(\mathrm{SL}_2(\mathbb{Z}))$. □

III.5.1 G_4 and G_6 generate modular forms

Theorem III.5.3. $M(\mathrm{SL}_2(\mathbb{Z})) = \displaystyle\bigoplus_{k \in \mathbb{Z}} M_k(\mathrm{SL}_2(\mathbb{Z}))$ *is freely generated by G_4 and G_6.*

We will soon show that G_4 and G_6 are algebraically independent. Assuming this, $\langle G_4, G_6 \rangle$ is a freely generated subring of $M(\mathrm{SL}_2(\mathbb{Z}))$. Recall that

$$\dim M_k(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}, \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \pmod{12}. \end{cases}$$

For example:

k	4	6	8	10	12	14	16
dim	1	1	1	1	2	1	2
gen	G_4	G_6	G_4^2	G_4G_6	G_4^3, G_6^2	$G_4^2G_6$	$G_4^4, G_4G_6^2$

Note that $\dim S_{k+12} + 1 = \dim M_{k+12} = \dim M_k + 1$. Consider the map

$$\begin{aligned} \delta : M_k &\xrightarrow{\simeq} S_{k+12} \\ f &\mapsto f \cdot \Delta. \end{aligned}$$

So we want to show there exists a non-cusp form in $\langle G_4, G_6 \rangle$ for all even $k \geq 4$. Indeed:

- If $k \equiv 0, 4, 8 \pmod{12}$, we have $G_4^{k/4}$.
- If $k \equiv 6 \pmod{12}$, we have $G_6^{k/6}$.
- If $k \equiv 2, 10 \pmod{12}$, we have $G_4^{\frac{k-6}{4}} \cdot G_6$.

Lemma III.5.4. *If $f_1, f_2 \in M_k(\Gamma)$ are non-zero and $f_1 \neq \lambda f_2$ for $\lambda \in \mathbb{C}$, then f_1 and f_2 are algebraically independent.*

Proof. If $F(x, y) \in \mathbb{C}[x, y]$ such that $F(f_1, f_2) = 0$, then $F(x, y) = \sum_{d \geq 0} F_d(x, y)$, where F_d is homogeneous of degree d. For each d, there exists $P_d(t)$ such that

$$\frac{F_d(f_1, f_2)}{f_2^d} = P_d(f_1/f_2).$$

Since P_d is a polynomial in one variable, it has finitely many roots. Thus, f_1/f_2 is constant, contrary to what was assumed. $\square$

Thus, G_4^3 and G_6^2 are algebraically independent, and hence so are G_4 and G_6.

III.5.2 Congruences

Observe that

$$E_{12} = \frac{691}{2 \cdot 12 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13} + \dots,$$
$$E_6 = \frac{-1}{2 \cdot 6 \cdot 2 \cdot 3 \cdot 7} + \dots,$$

so

$$\Delta = \frac{65}{65 + 691} E_{12} - \frac{691}{3} E_6^2.$$

We can write $\Delta = \sum_{n=1}^{\infty} \tau(n) q^n$, and

$$\Delta \equiv E_{12} \pmod{691},$$

i.e., if p is prime, then

$$\tau(p) \equiv 1 + p^{11} \pmod{691}.$$

III.6 2014-03-05: Dirichlet characters

III.6.1 Dirichlet characters

Let N be a positive integer. A *Dirichlet character* mod N is a homomorphism $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$.

Given two Dirichlet characters $\chi, \eta : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$, we can form the product

$$(\chi\eta)(a) = \chi(a)\eta(a).$$

There is also an identity character $\mathbb{1}_N = \mathbb{1}$ defined by $\mathbb{1}(a) = 1$, and inverses are given by complex conjugation. Hence, the Dirichlet characters form a group $\widehat{(\mathbb{Z}/N)^\times}$, the dual group of $(\mathbb{Z}/N)^\times$.

Dirichlet characters satisfy the usual orthogonality relations.

Note III.6.1. If $d \mid N$ and χ is a Dirichlet character mod d, then χ induces a Dirichlet character χ_N mod N.

The *conductor* c_χ of χ is the least positive integer such that χ factors through $(\mathbb{Z}/c_\chi)^\times$.

Example III.6.2. Let p be an odd prime. Then

$$a \mapsto \left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a solution,} \\ -1 & \text{otherwise} \end{cases}$$

is a Dirichlet character mod p.

If $c_\chi = N$, then χ is called *primitive*.

Remark III.6.3. By convention, we extend χ to $\mathbb{Z}/N\mathbb{Z}$ by defining $\chi(0) = 0$, and to $\mathbb{Z}$ via $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/N\mathbb{Z}$, so

$$\chi(a) = \chi(a \mod N).$$

Note III.6.4.

- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$.
- $\mathbb{1}_N(a) = \begin{cases} 1, & \gcd(a, N) = 1, \\ 0, & \gcd(a, N) \neq 1. \end{cases}$
- $\chi(0) = \begin{cases} 0, & N > 1, \\ 1, & N = 1. \end{cases}$

III.6.2 Gauss sums

Concretely: let $\mu_N := e^{2\pi i/N}$, a primitive N-th root of 1. Let χ be a Dirichlet character mod N. Then

$$\tau(\chi) := \sum_{a \in \mathbb{Z}/N} \chi(a)\mu_N^a \in \mathbb{C}.$$

Look at

$$\sum_{a \in \mathbb{Z}/N} \chi(a) \left(\mu_N^m\right)^a = \sum_{a \in \mathbb{Z}/N} \chi(a)\psi_m(a).$$

Conceptually: The map

$$\begin{aligned} \psi_m : \mathbb{Z}/N &\to \mathbb{C} \\ a &\mapsto (\mu_N^m)^a \end{aligned}$$

is an *additive character* of $\mathbb{Z}/N$. Note: $\psi_m(a) = \psi_1(ma)$.

Given any additive character $\psi : \mathbb{Z}/N \to \mathbb{C}$ and any $f : \mathbb{Z}/N \to \mathbb{C}$, we define the *Fourier transform*[1]

$$\widehat{f}(m) := \sum_{a \in \mathbb{Z}/N} f(a)\psi_m(-a).$$

Note that $\tau(\chi) = \tau(\chi, \psi_{-1}) = \widehat{\chi}(-1)$.

Facts:

- If χ is primitive mod N, then for all $m \in \mathbb{Z}$,

$$\tau(\chi, \psi_m) = \overline{\chi}(m)\tau(\chi),$$

  and $|\tau(\chi)| = \sqrt{N}$.

- Consider χ mod N of conductor N_0 coming from χ_0 mod N_0. Then

$$\tau(\chi) = \mu(N/N_0)\chi_0(N/N_0)\tau(\chi_0).$$

- $\tau(\overline{\chi}) = \chi(-1)\tau(\chi)$.

- If χ and χ' are characters mod N and N', respectively, and $\gcd(N, N') = 1$, then

$$\tau(\chi\chi') = \chi(N')\chi'(N)\tau(\chi)\tau(\chi').$$

  More generally, if $(N, N') \geq 1$, then

$$\tau(\chi, \chi') = \frac{\tau(\chi)\tau(\chi')}{J(\chi, \chi')},$$

  where $J(\chi, \chi')$ is the *Jacobi sum*.

- $\chi(-1) = \pm 1$. If $\chi(-1) = 1$ (resp. $\chi(-1) = -1$), then χ is *even* (resp. *odd*). If $N \geq 3$, then exactly half the Dirichlet characters are even and half are odd.

III.6.3 L-functions

Definition III.6.5. The *Dirichlet L-function* of a character χ is the function defined for $\mathrm{Re}(s) > 1$ by

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Since $|\chi(n)| = 1$, this converges absolutely for $\mathrm{Re}(s) > 1$.

Note III.6.6. If $\chi = \mathbb{1}_1$, then $L(s, \chi) = \zeta(s)$.

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[1] The name makes sense: If $f : \mathbb{R} \to \mathbb{C}$, the additive characters of $\mathbb{R}$ are $\psi_y(x) = e^{2\pi i y x}$, and

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x)\psi_y(-x)\, dx$$

is the usual Fourier transform.

Theorem III.6.7 (Euler). *Let $F(n)$ be a multiplicative function $\mathbb{Z}_{\geq 1} \to \mathbb{C}$ (i.e., $F(ab) = F(a)F(b)$ if $(a,b) = 1$) such that $\sum_{n=1}^{\infty} F(n)$ is absolutely convergent. Then*

$$\sum_{n=1}^{\infty} F(n) = \prod_{p \text{ prime}} \left(\sum_{m=0}^{\infty} F(p^m) \right),$$

and the product is absolutely convergent. If F is completely *multiplicative, then $F(p^m) = F(p)^m$, so*

$$\sum_{n=1}^{\infty} F(n) = \prod_{p \text{ prime}} \frac{1}{1 - F(p)}.$$

Applying this to $F(n) = \chi(n) n^{-s}$, we obtain

$$L(s, \chi) = \prod_p \frac{1}{1 - \chi(p) p^{-s}}.$$

III.7 2014-03-07: The functional equation

Today, we will prove the functional equation for $L(s, \chi)$.

Here's the idea for proving the functional equation: use "modular forms" of half-integer weight (theta series) and harmonic analysis for $\mathbb{R}_{>0}^{\times}$.

As motivation, consider the function Γ defined for $\mathrm{Re}(s) > 0$ by

$$\Gamma(s) = \int_0^{\infty} e^{-y} y^s \frac{dy}{y}.$$

This generalizes the factorial function, in the sense that $\Gamma(s+1) = s\Gamma(s)$ and $\Gamma(1) = 1$, so $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{Z}_{\geq 1}$.

Using the change of variables $y \mapsto \pi n^2 y$, we have

$$\int_0^{\infty} e^{-\pi n^2 y} y^2 \frac{dy}{y} = \pi^{-s} \Gamma(s) \frac{1}{n^{2s}}.$$

Summing over all $n \geq 1$,

$$\pi^{-s} \Gamma(s) \zeta(2s) = \int_0^{\infty} \left(\sum_{n=1}^{\infty} e^{-\pi n^2 y} \right) y^s \frac{dy}{y}.$$

Recall the function

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z} = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi n^2 z}.$$

The above function θ satisfies a functional equation coming from the fact that θ is a modular form. Define

$$\Lambda(s) := \pi^{-s/2} \Gamma\left(\tfrac{s}{2}\right) \zeta(s).$$

We will use "modularity" of θ to show that $\Lambda(s) = \Lambda(1-s)$.

III.7.1 The Mellin transform

Let us compare the topological groups $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}^{\times}, \cdot)$:

	$(\mathbb{R}, +)$	$(\mathbb{R}_{>0}^{\times}, \cdot)$
	locally compact	locally compact
Haar measure:	dx	$d^{\times}y = \frac{dy}{y}$
Characters:	$\psi_m(x) = e^{2\pi imx}$, $m \in \mathbb{R}$	$\psi_s(y) = y^s$, $s \in \mathbb{C}$
"Fourier" transform:	$\hat{f}(m) = \int_{\mathbb{R}} f(x)e^{-2\pi imx}dx$	Mellin transform: $\int_0^\infty F(y)y^s\frac{dy}{y}$

Definition III.7.1. The *Mellin transform* of a continuous function $F : \mathbb{R}_{>0}^{\times} \to \mathbb{C}$ is

$$\mathcal{M}(F)(s) := \int_0^\infty \big(F(y) - F(\infty)\big)y^s\frac{dy}{y}.$$

Example III.7.2. $\Gamma(s) = \mathcal{M}(e^{-y})(s)$.

Theorem III.7.3. *Let $F, G : \mathbb{R}_{>0}^{\times} \to \mathbb{C}$ be continuous functions such that*

$$F(y) = a_F + \mathrm{O}(e^{-cy^\alpha}),$$
$$G(y) = a_G + \mathrm{O}(e^{-cy^\alpha})$$

as $y \to \infty$. Suppose there exist $k \in \mathbb{R}_{>0}$ and $C \in \mathbb{C}^\times$ such that

$$F\big(\tfrac{1}{y}\big) = Cy^kG(y).$$

Then:

(i) $\mathcal{M}(F)(s)$ and $\mathcal{M}(G)(s)$ converge absolutely and uniformly on compact subsets of $\mathrm{Re}(s) >$ *k, and they have an analytic continuation to $\mathbb{C} \setminus \{0, k\}$.*

(ii) $\mathcal{M}(F)(s)$ and $\mathcal{M}(G)(s)$ have simple poles at 0 *and k with residues as follows:*

	0	k
$\mathcal{M}(F)$	$-a_F$	Ca_G
$\mathcal{M}(G)$	$-a_G$	$C^{-1}a_F$

(iii) $\mathcal{M}(F)(s) = C\mathcal{M}(G)(k-s)$.

Proof. As s varies over compact subsets of $\mathbb{C}$, for $y \geq 1$, $e^{-cy^\alpha}y^{\mathrm{Re}(s)+1}$ is $\mathrm{O}(1)$-independent of $\mathrm{Re}(s)$. Thus, for some constant β,

$$\int_1^\infty \big|\big(F(y) - a_F\big)y^{s-1}\big|\, dy \leq \int_1^\infty \beta e^{-cy^\alpha}y^{\mathrm{Re}(s)+1}\frac{dy}{y^2}.$$

So for $\mathrm{Re}(s) > k$, $\mathcal{M}(F)(s) = \int_1^\infty + \int_0^1$, and the $\int_1^\infty$ term converges absolutely and uniformly on compact subsets. Now we use the functional equation to handle the $\int_0^1$ term. Applying the change of variables $y \mapsto \frac{1}{y} = u$, $du = -\frac{1}{y^2}dy$,

$$\begin{aligned}\int_1^\infty F\big(\tfrac{1}{y}\big)y^{-s}\frac{dy}{y} - a_F\frac{y^s}{s}\bigg|_0^1 &= -\frac{a_F}{s} + C\int_1^\infty G(y)y^{k-s}\frac{dy}{y}\\ &= -\frac{a_F}{s} - \frac{Ca_G}{k-s} + C\int_1^\infty \big(G(y) - a_G\big)y^{k-s}\frac{dy}{y}.\end{aligned}$$

Note that $\int_1^\infty (G(y)-a_G)y^{k-s}\frac{dy}{y}$ converges absolutely and uniformly. The other two terms can give poles at 0 and k. In particular:

$$\mathcal{M}(F)(s) = \frac{-a_F}{s} + \frac{Ca_G}{s-k} + \mathcal{F}(s),$$

where

$$\mathcal{F}(s) = \int_1^\infty \big(F(y)-a_F\big)y^s\frac{dy}{y} + C\int_1^\infty \big(G(y)-a_G\big)y^{k-s}\frac{dy}{y}$$

is a holomorphic function that converges absolutely and uniformly. Applying the same argument for G, we have $G\big(\frac{1}{y}\big) = C^{-1}y^kF(y)$, so

$$\mathcal{M}(G)(s) = \frac{-a_G}{s} + \frac{C^{-1}a_F}{s-k} + \mathcal{G}(s),$$

where $\mathcal{G}(s)$ is a holomorphic function that converges absolutely and uniformly. Clearly,

$$\mathcal{F}(s) = C\mathcal{G}(k-s),$$

from which the functional equation follows immediately. □

III.7.2 Proof of the functional equation

Let χ be a primitive Dirichlet character mod N. We want to find F_χ and G_χ such that

$$\mathcal{M}(F_\chi)(s) = \Lambda(s,\chi)N^{\frac{s+\delta}{2}} =: \tilde{\Lambda}(s,\chi),$$
$$\mathcal{M}(G_\chi)(s) = \Lambda(s,\overline{\chi})N^{\frac{s+\delta}{2}} =: \tilde{\Lambda}(s,\overline{\chi}).$$

Here, we will have $C = \frac{\tau(\chi)}{i^\delta\sqrt{N}}$.

Observe that

$$\Lambda(s,\chi) = \Lambda\left(\frac{s+\delta}{2}\right)L(s,\chi)\pi^{-\frac{(s+\delta)}{2}}\chi(-1)$$

and

$$\Gamma\left(\frac{s+\delta}{2}\right) = \int_0^\infty e^{-y}y^{\frac{s+\delta}{2}}\frac{dy}{y}.$$

Taking the change of variables $y \mapsto \frac{\pi n^2}{N}y$, we obtain

$$\left(\frac{N}{\pi}\right)^{\frac{s+\delta}{2}}\Gamma\left(\frac{s+\delta}{2}\right)\frac{1}{n^s} = \int_0^\infty n^\delta e^{-\pi n^2 y/N}y^{\frac{s+\delta}{2}}\frac{dy}{y},$$

so

$$\begin{aligned}\Lambda(\tilde{s},\chi) &= \left(\frac{N}{\pi}\right)^{\frac{s+\delta}{2}}\Gamma\left(\frac{s+\delta}{2}\right)L(s,\chi)\\ &= \int_0^\infty \left(\sum_{n=1}^\infty \chi(n)n^\delta e^{-\pi n^s y/N}\right)y^{\frac{s+\delta}{2}}\frac{dy}{y}\\ &= \int_0^\infty \frac{1}{2}\big(\theta_\chi(iy)-\chi(0)\big)y^{\frac{s+\delta}{2}}\frac{dy}{y},\end{aligned}$$

where

$$\theta_\chi(z) := \sum_{n\in\mathbb{Z}} \chi(n) n^\delta e^{i\pi n^2 z/N}.$$

Theorem III.7.4. *The function θ_χ satisfies the functional equation*

$$\theta_\chi\left(\tfrac{-1}{z}\right) = \frac{\tau(\chi)}{i^\delta\sqrt{N}} \left(\frac{z}{i}\right)^{\delta+\frac{1}{2}} \theta_{\overline{\chi}}(z).$$

Assuming the above theorem for now, let $\alpha_\chi := \left(\frac{\pi}{N}\right)^{\delta/2}$. Then

$$F_\chi(y) = \theta_\chi(iy)\cdot\frac{\alpha_\chi}{2},$$
$$G_\chi(y) = \theta_{\overline{\chi}}(iy)\cdot\frac{\alpha_\chi}{2}.$$

Thus,

$$F_\chi\left(\frac{1}{y}\right) = \frac{\alpha_\chi}{2}\theta_\chi\left(\frac{i}{y}\right) = \frac{\alpha_\chi}{2}\theta_\chi\left(\frac{-1}{cy}\right) = w(\chi) y^{\delta+\frac{1}{2}}\theta_{\overline{\chi}}(iy)\frac{\alpha_\chi}{2},$$

where $w(\chi) := \frac{\tau(\chi)}{i^\delta\sqrt{N}}$ is the *root number*. Note that $|w(\chi)| = 1$.

By the previous theorem, assuming $\chi \neq 1$,

$$\tilde{\Lambda}(s,\chi) = \mathcal{M}(F_\chi)\left(\frac{s+\delta}{2}\right) = w(\chi)\mathcal{M}(G_\chi)\left(\delta+\frac{1}{2}-\frac{s+\delta}{2}\right) = w(\chi)\tilde{\Lambda}(1-s,\overline{\chi}).$$

III.8 2013-03-10: Eisenstein series of a character

Today, we'll talk more about the Eisenstein series $\mathcal{E}_k(\Gamma_0(N))$ and $\mathcal{E}_k(N,\chi)$, where χ is a Dirichlet character mod N.

Recall that

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi\in\widehat{(\mathbb{Z}/N)^\times}} M_k(N,\chi),$$

with an action of $\Gamma_0(N)/\Gamma_1(N)$.

Suppose $k \geq 3$. Recall that for $\bar{v} \in (\mathbb{Z}/N)^2$ of order N,

$$\tilde{E}_k^{\bar{v}}(\tau) = \delta(\bar{c}_v)\zeta^{\pm}_{\bar{d}_v}(k) + \frac{\Omega_k}{N^k}\sum_{n=1}^\infty \sigma^{\bar{v}}_{k-1}(n) q_N^n,$$

where

$$\sigma^{\bar{v}}_{k-1} = \sum_{\substack{m\mid n\\ \frac{n}{m}\equiv c_v}} \operatorname{sgn}(m) m^{k-1}\mu_N^{d_v m}.$$

Example III.8.1. Consider $\displaystyle\sum_{\substack{d\in(\mathbb{Z}/N)^\times\\(d,N)=1}} \tilde{E}_k^{\overline{(0,d)}}(\tau)$. Note that for $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma\end{pmatrix} \in \Gamma_0(N)$,

$$\overline{(0,d)\gamma} = \overline{(0,dd_\gamma)}.$$

Notation:

- φ is a Dirichlet character mod M_φ.
- ψ is a Dirichlet character mod M_ψ.
- $\varphi\psi$ is a Dirichlet character mod $M_\varphi M_\psi$.

Let φ be *primitive*, and ψ such that $(\varphi\psi)(-1) = (-1)^k$.
One can check that, for $\gamma \in \Gamma_0(N)$ (hence $c_\gamma \equiv 0$ and $a_\gamma d_\gamma \equiv 1 \bmod N$),

$$\overline{(cM_\varphi, d + eM_\varphi)\gamma} = \overline{(c'M_\varphi, d' + e'M_\varphi)},$$

where $c' = ca_\gamma$, $d' = dd_\gamma$, and $e' = (e + ca_\gamma b_\gamma)d_\gamma$. Moreover,

$$\psi(c')\overline{\varphi}(d') = \psi(c)\psi(a_\gamma)\overline{\varphi}(d)\overline{\varphi}(d_\gamma) = (\psi\varphi)(d_\gamma)^{-1}\psi(c)\overline{\varphi}(d).$$

This suggests that we take the following "symmetrized twisted sum":

Definition III.8.2. $\tilde{E}_k^{\psi,\varphi}(\tau) := \displaystyle\sum_{c \pmod{M_\psi}} \sum_{d \pmod{M_\varphi}} \sum_{e \pmod{M_\psi}} \psi(c)\overline{\varphi}(d)\tilde{E}_k^{\overline{(cM_\varphi, d+eM_\varphi)}}(\tau).$

If $\gamma \in \Gamma_0(N)$, then

$$\big(\tilde{E}_k^{\psi,\varphi}|_k\gamma\big)(\tau) = (\psi\varphi)(d_\gamma)\tilde{E}_k^{\psi,\varphi}(\tau).$$

Hence, $\tilde{E}_k^{\psi,\varphi} \in M_k(N, \psi\varphi)$.

III.8.1 Fourier expansions

We'll split this up into the constant and non-constant parts.
The non-constant part of $\tilde{E}_k^{\bar{v}}$ can be rewritten as

$$\frac{\Omega_k}{N^k} \sum_{\substack{mn>0 \\ n\equiv cM_\varphi}} \mathrm{sgn}(m)m^{k-1}\mu_N^{d_v m} q_N^{mn}.$$

For $\tilde{E}_k^{\psi,\varphi}$, we get

$$\frac{\Omega_k}{N^k} \sum_c \sum_d \sum_e \psi(c)\overline{\varphi}(d) \sum_{\substack{mn>0 \\ n\equiv cM_\varphi \ (N)}} \mathrm{sgn}(m)m^{k-1}\mu_N^{(d+eM_\varphi)m} q_N^{mn}$$
$$= \frac{\Omega_k}{N^k} \sum_c \sum_d \psi(c)\overline{\varphi}(d) \sum_{\substack{mn>0 \\ n\equiv cM_\varphi \ (N)}} \mathrm{sgn}(m)m^{k-1}\mu_N^{dm} \left(\sum_e \mu_{M_\psi}^{em}\right) q_N^{mn}.$$

Note that $\sum_e \mu_{M_\psi}^{em} = M_\psi$ if $M_\psi \mid m$, and 0 otherwise. Applying the change of variables $m \mapsto mM_\psi$ and $n \mapsto nM_\varphi$,

$$\begin{aligned}
&= \frac{\Omega_k}{M_\varphi^k} \sum_c \sum_d \psi(c)\overline{\varphi}(d) \sum_{\substack{mn>0 \\ n\equiv c \pmod{M_\psi}}} \operatorname{sgn}(m) m^{k-1} \mu_{M_\varphi}^{dm} q^{mn} \\
&= \frac{\Omega_k}{M_\varphi^k} \sum_c \psi(c) \sum_{\substack{mn>0 \\ n\equiv c}} \operatorname{sgn}(m) m^{k-1} \underbrace{\left(\sum_d \overline{\varphi}(d) \mu_{M_\varphi}^{dm} \right)}_{\varphi(m)\psi(\overline{\varphi})} \\
&= \frac{\Omega_k}{M_\varphi^k} \tau(\overline{\varphi}) \sum_{mn>0} \psi(n) \operatorname{sgn}(m) \varphi(m) m^{k-1} q^{mn} \\
&= 2\frac{\Omega_k}{M_\varphi^k} \tau(\overline{\varphi}) \sum_{m,n>0} \psi(n)\varphi(m) m^{k-1} q^{mn} \\
&= \frac{2\Omega_k}{M_\varphi^k} \tau(\overline{\varphi}) \sum_{n=1}^{\infty} \left(\sum_{\substack{m|n \\ m>0}} \psi\left(\frac{n}{m}\right) \varphi(m) m^{k-1} \right) q^n \\
&= \frac{2\Omega_k}{M_\varphi^k} \tau(\overline{\varphi}) \sum_{n=1}^{\infty} \sigma_{k-1}^{\psi,\varphi}(n) q^n,
\end{aligned}$$

where

$$\sigma_{k-1}^{\psi,\varphi}(n) := \psi * (\varphi m^{k-1}) = \sum_{\substack{m|n \\ m>0}} \psi\left(\frac{n}{m}\right) \varphi(m) m^{k-1}.$$

The constant term is

$$\sum_c \sum_d \sum_e \psi(c)\overline{\varphi}(d)\delta(\overline{cM_\varphi})\zeta^{\pm}_{\overline{d+eM_\varphi}}(k) = \psi(0) \sum_d \sum_e \overline{\varphi}(d)\zeta^{\pm}_{\overline{d+eM_\varphi}}(k).$$

Recall that $\psi(0) \neq 0$ iff $\psi = \mathbb{1}$. Take $\psi(0) \neq 0$, so $\psi(0) = 1$, $M_\psi = 1$, and $M_\varphi = N$, whence

$$\begin{aligned}
&= \psi(0) \sum_{d \pmod N} \overline{\varphi}(d) \sum_{\substack{m\equiv d \pmod N \\ m\neq 0}} \frac{1}{m^k} \\
&= 2\psi(0) L(k, \overline{\varphi})
\end{aligned}$$

by the functional equation for $L(s,\chi)$. Let

$$E_k^{\psi,\varphi}(\tau) = \frac{M_\varphi^k}{2\Omega_k \tau(\overline{\varphi})} \tilde{E}_k^{\psi,\varphi}(\tau).$$

Then:

Theorem III.8.3. $E_k^{\psi,\varphi}(\tau) = \psi(0)\dfrac{L(1-k,\varphi)}{2} + \displaystyle\sum_{n=1}^{\infty} \sigma_{k-1}^{\psi,\varphi}(n) q^n \in \mathcal{E}_k(N, \psi\varphi)$.

Definition III.8.4. Let $\mathscr{D}_{N,k}$ be the set of triples (ψ, φ, t) such that ψ, φ are primitive mod M_ψ, M_φ, respectively, $(\psi\varphi)(-1) = (-1)^k$, and $t \in \mathbb{Z}_{\geq 1}$ such that $tM_\psi M_\varphi \mid N$.

Let $e_k^{\psi,\varphi,t}(\tau) := E_k^{\psi,\varphi}(t\tau) \in M_k(\Gamma_1(tM_\psi M_\varphi)) \subseteq M_k(\Gamma_1(N))$.

Theorem III.8.5. *Let χ be a character mod N. Then*

$$\left\{E_k^{\psi,\varphi,t} : (\psi, \varphi, \tau) \in \mathscr{D}_{N,k} \text{ and } \psi\varphi = \chi\right\}$$

is a basis of $\mathcal{E}_k(N, \chi)$.

III.9 2014-03-24: Families of Eisenstein series [incomplete]

III.9.1 Weight 2 Eisenstein series

Let

$$\tilde{E}_2(\tau) := \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} \frac{1}{(c\tau+d)^2}.$$

This converges *conditionally.* Then

$$(\tilde{E}_2|_2\gamma)(\tau) = \tilde{E}_2(\tau) - \frac{2\pi i c}{c\tau+d}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, but $\tilde{E}_2(\tau) - \frac{\pi}{y}$ (where $\tau = x + iy$) is weight 2 invariant under $\mathrm{SL}_2(\mathbb{Z})$, but *not* holomorphic.

Note III.9.1. $M_2(\mathrm{SL}_2(\mathbb{Z})) = 0$.

Recall: $\dim \mathcal{E}_2(\Gamma) = \nu_\infty - 1$.

For $\nu_\infty \geq 2$, here's the idea: Say you find $E^{(1)}$ and $E^{(2)}$ weight 2 invariant, but not holomorphic. Then we hope that $E^{(1)} - E^{(2)}$ is holomorphic.

Fix $\bar{v} \in (\mathbb{Z}/N)^2$ of order N. Let

$$\tilde{E}_2^{\bar{v}}(\tau) := \delta(\tau_v)\zeta_{\bar{d}_v}^{\pm}(2) + \frac{\Omega_2}{N^2}\sum_{n=1}^{\infty} \sigma_1^{\bar{v}}(n) q_N^n,$$
$$g_2^{\nabla}(\tau) := \tilde{E}_2^{\bar{v}}(\tau) - \frac{\pi}{N^2 y}$$

weight 2 invariant under $\Gamma(N)$.

Theorem III.9.2. $\mathcal{E}_2(\Gamma(N)) = \left\{\sum_{\bar{v}} a_{\bar{v}}\tilde{E}_2^{\bar{v}} : \sum_{\bar{v}} a_{\bar{v}} = 0, a_{\bar{v}} \in \mathbb{C}\right\}$.

. . .

III.9.2 Families of Eisenstein series

(Miyake §7.2)

Hecke introduced a parameter $s \in \mathbb{C}$. Let $k \in \mathbb{Z}$, and let ψ, φ be primitive characters mod M_ψ, M_φ, respectively. Then

$$\tilde{E}_k(\tau, s; \psi, \varphi) := \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{\psi(m)\bar{\varphi}(n)}{(m\tau+n)^k \left|m\tau+n\right|^{2s}}$$

converges absolutely and uniformly on $\mathrm{Re}(k+2s) \geq 2+\varepsilon$ for all $\varepsilon > 0$, and so it's holomorphic. Let

$$\Gamma_0(M_\psi, M_\varphi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : b \equiv 0 \pmod{M_\varphi},\ c \equiv 0 \pmod{M_\psi} \right\}.$$

Then for all $\gamma \in \Gamma_0(M_\psi, M_\varphi)$,

$$\tilde{E}_k(\gamma \cdot z, s; \psi, \varphi) = (\psi\varphi)(d) j(cz+d)^d \left|cz+d\right|^{2s} \tilde{E}_k(z, s; \psi, \varphi).$$

So if $k \geq 3$, we can plug in $s = 0$ to obtain

$$\tilde{E}_k(z, 0; \psi\varphi)|_k\gamma = (\psi\varphi)(d)\tilde{E}_k(z, 0; , \psi, \varphi)$$

for all $\gamma \in \Gamma_0(M_\psi, M_\varphi)$. In fact, for $k \geq 3$,

$$\tilde{E}_k(M_\varphi z, 0; \psi, \varphi) = \tilde{E}_k^{\psi,\varphi}(z).$$

The idea is to analytically continued $\tilde{E}_k(z, s; \psi, \varphi)$ to $\mathrm{Re}(s) \geq -\varepsilon$ for some $\varepsilon > 0$ and $k = 1, 2$, and plug in $s = 0$.

Chapter IV

Hecke operators

IV.1 2014-03-26: Hecke operators

IV.1.1 Motivation

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Think of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$ as the space of 2-dimensional lattices (up to homothety, i.e., scaling and rotating). Let

$$\mathscr{L} := \{\text{full rank lattices } \Lambda \subseteq \mathbb{C}\},$$
$$\mathcal{B} := \left\{\text{pairs } (\omega_1, \omega_2) \in \mathbb{C}^\times \text{ such that } \mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0\right\}.$$

There is a natural surjection

$$\Phi : \mathcal{B} \twoheadrightarrow \mathscr{L},$$
$$(\omega_1, \omega_2) \mapsto \Lambda(\omega_1, \omega_2) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2.$$

Two pairs $\omega, \omega' \in \mathcal{B}$ give the same lattice iff there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\omega = \omega'$. Thus, $\mathscr{L} \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{B}$.

There is also a map

$$\Psi : \mathcal{B} \twoheadrightarrow \mathfrak{h},$$
$$(\omega_1, \omega_2) \mapsto \frac{\omega_1}{\omega_2}.$$

This commutes with the action of $\mathrm{SL}_2(\mathbb{Z})$.

Note that $\mathbb{C}^\times$ acts on $\mathscr{L}$ and $\mathcal{B}$ by scalar multiplication, and Ψ induces an $\mathrm{SL}_2(\mathbb{Z})$-equivariant isomorphism

$$\Psi : \mathcal{B}/\mathbb{C}^\times \xrightarrow{\simeq} \mathfrak{h}.$$

So we have an isomorphism

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{B}/\mathbb{C}^\times \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$$

between the space of lattices modulo homothety and the modular curve.

Remark IV.1.1. An elliptic curve $E/\mathbb{C}$ is a complex torus, i.e., there is a lattice $\Lambda \subseteq \mathbb{C}$ such that $E \cong \mathbb{C}/\Lambda$. Moreover, $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ as elliptic curves iff $\Lambda \sim \Lambda'$ (homothety). So we can think of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$ as the moduli space of complex elliptic curves.

IV.1.2 Connection to modular forms

Since $\mathbb{C}^\times$ acts on $\mathscr{L}$ and $\mathbb{C}$, it acts on the set of maps $\mathrm{Hom}_{\mathbf{Set}}(\mathscr{L}, \mathbb{C})$.

The characters of $\mathbb{C}^\times$ are of the form $z \mapsto z^s$. For $k \in \mathbb{Z}$, say F is of *weight* k if for all $\lambda \in \mathbb{C}^\times$ and $\Lambda \in \mathscr{L}$,

$$F(\lambda\Lambda) = \lambda^{-k} F(\Lambda).$$

Denote $F(\omega_1, \omega_2) := F(\Lambda(\omega_1, \omega_2))$. Weight k means:

$$F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-k} F(\omega_1, \omega_2).$$

We can dehomogenize in F: let

$$\tilde{F}(\omega_2, \omega_2) := \omega_2^k F(\omega_1, \omega_2).$$

If F has weight k, then $\tilde{F}(\omega_1, \omega_2) = F\left(\frac{\omega_1}{\omega_2}, 1\right)$. Let $z := \frac{\omega_1}{\omega_2}$. There is a function $f : \mathfrak{h} \to \mathbb{C}$ such that $\tilde{F}(\omega_1, \omega_2) = f(z)$.

For F to be $\mathrm{SL}_2(\mathbb{Z})$-invariant means that

$$(c\omega_1 + d\omega_2)^k F(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2) = (c\omega_1 + d\omega_2)^k F(\omega_1, \omega_2) = \left(c\frac{\omega_1}{\omega_2} + d\right)^k \omega_2^k F(\omega_1, \omega_2),$$

i.e., $f|_k\gamma = f$.

IV.1.3 Hecke operators

Let $\mathscr{D}$ be the free abelian group on $\mathscr{L}$. For $n \in \mathbb{Z}_{\geq 1}$, we have $\mathbb{Z}$-linear operators

$$\begin{aligned} T(n) : \mathscr{D} &\to \mathscr{D} \\ [\Lambda] &\mapsto \sum_{\substack{\Lambda' \subseteq \Lambda \\ \text{index } n}} [\Lambda'] \end{aligned}$$

and

$$\begin{aligned} S(n) : \mathscr{D} &\to \mathscr{D} \\ [\Lambda] &\mapsto [n\Lambda]. \end{aligned}$$

Now any map $F : \mathscr{L} \to \mathbb{C}$ yields $F : \mathscr{D} \to \mathbb{C}$, and for any linear $T : \mathscr{D} \to \mathscr{D}$, we can define

$$(T \cdot F)([\Lambda]) = F(T \cdot [\Lambda]).$$

So we can define $T(n)$ on $f : \mathfrak{h} \to \mathbb{C}$ by

$$(T(n) \cdot f)(z) = \text{function on } \mathfrak{h} \text{ corresponding to } n^{k-1} T(n) \cdot F.$$

This is a Hecke operator.

Let's rephrase this definition in terms of group theory. For $n \geq 1$, let

$$M_2(n) := \left\{\alpha \in M_2(\mathbb{Z}) \;\middle|\; \det(\alpha) = n\right\}.$$

Let $\alpha \in M_2(n)$ and $\Lambda \in \mathscr{L}$. Then $\Lambda' := \alpha\Lambda \subseteq \Lambda$ has index n. Conversely, if $\Lambda' \subseteq \Lambda$ has index n, then there exists $\alpha \in M_2(n)$ such that $\Lambda' = \alpha\Lambda$. Moreover, $\alpha\Lambda = \beta\Lambda$ iff there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\beta = \gamma\alpha$.

Lemma IV.1.2. $M_2(n) = \bigsqcup_{\substack{a \geq 1 \\ 0 \leq b < d \\ ad=n}} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$

So sublattice of $\Lambda(\omega_1, \omega_2)$ of index n are exactly $\Lambda(a\omega_1 + b\omega_2, d\omega_2)$, where a, b, d are as above.

Fact IV.1.3 (Structure theorem of finitely-generated abelian groups). Let $\alpha \in M_2(n)$ and $\Lambda' = \alpha\Lambda$. We can choose bases ω and ω' of Λ and Λ' such that $\omega_1' = a\omega_1$ and $\omega_2' = d\omega_2$, where $a \geq 1$, $a \mid d$, and $ad = n$. That is,

$$M_2(n) = \bigsqcup_{\substack{a \geq 1 \\ a \mid d \\ ad=n}} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mathrm{SL}_2(\mathbb{Z}).$$

Remark IV.1.4. Next time, we'll discuss "abstract Hecke algebras", first introduced by Shimura. The idea is, given $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, and $T(n)$ acting on $M_k(\Gamma)$, we need to think of $T(n)$ as a formal sum over double cosets.

IV.2 2014-03-28: Abstract Hecke algebras

Let G be a group (e.g., $\mathrm{GL}_2^+(\mathbb{R})$) and $\Gamma, \Gamma' \leq G$ subgroup (e.g., congruence subgroups).

Definition IV.2.1. We say that Γ and Γ' are *commensurable*, denoted $\Gamma \approx \Gamma'$, if $[\Gamma : \Gamma \cap \Gamma']$ and $[\Gamma' : \Gamma \cap \Gamma']$ are both finite. The *commensurator* of Γ is G is

$$\tilde{\Gamma} := \left\{ g \in G \;\middle|\; g\Gamma g^{-1} \approx \Gamma \right\} \leq G.$$

(If $\Gamma \leq \mathrm{GL}_2^+(\mathbb{R})$ is a congruence subgroup, then $\tilde{\Gamma} = \mathbb{R}^\times \cdot \mathrm{GL}_2^+(\mathbb{Q})$.)

Lemma IV.2.2. *Suppose* $\Gamma \approx \Gamma'$. *Then:*

(i) $\tilde{\Gamma} = \tilde{\Gamma}'$.

(ii) For all $\alpha \in \tilde{\Gamma}$,

$$\Gamma\alpha\Gamma' = \bigsqcup_{i=1}^{d} \Gamma\alpha\gamma_i = \bigsqcup_{j=1}^{e} \delta_j\alpha\Gamma',$$

where the γ_i *and* δ_j *are coset representatives of* $(\Gamma' \cap \alpha^{-1}\Gamma\alpha)\backslash\Gamma'$ *and* $\Gamma/(\Gamma \cap \alpha\Gamma'\alpha^{-1})$, *respectively.*

Definition IV.2.3 (Abstract Hecke algebras)**.** Let $\Gamma \approx \Gamma'$, and let $\Delta \subseteq \tilde{\Gamma}$ be a submonoid such that $\Delta \supseteq \Gamma, \Gamma'$. Let R be a commutative ring. Then we define $\mathcal{H}_R(\Gamma, \Gamma'; \Delta)$ to be the free R-module generated by double cosets $\Gamma\alpha\Gamma'$ for $\alpha \in \Delta$. (We also write $\mathcal{H}_R(\Gamma; \Delta)$ when $\Gamma = \Gamma'$, and we drop the subscript when $R = \mathbb{Z}$.)

Let $\Gamma \approx \Gamma' \approx \Gamma''$, and let $\Delta \supseteq \Gamma''$. Define a multiplication map

$$\mathcal{H}_R(\Gamma, \Gamma'; \Delta) \times \mathcal{H}_R(\Gamma', \Gamma''; \Delta) \to \mathcal{H}_R(\Gamma, \Gamma''; \Delta)$$

as follows: for $\alpha, \beta \in \Delta$, write

$$\Gamma\alpha\Gamma' = \bigsqcup_i \Gamma\alpha_i,$$
$$\Gamma'\beta\Gamma'' = \bigsqcup_j \Gamma'\beta_j.$$

Then

$$\Gamma\alpha\Gamma'\beta\Gamma'' = \bigcup_j \Gamma\alpha\Gamma'\beta_j = \bigcup_{i,j} \Gamma\alpha_i\beta_j = \bigsqcup_k \Gamma\gamma_k\Gamma'',$$

so we define

$$(\Gamma\alpha\Gamma') \cdot (\Gamma'\beta\Gamma'') \stackrel{\text{def}}{=} \sum_{\Gamma\gamma\Gamma'' \subseteq \Gamma\alpha\Gamma'\beta\Gamma''} c_\gamma \Gamma\gamma\Gamma'',$$

where $c_\gamma := \#\{(i,j) : \Gamma\alpha_i\beta_j = \Gamma\gamma\}$. This is well-defined and associative.

Lemma IV.2.4. *Let $\alpha \in \tilde{\Gamma}$. If $\#(\Gamma\backslash\Gamma\alpha\Gamma) = \#(\Gamma\alpha\Gamma/\Gamma) =: d$, then there exist α_i such that*

$$\Gamma\alpha\Gamma = \bigsqcup_{i=1}^{d} \Gamma\alpha_i = \bigsqcup_{i=1}^{d} \alpha_i\Gamma.$$

Proof. Choose α_i, β_j such that $\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i = \bigsqcup_j \beta_j\Gamma$ and $\Gamma\alpha_i \cap \beta_j\Gamma \neq \varnothing$ for all i, j. Let $\delta_i \in \Gamma\alpha_i \cap \beta_i\Gamma$. Then $\Gamma\alpha_i = \Gamma\delta_i$ and $\beta_i\Gamma = \delta_i\Gamma$. □

Remark IV.2.5. The multiplication defined above makes $\mathcal{H}_R(\Gamma; \Delta)$ into a unital associative algebra.

Theorem IV.2.6. *Suppose there is an involution $c : \Delta \to \Delta$ such that:*

- $(\alpha\beta)^c = \beta^c\alpha^c$
- $(a^c)^c = \alpha$
- $\Gamma^c = \Gamma$
- $\Gamma\alpha^c\Gamma = \Gamma\alpha\Gamma$ *for all* $\alpha \in \Delta$

Then for all $\alpha \in \Delta$, $\Gamma\backslash\Gamma\alpha\Gamma$ and $\Gamma\alpha\Gamma/\Gamma$ have a common set of representatives, and $\mathcal{H}_R(\Gamma; \Delta)$ is commutative.

Let M be an R-linear right Δ-module. Then M^Γ has a natural $\mathcal{H}_R(\Gamma, \Gamma'; \Delta)$ "action". Write $\Gamma\alpha\Gamma' = \bigsqcup_{i=1}^d \Gamma\alpha_i$, and define

$$m|\Gamma\alpha\Gamma' \stackrel{\text{def}}{=} \sum_{i=1}^{d} m|\alpha_i \in M^{\Gamma'}.$$

IV.2.1 Concrete example

Now let $\Gamma = \Gamma_0(N)$, and consider

$$\Delta_0(N) := \left\{\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \pmod{N},\ \gcd(a,N)=1,\ \det A > 0\right\},$$
$$\Delta_0^*(N) := \left\{\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \pmod{N},\ \gcd(d,N)=1,\ \det A > 0\right\}.$$

Let

$$\mathcal{H}(N) := \mathcal{H}(\Gamma_0(N), \Delta_0(N)),$$
$$\mathcal{H}^*(N) := \mathcal{H}(\Gamma_0(N), \Delta_0^*(N)).$$

Fact IV.2.7. If $\alpha \in \Delta_0(N)$ (resp. $\Delta_0^*(N)$), then there are unique $a, d \in \mathbb{Z}_{\geq 1}$ such that $a \mid d$, $\gcd(a,N)=1$, and $\Gamma\alpha\Gamma = \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma$ (resp. $\Gamma\alpha\Gamma = \Gamma \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} \Gamma$).

Theorem IV.2.8. *$\mathcal{H}(N)$ and $\mathcal{H}^*(N)$ are commutative.*

Proof. Apply Theorem IV.2.6, using the following involution:

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix}^c = \begin{pmatrix} a & c \\ bN & d \end{pmatrix}.$$

□

For $\gcd(a,N)=1$, let

$$\Gamma(a,d) := \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma \in \mathcal{H}(N),$$
$$\Gamma^*(a,d) := \Gamma \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} \Gamma \in \mathcal{H}^*(N).$$

For χ a Dirichlet character mod N, and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0(N)$, extend χ to a map $\Gamma_0(N) \to \mathbb{C}^\times$ by $\chi(\alpha) := \overline{\chi}(a)$. If $\alpha \in \Delta_0^*(N)$, extend to $\chi^*(\alpha) := \chi(d)$. If $\alpha \in \Gamma_0(N)$, then $\chi(\alpha) = \chi^*(\alpha)$.

For G any of A, M, S, and ? either nothing or $*$, we have the *Hecke action*

$$\mathcal{H}^?(N) \circlearrowleft G_k(N,\chi),$$

defined by

$$f|_K \Gamma\alpha\Gamma := \det(\alpha)^{\frac{k}{2}-1} \sum_i \overline{\chi^?}(\alpha_i) f|_k \alpha_i,$$

where $\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i$.

IV.3 2014-03-31: Hecke actions

Continuing with the notation from last time: For $n \geq 1$, let

$$T^?(n) := \sum_{\substack{\Gamma\alpha\Gamma \in \mathcal{H}^?(N) \\ \det(\alpha)=n}} .$$

Then:

$$T(n) = \sum_{\substack{ad=n \\ (a,N)=1 \\ a|d}} T(a,d),$$

$$T^*(n) = \sum_{\substack{ad=n \\ (a,N)=1 \\ a|d}} T^*(d,a).$$

So, $T(p) = T(1,p)$ and $T^*(p) = T^*(p,1)$ for p prime.

Theorem IV.3.1.
- *If $(m,n) = 1$, then $T(m)T(n) = T(mn)$.*
- *If $p \mid N$, then $T(p^e) = T(p)T(p^{e-1})$.*
- *Otherwise, $T(p^e) = T(p)T(p^{e-1}) - pT(p,p)T(p^{e-2})$.*
- *Generally,*

$$T(m)T(n) = \cdots$$

  ...

IV.3.1 Petersson inner product

This is an inner product $\langle \cdot, \cdot \rangle$ on $S_k(\Gamma)$.

Recall that $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$. Since $\mathbb{P}^1(\mathbb{Q})$ is countable, it has measure zero. We have

$$\mathfrak{h} \cong \mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^\times_{>0} \, \mathrm{O}_2(\mathbb{R}),$$

which is a locally compact group, and thus has a Haar measure μ_G such that $\mu_G(\alpha X) = \mu_G(X)$ for all $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ and all Borel $X \subseteq \mathrm{GL}_2^+(\mathbb{R})$. This induces a $\mathrm{GL}_2^+(\mathbb{R})$-invariant measure on $\mathfrak{h}$. If $\tau = x + iy$, then $d\mu(\tau) = \frac{dx\,dy}{y^2}$.

Recall that $\mathcal{F}^* = \left\{|\mathrm{Re}(\tau)| \leq \frac{1}{2},\ |\tau| \geq 1\right\} \cup \{\infty\}$.

Lemma IV.3.2. *If $\varphi : \mathfrak{h} \to \mathbb{C}$ is continuous and bounded, then for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$,*

$$\int_{\mathcal{F}^*} \varphi(\alpha\tau) d\mu(\tau)$$

converges.

Example IV.3.3. For $\varphi = 1$, we have

$$\int_{\mathcal{F}^*} d\mu(\tau) = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} dx = 2\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx = 2\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{3} = \zeta(2) \cdot \frac{2}{\pi}.$$

Thus, $\mathrm{Vol}(X(\mathrm{SL}_2(\mathbb{Z}))) = \frac{\pi}{3}$.

IV.4 2014-04-09: Newforms

(1) We've seen that $S_k(N,\chi)$ has basis of eigenvectors for $\mathbb{T}^{(N)}_{\mathrm{cusp}}(N,\chi)$. Why not all of $\mathbb{T}_{\mathrm{cusp}}(N,\chi)$?

(2) Let f be a newform eigenvector for all Hecke operators. Then $L(s,f)$ has an Euler product.

(3) Theory of conductors for modular forms.

Hecke showed that, if $\chi = \mathbb{1}$ and N is prime and $k < 12$ or $k = 14$, or if χ is primitive, then $S_k(N,\chi)$ *does* have a basis for eigenvectors for the full Hecke algebra.

However: suppose, for example, that $f \in S_k(N,\chi)$, $p \nmid N$, $f|_k T(p) = \lambda_p f$, and $f \in S_k(Np^r,\chi)$ for all $r \geq 0$. Moreover, write $f_j(z) := f(p^j z) \in S_k(Np^r,\chi)$ for $0 \leq j \leq r$ and $\delta_j = \begin{pmatrix} p^j & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$(f|_k\delta_j)(z) = p^{jk/2} f(p^j z),$$

and $\delta_j \Gamma_0(N) \delta_j^{-1} \subseteq \Gamma_0(N)$.

Consider the actions $T(p) \circlearrowright S_k(N,\chi)$ and $U(p) \circlearrowright S_k(Np^r,\chi)$ for $r \geq 1$. Then

$$\begin{aligned} T(p) &= U(p) \sqcup \Gamma_0(Np^r) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \\ f_0|_k U(p) &= f|_k T(p) - p^{\frac{k}{2}-1} f|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \lambda_p f - p^{k-1}\chi(p) f. \end{aligned}$$

If $j \geq 1$, then

$$\begin{aligned} f_j|_k U(p) &= p^{\frac{k}{2}-1} \sum_{m=0}^{p-1} \overline{\chi \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix}} f_j|_k \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix} \\ &= p^{-1} \sum_{m=0}^{p-1} f_{j-1}|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix} \\ &= p^{-1} \sum_{m=0}^{p-1} f_{j-1}|_k \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = f_{j-1}. \end{aligned}$$

So the action $U(p) \circlearrowright V := \langle f_0, f_1, \ldots, f_r \rangle$ is given by the matrix

$$\begin{pmatrix} \lambda_p & 1 & 0 & 0 & \ldots \\ -p^{k-1}\chi(p) & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Note that $W := \langle f_0, f_1 \rangle \subseteq V$ is $U(p)$-stable, with characteristic polynomial $x^2 - \lambda_p x + p^{k-1}\chi(p)$.

Lemma IV.4.1. *If $T : V \to V$ is a diagonalizable linear operator and $W \leq V$ is T-stable, then $\bar{T} : V/W \to V/W$ is diagonalizable.*

Hence, if $r \geq 3$, then $U(p)$ is *not* diagonalizable on $\langle f_0, \ldots, f_r \rangle$.

We want to develop a theory of "primitive" modular forms and "conductors". For $t \in \mathbb{Z}_{\geq 1}$, $\delta_t = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ and $f(tz) = t^{-k/2} f|_k \delta_t$, so if $f \in M_k(N, \chi)$, then $f(tz) \in M_k(Nt, \chi)$.

Let M, N be positive integers with $M \mid N$, so $S_k(M, \chi) \subseteq S_k(N, \chi)$. For all $t \mid \frac{N}{M}$, we have a "*degeneracy map*"

$$\begin{aligned} \iota_{M,N,t} : S_k(M, \chi) &\to S_k(N, \chi), \\ f &\mapsto f|_k \delta_t \in S_k(Mt, \chi). \end{aligned}$$

Fact IV.4.2. Let $T = T(n)$ or $T(n, n)$, where $(n, T) = 1$. (For $T(n, n)$, also require $(n, N) = 1$.) Then the following diagram commutes:

$$\begin{tikzcd} S_k(M, \chi) \arrow[r, "T"] \arrow[d, "\iota_{M,N,t}"'] & S_k(M, \chi) \arrow[d, "\iota_{M,N,t}"] \\ S_k(N, \chi) \arrow[r, "T"] & S_k(N, \chi) \end{tikzcd}$$

Define the *old subspace* by

$$S_k^{\text{old}}(N, \chi) \overset{\text{def}}{=} \bigcup_{\substack{M \mid N \\ M \neq N}} \bigcup_{t \mid \frac{N}{M}} \iota_{M,N,t}(S_k(M, \chi)) \subseteq S_k(N, \chi).$$

Proposition IV.4.3. *$S_k^{\text{old}}(N, \chi)$ is Hecke stable.*

Proof. The fact for $p \mid N$ is similar to computations of $U(p) \circlearrowright f$. □

The *new subspace* is

$$S_k^{\text{new}}(N, \chi) \overset{\text{def}}{=} S_k^{\text{old}}(N, \chi)^{\perp} = \left\{ f \in S_k(N, \chi) \;\middle|\; \langle f, g \rangle = 0 \ \forall g \in S_k^{\text{old}}(N, \chi) \right\}.$$

A form $f \in S_k(N, \chi)$ is called *new* or *primitive* if $f \in S_k^{\text{new}}(N, \chi)$.

Proposition IV.4.4. *$S_k^{\text{new}}(N, \chi)$ is Hecke stable.*

Proof. Observe that $\langle f|_k T(n), g \rangle = \langle f, g|_k T^*(n) \rangle$, where $T^*(n) = \overline{\chi}(n) T(n)$. □

Corollary IV.4.5. *Both S_k^{new} and S_k^{old} have a basis of eigenvectors for the* anemic *Hecke algebra.*

Our goal is to show $S_k^{\text{new}}(N, \chi)$ has a basis of eigenvectors for the *full* Hecke algebra.

IV.4.1 Atkin–Lehner–Li theory

Note IV.4.6. If $f = \sum_{n\geq 1} a_n q^n \in S_k(M,\chi)$ and $d \mid \frac{N}{M}$, then

$$\iota_{M,N,d}(f) = d^{\frac{k}{2}} \sum_{n\geq 1} a_n q^{dn},$$

so $\iota_{M,N,d}(f)$ has $a_n = 0$ whenever $(n,d) = 1$.

Lemma IV.4.7 (Main lemma of Atkin–Lehner–Li theory)**.** *If $f \in S_k(\Gamma,(N))$ and $a_n(f) = 0$ for all $(n,N) = 1$, then for all $p \mid N$, there is a modular form $f^{(p)} \in S_k(\Gamma, \frac{N}{p})$ such that*

$$f = \sum_{p\mid N} \iota_{N/p,N,p}(f^{(p)}).$$

Proof. See §5.7 of Diamond–Shurman or §4.6 of Miyake. ☐

IV.5 2014-04-11: More about newforms

If $f \in S_k(\Gamma_1(N))$ with $a_n(f) = 0$ for all $(n,N) = 1$, then f is old.

Definition IV.5.1. If $f \neq 0$ in $M_k(\Gamma_1(N))$ and f is an eigenvector for all $T \in \mathcal{H}(N)$, then f is called an *eigenform*. An eigenform $f = \sum_{n\geq 0} a_n q^n$ is *normalized* if $a_1(f) = 1$. A *newform* is a normalized eigenform in $S_k^{\text{new}}(\Gamma_1(N))$.

Let $f \in S_k(N,\chi)$ be an eigenvector for $\mathcal{H}^{(N)}(N)$. Then for all n with $(n,N) = 1$, there are $\lambda_n, d_n \in \mathbb{C}$ such that $f|_k T(n) = \lambda_n f$ and $f|_k T(n,n) = n^{k-1} d_n f$. One can check that $\chi : n \mapsto d_n$ is a Dirichlet character of level N, so $f \in S_k(N,\chi)$.

For all n with $(n,N) = 1$, we know $a_n(f) = \lambda_n a_1(f)$. So if $a_1(f) = 0$, then f is old by the main lemma. Hence, if $f \in S_k^{\text{new}}(N,\chi)$, then $a_1(f) \neq 0$, so we can normalize to $a_1(f) = 1$ without loss of generality. Then $a_n(f) = \lambda_n$.

For $m \geq 1$, let $g_m = f|_k T(m) - a_m(f) f \in S_k^{\text{new}}(N,\chi)$. Then g_m is also an eigenvector for $T(n)$ and $T(n,n)$ whenever $(n,N) = 1$. We compute $a_1(g_m)$:

$$a_1(g_m) = a_1\big(f|_k T(m)\big) - a_1\big(a_m(f) f\big) = a_m(f) - a_m(f) = 0.$$

So g_m is both old and new, hence $g_m = 0$. Thus, $f|_k T(m) = \lambda_m f$ (where $\lambda_m = a_m(f)$) for all $m \geq 1$.

Theorem IV.5.2. *If $f \in S_k^{\text{new}}(\Gamma_1(N))$ with $f \neq 0$ is an eigenvector for $T(n)$ and $T(n,n)$ whenever $(n,N) = 1$, then:*

(i) f is an eigenform and a multiple of a newform.

(ii) (Multiplicity one): If $0 \neq \tilde{f} \in S_k^{\text{new}}(\Gamma_1(N))$ with the same Hecke eigenvalues for the full Hecke algebra, then $\tilde{f} = cf$ for some $c \in \mathbb{C}^\times$.

(iii) The set of newforms of level N and character χ are an orthogonal basis of $S_k(N,\chi)$.

We can prove a stronger form of the above "multiplicity one" result:

Theorem IV.5.3. *Suppose $f, g \in S_k^{\mathrm{new}}(\Gamma_1(N))$ and $D \in \mathbb{Z}_{\geq 1}$ such that f, g are eigenvectors for $\mathbb{T}_{cusp}^{(ND)}(N,\chi)$ with $\lambda_p(f) = \lambda_p(g)$ for all prime $p \nmid ND$. Then $f = cg$ for some $c \in \mathbb{C}^\times$.*

Theorem IV.5.4. *The set $\{f(tz) : f \in S_k^{\mathrm{new}}(M,\chi),\ tM \mid N\}$ is a basis of $S_k(N,\chi)$.*

Theorem IV.5.5. *If $g \in S_k(\Gamma_1(N))$ is a normalized eigenform, then there exists $M \mid N$ and $f \in S_k^{\mathrm{new}}(\Gamma_1(M))$ such that $a_p(f) = a_p(g)$ for all $p \nmid N$. We define* $\mathrm{cond}(g) := M$.

IV.5.1 Euler products

Next time, we'll talk about L-functions properly. Today, we'll just briefly remark on Euler products.

Given $f = \sum_{n\geq 0} a_n q^n$, define

$$L(s,f) = \sum_{n\geq 1} \frac{a_n}{n^s}.$$

When is there an Euler product?

Theorem IV.5.6. *Let R be a commutative ring and $t(n), d(n) \in R$ such that $t(1) = d(1) = 1$ and $d(mn) = d(m)d(n)$ for all $m, n \in \mathbb{Z}_{\geq 1}$. Then the following are equivalent:*

(i) $t(mn) = t(m)t(n)$ *whenever* $(m,n) = 1$, *and* $t(p)t(p^e) = t(p^{e+1})pd(p)t(p^{e-1})$ *for all primes p and $e \geq 1$.*

(ii) $\displaystyle\sum_{n\geq 1} \frac{t(n)}{n^s} = \prod_p \left(1 - t(p)p^{-s} + pd(p)p^{-2s}\right)^{-1}.$

IV.6 2014-04-14: L-functions

Let $f = \sum_{n\geq 0} a_n q^n \in M_k(N,\chi)$. Define

$$L(s,f) = \sum_{n\geq 1} \frac{a_n}{n^s}.$$

The Dirichlet series converges for $\mathrm{Re}(s) > C_f$. Indeed, if $|a_n| = \mathrm{O}(n^\nu)$, then

$$\left|\frac{a_n}{n^s}\right| \leq Cn^{\nu - s},$$

so it converges absolutely and uniformly on compact subsets of $\mathrm{Re}(s) > \nu + 1$.

Lemma IV.6.1. *If f is a cusp form of weight k, then $|a_n| = \mathrm{O}(n^{k/2})$. Otherwise, $|a_n| = \mathrm{O}(n^{k-1+\varepsilon})$ for all $\varepsilon > 0$.*

In fact, Deligne proved the following theorem, conjectured by Ramanujan and Petersson:

Theorem IV.6.2. *If f is a newform of level N, then for all n with $(n, N) = 1$,*

$$|a_n| = \mathrm{O}\big(n^{\frac{k-1}{2}}\big).$$

The idea is to consider $x^2 - a_p x - p^{k-1}\chi(p) = (x - \alpha_p)(x - \beta_p)$. The modular form is attached to a 2-dimensional Galois representation coming from the ℓ-adic cohomology of a variety; a_p and $p^{k-1}\chi(p)$ are the trace and determinant, respectively, of this Galois representation. In particular, $L(s, f)$ is a factor in the zeta function of some proper smooth variety, occurring in H^{k-1}, so the Weil conjectures say that $|\alpha_p| = |\beta_p| = p^{\frac{k-1}{2}}$.

There is a converse:

Proposition IV.6.3. *If $f : \mathfrak{h} \to \mathbb{C}$ is a holomorphic function such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$, then the following are equivalent:*

(i) $f|_k\alpha$ is holomorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ (i.e., $f \in M_k(\Gamma)$).

(ii) There exists ν such that $|a_n| = \mathrm{O}(n^\nu)$.

IV.6.1 Euler products

Does $L(s, f)$ have an Euler product?

Recall from last time:

Theorem IV.6.4. *Let R be a commutative ring and $t(n), d(n) \in R$ such that $t(1) = d(1) = 1$ and $d(mn) = d(m)d(n)$ for all $m, n \in \mathbb{Z}_{\geq 1}$. Then the following are equivalent:*

(i) $t(mn) = t(m)t(n)$ whenever $(m, n) = 1$, and $t(p)t(p^e) = t(p^{e+1})pd(p)t(p^{e-1})$ for all primes p and $e \geq 1$.

(ii) $\displaystyle\sum_{n\geq 1} \frac{t(n)}{n^s} = \prod_p \big(1 - t(p)p^{-s} + pd(p)p^{-2s}\big)^{-1}$.

Proof. Suppose (i). Then

$$\sum_{n\geq 1} \frac{t(n)}{n^s} = \prod_p \left(\sum_{e\geq 0} \frac{t(p^e)}{p^{es}}\right).$$

We know $t(p^{e+1}) - t(p)t(p^e) + pd(p)t(p^{e-1}) = 0$. So

$$\begin{aligned}\big(1 - t(p)p^{-s} + pd(p)p^{-2s}\big) \sum_{e\geq 0} \frac{t(p^e)}{p^{es}} &= t(1) + \frac{t(p)}{p^s} + \frac{t(p^2)}{p^{2s}} + \frac{t(p^3)}{p^{3s}} + \dots \\ &\quad - \left(\frac{t(p)}{p^s} + \frac{t(p)^2}{p^{2s}} + \frac{t(p)t(p^2)}{p^{3s}} + \dots\right) \\ &\quad + \frac{pd(p)}{p^{2s}} + \frac{pd(p)t(p)}{p^{3s}} + \dots,\end{aligned}$$

and adding diagonally the terms with the same p^{es} term in the denominator, everything but the initial $t(1) = 1$ cancels, yielding

$$\sum_{n\geq 1} \frac{t(n)}{n^s} = \prod_p \left(\sum_{e\geq 0} \frac{t(p^e)}{p^{es}} \right) = \prod_p \left(1 - t(p)p^{-s} + pd(p)p^{-2s}\right)^{-1}.$$

For the converse, see Diamond–Shurman. □

For our application, let $R = \mathcal{H}(N)$, and let

$$t(n) = T(n),$$
$$d(n) = \begin{cases} T(n,n) & \text{if } (n,N) = 1, \\ 0 & \text{if } (n,N) > 1. \end{cases}$$

If $f \in M_k(N,\chi)$ and $\mathscr{P}$ is a set of primes such that $f|_k T(p) = \lambda(p) f$ for all $p \in \mathscr{P}$, then

$$L(s,f) = \left(\prod_{p\in\mathscr{P}} \left(1 - \lambda(p)p^{-s} + \chi(p)p^{k-1-2s}\right)^{-1} \right) \cdot \sum_{(n,\mathscr{P})=1} \frac{a_n}{n^s}.$$

So if f is a newform, then

$$L(s,f) = \prod_p \left(1 - a_p p^{-s} + \chi(p)p^{k-1-2s}\right)^{-1}.$$

IV.6.2 Analytic continuation and functional equation

Our next goal is to show $L(s,f)$ has analytic continuation, a functional equation, and other analytic properties.

The basic tool is the *Mellin transform*: Given $f = \sum_{n\geq 0} a_n q^n \in M_k(N,\chi)$, let

$$\Lambda(s,f) := \mathcal{M}\big(f(it)\big)(s) \overset{\text{def}}{=} \int_0^\infty \big(f(it) - a_0\big) t^s \frac{dt}{t}.$$

Recall that $a_n = \mathrm{O}(n^\nu)$, so if $\mathrm{Re}(s) > \nu + 1$, then

$$\begin{aligned} \int_0^\infty \big(f(it) - a_0\big) t^s \frac{dt}{t} &= \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi n t} t^s \frac{dt}{t} \\ &= \sum_{n=1}^\infty \int_0^\infty a_n e^{-t} t^s (2\pi n)^{-s} \frac{dt}{t} \\ &= \frac{1}{(2\pi)^s} \left(\sum_{n=1}^\infty \frac{a_n}{n^s} \right) \int_0^\infty e^{-t} t^s \frac{dt}{t} \\ &= \frac{1}{(2\pi)^s} \Gamma(s) L(s,f). \end{aligned}$$

Remark IV.6.5. There is an *inverse Mellin transform*: if $L(s) = \mathcal{M}(F)(s) = \int_0^\infty (F(s) - F(\infty))t^s \frac{dt}{t}$, then (under analytic assumptions),

$$F(s) - F(\infty) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} L(s)t^{-s}\,ds \stackrel{\text{def}}{=} \mathcal{M}^{-1}(L)(s).$$

So if we have $L(s)$, we can ask when

$$\mathcal{M}^{-1}\left(\frac{(2\pi)^s}{\Gamma(s)}L(s)\right)$$

is a modular form.

IV.7 2014-04-16

The contents of today's lecture is given in more detail in Miyake, §4.3 and §4.7.

Let $\{a_n\}_{n=0}^\infty$ be a sequence in $\mathbb{C}$, and let

$$f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$$

for $z \in \mathfrak{h}$.

Fact IV.7.1 (†). If $a_n = \mathrm{O}(n^\nu)$ for some $\nu > 0$, then the series defining f converges absolutely and uniformly on compact subsets (of $\mathfrak{h}$), so $f(z)$ is holomorphic on $\mathfrak{h}$. Moreover, writing $z = x + iy$, we have $|f(z)| = \mathrm{O}(y^{-\nu-1}$ as $y \to 0$ and $|f(z) - a_0| = \mathrm{O}(e^{-2\pi y})$ as $y \to \infty$, both uniformly in x.

Conversely, let $f(z)$ be holomorphic on $\mathfrak{h}$ such that there exist $a_n \in \mathbb{C}$ with $f(z) = \sum_{n\geq 0} a_n q^n$ converging absolutely and uniformly on compact subsets, and such that there exists $\nu > 0$ with $|f(z)| = \mathrm{O}(y^{-\nu-1})$ as $y \to 0$. Then $|a_n| = \mathrm{O}(n^\nu)$ (and so $|f(z) - a_0| = \mathrm{O}(e^{-2\pi y})$ as $y \to \infty$).

Suppose f satisfies the above conditions (†). Then

$$L(s, f) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

converges absolutely and uniformly on compact subsets for $\mathrm{Re}(s) > \nu + 1$.

Let $N \in \mathbb{Z}_{\geq 1}$, and define

$$\tilde{\Lambda}_N(s, f) := \frac{N^{s/2}}{(2\pi)^s}\Gamma(s)L(s, f) = \int_0^\infty \left(f\left(\frac{it}{\sqrt{N}}\right) - a_0\right) t^s \frac{dt}{t}.$$

We now define the *Atkin–Lehner operator*: let $W_n = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Then $W_N\Gamma_0(N)W_N^{-1} = \Gamma_0(N)$.

Fact IV.7.2. The map $f \mapsto f|_k W_N$ gives an isomorphism

$$G_k(N, \chi) \xrightarrow{\simeq} G_k(N, \overline{\chi}),$$

where G is any of M, S, or $\mathcal{E}$.

Theorem IV.7.3 (Hecke)**.** *Let $k, N \in \mathbb{Z}_{\geq 1}$, and let $f = \sum_{n \geq 0} a_n q^n$ and $g = \sum_{n \geq 0} b_n q^n$ be functions satisfying (†). The following are equivalent:*

(A) $g(z) = (f|_k W_N)(z)$.

(B) Both $\tilde{\Lambda}_N(s, f)$ and $\tilde{\Lambda}_N(s, g)$ can be analytically continued to all $s \in \mathbb{C}$,

$$\tilde{\Lambda}_N(s, f) = i^k \tilde{\Lambda}_N(k - s, g),$$

and the function

$$\tilde{\Lambda}_N(s, f) + \frac{a_0}{s} + \frac{i^k b_0}{k - s}$$

is entire and bounded in vertical strips.

Remark IV.7.4. The proof (which we won't present here) uses the *Phragmén–Lindelöf principle*: Let $F(s)$ be holomorphic on the vertical strip $a \leq \mathrm{Re}(s) \leq b$, and suppose $|F(s)| = \mathrm{O}(e^{|s|^\delta})$ for some $\delta \geq 0$. If there are constants M_a and M_b such that, for all $t \in \mathbb{R}$,

$$|F(a + it)| \leq M_a \cdot (1 + |t|)^\alpha,$$
$$|F(b + it)| \leq M_b \cdot (1 + |t|)^\beta,$$

then for all $a \leq \sigma \leq b$,

$$|F(\sigma + it)| \leq M_a^{\ell(\sigma)} M_b^{1-\ell(\sigma)} (1 + |t|)^{\alpha \ell(\sigma) + \beta(1 - \ell(\sigma))},$$

where $\ell(\sigma) = 1 - \frac{\sigma - a}{b - a}$.

Corollary IV.7.5. *If $f \in S_k(N, \chi)$, then $\tilde{\Lambda}_N(s, f)$ is entire and satisfies*

$$\tilde{\Lambda}_N(s, f) = i^k \tilde{\Lambda}_N(k - s, f|_k W_N).$$

Corollary IV.7.6 ($N = 1$)**.** *Recall that* $\mathrm{SL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = W_1 \right\rangle$. *Let $k \in \mathbb{Z}_{\geq 2}$ be even, and suppose f satisfies (†). Then $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ if and only if:*

- *$\Lambda(s, f)$ can be analytically continued to all $s \in \mathbb{C}$.*
- *$\Lambda(s, f) = (-1)^{k/2} \Lambda(k - s, f)$.*
- *$\Lambda(s, f) + \dfrac{a_0}{s} + \dfrac{(-1)^{k/2} a_0}{k - s}$ is entire and bounded in vertical strips.*

Furthermore, if $a_0 = 0$, then $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$.

Proof. By definition, $f|_k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = f$. Then $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ iff $f|_k W_1 = f$. Apply Hecke's theorem with $N = 1$ and $f = g$. □

To generalize to $M_k(N, \chi)$, let f satisfy (†), and let ψ be a Dirichlet character. Let $f_\psi = \sum_{n \geq 0} \psi(n) a_n q^n$ (which also satisfies (†)) and

$$L(s, f, \psi) := L(s, f_\psi) = \sum_{n \geq 1} \frac{\psi(n) a_n}{n^s}.$$

Let m_ψ be the conductor of ψ, and let

$$\tilde{\Lambda}_N(s, f, \psi) := \tilde{\Lambda}_{N m_\psi^2}(s, f_\psi) = \frac{(m\psi\sqrt{N})^s}{(2\pi)^s} \Gamma(s) L(s, f, \psi).$$

Lemma IV.7.7. *If f, g satisfy (†) and ψ is primitive of conductor $m > 1$, then the following are equivalent:*

(A)$_\psi$ There is a constant C_ψ such that $f_\psi|_k W_{Nm^2} = C_\psi g_{\overline{\psi}}$.

(B)$_\psi$ $\tilde{\Lambda}_N(s, f, \psi)$ has analytic continuation and is bounded in vertical strips, and

$$\tilde{\Lambda}_N(s, f, \psi) = i^k C_\psi \tilde{\Lambda}_N(k - s, g, \overline{\psi}).$$

Lemma IV.7.8. *Let $f \in M_k(N, \chi)$, ψ primitive of conductor m, and $M = \mathrm{lcm}(N, m^2, m m_\chi)$. Then $f_\psi \in M_k(M, \chi\psi^2)$.*

Let $\mathscr{P}$ be any set of odd primes and 4 such that, for all $p \in \mathscr{P}$, we have $(p, N) = 1$ and

$$\mathscr{P} \cap \{a + nb : n \in \mathbb{Z}\} \neq \varnothing$$

for all $a, b \geq 1$ with $(a, b) = 1$. (For example, we could take $\mathscr{P}$ to be all odd primes $p \nmid N$.)

Theorem IV.7.9 (Weil's converse theorem)**.** *Let $k, N \in \mathbb{Z}_{\geq 1}$, χ a Dirichlet character mod N such that $\chi(-1) = (-1)^k$, and $a_n, b_n \in \mathbb{C}$ both $\mathrm{O}(n^\nu)$ (for some $\nu > 0$). Let $f = \sum_{n \geq 0} a_n q^n$ and $g = \sum_{n \geq 0} b_n q^n$. Then $f \in M_k(N, \chi)$, $g \in M_k(N, \overline{\chi})$, and $f|_k W_N = g$ if and only if:*

- *$\tilde{\Lambda}_N(s, f)$ and $\tilde{\Lambda}_N(s, g)$ satisfy (B).*
- *For all primitive Dirichlet characters ψ of conductor $m \in \mathscr{P}$, $\tilde{\Lambda}_N(s, f, \psi)$ and $\tilde{\Lambda}_N(s, g, \psi)$ satisfy (B)$_\psi$ with*

$$C_\psi = \chi(m) \psi(-N) \frac{\tau(\psi)}{\tau(\overline{\psi})},$$

  where τ denotes the Gauss sum.

Furthermore, if $L(s, f)$ is absolutely convergent at $s = k - \delta$ for any $\delta > 0$, then f, g are cusp forms.

IV.8 2014-04-21

Theorem IV.8.1. *Let $f \in M_k(N, \chi)$ be a newform. Then there is a number field $K_f/\mathbb{Q}$ such that $a_n(f) \in \mathcal{O}_{K_f}$ for all $n \geq 0$. In fact, we can take $K_f = \mathbb{Q}(a_n : n \geq 1)$, and $\mathbb{Q}(\chi) \subseteq K_f$ and K_f is totally real or CM.*

Let $\mathbb{T} := \mathbb{T}(N, \chi) = \operatorname{im}(\mathcal{H}(N) \hookrightarrow \operatorname{End}_{\mathbb{C}}(M_k(N, \chi)))$. Note that $M_k(N, \chi, \mathbb{Z}[\chi])$ is Hecke stable (since $a_n(f|_k T(m)) = \sum_{0<d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f))$, and it contains a basis of $M_k(N, \chi)$. Hence, $\mathbb{T} \subseteq \operatorname{End}_{\mathbb{Z}[\chi]}(M_k(N, \chi, \mathbb{Z}[\chi]))$, recalling that $\mathcal{H}(N) = \mathbb{Z}[T(n), T(n, n)]$. We have proved the following:

Proposition IV.8.2. *$\mathbb{T}$ is a finitely-generated free $\mathbb{Z}[\chi]$-module.*

Let f be an eigenvector for all $T(m)$. We can define a homomorphism $\theta_f : \mathbb{T} \to \mathbb{C}$ by $f|_k T = \theta_f(T) f$ for all $T \in \mathbb{T}$. The image of θ_f is a finite module over $\mathbb{Z}[\chi]$, so it's in the ring of integers of some K_f.

Let $K'_f := \mathbb{Q}(a_n : n \geq 1)$. We want to show that we can take $K_f = K'_f$, for which it suffices to show $\chi(d) \in K'_f$ whenever $(d, N) = 1$.

Lemma IV.8.3. *Let $K/\mathbb{Q}$ be a finite extension. If $a_n \in K$ for all $n \geq 1$, then so is a_0.*

Fact IV.8.4. A number field $K/\mathbb{Q}$ has a well-defined complex conjugation iff K is totally real or CM.

Lemma IV.8.5. *For all n with $(n, N) = 1$, $a_n = \overline{a_n} \chi(n)$.*

Proof. We have $T^*(n) = \overline{\chi}(n) T(n)$, so

$$a_n \langle f, f \rangle = \langle f|_k T(n), f \rangle = \langle f, f|_k T^*(n) \rangle = \langle f, \overline{\chi}(n) a_n f \rangle = \chi(n) \overline{a_n} \langle f, f \rangle . \qquad \square$$

Corollary IV.8.6. *For all $\sigma \in \operatorname{Aut}_{\mathbb{Q}}(\mathbb{C})$ and all $(n, N) = 1$,*

$$\sigma(a_n) = \overline{\sigma(a_n)} \chi^\sigma(n).$$

Proof. Apply the above lemma to f^σ. $\square$

So for all $(n, N) = 1$,

$$\sigma(\overline{a}_n) \chi^\sigma(n) = \sigma(a_n) = \overline{\sigma(a_n)} \chi^\sigma(n),$$

so $\overline{\sigma(a_n)} = \sigma(\overline{a}_n)$. Thus, $\overline{a}_n$ is well-defined for $(n, N) = 1$. The existence of a well-defined complex conjugation on K_f shows that it is totally real or CM. $\square$

When is K_f totally real?

Theorem IV.8.7. *K_f is totally real iff $a_p = \chi(p) a_p$ for all $p \nmid N$. In this case, $\chi^2 = \mathbb{1}$.*

IV.8.1 CM forms

Let $f \in S_k(N, \chi)$ be a newform, and φ a Dirichlet character mod M. Let

$$f_\varphi = \sum_{n \geq 1} \varphi(n) a_n q^n \in S_k(NM^2, \chi\varphi^2),$$
$$f_\varphi|_k T(p) = (\varphi(p) a_p) f_\varphi \qquad \forall p \nmid NM.$$

Definition IV.8.8. Let φ be a non-trivial Dirichlet character. Say f has *CM* by φ if $\varphi(p) a_p = a_p$ for all $p \nmid NM$.

Note IV.8.9.

- $\varphi^2 = \mathbb{1}$
- φ corresponds to an imaginary quadratic $F/\mathbb{Q}$, i.e.,

$$\varphi(p) = \begin{cases} 1 & \text{if } p \text{ split in } F, \\ -1 & \text{if } p \text{inert in } F. \end{cases}$$

- If f is a newform, $\chi \neq \mathbb{1}$, and K_f is totally real, then f has CM by its *Nebentypus* χ.

Chapter V

Automorphic forms

V.1 2014-04-23: Automorphic forms and representations

Today, we will put modular forms into the larger context of automorphic forms and automorphic representations of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{A})$. The first step is to go from $S_k(\Gamma)$ to automorphic forms on $\mathrm{SL}_2(\mathbb{R})$ (Gelbart, chapter 2).

Recall: $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \xrightarrow{\simeq} \mathfrak{h}$ by the map $g \mapsto g \cdot i$.

The idea: given $f : \mathfrak{h} \to \mathbb{C}$, "lift" this to a map $\varphi_f : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$. Specifically, for $f \in S_k(\Gamma)$ (where Γ is a congruence subgroup and k is even), define

$$\varphi_f(g) := f(g \cdot i) j(g, i)^k.$$

Then for all $\gamma \in \Gamma$, let $z = g \cdot i$, whence

$$\begin{aligned}
\varphi_f(\gamma g) &= f(\gamma \cdot (g \cdot i)) j(\gamma g, i)^{-k} = f(\gamma \cdot z) j(\gamma, z)^{-k} j(g, i)^{-k} \\
&= (f|_k \gamma)(z) j(g, i)^{-k} = f(g \cdot i) j(g, i)^{-k} = \varphi_f(g).
\end{aligned}$$

Thus, φ_f is left-Γ-invariant.

What properties characterize the image of $f \mapsto \varphi_f$?

There are nice coordinates on $G := \mathrm{SL}_2(R)$: let

$$\begin{aligned}
A &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}_{>0} \right\}, \\
N &= \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{R} \right\}, \\
K = \mathrm{SO}_2(\mathbb{R}) &= \left\{ \kappa_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}.
\end{aligned}$$

Note that $\mathrm{SO}_2(\mathbb{R}) \subseteq \mathrm{SL}_2(\mathbb{R})$ is a maximal compact subgroup.

The *Iwasawa decomposition* of $\mathrm{SL}_2(\mathbb{R})$ is:

$$G = NAK.$$

Let $B = NA$ be the upper-triangular subgroup, called the *Borel subgroup*. Under the action $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathfrak{h}$, B acts transitively: for $z = x + iy \in \mathfrak{h}$, let

$$b_z := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}.$$

Then $b_z \cdot i = x + iy = z$.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = b_z \kappa_\theta$, then $z = g \cdot i$ and $\theta = \arg(ci + d)$. Then:

Proposition V.1.1. *Let $f \in S_k(\Gamma)$.*

(i) φ_f is left-Γ-invariant.

(ii) $\varphi_f(g\kappa_\theta) = e^{-ik\theta}\varphi_f(g)$.

(iii) $\varphi_f(g)$ is bounded and in $L^2(\Gamma\backslash G)$.

(iv) φ_f is cuspidal, *i.e., for all $g \in \mathrm{SL}_2(\mathbb{R})$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$,*

$$\int_0^1 \varphi_f\left(\alpha \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g\right) dx = 0,$$

where h is the width of the cusp $\alpha \cdot \infty$.

Proof. (i) Already proven.

(ii) Observe that

$$\varphi_f(g\kappa_\theta) = f(g\kappa_\theta \cdot i)j(g\kappa_\theta, i)^{-k} = f(g \cdot i)j(g,i)^{-k}j(\kappa_\theta, i)^{-k} = \varphi_f(g)\big((\sin\theta)i + \cos\theta\big)^{-k}.$$

(iii) Note that $f \in S_k(\Gamma)$ iff $y^{k/2}\,|f(z)|$ is bounded. For $\varphi : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$, define

$$\int_G \varphi(g)\,dg := \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty\int_{-\infty}^\infty \varphi(x,y,\theta)\frac{dx\,dy}{y^2}d\theta.$$

Then

$$\int_{\Gamma\backslash G} |\varphi_f(g)|^2\,dg = \iint_{\Gamma\backslash\mathfrak{h}} |f(z)|^2\,y^k\frac{dx\,dy}{y^2} = \langle f, f\rangle_\Gamma \operatorname{Vol}(X(\Gamma)).$$

(iv) By definition, f is a cusp form $\iff$ $a_0(f|_k\alpha) = 0$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Thus, for all

$z \in \mathfrak{h}$,

$$\begin{aligned}
0 = a_0(f|_k\alpha) &= \int_0^h (f|_k\alpha)(x+z)\,dx \\
&= h\int_0^1 (f|_k\alpha)(hx+z)\,dx \\
&= h\int_0^1 f(\alpha\cdot(hx+z))j(\alpha, hx+z)^{-k}\,dx \\
&= h\int_0^1 f\left(\alpha\begin{pmatrix}1 & hx\\ 0 & 1\end{pmatrix} g\cdot i\right)\underbrace{j\left(\alpha, \begin{pmatrix}1 & hx\\ 0 & 1\end{pmatrix} g\cdot i\right)^{-k}}_{\frac{j\left(\alpha\left(\begin{smallmatrix}1 & hx\\ 0 & 1\end{smallmatrix}\right)g, i\right)^{-k}}{j\left(\left(\begin{smallmatrix}1 & hx\\ 0 & 1\end{smallmatrix}\right), z\right)^{-k}}} dx \\
&= hj(g,i)^{-k}\int_0^1 \varphi_f\left(\alpha\begin{pmatrix}1 & hx\\ 0 & 1\end{pmatrix} gi\right) dx. \qquad \square
\end{aligned}$$

So we have a map

$$S_k(\gamma) \to L^2(\Gamma\backslash G)$$
$$f \mapsto \varphi_f.$$

A function f is holomorphic $\iff$ $\partial_{\overline{z}} f = 0$, where

$$\partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

To characterize the image of the above map, we now define the *Laplace–Beltrami operator* on $\mathrm{SL}_2(\mathbb{R})$: in terms of (x, y, θ),

$$\Delta = -y^2\left(\partial_x^2 + \partial_y^2\right) - y\partial x\partial_\theta.$$

Facts:

- Δ is self-adjoint.
- Δ is non-negative.
- Δ commutes with $R(g)$ for all $g \in G$, where R denotes the right regular representation of G on $L^2(\Gamma\backslash G)$, i.e., for all $g \in G$ and $\varphi : G \to \mathbb{C}$ in L^2,

$$\big(R(g)\varphi\big)(h) = \varphi(hg).$$

  This is a unitary representation of G.

Theorem V.1.2. *Let*

$$A_k^2(\Gamma) := \left\{\varphi \in L^2(\Gamma\backslash G) : \varphi \text{ satisfies (i) to (iv) above, and } \Delta\varphi = -\frac{k}{2}\left(\frac{k}{2} - 1\right)\varphi\right\}.$$

Then the map $f \mapsto \varphi_f$ is an isomorphism $S_k(\Gamma) \cong A_k^2(\Gamma)$.

Proof sketch. If $g = b_z\kappa_\theta$, then $\varphi_f(g) = y^{k/2}f(z)e^{-ik\theta}$, so

$$(\Delta\varphi_f)(g) = (\dots)\left(\partial_x^2 + \partial_y^2\right)f + (\dots)\partial_{\overline{z}} f - \frac{k}{2}\left(\frac{k}{2} - 1\right)\varphi_f. \qquad \square$$

V.1.1 Automorphic forms

Let us generalize the above situation.

Definition V.1.3. An *automorphic form* on $G = \mathrm{SL}_2(\mathbb{R})$ of level Γ is a map $\varphi : G \to \mathbb{C}$ such that:

(0) $\varphi \in C^\infty(G, \mathbb{C})$.

(i) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$.

(ii) φ is right K-finite, i.e., $g \mapsto \varphi(gk)$ for $k \in K$ span a finite-dimensional vector space.

(iii) φ is *slowly increasing*, i.e., $|\varphi(z, \theta)| = \mathrm{O}(y^N)$ for some N as $y \to \infty$.

(iv) φ is an eigenfunction for Δ.

We say φ is *cuspidal* if

(v) $\displaystyle\int_0^1 \varphi\left(\alpha \begin{pmatrix} 1 & hx \\ 0 & 1 \end{pmatrix} g\right) dx = 0$ for all $g \in G$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, where h is the width of $\alpha \cdot \infty$.

V.2 2014-04-25: Adelic stuff

Define the ring of *adeles* of $\mathbb{Q}$ by

$$\mathbb{A} \stackrel{\text{def}}{=} \mathbb{R} \times \prod_p{}' \mathbb{Q}_p,$$

where the product is the restricted direct product

$$\mathbb{A}_f \stackrel{\text{def}}{=} \prod_p{}' \mathbb{Q}_p \stackrel{\text{def}}{=} \left\{ (b_p) \in \textstyle\prod_p \mathbb{Q}_p : b_p \in \mathbb{Z}_p \text{ for all but finitely many } p \right\}.$$

We also define the *ideles* $\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}^\times$.

A *Hecke character* of $\mathbb{Q}$ is a continuous map $\psi : \mathbb{A}^\times \to \mathbb{C}^\times$ such that $\psi(\mathbb{Q}^\times) = 1$. The *idele class group* is $\mathbb{Q}^\times \backslash \mathbb{A}^\times$.

We have a strong approximation theorem

$$\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_{>0}^\times \widehat{\mathbb{Z}}^\times,$$

so $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ induces a Hecke character $\chi_{\mathbb{A}}$, and every ψ is equal to $\chi_{\mathbb{A}} \left|\cdot\right|_a dele^s$ for some χ and some $s \in \mathbb{C}$, where

$$|z|_{\mathbb{A}} = |z_\infty|_\infty \cdot \prod_p |z_p|_p$$

for $z \in \mathbb{A}^\times$.

Let $G = \mathrm{GL}_2$. We will study

$$G_{\mathbb{A}} = \mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{R}) \times \prod_p{}' \mathrm{GL}_2(\mathbb{Q}_p).$$

Let $N \in \mathbb{Z}_{\geq 1}$, and let $K_{p,0}(N) \subseteq G_p := \mathrm{GL}_2(\mathbb{Q}_p)$ be the compact open subgroup

$$K_{p,0}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv 0 \pmod N \right\}.$$

Let $K_{f,0}(N) = \prod_p K_{p,0}(N) \leq G_f := \mathrm{GL}_2(\mathbb{A}_f)$. Then we have a strong approximation theorem:

$$G_{\mathbb{A}} = G_{\mathbb{Q}} G_\infty^+ K_{f,0}(N).$$

Note that $\Gamma_0(N) = G_{\mathbb{Q}} \cap G_\infty^+ K_{f,0}(N)$.

Let $f \in S_k(N, \chi)$, and let $\varphi_f : G_{\mathbb{A}} \to \mathbb{C}$ be given by

$$\varphi_f(g) = f(g_\infty \cdot i) j(g_\infty, i)^{-k} \chi_{\mathbb{A}}(k_0),$$

where $g = \gamma g_\infty k_0$.

Note V.2.1. Let $Z_{\mathbb{A}} :=$ center of $G_{\mathbb{A}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{A}^\times \right\}$. Then

$$G_{\mathbb{Q}} \backslash G_{\mathbb{A}} / \big(Z_{\mathbb{A}} K_\infty K_{f,0}(N) \big) \cong \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R}) \cong Y_0(N).$$

Definition V.2.2. An *automorphic form* on $\mathrm{GL}_2(\mathbb{Q})$ is a function $\varphi : G_{\mathbb{A}} \to \mathbb{C}$ such that:

(0) $\varphi(\gamma g_\infty k_0)$ is smooth as a function of g_∞.

(i) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G_{\mathbb{Q}}$.

(ii) φ is right $K = K_\infty K_f$-finite.

(iii) φ is slowly-increasing, i.e., for all $c > 0$ and compact $C \subseteq G_{\mathbb{A}}$, there is a constant M such that

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) = \mathrm{O}(|a|^M)$$

for all $a \in \mathbb{A}^\times$ with $|a| > c$ and all $g \in C$.

(iv) φ is $\mathscr{Z}$-finite as a function of g_∞.

(v) There is a Hecke character ψ such that $\varphi(zg) = \psi(z)\varphi(g)$ for all $z \in Z_{\mathbb{A}}$ and $g \in G_{\mathbb{A}}$.

Remark V.2.3. To explain (iv), let $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ be the Lie algebra. Any $X \in \mathfrak{g}$ acts as a first-order differential operator on $C^\infty(\mathrm{GL}_2(\mathbb{R}))$. Given a Lie algebra $\mathfrak{g}$, we have the "universal enveloping algebra" $U(\mathfrak{g})$, a unital associative algebra.

The representation theory of $\mathfrak{g}$ is given by the representation theory of $U(\mathfrak{g})$. Elements of $U(\mathfrak{g})$ are higher-order differential operators on $C^\times(\mathrm{GL}_2(\mathbb{R}))$, and

$$\mathscr{Z} := \text{center of } U(\mathfrak{g}).$$

Let $\mathcal{A}(G)$ be the set of automorphic forms on $\mathrm{GL}_2(\mathbb{Q})$.

Definition V.2.4. An automorphic form $\varphi \in \mathcal{A}(G)$ is *cuspidal* if

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0$$

for all but finitely many g.

Definition V.2.5. An *automorphic representation* of $\mathrm{GL}_2(\mathbb{Q})$ is (roughly speaking) an irreducible constituent of $\mathrm{GL}_2(\mathbb{A}) \circlearrowright \mathcal{A}(\mathrm{GL}_2)$.

Definition V.2.6 (Admissible representations)**.** Say G is a locally profinite group (like G_f or G_p), and $\pi : G \to \mathrm{GL}(V)$ is a representation of G on some $\mathbb{C}$-vector space V. We say π is *smooth* if it is locally constant, i.e., for all $v \in V$, there is a compact open subgroup $K \leq G$ such that $Kv = v$.

We say π is *admissible* if π is smooth and $\dim_{\mathbb{C}} V^K < \infty$ for all compact open $K \leq G$.

Fact V.2.7. $\mathcal{A}(G)|_{G_f}$ is admissible.

The Hecke algebra is the algebra $\mathcal{H}(G) = C_c^\infty(G, \mathbb{Z})$, consisting of smooth, compactly supported functions $G \to \mathbb{Z}$ with the operation given by convolution:

$$(\varphi_1 * \varphi_2)(g) = \int_G \varphi_1(x)\varphi_2(x^{-1}g)\, dx.$$

Inside $\mathcal{H}(G_p)$, the characteristic function of the double coset $K_{p,0}(N)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K_{p,0}(N)$ corresponds to $T(p)$, and that of $K_{p,0}(N)\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K_{p,0}(N)$ corresponds to $T(p,p)$.

If π is an admissible representation of G, and $\varphi \in \mathcal{H}(G)$, then π induces an action of $\mathcal{H}(G)$ on π given by

$$\pi(\varphi) \cdot v = \int_G \varphi(g)\pi(g) \cdot v\, d_G g.$$

There's also a Hecke algebra at ∞, basically $U(\mathfrak{g})$.

Remark V.2.8.
(1) Admissible representations of G_p or G_f correspond to admissible representations of $\mathcal{H}(G_p)$ or $\mathcal{H}(G_f)$. (The Hecke algebra is like a group ring construction.)

(2) $\mathcal{H}(G_{\mathbb{A}}) = \left(\bigotimes' \mathcal{H}(G_p)\right) \otimes \mathcal{H}(G_\infty)$.

(3) Fact: for any irreducible admissible representation π of $(\mathfrak{g}, K_\infty) \times G_f$, there exists π_∞ admissible of $(\mathfrak{g}, K_\infty)$ and π_p admissible of G_p such that

$$\pi \cong \pi_\infty \otimes {\bigotimes_p}' \pi_p.$$

Chapter VI

Elliptic curves

VI.1 2014-04-28: Algebro-geometric perspective

Recall that $\mathcal{A}_k(\Gamma)$ are sections of $\Omega^{k/2}_{X(\Gamma)}$ over $X(\Gamma)$.

To view modular forms over $\mathbb{Z}$, we want to define $X(\Gamma)$ over $\mathbb{Z}$. The idea is that $X(1)$ parametrizes elliptic curves over $\mathbb{C}$, so to define $X(1)_{/\mathbb{Z}}$, we write down a "moduli problem" of elliptic curves over $\mathbb{Z}$ and show it is "represented" by some curve.

Let S be a scheme, and let $\mathbf{Sch}_{/S}$ be the category of schemes over S. By the Yoneda Lemma, $\mathbf{Sch}_{/S}$ embeds into $\mathbf{Fun}^{\mathrm{op}}_{/S}$, i.e., we can identify $X \in \mathbf{Sch}_{/S}$ with its functor of points $\underline{X}$.

Example VI.1.1. Let $S = \operatorname{Spec}\mathbb{Z}$, $X = \operatorname{Spec}\mathbb{Z}[t, t^{-1}]$, R any ring, and $T = \operatorname{Spec} R$. Then

$$X(T) = \operatorname{Hom}_{\mathbf{Sch}}(T, X) = \operatorname{Hom}_{\mathbf{Rings}}(\mathbb{Z}[t, t^{-1}], R) = R^\times.$$

We could ask if there is $X \in \mathbf{Sch}_{/\mathbb{Z}}$ such that $X(\operatorname{Spec} R) = R^\times$. Indeed, there is; X is usually denoted $\mathbb{G}_m$ and called the "multiplicative group". In other words, the functor $\operatorname{Spec} R \mapsto R^\times$ is representable by $X = \operatorname{Spec}\mathbb{Z}[t, t^{-1}]$.

Over $\mathbb{C}$, an elliptic curve is a genus 1 compact Riemann surface, i.e., a compact smooth algebraic curve of genus 1. Recall:

$$\operatorname{genus}(X) = \dim_{\mathbb{C}}(\text{differentials}).$$

Definition VI.1.2 (Dimension and relative dimension)**.** Given an S-scheme $X \to S$, the *relative dimension* of X at $s \in S$ is $\dim X_s$, where $X_s := X \times_S s$ is the fiber over s. If this is independent of s, we say X has relative dimension $\dim X_s$.

Definition VI.1.3. An S-scheme $f : X \to S$ is *smooth* of relative dimension n if

(i) f is locally of finite presentation.

(ii) f has relative dimension n.

(iii) The sheaf of relative differentials $\Omega_{X/S}$ is locally free of rank n.

If $n = 1$, we call X a *smooth curve*.

Definition VI.1.4. An elliptic curve $\pi : E \to S$ is a proper smooth curve such that:

(i) For any $\mathrm{Spec}(\overline{k}) \hookrightarrow S$ (where $\overline{k}$ is an algebraically closed field), $E_{\overline{k}} := E \times_S \mathrm{Spec}(\overline{k})$ is connected, and $\dim_{\overline{k}} \pi_* \Omega_{E_{\overline{k}}/\mathrm{Spec}(\overline{k})} = 1$.

(ii) There is a section $0 : S \to E$.

VI.2 2014-04-30: Elliptic curves

Let $\pi : E \to S$ be an elliptic curve with identity $0 : S \to E$, i.e., for all $\mathrm{Spec}(\overline{k}) \to S$, $E_{\overline{k}}$ is connected and $\pi_* \Omega_{E_{\overline{k}}/\mathrm{Spec}\,\overline{k}}$ is 1-dimensional. Hence, E is a commutative group scheme over S.

Given $E \xrightarrow{\pi} S$, let $\underline{\omega} := \underline{\omega}_{E/S} := \pi_* \Omega_{E/S}$; this is an invertible $\mathcal{O}_S$-module. By Serre duality, $\pi_* \Omega_{E/S} \cong R^1 \pi_* \mathcal{O}_E$.

VI.2.1 Geometric definition of modular forms of level 1

This definition is due to Katz.

Definition VI.2.1. Let $k \in \mathbb{Z}$. A (meromorphic) modular form of weight k (and level 1) is a rule f which assigns to any E/S an $f(E/S) \in H^0(S, \underline{\omega}^{\otimes k}) = \Gamma(S, \omega^{\otimes k})$ satisfying the following properties:

(i) isomorphism independence: if $E \xrightarrow{\simeq} E'$ (as S-schemes), then $f(E/S) = f(E'/S)$.

(ii) independence of base change: if $g : S' \to S$, then $f(E/S) = f(E_{S'}/S')$.

Let $\mathcal{A}_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ be the $\mathbb{Z}$-module of such rules.

Remark VI.2.2.

- If we fix $S_0 = \mathrm{Spec}\, R_0$ and consider S over S_0, then we get $\mathcal{A}_k(\mathrm{SL}_2(\mathbb{Z}), R_0)$.
- If A is a commutative ring that is flat over $\mathbb{Z}$, then
$$\mathcal{A}_k(\mathrm{SL}_2(\mathbb{Z}), A) = \mathcal{A}_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}) \otimes_{\mathbb{Z}} A.$$
- $\mathcal{A}_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}) = \mathcal{A}_k(\mathrm{SL}_2(\mathbb{Z}))$.

Here is an equivalent definition:

Definition VI.2.3. Consider pairs $(E/R, \omega)$ with R a ring, E an elliptic curve over R, and ω a nowhere vanishing section of $\underline{\omega}$. Say $\varphi : (E, \omega) \xrightarrow{\simeq} (E', \omega')$ if $\varphi : E \xrightarrow{\simeq} E$ and $\omega = \varphi^* \omega'$.

Then we define $\mathcal{A}_k(\mathrm{SL}_2(|Z), \mathbb{Z})$ to be the set of rules $(E/R, \omega) \mapsto f(E, \omega)$ which are:

(i) isomorphism independent;

(ii) invariant under base change;

(iii) for all $\lambda \in R^\times$, $f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$.

VI.2.2 Tate curves

The behavior at infinity is described by Tate curves. Let

$$E_{\mathrm{Tate}/\mathbb{Z}((q))} : y^2 + xy = x^3 - b_2(q)x - b_3(q),$$

where $b_2, b_3 \in q\mathbb{Z}[[q]]$ are defined by

$$b_2(q) = 5\sum_{n\geq 1} \frac{n^3 q}{1-q^n},$$
$$b_3(q) = \sum_{n\geq 1} \frac{7n^5+5n^2}{12}\cdot\frac{q^n}{1-q^n}.$$

The discriminant and j-invariant of the Tate curve are

$$\Delta(E_{\mathrm{Tate}}) = q\prod_{n\geq 1}(1-q^n)^{24},$$
$$j(E_{\mathrm{Tate}}) = \frac{1}{q} + 744 + 196884q + \ldots$$

Geometrically, we can think of $\mathbb{Z}((q))$ as a formal neighborhood of the cusp of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$, and E_{Tate} is an elliptic curve over that neighborhood. If $f \in \mathcal{A}_k(\mathrm{SL}_2(\mathbb{Z}), R_0)$, then its q-expansion is $f(E_{\mathrm{Tate}}, \omega_{\mathrm{can}}) \in \mathbb{Z}((q)) \otimes_{\mathbb{Z}} R_0$, where $\omega_{\mathrm{can}} = \frac{dx}{y}$ is the "canonical differential".

Aside VI.2.4. Say E is an elliptic curve over $\mathbb{Z}_p$ with multiplicative reduction. Then *Tate's uniformization theorem* says that $E \cong \mathbb{Q}_p^\times/q^{\mathbb{Z}}$ for some q. (This is a p-adic, multiplicative analogue of a complex torus $\mathbb{C}/\Lambda$.)

VI.2.3 Moduli

Let

$$\mathcal{B} = \left\{(\omega_1,\omega_2) \in \mathbb{C}^\times \times \mathbb{C}^\times : \frac{\omega_2}{\omega_1} \in \mathfrak{h}\right\}.$$

If $\omega = (\omega_1, \omega_2)$, and if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then

$$\omega' = \begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix} = \gamma\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix}.$$

We have

$$\mathscr{L}/\mathbb{C}^\times \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{B}/\mathbb{C}^\times \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}.$$

Take $\Gamma = \Gamma_0(N)$. What does $\Gamma\backslash\mathfrak{h}$ parametrize?

Let $\Lambda = \Lambda(\omega_1,\omega_2)$, $\gamma = \begin{pmatrix} a & b \\ Nc & d\end{pmatrix}$, and $\omega' = \gamma\omega = \begin{pmatrix} a\omega_1 + b\omega_2 \\ Nc\omega_1 + d\omega_2\end{pmatrix}$, so

$$\frac{\omega_2'}{N} \equiv \frac{d\omega_2}{N} \pmod{\Lambda}.$$

We have $(d, N) = 1$, so $\frac{\omega_2}{N}$ and $\frac{\omega_2'}{N}$ generate the same cyclic subgroup of order N in $\mathbb{C}/\Lambda$.

Lemma VI.2.5. $\Gamma_0(N)\backslash\mathcal{B} \cong \mathscr{L}_0(N) := \{(\Lambda, C_N) \mid C_N \subseteq \Lambda \text{ cyclic of order } N\}$.

Call the choice of $C_N \le E$ (cyclic of order N) a *level* $\Gamma_0(N)$*-structure*.

The following is due to work of Deligne–Rapoport, Drinfeld, and Katz–Mazur. Given $E \xrightarrow{\pi} S$, we can define $C_N \le E$. To formalize this, let (**Ell**) be the category whose objects are elliptic curves $E \xrightarrow{\pi} S$, and whose maps are commutative squares

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

such that $E' \cong E \times_S S'$. A "*moduli problem*" is a contravariant functor

$$\mathcal{P} : (\mathbf{Ell}) \to (\mathbf{Sets}).$$

An element of $\mathcal{P}(E/S)$ is called a level $\mathcal{P}$ structure on E/S.

Definition VI.2.6. $\mathcal{A}_k(\mathcal{P}, \mathbb{Z})$ is a rule that assigns to $(E/S, \alpha \in \mathcal{P}(E/S))$ some $f(E, \alpha) \in H^0(S, \underline{\omega}^{\otimes k})$ independent under isomorphism and base change.

Say $\mathcal{P}$ is representable, i.e., $\mathcal{E}_\mathcal{P} \to \mathcal{M}_\mathcal{P}$ and $\alpha_\mathcal{P} \in \mathcal{P}(\mathcal{E}_\mathcal{P}, \mathcal{M}_\mathcal{P})$ such that any $(E/S, \alpha)$ is a "pullback" of $(\mathcal{E}_\mathcal{P}/\mathcal{M}_\mathcal{P}, \alpha_\mathcal{P})$, i.e., $E \cong \mathcal{E}_\mathcal{P} \times_{\mathcal{M}_\mathcal{P}} S$. Then there is an isomorphism

$$\begin{aligned} \varphi : \mathcal{A}_k(\mathcal{P}, \mathbb{Z}) &\xrightarrow{\cong} H^0\big(\mathcal{M}_\mathcal{P}, (\underline{\omega}_{\mathcal{E}_\mathcal{P}/\mathcal{M}_\mathcal{P}})^{\otimes k}\big), \\ \varphi(f) &= f(\mathcal{E}_\mathcal{P}/\mathcal{M}_\mathcal{P}, \alpha_\mathcal{P}) \qquad \forall f \in \mathcal{A}_k(\mathcal{P}, \mathbb{Z}). \end{aligned}$$

VI.3 2014-05-05: Galois representations attached to modular forms

Say f is a newform in $S_k(N, \chi)$. Then

$$L(s, f) = \prod_p \left(1 - a_p p^{-s} + \chi(p) p^{k-1-2s}\right)^{-1}.$$

This looks like the L-function of a 2-dimensional Galois representation.

If A is an abelian variety of dimension g (e.g., if $g = 1$, then E is an elliptic curve). If $\overline{k}$ is algebraically closed and $(n, \operatorname{char} \overline{k}) = 1$, then

$$A(\overline{k})[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

If A is defined over $\mathbb{Q}$, and $G_\mathbb{Q} := \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $A(\overline{Q})$ and $A(\overline{Q})[n]$ for all n, then for ℓ prime, we define the *ℓ-adic Tate module* of A to be

$$T_\ell(A) \stackrel{\text{def}}{=} \varprojlim_n A(\overline{Q})[\ell^n].$$

Note that $T_\ell(A) \cong \mathbb{Z}_\ell^{2g}$ as modules. So we get a $2g$-dimensional Galois representation

$$\rho_{A,\ell} : G_\mathbb{Q} \to \mathrm{GL}(T_\ell(A)) \cong \mathrm{GL}_{2g}(\mathbb{Z}_\ell).$$

Let $p \neq \ell$ be prime. Define

$$P_{A,p}(x) \stackrel{\text{def}}{=} \det\left(1 - x\rho_{A,\ell}(\mathrm{Frob}_p)|V_\ell(A)^{I_p}\right).$$

Here, $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ induces an inclusion $G_{\mathbb{Q}_p} \hookrightarrow G_\mathbb{Q}$. There is an exact sequence

$$1 \to I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1,$$

and $G_{\mathbb{F}_p} \cong \hat{\mathbb{Z}} = \langle \widehat{a \mapsto a^p} \rangle$, the profinite completion of the free cyclic group generated by the Frobenius map $\mathrm{Frob}_p : a \mapsto a^p$. Hence, Frob_p is well-defined in $G_{\mathbb{Q}_p}$ up to the inertia group I_p.

Moreover, the inclusion $G_\mathbb{Q} \hookrightarrow G_{\mathbb{Q}_p}$ is well-defined up to conjugation, so Frob_p is well-defined in $G_\mathbb{Q}$ up to inertia and conjugation. Since I_p acts trivially on $V_\ell(A)^{I_p}$ (by definition), and characteristic polynomials are invariant under conjugation, $P_{A,p}(x)$ is well-defined.

Theorem VI.3.1. *$P_{A,p}(x)$ is independent of ℓ (assuming $\ell \neq p$).*

Example VI.3.2. Let E be an elliptic curve, and assume $\rho_{A,\ell}$ is unramified at p, i.e., $V_\ell(A)^{I_p} = V_\ell(A)$. Then

$$\det(1 - x \cdot \rho_{A,\ell}(\mathrm{Frob}_p)) = 1 - a_p x + p x^2,$$

where $a_p = p + 1 - \#E(\mathbb{F}_p)$. Furthermore, if N is the conductor of E and $p \mid N$, then

$$P_{E,p}(x) = 1 - a_p x,$$

where $a_p = 0$ if $p^2 \mid N$ (additive reduction), and $a_p = \pm 1$ otherwise (multiplicative reduction); $a_p = 1$ corresponds to split multiplicative reduction, and $a_p = -1$ nonsplit.

Returning to the general case, define

$$L(s, A) \stackrel{\text{def}}{=} \prod_p P_{A,p}(p^{-s})^{-1}.$$

For example, if $A = E$, then

$$L(s, E) = \prod_p \left(1 - a_p p^{-s} + \mathbb{1}_N(p) p^{1-2s}\right)^{-1},$$

where $\mathbb{1}_N$ is the trivial Dirichlet character mod N.

For an elliptic curve $E/\mathbb{F}_p$, the numbers $\#E(\mathbb{F}_{p^n})$ can be put together into a generating series $Z(x, E_{/\mathbb{F}_p})$. Weil proved that

$$Z(x, E_{/\mathbb{F}_p}) = \frac{1 - a_p x + x^2}{(1 - x)(1 - px)}.$$

Weil also proved the Riemann hypothesis for elliptic curves, i.e., $|\alpha_p| = |\beta_p| = p^{1/2}$, where

$$1 - a_p x + x^2 = (1 - \alpha_p x)(1 - \beta_p x).$$

This generating function is closely related to the L-function:

$$L(s, E_{/\mathbb{F}_p}) = Z(p^{-s}, E_{/\mathbb{F}_p}).$$

We can attach Galois representations to newforms:

Theorem VI.3.3. *If $f = \sum_{n\geq 1} a_n q^n \in S_k(N, \chi)$ is a newform, then for all primes ℓ, there is a continuous, irreducible Galois representation*

$$\rho_{f,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell}),$$

unramified at p for all $p \nmid \ell N$, such that

$$P_{\rho_{f,\ell},p}(x) = 1 - a_p x + \chi(p) p^{k-1} x^2.$$

Remark VI.3.4. The Galois representation $\rho_{f,\ell}$ is uniquely defined by the collection of polynomials $\left\{P_{\rho_{f,\ell},p} : p \nmid \ell N\right\}$. This is because $\{\mathrm{Frob}_p : p \nmid \ell N\}$ is dense in $G_{\mathbb{Q}}$ (by Čebotarev density), and we can recover an irreducible representation from characteristic polynomials.

This was proved for $k = 2$ by Eichler, Shimura, and Igusa: the Hecke operators have a geometric interpretation as well. Let $J_1(N) = \mathrm{Jac}(X_1(N)/\mathbb{Q})$ over $\mathbb{Q}$. The Hecke algebra $\mathbb{T}$ acts on $J_1(N)$. For all $p \nmid \ell N$, by the *Eichler–Shimura congruence*, $\rho_{J_1(N),\ell}(\mathrm{Frob}_p)$ satisfies $x^2 - T(p)x + T(p,p)$.

Under this correspondence, $f \longleftrightarrow \lambda_f : \mathbb{T} \to K_f = \mathbb{Q}(a_n)$. Let $I_f := \ker \lambda_f$. Then

$$\mathbb{T}/I_f \xrightarrow{\simeq} \mathbb{Z}[a_n] =: \mathcal{O}_f,$$

and $A_f := J_1(N)/I_f J_1(N)$ is an abelian variety of dimension $d_f := [K_f : \mathbb{Q}]$. Then $\rho_{f,\ell} := T_\ell(A_f)$ is a $2d_f$-dimensional $\mathbb{Z}_\ell$-module, and hence a 2-dimensional $\mathcal{O}_f$-module.

VI.4 2014-05-07: Galois representations of higher weight

The following construction of Galois representations for $k \geq 2$ is due to Deligne and Shimura (unpublished).

Let $\mathcal{E} \xrightarrow{\pi} X_1(N)$ be the universal elliptic curve over $\mathbb{Q}$, and consider the *Kuga–Sato variety*

$$\mathcal{E}^{(n)} := \underbrace{\mathcal{E} \times_{X_1(N)} \cdots \times_{X_1(N)} \mathcal{E}}_{n \text{ copies}}.$$

Deligne defines a "canonical desingularization" $\tilde{\mathcal{E}}^{(n)}$. Consider the Galois representation

$$H^{k-1}_{\text{ét}}(\tilde{\mathcal{E}}^{(k-2)} \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}_\ell})$$

with actions of $G_{\mathbb{Q}}$ and $\mathbb{T}_1(N)$. Let $\rho_{f,\ell}$ be the piece on which $\mathbb{T}_1(N)$ acts like it acts on f.

For $k = 2$, we have $\tilde{\mathcal{E}}^{(0)} = X_1(N)$ and

$$H^1_{\text{ét}}(X_1(N), \overline{\mathbb{Q}_\ell})^\vee \cong T_\ell(J_1(N)).$$

Note VI.4.1. This attaches a *motive* $M(f)$ to f. (Deligne looked at

$$H^1_{\text{ét}}\big(X_1(N), \operatorname{Sym}^{k-2} R^1\pi_*(\overline{\mathbb{Q}_\ell})\big),$$

which is isomorphic.)

For $k = 1$, this is a theorem of Deligne–Serre by looking at congruences with higher weight forms.

Chapter VII

Generalizations of modular forms

VII.1 2014-05-07, continued

For the rest of today, we'll talk about some classical generalizations of modular forms.

VII.1.1 Hilbert modular forms

Let F be a totally real number field of degree d, let $\sigma_1, \ldots, \sigma_d : F \hookrightarrow \mathbb{R}$ be the d distinct embeddings into $\mathbb{R}$, and let $\mathcal{O}_F$ be its ring of integers. This yields

$$\mathrm{GL}_2(\mathcal{O}_F) \subseteq \mathrm{GL}_2(F) \overset{\prod \sigma_i}{\hookrightarrow} \mathrm{GL}_2(\mathbb{R})^d.$$

If $a \in F$, we say a is *totally positive* (denoted $a \gg 0$) if $\sigma_i(a) > 0$ for all i. Let

$$\mathrm{GL}_2^+(F) := \{\alpha \in \mathrm{GL}_2(F) : \det(\alpha) \gg 0\}.$$

If $\alpha \in \mathrm{GL}_2^+(F)$ and $\underline{z} = (z_1, \ldots, z_d) \in \mathfrak{h}^d$, let

$$\alpha \cdot \underline{z} := (\sigma_i(\alpha) \cdot z_i)_{i=1,\ldots,d},$$
$$j(\alpha, \underline{z} := \prod_{i=1}^{d} j(\sigma_i(\alpha), z_i).$$

For $\underline{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and $a \in F$, let

$$a^{\underline{k}} := \prod_{i=1}^{d} \sigma_i(a)^{k_i}.$$

For $f : \mathfrak{h}^d \to \mathbb{C}$, let

$$(f|_k\alpha)(\underline{z}) := \det(\alpha)^{-\underline{k}/2} j(\alpha, \underline{z}) f(\alpha \cdot \underline{z})$$

for $\alpha \in \mathrm{GL}_2^+(F)$.

Let $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_F)$ be a subgroup of finite index.

Definition VII.1.1. A *Hilbert modular form* of weight $\underline{k}$ and level Γ is a function $f : \mathfrak{h}^d \to \mathbb{C}$ such that:

(i) f is holomorphic.

(ii) $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

(iii) $f|_k\alpha$ is holomorphic at ∞ for all $\alpha \in \mathrm{GL}_2^+(F)$, in the following sense: Let

$$M_\Gamma := \left\{ m \in \mathcal{O}_F : \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \Gamma \right\},$$
$$M_\Gamma^\vee := \left\{ \nu \in F : \mathrm{Tr}_{F/\mathbb{Q}}(\nu m) \in \mathbb{Z} \ \forall m \in M_\Gamma \right\}.$$

Then f satisfying (i) and (ii) can be written as

$$f(\underline{z}) = \sum_{\nu \in M_\Gamma^\vee} a_\nu e^{2\pi i \,\mathrm{Tr}(\nu \underline{z})},$$

where $\mathrm{Tr}(\nu\underline{z}) := \sum_{i=1}^d \sigma_i(\nu) z_i$.

We say f is holomorphic at ∞ if $a_\nu = 0$ unless $\nu = 0$ or $\nu \gg 0$.

Theorem VII.1.2 (Köcher's principle)**.** *If $d \geq 2$, if f satisfies (i) and (ii), then it satisfies (iii).*

Definition VII.1.3. As before, f is a cusp form if $a_0(f|_k\alpha) = 0$ for all α.

If $\underline{k} = (k, k, \ldots, k)$, we say f has *parallel weight* k.

Theorem VII.1.4. *Some facts about Hilbert modular forms:*

- $\dim_{\mathbb{C}} M_{\underline{k}}(\Gamma) < \infty$.
- $M_{\underline{0}}(\Gamma) = \mathbb{C}$.
- $S_{\underline{0}}(\Gamma) = 0$.
- *If $k_i \leq 0$ for some i and $\underline{k} \neq \underline{0}$, then $M_{\underline{k}}(\Gamma) = 0$.*
- *If $\underline{k}$ is* not *parallel, then $M_{\underline{k}}(\Gamma) = S_{\underline{k}}(\Gamma)$.*

Remark VII.1.5. In the representation theory of $\mathrm{GL}_2(\mathbb{Q})$, every irreducible representation is of the form $\mathrm{Sym}^{k_0}\mathbb{Q}^2 \otimes \det^{k_1}$ for some k_0, k_1. The weight of the representation is (k_0, k_1), corresponding to characters of the maximal torus $\begin{pmatrix} \mathbb{Q}^\times & 0 \\ 0 & \mathbb{Q}^\times \end{pmatrix}$. In our context, only $k_0 - k_1 =: k$ matters; this is the weight k of a modular form.

For Hilbert modular forms, the weights of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2(F)$ correspond to $(k_0, \ldots, k_d)$, and $\underline{k} = (k_1 - k_0, k_2 - k_0, \ldots, k_d - k_0)$.

Let $Y(\Gamma) := \Gamma\backslash\mathfrak{h}^d$; this is a d-dimensional complex manifold and a d-dimensional algebraic variety over $\mathbb{C}$. Moreover, $Y(1)$ parametrizes principally polarized d-dimensional abelian varieties with RM (real multiplication) by $\mathcal{O}_F$, i.e., a triple (A, λ, ι) with A a d-dimensional abelian variety, $\lambda : A \to A^\vee$, and $\iota : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$.

If $\underline{k}$ is paritious (i.e., $k_i = k_j \bmod 2$), then we get a Galois representation

$$\rho_{f,\ell} : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$$

(Wiles, Taylor, Carayol, Blasius–Rogowski, Ohta, Janis, Tunnell–?).

Next time, we'll talk about Siegel modular forms and other generalizations.

VII.2 2014-05-09: Siegel modular forms, CFT, and Langlands

VII.2.1 Siegel modular forms

Let $\mathrm{Sp}_{2g}(\mathbb{R}) = \left\{\alpha \in \mathrm{GL}_{2g}(\mathbb{R}) : \alpha^T J \alpha = J\right\}$, where $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. (Note that $\mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$, so the $g = 1$ case recovers classical modular forms.) This has a maximal compact subgroup $K_{2g} := \mathrm{O}_{2g}(\mathbb{R}) \cap \mathrm{Sp}_{2g}(\mathbb{R})$.

The *Siegel upper-half space* of degree g is

$$\mathrm{Sp}_{2g}(\mathbb{R})/K_{2g} \cong \mathfrak{h}_g := \left\{Z = X + iY \in M_g(\mathbb{C}) : Z^T = Z \text{ and } Y > 0\right\}.$$

This is a $\frac{1}{2}g(g+1)$-dimensional complex manifold with an action of $\mathrm{Sp}_{2g}(\mathbb{R})$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

Congruence subgroups are defined by

$$\Gamma(N) := \left\{\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) : \gamma \equiv I_{2g} \mod N\right\}.$$

Definition VII.2.1. A *Siegel modular form* is a function $f : \mathfrak{h}_g \to \mathbb{C}$ such that:

(i) f is holomorphic.

(ii) $f|_k\gamma = f$ for all $\gamma \in \Gamma$, where

$$(f|_k\gamma)(z) := j(\gamma, z)^{-k} f(\gamma \cdot z)$$

and

$$j\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z\right) := \det(CZ + D).$$

(iii) If $g = 1$, then f is holomorphic at ∞. (If $g \geq 2$, this is automatic.)

We get a Fourier expansion

$$f(Z) = a_0 + \sum_{Q>0} a_Q e^{2\pi i \operatorname{Tr}(QZ)},$$

where Q ranges over positive-definite, symmetric $g \times g$ matrices such that the associated quadratic form $(x_1, \ldots, x_g)Q(x_1, \ldots, x_g)^T$ has integer coefficients.

Let $Y(\Gamma) := \Gamma\backslash\mathfrak{h}_g$. Then, in particular, $Y(1)$ parametrizes all principally polarized abelian varieties of dimension g.

We also have a notion of Eisenstein series. For $\mathrm{SL}_2(\mathbb{R})$, Dirichlet characters φ, ψ yield $E_k^{\varphi,\psi}$. For $g = 2$, given a Siegel modular form f, we have an attached Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}_\ell})$.

VII.2.2 Class field theory

Let F be a number field (or a global field). Let $\mathbb{A}_F$ be the ring of adeles of F, and let $C_F := F^\times \backslash \mathbb{A}_F^\times$ be the idele class group.

Class field theory consists of an Artin reciprocity map

$$\begin{aligned} \mathrm{rec}_F : C_F &\to G_F^{\mathrm{ab}} = \text{maximal abelian quotient of } \mathrm{Gal}(\overline{F}/F) \\ p &\mapsto \mathrm{Frob}_p \end{aligned}$$

with dense image. We can think of this as a bijection

$$\left\{ \begin{array}{c} \text{continuous characters} \\ \chi : G_F \to \mathbb{C}^\times \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{finite order continuous} \\ \text{characters } C_F \to \mathbb{C}^\times \end{array} \right\}.$$

(Similarly, if F is a local field, an analogous statement is true with $C_F := F^\times$.)

What if we consider *all* continuous characters, rather than only those of finite order? Class field theory provides the *Weil group* of F, a group W_F with a map $\varphi_F : W_F \to G_F$ with dense image.

Let E/F be a finite extension. Then $W_E = \varphi_F^{-1}(G_E)$. Define the *relative Weil group*

$$W_{E/F} := W_F / \overline{[W_E, W_E]}.$$

If E/F is Galois, we obtain a short exact sequence

$$1 \to C_E \to W_{E/F} \to \mathrm{Gal}(E/F) \to 1.$$

Now the reciprocity map provides a topological isomorphism

$$\mathrm{rec}_F : C_F \xrightarrow{\simeq} W_F^{\mathrm{ab}},$$

giving a bijection

$$\left\{ \begin{array}{c} \text{1-dimensional irreducible} \\ \text{representations of } C_F \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{1-dimensional irreducible} \\ \text{representations of } W_F \end{array} \right\}.$$

VII.2.3 Langlands reciprocity

What if we look at higher-dimensional irreducible representations of W_F? Langlands conjectured a bijection

$$\left\{ \begin{array}{c} \text{automorphic irreducible} \\ \text{representations of } \mathrm{GL}_n(\mathbb{A}_F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} n\text{-dimensional irreducible} \\ \text{representations of } W_F \end{array} \right\}.$$

Langlands proved this conjecture for $F = \mathbb{R}$ and $G = \mathrm{GL}_1$ or any reductive group over $\mathbb{R}$, giving a natural correspondence

$$\left\{ \begin{array}{c} \text{admissible representation of} \\ G(\mathbb{R}) \end{array} \right\} \longleftrightarrow \left\{ \rho : W_{\mathbb{R}} \to {}^L G(\mathbb{C}) \right\},$$

where ${}^L G$ denotes the *Langlands dual* of G. Langlands predicted that generalizing this correspondence should provide insight into global and local fields.

Here's the general idea: The collection of automorphic representations of $G(\mathbb{A}_F)$ should be a *neutral Tannakian category*. As such, there should exist a group L_F (with a map $L_F \to W_F \to G_F$), the *Langlands group*, such that this category is the category of representations of L_F.

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