Math 863 Notes
Toric Varieties

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Figure 1: The torii of Itsukushima Shrine on Miyajima.

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Chapter I

Affine toric varieties

I.1 2014-09-02

I.1.1 Course information

Course website: `http://www.math.wisc.edu/~derman/863.html`

Textbook: [CLS].

Grading is entirely based on 6 homework assignments. Homework is required to be done in groups of 3 to 4. (No exams.)

There will be an AMS session on toric geometry at UW–Eau Claire on September 20–21. (No funding available. Carpools will be set up.)

I.1.2 Motivation and remarks

We can study varieties $X \subseteq \mathbb{P}^n$ extrinsically, or we can study X intrinsically. Embeddings $X \hookrightarrow \{\text{toric variety}\}$ provide a richer context for extrinsic study.

The whole course will be over $\mathbb{C}$; however, much of the theory works over arbitrary fields, and almost all over arbitrary algebraically closed fields of any characteristic.

In this course, all varieties are assumed to be integral.

I.1.3 The torus

We define the *torus*

$$T := (\mathbb{C}^*)^n = \operatorname{Spec} \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}] = \operatorname{Spec} \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}].$$

Points of T correspond to $(t_1, \dots, t_n)$ with $t_i \in C^*$. There is a multiplication map

$$T \times T \to T$$

given by componentwise multiplication, corresponding to the ring map

$$\begin{aligned} \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}] &\to \mathbb{C}[y_1^{\pm}, \dots, y_n^{\pm}, z_1^{\pm}, \dots, z_n^{\pm}], \\ x_i &\mapsto y_i z_i. \end{aligned}$$

Proposition I.1.1 ([CLS], 1.1.1)**.** *(1) Let $\Phi : T_1 \to T_2$ be a morphism of tori (i.e., a morphism of varieties and of groups). Then $\varphi(T_1)$ is a torus and is closed in T_2.*

(2) Let T be a torus and $H \subseteq T$ an algebraic subgroup (i.e., a subvariety and a subgroup). Then H is a torus.

I.1.4 Characters

Definition I.1.2. A *character* of a torus T is a map $\chi : T \to \mathbb{C}^*$ of groups.

Example I.1.3. Let $T = (\mathbb{C}^*)^n$ and $m = (a_1, \dots, a_n) \in \mathbb{Z}^n$. Then

$$\chi^m(t_1, \dots, t_n) := t_1^{a_1} \cdot \ldots \cdot t_n^{a_n}$$

is a character.

Fact I.1.4. All characters arise in the above way. In particular,

$$M := \text{character lattice of } T = \mathrm{Hom}_{\mathbf{Grp}}(T, \mathbb{C}^*) \cong \mathbb{Z}^n.$$

I.1.5 Linear actions

Suppose a torus T acts linearly on a finite-dimensional vector space W. Then

$$W = \bigoplus_{m \in M} W_m$$

as a T-representation, where

$$W_m := \left\{ w \in W \;\middle|\; t \cdot w = \chi^m(t) w \; \forall t \in T \right\}.$$

Example I.1.5. Let $A = \mathbb{C}[x^{\pm}, y^{\pm}]$ and $T = \operatorname{Spec} A = (\mathbb{C}^*)^2$. Then $M \cong \mathbb{Z}^2$ and

$$A = \bigoplus_{(m_1, m_2)} A_{(m_1, m_2)} = \bigoplus_{(m_1, m_2) \in \mathbb{Z}^2} \mathbb{C} \cdot x^{m_1} y^{m_1}.$$

I.1.6 One-parameter subgroups

Definition I.1.6. A *one-parameter subgroup* of T is a group homomorphism

$$\lambda : \mathbb{C}^* \to T.$$

Example I.1.7. Let $T = (\mathbb{C}^*)^n$ and $u = (b_1, \dots, b_n) \in \mathbb{Z}^n$. Then

$$\lambda^u(t) = (t^{b_1}, \dots, t^{b_n})$$

defines a one-parameter subgroup.

All one-parameter subgroups arise in this way. We denote

$$N = \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{C}^*, T) \cong \mathbb{Z}^n.$$

I.1.7 Duality of lattices

The lattices M and N are dual in the sense that there is a perfect bilinear pairing

$$M \times N \to \mathbb{Z}.$$

There are two ways to define this:

Intrinsic For $m \in M$ and $u \in N$,

$$\chi^m \circ \lambda^u : \mathbb{C}^* \to \mathbb{C}^*$$

is a character for $\mathbb{C}^*$, and hence has the form $t \mapsto t^\ell$ for some $\ell \in \mathbb{Z}$. Define the pairing by

$$\langle m, n \rangle := \ell.$$

Extrinsic A choice of isomorphism $T \cong (\mathbb{C}^*)^n$ induces bases $M \cong \mathbb{Z}^n$ and $N \cong \mathbb{Z}^n$. For $m = (m_1, \ldots, m_n) \in M$ and $u = (u_1, \ldots, u_n) \in N$, define

$$\langle m, n \rangle := (m_1, \ldots, m_n) \cdot (u_1, \ldots, u_n)$$

to be the dot product.

In summary,

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) &\cong N, \\ \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) &\cong M. \end{aligned}$$

Moreover, there is a natural isomorphism

$$\begin{aligned} N \otimes_{\mathbb{Z}} \mathbb{C}^* &\xrightarrow{\simeq} T, \\ u \otimes t &\mapsto \lambda^u(t). \end{aligned}$$

Hence, we can think of $T = T_N$ with N being the lattice points inside the torus T.

I.1.8 Affine toric varieties

Definition I.1.8. An *affine toric variety* is an (integral) affine variety $V = \operatorname{Spec} A$ containing a torus $T \cong (\mathbb{C}^*)^n$ as an open subset, such that the action of T on itself extends to an action $T \times V \to V$. (The embedding of the torus is part of the data of the toric variety.)

Example I.1.9. The cuspidal curve $V(x^3 - y^2) \cong \{(t^2, t^3) \mid t \in \mathbb{C}\} \subseteq \mathbb{A}^2$ is an affine toric variety:

$$\mathbb{C}^* = \left\{(t^2, t^3) \;\middle|\; t \neq 0\right\} \subseteq V(x^3 - y^2).$$

Example I.1.10. $V(xy - zw) \subseteq \mathbb{C}^4$ is a 3-dimensional affine toric variety. The torus is

$$T := \left\{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \;\middle|\; t_i \in \mathbb{C}^*\right\},$$

and the action is

$$\begin{aligned} T \times V(xy - zw) &\to V(xy - zw), \\ (s_1, s_2, s_3), (x, y, z, w) &\mapsto (s_1 x, s_2 y, s_3 z, s_1 s_2 s_3^{-1} w). \end{aligned}$$

Remark I.1.11. The interesting, hard part of the variety is at the "boundary", outside the torus. For example,

$$\begin{aligned} \mathbb{P}^2 \setminus (\mathbb{C}^*)^2 &= \{\text{union of three lines}\}, \\ (\mathbb{P}^1 \times \mathbb{P}^1) \setminus (\mathbb{C}^*)^2 &= \{\text{union of four lines}\}. \end{aligned}$$

I.2 2014-09-04

Today: Finishing section 1.1.

I.2.1 Example: A torus action that doesn't extend

Consider the embedding

$$\begin{aligned} (\mathbb{C}^*)^2 &\subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2, \\ (s,t) &\mapsto (s, t+s^2). \end{aligned}$$

We want to find an action

$$(\mathbb{C}^*)^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^2$$

extending

$$(s,t), \left[\frac{x}{z} : \frac{y}{z} : 1\right] \mapsto \left[\frac{sx}{z} : \frac{ty}{z} + \frac{s^2x^2}{z^2} : 1\right] = \left[sxz : tyz + s^2x^2 : z^2\right].$$

But this can't be extended to $[0:1:0]$.

I.2.2 Constructing affine toric varieties

Fix a finite set $\mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M$, and consider the map

$$\begin{aligned} \Phi_{\mathcal{A}} : T_N &\to (\mathbb{C}^*)^s, \\ t &\mapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)). \end{aligned}$$

Define $Y_{\mathcal{A}}$ to be the Zariski closure of the image of $\Phi_{\mathcal{A}}$.

Write $\mathbb{Z}\mathcal{A} \subseteq M$ for the sublattice generated by $\mathcal{A}$.

Proposition I.2.1 ([CLS], Prop. 1.1.8)**.** *$Y_{\mathcal{A}}$ is an affine toric variety with character lattice $\mathbb{Z}\mathcal{A}$. (Thus,* $\dim Y_{\mathcal{A}} = \operatorname{rank} \mathbb{Z}\mathcal{A}$*.)*

Example I.2.2. If $\mathcal{A} = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, then

$$Y_{\mathcal{A}} = \text{closure of} \begin{pmatrix} (\mathbb{C}^*)^2 \to \mathbb{C}^4 \\ (s,t) \mapsto (s^3, s^2t, st^2, t^3) \end{pmatrix},$$

which is the affine cone over the twisted cubic.

I.2.3 Toric ideals

Consider the exact sequence
$$0 \to L \to \mathbb{Z}^s \xrightarrow{[\mathcal{A}]} \to M.$$
For any $\ell = (\ell_1, \ldots, \ell_s) \in L$, we have $\sum_i \ell_i m_i = 0$. Define $\ell_+ \in \mathbb{Z}^s$ to consist of all strictly positive entries of ℓ, and $-\ell_-$ all strictly negative entries, so that $\ell = \ell_+ - \ell_-$ and $\ell_+, \ell_- \in \mathbb{N}^s$.

In particular, X^{ℓ_+} and X^{ℓ_-} are both monomials.

Lemma I.2.3. *The binomial $X^{\ell_+} - X^{\ell_-}$ vanishes on $Y_{\mathcal{A}}$.*

Proof. It's enough to show that $X^{\ell_+} - X^{\ell_-}$ vanishes on the torus $\Phi((\mathbb{C}^*)^n)$. We have
$$\Phi_{\mathcal{A}}(t) = (t^{m_1}, \ldots, t^{m_s}).$$
Evaluating $X^{\ell_+} - X^{\ell_-}$ on this, we get
$$t^{\ell_+ \cdot [\mathcal{A}]} - t^{\ell_- \cdot [\mathcal{A}]}.$$
But $\ell_+ \cdot [\mathcal{A}] = \ell_- \cdot [\mathcal{A}]$. ☐

Proposition I.2.4 ([CLS], 1.1.9)**.** *The defining ideal of $Y_{\mathcal{A}} \subseteq \mathbb{C}^s$ is*
$$\left\langle X^{\ell_+} - X^{\ell_-} \;\middle|\; \ell \in L \right\rangle.$$

The proof, which we omit, uses Gröbner bases.

Definition I.2.5. An ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_s]$ is *toric* if I is prime and generated by binomials.

I.2.4 Affine semigroups

Definition I.2.6. A *semigroup* S is a set with an associative binary operation and an identity element.

An *affine semigroup* is a commutative semigroup, finitely generated over $\mathbb{N}$, that can be embedded into a lattice.

Example I.2.7. $\mathbb{N}^s$ and $\mathbb{Z}^s$ are affine semigroups.

Definition I.2.8. Given an affine semigroup S, we define the *semigroup algebra*
$$\mathbb{C}[S] := \left\{ \sum_{m \in S} c_m \chi^m \;\middle|\; c_m \in \mathbb{C},\ \{m \in S : c_m \neq 0\} \text{ finite} \right\}$$
with the multiplication induced by $\chi^m \cdot \chi^{m'} := \chi^{m+m'}$.

Example I.2.9.
- $\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \ldots, x_n]$.
- $\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$.
- If $S = \langle 2, 3 \rangle \subseteq \mathbb{Z}$, then
$$\mathbb{C}[S] = \mathbb{C}[t^2, t^3] \subseteq \mathbb{C}[t].$$

- If $S = \mathbb{N}\mathcal{A}$ with $\mathcal{A} = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, then

$$\mathbb{C}[S] = \mathbb{C}[s^3, s^2t, st^2, t^3] \subseteq \mathbb{C}[s,t] \subseteq \mathbb{C}[s^{\pm}, t^pm].$$

Proposition I.2.10 ([CLS], 1.1.14)**.** *Let $S = \mathbb{N}\mathcal{A}$ be an affine semigroup. Then*

(1) $\mathbb{C}[S]$ is a finitely-generated integral $\mathbb{C}$-algebra.

(2) $\operatorname{Spec}\mathbb{C}[S] = Y_{\mathcal{A}}$.

Proof. Write $\mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M$. Then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}] \subseteq \mathbb{C}[M]$. Moreover,

$$\begin{aligned} \pi : \mathbb{C}[x_1, \ldots, x_s] &\to \mathbb{C}[M], \\ x_i &\mapsto \chi^{m_i} \end{aligned}$$

corresponds to $\mathbb{A}^s \leftarrow T$ on the level of spectra, so

$$\mathbb{C}[Y_{\mathcal{A}}] = \mathbb{C}[x_1, \ldots, x_s]/I(Y_{\mathcal{A}}) = \mathbb{C}[x_1, \ldots, x_s]/(\ker\pi) = \operatorname{im}\pi = \mathbb{C}[S].$$ □

I.3 2014-09-09

Theorem I.3.1. *Let V be an affine variety. The following are equivalent:*

(1) V is affine toric.

(2) $V = Y_{\mathcal{A}}$ for some $\mathcal{A} \subseteq M$.

(3) V is defined by a toric ideal.

(4) $V = \operatorname{Spec}\mathbb{C}[S]$ for some affine semigroup S.

I.3.1 Cones

Philosophy: Semigroup are bad. Cones are good.

The idea is to study $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ instead of $\operatorname{Pic}(X)$. This is similar to studying rational homology instead of integral homology.

Definition I.3.2. A *convex polyhedral cone* in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a set

$$\sigma = \operatorname{Cone}(S) := \left\{ \sum_{u \in S} \lambda_u \cdot u \;\middle|\; t_u \in \mathbb{R}_{\geq 0} \right\} \subseteq N_{\mathbb{R}},$$

where $S \subseteq N_{\mathbb{R}}$ is finite. By convention, $\operatorname{Cone}(\varnothing) = \{0\}$.

Definition I.3.3. A *polytope* is a set

$$P = \operatorname{Conv}(S) := \left\{ \sum_{u \in S} \lambda_u \cdot u \;\middle|\; \lambda_u \in \mathbb{R}_{\geq 0} \text{ and } \sum_u \lambda_u = 1 \right\} \subseteq N_{\mathbb{R}},$$

where S is finite.

Given a polytope $P \subseteq N_{\mathbb{R}}$, define the cone

$$C(P) := \left\{(\lambda u, \lambda) \in N_{\mathbb{R}} \times \mathbb{R} \;\middle|\; u \in P,\ \lambda \in \mathbb{R}_{\geq 0}\right\} \subseteq N_{\mathbb{R}} \times \mathbb{R}.$$

The duality $M \times N \to \mathbb{Z}$ induces a pairing

$$M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}.$$

Definition I.3.4. Given a cone $\sigma \in N_{\mathbb{R}}$, define the *dual cone*

$$\sigma^{\vee} := \left\{m \in M_{\mathbb{R}} \;\middle|\; \langle m, u\rangle \geq 0\ \forall u \in \sigma\right\}.$$

Note that σ is polyhedral if and only if $\sigma^{\vee}$ is polyhedral, and $\sigma^{\vee\vee} = \sigma$.
For each $m \in M_{\mathbb{R}}$, we have the linear functional

$$\begin{aligned}\langle m, -\rangle : N_{\mathbb{R}} &\to \mathbb{R},\\ u &\mapsto \langle m, u\rangle .\end{aligned}$$

So we can define a hyperplane H_m and a closed half-space H_m^+ by

$$\begin{aligned}H_m &:= \left\{u \in N_{\mathbb{R}} \;\middle|\; \langle m, u\rangle = 0\right\},\\ H_m^+ &:= \left\{u \in N_{\mathbb{R}} \;\middle|\; \langle m, u\rangle \geq 0\right\}.\end{aligned}$$

Definition I.3.5. A *face* of a polyhedral cone is $\tau = H_m \cap \sigma$ for some $m \in \sigma^{\vee}$, in which case we write $\tau \preceq \sigma$.

A *facet* is a codimension 1 face, and a *ray* is a dimension 1 face.

Note I.3.6. The intersection of two faces is a face. A face of a face is a face.

We can describe a cone either by rays than span it, or by facets/half-spaces. These are dual to each other in a precise sense:

Proposition I.3.7 (Fourier–Motzkin duality)**.** *Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone. Then:*

(1) If $\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+$ for $m_i \in \sigma^{\vee}$, then $\sigma^{\vee} = \mathrm{Cone}(m_1, \ldots, m_s) \subseteq M_{\mathbb{R}}$.

(2) If $\dim \sigma = n$, then each facet of σ is a half-space from (1).

Given a face $\tau \preceq \sigma \subseteq N_{\mathbb{R}}$, define

$$\begin{aligned}\tau^{\perp} &:= \left\{m \in M_{\mathbb{R}} \;\middle|\; \langle m, u\rangle = 0\ \forall u \in \tau\right\},\\ \tau^{*} &:= \left\{m \in \sigma^{\vee} \;\middle|\; \langle m, u\rangle = 0\ \forall u \in \tau\right\} = \tau^{\perp} \cap \sigma^{\vee}.\end{aligned}$$

We can now formulate a stronger form a Fourier–Motzkin duality:

Theorem I.3.8 (Stronger Fourier–Motzkin duality)**.** *With notation as above, $\tau^* \preceq \sigma^{\vee}$, and the map $\tau \mapsto \tau^*$ is a bijective, inclusion-reversing correspondence between faces of σ and faces of $\sigma^{\vee}$.*

Lemma I.3.9. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone. The following are equivalent:*

(1) σ *is* strongly convex.

(2) $\{0\}$ *is a face of* σ.

(3) σ *contains no positive-dimensional linear space.*

(4) $\sigma \cap (-\sigma) = \{0\}$.

(5) $\dim \sigma^\vee = n$.

Definition I.3.10. We say a cone is *rational* if it has the form $\mathrm{Cone}(S) \subseteq N_\mathbb{R}$, where S consists of lattice points.

For a rational cone, we can talk about minimal generators.

Theorem I.3.11. *Let* $\sigma \subseteq N_\mathbb{R}$ *be a rational, polyhedral cone. Then*

$$U_\sigma := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

is an affine toric variety.

This is the best way to define affine toric varieties. Not all affine semigroups arise in this way; we'll see next time that the ones that do are exactly the ones corresponding to *normal* affine toric varieties.

I.4 2014-09-11

Approaches to toric varieties:

- $Y_\mathcal{A} = \overline{\mathrm{im}\left((\mathbb{C}^*)^n \xrightarrow{[\mathcal{A}]} (\mathbb{C})^s\right)}$, $\mathcal{A} \subseteq M$.
- a prime binomial ideal in $\mathbb{C}[x_1, \ldots, x_s]$.
- $\mathrm{Spec}\,\mathbb{C}[\text{affine semigroup}]$
- $\sigma \subseteq N_\mathbb{R}$ rational polyhedral cone. (Note that $\sigma^\vee \cap M$ is an affine semigroup.)

Definition I.4.1. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex, rational, polyhedral cone.

(1) σ is *smooth* if its minimal generators can be extended to a lattice basis of N.

(2) σ is *simplicial* if its minimal generators can be extended to a vector space basis for $N_\mathbb{R}$.

Note that smooth cones are also simplicial.

Theorem I.4.2. *Let* $\sigma \subseteq N_\mathbb{R}$ *be a rational polyhedral cone. Then*

$$U_\sigma := \mathrm{Spec}\,\mathbb{C}[\sigma^\vee \cap M]$$

is an affine toric variety. (The content of this statement is that $\sigma^\vee \cap M$ *is finitely-generated.) Furthermore,*

$$U_\sigma = n \iff T_n = \textit{torus of } U_\sigma \iff \sigma \textit{ is strongly convex.}$$

Example I.4.3. If $\sigma = \mathrm{Cone}(e_1, e_2 - e_1)$, then $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[y, xy] \subseteq \mathbb{C}[x^\pm, y^\pm]$.

Proof. Write $S_\sigma := \sigma^\vee \cap M$. By Gordon's lemma, S_σ is finitely-generated. Thus, U_σ is an affine toric variety with character lattice $\mathbb{Z} \cdot S_\sigma \subseteq M$, where

$$\mathbb{Z} \cdot S_\sigma = \left\{ m_1 - m_2 \;\middle|\; m_i \in S_\sigma \right\}.$$

We claim that $M/\mathbb{Z}S_\sigma$ is torsion-free. Indeed, let $m \in M$ where $km \in \mathbb{Z}S_\sigma$. Then $km = m_1 - m_s$ for some $m_i \in S_\sigma$, so

$$m + m_2 = \frac{1}{k} m_1 + \frac{k-1}{k} m_2 \in \sigma^\vee \cap M = S_\sigma.$$

Hence,

$$m = (m + m_2) - m_2 \in \mathbb{Z}S_\sigma,$$

proving the claim. Thus, the following are equivalent:

(1) The rank of $\mathbb{Z}S_\sigma$ is n.

(2) The character lattice of $\mathbb{Z}S_\sigma$ is all of M.

(3) The torus of U_σ is T_N.

(4) $\dim U_\sigma = n$.

Moreover, σ is strongly convex if and only if $\dim \sigma^\vee = n$. □

Remark I.4.4. All cones in $N_\mathbb{R}$ henceforth will be strongly convex.

Proposition I.4.5. *Let $V = \operatorname{Spec} \mathbb{C}[S]$. There are natural bijections between:*

(1) Closed points $p \in V$.

(2) Maximal ideals $\mathfrak{m} \subset \mathbb{C}[S]$.

(3) Semigroup homomorphisms $\gamma : S \to \mathbb{C}$, where $\mathbb{C}$ is considered as a semigroup under multiplication.

Proof. The equivalence of (1) and (2) is classical algebraic geometry. To show (1) implies (3), given closed $p \in V$, define

$$\begin{aligned} \gamma : S &\to \mathbb{C}, \\ m &\mapsto \chi^m(p). \end{aligned}$$

To show (3) implies (2), for any semigroup homomorphism $\gamma : S \to \mathbb{C}$, since $\{\chi^m \mid m \in S\}$ is a $\mathbb{C}$-basis of $\mathbb{C}[S]$, this extends to a $\mathbb{C}$-algebra homomorphism $\chi^m \mapsto \gamma(m) : \mathbb{C}[S] \twoheadrightarrow \mathbb{C}$, the kernel of which is a maximal ideal of $\mathbb{C}[S]$. □

The advantage of the semigroup homomorphism perspective is that we can see the toric action intrinsically: given

$$\begin{aligned}\gamma : S &\to \mathbb{C},\\ m &\mapsto \gamma(m),\end{aligned}$$

the torus acts by

$$\begin{aligned}t \cdot \gamma : S &\to \mathbb{C},\\ m &\mapsto \chi^m(t) \cdot \gamma(m).\end{aligned}$$

Definition I.4.6. An affine semigroup S is *pointed* if $S \cap -S = \{0\}$.

I.4.1 Torus-fixed points

For example, $0 \in \mathbb{A}^n$ is fixed by the action of the torus, while $(\mathbb{C}^*)^n$ has no fixed point.

Proposition I.4.7. *(1)* $\operatorname{Spec}\mathbb{C}[S]$ *has a torus-fixed point iff* S *is pointed.*

(2) $Y_{\mathcal{A}} \subseteq \mathbb{A}^s$ *has a torus fixed point iff* $0 \in Y_{\mathcal{A}}$.

Proof. Fix $N \cong \mathbb{Z}^n$, $T_N \cong (\mathbb{C}^*)^n$, and $M \cong \mathbb{Z}^n$. Let

$$\begin{aligned}\gamma : S &\to \mathbb{C},\\ (m_1, \ldots, m_n) &\mapsto \gamma(m_1, \ldots, m_n)\end{aligned}$$

represent a point p. Then

$$t \cdot \gamma(m_1, \ldots, m_n) \mapsto (t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}) \cdot \gamma(m_1, \ldots, m_n).$$

Thus, $t\gamma(m_1, \ldots, m_n) = \gamma(m_1, \ldots, m_n)$ if and only if either $t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n} = 1$ or $\gamma(m_1, \ldots, m_n) = 0$. In other words, $\gamma = t\gamma$ for all t if and only if

$$\gamma(m_1, \ldots, m_n) = \begin{cases} 0 & \text{if } m \neq 0, \\ 1 & \text{if } m = 0. \end{cases}$$

This is a semigroup homomorphism iff S is pointed. □

Definition I.4.8. $S \subseteq M$ is *saturated* if for all integer $k \geq 1$ and $m \in M$, if $km \in S$, then $m \in S$.

Example I.4.9. The semigroup $\mathbb{Z}S_\sigma$ we saw earlier is saturated.

Theorem I.4.10. *Let* V *be an affine toric variety. The following are equivalent:*

(1) V *is normal.*

(2) $V = \operatorname{Spec}\mathbb{C}[S]$ *for some saturated affine semigroup* $S \subseteq M$.

(3) $V = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$, *where* $\sigma \subseteq N_{\mathbb{R}}$ *is strongly convex.*

I.5 2014-09-16

Example I.5.1. Consider the map

$$\begin{aligned}(\mathbb{C}^*)^2 &\to \mathbb{C}^4\\ (s,t) &\mapsto \left(s^4, s^3t, st^3, t^4\right).\end{aligned}$$

Let $\mathcal{A} = \{(4,0),(3,1),(1,3),(0,4)\}$. Is $\mathbb{N}\mathcal{A}$ saturated in $\mathbb{Z}\mathcal{A}$? No: $(4,4) \in \mathbb{N}\mathcal{A}$, but $(2,2) \notin \mathbb{N}\mathcal{A}$.

Theorem I.5.2. *Let V be an affine toric variety. The following are equivalent:*

(1) V is normal.

(2) $V = \operatorname{Spec}\mathbb{C}[S]$,w here $S \subseteq M$ is a saturated *affine semigroup.*

(3) $V = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$, where $\sigma \subseteq N$ is a strongly convex cone.

Proof. **(3) $\iff$ (2)** Last time, we showed that $\sigma^\vee \cap M$ is saturated when $\sigma \subseteq N$ is strongly convex.

(2) $\implies$ (2) $\mathbb{C}[S]$ is integrally closed in its field of fractions. Suppose $km = m' \in S$. Then $\chi^m \in \mathbb{C}(S)$ satisfies the integral equation $z^k - \chi^{m'} = 0$. Hence, $\chi^m \in \mathbb{C}[S]$ since $\mathbb{C}[S]$ is normal.

(2) $\implies$ (1) (sketch) Let A be the normalization of $\mathbb{C}[S]$. Note that $\mathbb{C}[S] \subseteq A \subseteq \mathbb{C}[M]$.

Idea: $\mathbb{C}[S]$ has a $\mathbb{Z}^n = M$-grading, i.e.,

$$\mathbb{C}[S] = \bigoplus_{\alpha\in\mathbb{Z}^n} \mathbb{C}[S]_\alpha,$$

where each $\mathbb{C}[S]_\alpha$ has dimension ≤ 1.

Lemma. *A also has a $\mathbb{Z}^n$-grading.*

Idea: Let $a \in A \setminus \mathbb{C}[S]$. Then each graded component of a lies in $A \setminus \mathbb{C}[S]$. Induct on the number of graded components.

Going back to the theorem, we may assume the extension $\mathbb{C}[S] \subsetneq A$ is generated by $\mathbb{Z}^n$-homogeneous elements, i.e., by monomials χ^m. Assume χ^m satisfies

$$f(z) = z^k + c_{k-1}z^{k-1} + \cdots + c_0 = 0.$$

Then, writing c_i' for the degree $i\cdot\alpha$ part of c_i. Want $c_i' \cdot (\chi^m)^i$ to have same degrees as $(\chi^m)^k$. Then χ^m also satisfies

$$g(z) = z^k + c_{k-1}'z^{k-1} + \cdots + c_0'.$$

Then do something... (See [CLS].) □

Corollary I.5.3. *An affine toric variety V can be realized as $V = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$ for σ a strongly convex cone iff V is normal.*

I.5.1 Smooth cones

Recall that a rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is *smooth* if its minimal generators form *part of* a lattice basis.

Theorem I.5.4. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational, polyhedral, strongly convex cone. Then $Y = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$ is smooth if and only if σ is smooth.*

Definition I.5.5. For an affine semigroup S, a *Hilbert basis* is a set of minimal semigroup generators. (Unique if S is pointed?)

Recall: if σ is strongly convex, then Y has a unique torus fixed point $p_\sigma \in Y$. Let H be the Hilbert basis of S_σ.

Lemma I.5.6. *The Zariski tangent space to Y at p_σ has dimension $|H|$.*

Proof. Let $P = \langle \chi^m \mid m \in S_\sigma \rangle$ be the maximal ideal of $\mathbb{C}[S_\sigma]$ corresponding to p_σ. Say m is *decomposable* in S_σ if there exist $m', m'' \in S_\sigma$ such that $m = m' + m''$ and $m', m'' \neq 0$; say m is *indecomposable* otherwise. Then P/P^2 is naturally spanned by

$$\left\{\chi^m \;\middle|\; m \text{ is indecomposable in } S_\sigma\right\},$$

which is in bijection with the set of minimal generators H of S_σ. □

Remark I.5.7. In fact, the Zariski tangent space to Y at p_σ has a natural basis that's canonically in bijection with H.

Proof of theorem. Suppose σ is smooth, and write $\sigma = \operatorname{Cone}(e_1, e_2, \ldots, e_r)$. Then $\sigma^\vee = \operatorname{Cone}(e_1, \ldots, e_r, \pm e_{r+1}, \ldots, \pm e_n)$, so $\sigma^\vee \cap M = \mathbb{N}^r \oplus \mathbb{Z}^{n-r}$. Hence,

$$Y = \operatorname{Spec}\mathbb{C}[x_1, \ldots, x_r, x_{r+1}^{\pm}, \ldots, x_n^{\pm}] = \mathbb{A}^r \times (\mathbb{C}^*)^{n-r},$$

which is clearly smooth.

Conversely, suppose Y is smooth. Then Y is normal, so we can write $Y = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$. Decompose Y into torus orbits. On each torus orbit, the Zariski tangent space has constant dimension. Since σ is rational, polyhedral, and strongly convex, there is a torus fixed point p_σ.

Claim. *p_σ is in the closure of each torus orbit.*

Granting this, Y is smooth $\iff$ Y is smooth at p_σ $\iff$ $|H| = n$ $\iff$ the semigroup S_σ has n minimal generators $\iff$ σ is smooth. □

I.6 2014-09-18

Should have done last time:

Proposition I.6.1. *Let $S \subseteq M$ be an affine semigroup and $S_{sat} \subseteq M$ its saturation. Then $\mathbb{C}[S_{sat}]$ is the normalization of $\mathbb{C}[S]$.*

To get an equivalence of categories, we must answer: Let $f : X \to Y$ be a map of affine toric varieties. When should we say that f is a "toric morphism"? Ideas:

- torus-equivariant
- $f(T_X) \subseteq T_Y$ and $f|_{T_X} \to T_Y$ is a group homomorphism.
- Comes from a semigroup map.

More precisely:

Definition I.6.2. Write $V_1 = \operatorname{Spec} \mathbb{C}[S_1]$ and $V_2 = \operatorname{Spec} \mathbb{C}[S_2]$, where $S_i \subseteq M_i$ are affine semigroups. We say $f : V_1 \to V_2$ is a *toric morphism* if it is induced by a semigroup homomorphism $\hat{f} : S_2 \to S_1$.

Example I.6.3. Let's classify toric maps $\mathbb{A}^1 \to \mathbb{A}^2$. This means giving a map $k[\mathbb{N}^2] \to k[\mathbb{N}^1]$ coming from a semigroup homomorphism $\hat{f} : \mathbb{N}^2 \to \mathbb{N}^1$. Such a map is characterized by $\hat{f}(1,0), \hat{f}(0,1) \in \mathbb{N}$, so the set of all such maps is in natural bijection with $\mathbb{N}^2$.

Assume $\gcd(a,b) = 1$. Then the toric map $f_{(a,b)} : \mathbb{A}^1 \to \mathbb{A}^2$ corresponding to $(a,b) \in \mathbb{N}^2$ sends $\mathbb{A}^1$ to the curve $V(x^b - y^a) = \operatorname{Spec} \mathbb{C}[t^a, t^b] \subseteq \mathbb{A}^2$.

Theorem I.6.4. *Let V_1, V_2 be affine toric varieties with tori T_1, T_2, and let $\phi : V_1 \to V_2$ be a map of varieties. Then:*

(1) ϕ is toric $\iff$ $\phi(T_1) \subseteq T_2$ and $\phi|_{T_1} : T_1 \to T_2$ is a group homomorphism.

(2) If ϕ is toric, then ϕ is torus-equivariant: $\phi(t \cdot p) = \phi(t) \cdot \phi(p)$ for all $t \in T_1$, $p \in V_1$.

Proof. Write $V_i = \operatorname{Spec} \mathbb{C}[S_i]$. Then $\mathbb{Z}S_i = M_i$ is the character lattice of V_i. If ϕ is toric, then ϕ is induced by $\hat{\phi} : S_2 \to S_1$, which induces a group homomorphism $\hat{\phi} : M_2 \to M_1$, and we get commutative diagrams:

$$\begin{tikzcd} \mathbb{C}[S_2] \arrow[r, "\hat{\phi}"] \arrow[d, "\subseteq"'] & \mathbb{C}[S_1] \arrow[d, "\subseteq"] \\ \mathbb{C}[M_2] \arrow[r, "\hat{\phi}"] & \mathbb{C}[M_1] \end{tikzcd} \qquad \begin{tikzcd} V_2 & V_1 \arrow[l, "\phi"'] \\ T_2 \arrow[u] & T_1 \arrow[l, "\phi|_{T_1}"'] \arrow[u] \end{tikzcd}$$

Recall that $T_i = \operatorname{Hom}_{\mathbb{Z}}(M_i, \mathbb{C}^*)$. Since $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ is a functor, $\phi|_{T_1}$ is a map of abelian groups.

Conversely, if $\phi|_{T_1} : T_1 \to T_2$ is a group homomorphism, then $\hat{\phi} : M_2 \to M_1$ is a group homomorphism, so $\hat{\phi}|_{S_2} : S_2 \to S_1 \subseteq M_1$ is a semigroup homomorphism.

It remains to show that a torus map is equivariant. We must show commutativity of the diagram

$$\begin{tikzcd} T_1 \times V_1 \arrow[r] \arrow[d, "(\phi|_{T_1}, \phi)"'] & V_1 \arrow[d, "\phi"] \\ T_2 \times V_2 \arrow[r] & V_2. \end{tikzcd}$$

To see this, note that if V_i is replaced with T_i, this just expresses that the torus map induces a group homomorphism of the torus. Since equivariance holds on a Zariski-dense subset, it holds everywhere. $\square$

Proposition I.6.5. *Let $\sigma_i \subseteq (N_i)_{\mathbb{R}}$ be a strongly convex, rational, polyhedral cone. Consider a map $\phi : N_1 \to N_2$. Then $\phi : T_1 \to T_2$ extends to a toric map $U_{\sigma_1} \to U_{\sigma_2}$ (where $U_{\sigma_i} = \operatorname{Spec} \mathbb{C}[\sigma_i^\vee \cap M_i]$) if and only if $\phi(\sigma_1) \subseteq \sigma_2$.*

Example I.6.6. There is a toric map $\operatorname{Spec} \mathbb{C}[y, xy^{-1}] \to \operatorname{Spec} \mathbb{C}[x, y]$, but not in the other direction.

Example I.6.7. What if you intersect a semigroup with a sublattice? For example, consider the inclusions of lattices

$$(2\mathbb{N})^2 \subseteq \left\{(a,b) \in \mathbb{N}^2 \;\middle|\; a+b \in 2\mathbb{N}\right\} \subseteq \mathbb{N}^2.$$

This corresponds to $\mathbb{C}[x^2, y^2] \subseteq \mathbb{C}[x^2, xy, y^2] \subseteq \mathbb{C}[x, y]$; the latter inclusion is the second Veronese embedding. In fact, every Veronese embedding can be realized by a similar inclusion of lattices.

I.7 2014-09-23

Setup:

$$0 \to L \to \mathbb{Z}^s \xrightarrow{[\mathcal{A}]} \mathbb{Z}^n = M,$$

$\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M = \mathbb{Z}^n$. How do we actually compute

$$I_L = \left\langle x^\alpha - x^\beta \;\middle|\; \alpha - \beta \in L \right\rangle?$$

Idea: let

$$J := \left\{x^{\alpha_i} - x^{\beta_i} \;\middle|\; \{\alpha_i - \beta_i\} \text{ is a } \mathbb{Z}\text{-basis for } L\right\}.$$

Then $V(J) \cap (\mathbb{C}^*)^n = V(I_L) \cap (\mathbb{C}^*)^n$. Indeed, on $(\mathbb{C}^*)^n$, monomials are invertible, so we can rescale $x^{\alpha_i} - x^{\beta_i}$ to either $x^{\alpha_i - \beta_i} - 1$ or $1 - x^{\beta_i - \alpha_i}$.

Let $x^{\alpha'} - x^{\beta'}$ be a generator of I_L but not in J. Since $\alpha' - \beta' \in L$, we can write $\alpha' - \beta' = \sum_i c_i(\alpha_i - \beta_i)$ for some $c_i \in \mathbb{Z}$. Hence,

$$x^{\alpha' - \beta'} - 1 = \pm \prod_{c_i > 0} (x^{\alpha_i - \beta_i} - 1)^{c_i} \cdot \prod_{c_i < 0} (1 - x^{\beta_i - \alpha_i})^{c_i}.$$

Since $V(J) \cap (\mathbb{C}^*)^n = V(I_L) \cap (\mathbb{C}^*)^n$, $I_L = J : (x_1 \cdot \ldots \cdot x_n)^\infty$.

Chapter II

Projective toric varieties

II.1 2014-09-23, continued

II.1.1 Toric varieties

Definition II.1.1. A *toric variety* is an (integral) variety X with a dense torus $T \subseteq X$ such that the action of T on itself extends to an action on X.

Example II.1.2. Let $\mathcal{A} = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$. Then $X_{\mathcal{A}} = V(I_L) \subseteq \mathbb{P}^3$ is the twisted cubic.

Example II.1.3. The cone for $\mathbb{P}^1$ is the union of two rays σ_1, σ_2 in opposite directions. The σ_i are smooth, strongly convex, rational polyhedral cones, and so is $\sigma_1 \cap \sigma_2$. This is a "fan" structure, which gives gluing data.

When does $\mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M$ product a projective toric variety?

Proposition II.1.4. *The following are equivalent:*

(1) $Y_{\mathcal{A}}$ is the affine cone over a projective toric variety (herein denoted $X_{\mathcal{A}}$).

(2) I_L is homogeneous.

(3) There exists $u \in N$ and $k \in \mathbb{Z}_{>0}$ such that $\langle m_i, u \rangle = k$ for $i = 1, \ldots, s$.

Remark II.1.5. If we set $M = \mathbb{Z}^n$, then (3) is equivalent to $(1, 1, \ldots, 1)$ being in the row space of $[\mathcal{A}] \subseteq \mathbb{Q}^s$.

Proof. **(1)** $\implies$ **(2)** Straightforward.

(2) $\implies$ **(3)** Let $x^\alpha - x^\beta \in I_L$. If $\deg(x^\alpha) \neq \deg(x^\beta)$, then $x^\alpha, x^\beta \in I_L$. But $(1, 1, \ldots, 1) \in Y_{\mathcal{A}}$, giving a contradiction. Hence, $x^\alpha - x^\beta$ is homogeneous, and so, for $\ell := \alpha - \beta \in L$, we have $\ell \cdot (1, \ldots, 1) = 0$ for any $\ell \in L$. Consider

$$0 \to L \to \mathbb{Z}^s \to M,$$

and tensor with $\mathbb{Q}$ and dualize to obtain

$$N_{\mathbb{Q}} \to \mathbb{Q}^s \xrightarrow{\varphi} \mathrm{Hom}_Q(L_{\mathbb{Q}}, \mathbb{Q}) \to 0.$$

Thus, $(1, \ldots, 1) \in \ker \varphi$, hence there exists $\tilde{u} \in N_{\mathbb{Q}}$ mapping to $(1, \ldots, 1)$. Clearing denominators yields $u \in N$ mapping to $(k, \ldots, k)$. So $u \cdot m_i = k$ for all $i = 1, \ldots, s$.

□

II.2 2014-09-25

Continuing from last time, let $\mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M$.

Proposition II.2.1. *The following are equivalent:*

(1) $Y_{\mathcal{A}} \subseteq \mathbb{A}^s$ is the cone over a projective toric variety (denoted $X_{\mathcal{A}} \subseteq \mathbb{P}^{s-1}$).

(2) I_L is homogeneous.

(3) There exist $u \in N$ and $k \in \mathbb{Z}_{>0}$ such that $\langle m_i, u \rangle = k$ for all i.

It remains to show that (3) implies (1).

Remark II.2.2. Let $M = \mathbb{Z}^n$. Then (3) is equivalent to $(1, 1, \ldots, 1) \in \mathbb{Z}^s$ lying in the row space over $\mathbb{Q}$ of $[\mathcal{A}] = \begin{bmatrix} m_1 & \ldots & m_s \end{bmatrix}$.

First, we claim $Y_{\mathcal{A}}$ is a cone. Consider $y \in Y_{\mathcal{A}} \subseteq \mathbb{A}^s$. Since $Y_{\mathcal{A}}$ is equivariant with respect to the T_Y-action, it follows that $Y_{\mathcal{A}}$ is equivariant with respect to any 1-parameter subgroup. Let $u \in N$ as in (3) and consider

$$\begin{aligned} \mathbb{C}^* &\xrightarrow{\lambda^u} (\mathbb{C}^*)^n \\ \tau &\mapsto \lambda^u(\tau). \end{aligned}$$

For $t \in T_Y$, $t \cdot y = (t^{m_1} y_1, t^{m_2} y_2, \ldots, t^{m_s} y_s)$. Hence,

$$\lambda^u(\tau) \cdot y = (\tau^{\langle m_1, u \rangle} y_1, \ldots, \tau^{\langle m_s, u \rangle} y_s) = (\tau^k y_1, \ldots, \tau^k y_s).$$

So $Y_{\mathcal{A}}$ is equivariant with respect to the dilation action on $\mathbb{A}^s$. Thus, $Y_{\mathcal{A}}$ is a cone over a projective variety $X_{\mathcal{A}} \subseteq \mathbb{P}^{s-1}$.

Claim II.2.3. *$X_{\mathcal{A}}$ is a toric variety with torus $T_X = T_Y / \lambda^u(\mathbb{C}^*)$.*

Elements $t, t' \in T_Y$ coincide in T_X if and only if $t' = \lambda^u(\tau) \cdot t$ for some $\tau \in \mathbb{C}^*$. Write $[y] \in \mathbb{P}^{s-1}$ for the class of $y \in \mathbb{A}^s$ and $[t] \in T_X$ for the class of $t \in T_Y$.

Key observation: the map

$$\begin{aligned} T_X \times X_{\mathcal{A}} &\to X_{\mathcal{A}} \\ [t], [y] &\mapsto [t \cdot y] \end{aligned}$$

is well-defined.

Lemma II.2.4. *Let $U_i := \mathbb{P}^{s-1} \setminus V(x_i)$. Then $U_i \cap X_\mathcal{A}$ is an affine toric variety.*

Certainly, $U_i \cap X_\mathcal{A}$ is an affine variety. Note that T_Y lies entirely in $\mathbb{A}^s \setminus \bigcup_{i=1}^s V(X_i)$. Hence, $T_X \subseteq \mathbb{P}^{s-1} \setminus \bigcup_{i=1}^s V(x_i) = \bigcap_{i=1}^s U_i$. So

$$T_X \subseteq U_i \cap X_\mathcal{A} = (\text{Zariski closure of } T_X \text{ in } U_i).$$

Where is the semigroup?[1] Identify

$$\begin{aligned} U_i &\xrightarrow{\simeq} \mathbb{A}^{s-1} \\ [a_1, \ldots, a_s] &\mapsto \left(\frac{a_1}{a_i}, \frac{a_2}{a_i}, \ldots, \frac{a_s}{a_i}\right). \end{aligned}$$

Then $U_i \cap X_\mathcal{A}$ is the closure of the image of

$$\begin{aligned} T_X &\to U_i \\ t &\mapsto \left(\chi^{m_1 - m_i}(t), \ldots, \chi^{m_s - m_i}(t)\right), \end{aligned}$$

and $\mathcal{A}_i := \mathcal{A} - m_i = \{m_1 - m_i, m_2 - m_i, \ldots, m_s - m_i\}$. Hence

$$U_i \cap X_\mathcal{A} = \operatorname{Spec} \mathbb{C}[\mathbb{N}\mathcal{A}_i].$$

□

The convex hull of $\mathcal{A}$,

$$P = \operatorname{Conv}(\mathcal{A}) = \left\{ \sum_{i=1}^s r_i m_i \;\middle|\; r_i \in \mathbb{R}_{\geq 0},\ \sum r_i = 1 \right\} \subseteq M_\mathbb{R}$$

is a polytope in $M_\mathbb{R}$.

Theorem II.2.5. *(1)* $\dim X_\mathcal{A} = \dim P$.

(2) If $J = \{j \in \{1, \ldots, s\} \mid m_j$ is a vertex of $P\}$, then $X_\mathcal{A} = \bigcup_{j \in J} (X_\mathcal{A} \cap U_j)$.

(3) $X_\mathcal{A} = X_{\mathcal{A}-m}$ for any $m \in M$. ("$X_\mathcal{A}$ only depends on the polytope $P = \operatorname{Conv}(\mathcal{A})$.")

II.2.1 Summary of chapter 2

Here's what we'll do on the rest of chapter 2:

- Facts about polytopes[2]
- Define a toric variety X_P for *any* polytope P.
- Later: a polytope P gives rise to a pair (X_P, D_P), where X_P is a toric variety and D_P is an ample line bundle on X_P.

---

[1] Where have all the monoids gone?
Long time passing
Where have all the monoids gone?
Long time ago

[2] Thanks for signing up for Polytope Facts! You will now receive fun daily facts about POLYTOPES!

II.3 2014-09-30: Fans and gluing

Here's the data we need to glue:

- affine schemes $\{V_\alpha\}$;
- for each pair (α, β), an open subset $V_{\alpha\beta} \subseteq V_\alpha$;
- isomorphisms $g_{\beta\alpha} : V_{\beta\alpha} \xrightarrow{\simeq} V_{\alpha\beta}$ satisfying:
  (1) $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$
  (2) cocycle condition

Definition II.3.1. A *fan* $\Sigma = \{\sigma\}$ in $N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ satisfying:

(1) Every $\sigma \in \Sigma$ is strongly convex, rational, and polyhedral.

(2) For all $\sigma \in \Sigma$, each face of σ is in Σ.

(3) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of σ_1 and of σ_2.

The *support* of Σ is $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subseteq N_\mathbb{R}$. We write

$$\Sigma(r) := \left\{\sigma \in \Sigma \;\middle|\; \dim \sigma = r\right\}.$$

Recall: Given $\sigma \in \Sigma$, we get a (normal) affine toric variety $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$ where $S_\sigma = \sigma^\vee \cap M$. If $\tau \preceq \sigma$, then $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$, where $H_m = \{u \in N_\mathbb{R} \mid \langle u, m\rangle = 0\}$.

Fact II.3.2. If $\tau = \sigma \cap H_m$, then $S_\tau = S_\sigma + \mathbb{Z}(-m)$.

Remark II.3.3. The above fact implies that $U_\tau = \operatorname{Spec} \mathbb{C}[S_\tau] = \operatorname{Spec} \mathbb{C}[S_\sigma]_{\chi^m} = (U_\sigma)_{\chi^m}$.

Fact II.3.4. If $\tau = \sigma_1 \cap \sigma_2$, then there exists $m \in \sigma_1^\vee \cap \sigma_2^\vee \cap M$ such that $\sigma_2 \cap H_m = \tau = \sigma_1 \cap H_m$ (i.e., we can use the same m).

Remark II.3.5. Hence, U_τ can be considered as a subset of U_{σ_1} or of U_{σ_2}. We will use this to glue.

Lemma II.3.6. *Fix a fan $\Sigma \subseteq N_\mathbb{R}$. Then $\{U_\sigma\}_{\sigma\in\Sigma}$ together with natural inclusion maps $U_{\sigma_2} \supseteq U_\tau \subseteq U_{\sigma_1}$ for all pairs σ_1, σ_2 and $\tau = \sigma_1 \cap \sigma_2$, glues to give a scheme X_Σ.*

Theorem II.3.7. *X_Σ is a normal toric variety.*

Proof. Since each cone is strongly convex, $\{0\}$ is a face of σ for all $\sigma \in \Sigma$, and hence $U_{\{0\}} = T_N \subseteq U_\sigma \subseteq X_\Sigma$ for all $\sigma \in \Sigma$. The torus actions are compatible on $U_{\sigma_1} \cap U_{\sigma_2}$ for all pairs σ_1, σ_2. So they glue to give $T_N \times X_\Sigma \to X_\Sigma$. The scheme X_Σ is integral and normal since each U_σ is. (See book for separatedness.) □

Remark II.3.8. It's not the case that $X_\mathcal{A} = X_{r\mathcal{A}}$ for all $r \in \mathbb{N}$. But it is the case that $X_{r\mathcal{A}} = X_{r'\mathcal{A}}$ for all $r, r' \gg 0$.

Theorem II.3.9. *Every normal toric variety equals X_Σ for some fan $\Sigma \subseteq N_\mathbb{R}$.*

II.4 2014-10-02: Polytope Facts

One preview theorem:

Theorem II.4.1. *Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan and X_Σ the toric variety. Then:*

(1) X_Σ is smooth $\iff$ each $\sigma \in \Sigma$ is smooth.

(2) X_Σ is an orbifold (has finite quotient singularities) $\iff$ each $\sigma \in \Sigma$ is simplicial.

(3) X_Σ is proper $\iff$ $|\Sigma| = N_{\mathbb{R}}$.

(4) X_Σ is projective $\iff$ Σ is the normal fan of a polytope P.

Definition II.4.2. Given $u \in N_{\mathbb{R}}$ and $b \in \mathbb{R}$, let

$$H_{u,b} := \left\{ m \in M_{\mathbb{R}} \;\middle|\; \langle u, m \rangle = b \right\},$$
$$H_{u,b}^{+} := \left\{ m \in M_{\mathbb{R}} \;\middle|\; \langle u, m \rangle \geq b \right\}.$$

We say Q is a *face* of P, written $Q \preceq P$, if $Q = P \cap H_{u,b}$ and $P \subseteq H_{u,b}^{+}$. Call $H_{u,b}$ a *supporting hyperplane*.

Note II.4.3. If P is full-dimensional and $F \preceq P$ is a facet, then $F = P \cap H_{u_F, -a_F}$, where $(u_F, -a_F)$ is unique up to scalar multiple.

Fact II.4.4.

- $P = \operatorname{Conv}(\text{vertices of } P)$.
- If $P = \operatorname{Conv}(S)$, then each vertex of P lies in S.
- If $Q \preceq P$, then Q is a polytope and the faces of Q are exactly those faces of P lying in Q.
- Every proper face $Q \prec P$ is the intersection of the facets containing Q.

Usually, P is given as $\operatorname{Conv}(S)$, where S is a finite set. If P is full-dimensional, then

$$P = \bigcap_{F \text{facet of } P} H_{u_F, -a_F}^{+} = \bigcup_{F} \left\{ m \mid \langle u_F, m \rangle \geq -a_F \right\}.$$

Let P be a d-dimensional polytope.

- P is a *d-simplex* if P has $(d+1)$ vertices.
- P is *simple* if every vertex is the intersection of precisely d facets.
- P is *simplicial* if all facets are simplices.

Example II.4.5. An octahedron is simplicial, but not simple. A cube is simple, but not simplicial.

Polytopes are *combinatorially equivalent* if there is a bijection between their faces preserving intersections, inclusions, and dimensions.

Sums or *Minkowski sums*: Let $P, Q \subseteq M_{\mathbb{R}}$ be polytopes. Define

$$P + Q = \{p + q \mid p \in P,\ q \in Q\}.$$

Multiples: Let $r \in \mathbb{R}$ and $P = \mathrm{Conv}(S)$. Define $rP := \mathrm{Conv}(rS)$. If

$$P = \bigcap_{i=1}^{s} \{\langle m, u_i \rangle \geq -a_i\},$$

then

$$rP = \bigcap_{i=1}^{s} \{\langle m, u_i \rangle \geq -ra_i\}.$$

Duals: if P is full-dimensional and $0 \in P$, then

$$P^\circ := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq -1 \text{ for all } m \in P\}.$$

Example II.4.6. The dual of the square $P = [-1,1]^2 = \mathrm{Conv}(\{(-1,1),(1,1),(1,-1),(-1,-1)\})$ is the diamond

$$\begin{aligned} P^\circ &= \{(u_1,u_2) \mid u_1 m_1 + u_2 m_2 \geq -1\ \forall (m_1,m_2) \in [-1,1]^2\} \\ &= \{u_1 + u_2 \geq -1\} \cap \{u_1 - u_2 \geq -1\} \cap \{-u_1 + u_2 \geq -1\} \cap \{-u_1 - u_2 \geq -1\} \\ &= \mathrm{Conv}(\{(1,0),(-1,0),(0,1),(0,-1)\}). \end{aligned}$$

A *lattice polytope* is a polytope whose vertices are lattice points. If P is a full-dimensional lattice polytope and $F \preceq P$ is a facet, then $F = P \cap H_{u_F, -a_F}$, where $u_F, -a_F$ are chosen to be integers (and hence unique if written in lowest terms).

Note II.4.7. For any $k, \ell \in \mathbb{N}$,

$$(kP) \cap M + (\ell P) \cap M \subseteq ((k+\ell)P) \cap M.$$

Definition II.4.8. P is *normal* if this is an equality for all $k, \ell \in \mathbb{N}$, or equivalently if

$$\underbrace{(P \cap M) + (P \cap M) + \cdots + (P \cap M)}_{k \text{ times}} = (kP) \cap M.$$

Recall that $C(P) = \mathrm{Cone}(P \times \{1\}) \subseteq M_{\mathbb{R}} \times \mathbb{R}$.

Claim II.4.9. *P is normal $\iff$ $(P \cap M) \times \{1\}$ generates the semigroup $C(P) \cap (M \times \mathbb{Z})$.*

Proof. Consider the bijection

$$\begin{aligned} (kP) \cap M &\longleftrightarrow C(P) \cap (M \times \{k\}) \\ \lambda &\mapsto (\lambda, k). \end{aligned}$$

Let $\{\lambda_1, \ldots, \lambda_s\} = P \cap M$. Then $\lambda = \sum_i a_i \lambda_i$ for some $a_i \in \mathbb{N}$ if and only if $(\lambda, k) = \sum_i a_i (\lambda_i, 1)$ for some $a_i \in \mathbb{N}$. $\square$

Theorem II.4.10. *Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional lattice polytope of dimension $n \geq 2$. Then kP is normal for all $k \geq n-1$.*

Remark II.4.11 (Future picture). P corresponds to a line bundle $\mathscr{L}$ on X_P, and $P \cap M$ corresponds to a basis of $H^0(\mathscr{L})$.

Definition II.4.12. P is *very ample* if for every vertex $m \in P$, the semigroup

$$S_{P,m} := \mathbb{N}\left\{m' - m \;\middle|\; m' \in P \cap M\right\}$$

is saturated in M.

II.5 2014-10-07: Line bundles and polytopes

Definition II.5.1. A polytope P is:

- *normal* if $(kP) \cap M + (\ell P) \cap M = ((k+\ell)P) \cap M$ for all k, ℓ.
- *very ample* if for all vertices $m \in P$, $S_{P,m} := \mathbb{N}\{m' - m : m \in P \cap M\}$ is saturated in M.

Theorem II.5.2. *Normal $\implies$ very ample.*

Proof. Choose a vertex $m_0 \in P$, and let $S = S_{P,m_0}$. Assume $km \in S$ for some $k \in \mathbb{Z}_{\geq 1}$. Then $km = \sum_{m' \in P \cap M} a'_m(m' - m_0)$ for some $a'_m \in \mathbb{N}$. Pick d so that $kd \geq \sum a'_m$. Then

$$km + kdm_0 = \left(\sum_{m' \in P \cap M} a'_m m'\right) + \left(kd - \sum a'_m\right) m_0 \in kdP.$$

By normality, $m + dm_0 = \sum_{i=1}^{d} m_i$, where $m_i \in P \cap M$. So

$$m = (m + dm_0) - dm_0 = \sum_{i=1}^{d} (m_i - m_0)$$

and $m_i - m_0 \in S$. $\square$

Definition II.5.3. If P is a full-dimensional, very ample, lattice polytope, then $X_P := X_{\mathcal{A}} \subseteq \mathbb{P}^{\#\mathcal{A}-1}$, where $\mathcal{A} = P \cap M$.

Example II.5.4. If $P = \operatorname{Conv}\{(0,0),(1,0),(0,1)\}$, then P is very ample, and X_P is the closure of

$$\begin{aligned} (\mathbb{C}^*)^2 &\to \mathbb{P}^2, \\ (s,t) &\mapsto [1 : s : t]. \end{aligned}$$

This is just the identity embedding $\mathbb{P}^2 \subseteq \mathbb{P}^2$ given by $\mathcal{O}(1)$.

Example II.5.5. We have $2P = \operatorname{Conv}\{(0,0),(2,0),(0,2)\}$, which is still very ample, and X_{2P} is the Zariski closure of

$$\begin{aligned}(\mathbb{C}^*)^2 &\to \mathbb{P}^5,\\ (s,t) &\mapsto [1:s:t:s^2:st:t^2].\end{aligned}$$

This is the Veronese embedding $\mathbb{P}^2 \subseteq \mathbb{P}^5$, given by $\mathcal{O}(2)$.

Recall: If P is a full-dimensional lattice polytope, then

$$P = \bigcap_{F \text{ facet}} \left\{m \in M_{\mathbb{R}} \;\middle|\; \langle m, u_F\rangle \geq -a_F\right\}.$$

For each vertex v of P, define

$$C_v := \operatorname{Cone}(P \cap M - v) = \operatorname{Cone}(\left\{m' - v \;\middle|\; m' \in P \cap M\right\}),$$

and $\sigma_v = C_v^\vee \subseteq N_{\mathbb{R}}$.

There's a dimension- and inclusion-reversing correspondence between faces of P and cones in Σ in $N_{\mathbb{R}}$, sending a face $Q \preceq F$ to

$$\sigma_Q := \operatorname{Cone}(u_F \mid F \text{ contains } Q).$$

Definition II.5.6. The *normal fan* Σ (or Σ_P or $\Sigma(P)$) of a full-dimensional lattice polytope is

$$\Sigma = \left\{\sigma_Q \;\middle|\; Q \preceq P\right\}.$$

Lemma II.5.7. *Σ is a fan.*

Theorem II.5.8. *Let P be a full-dimensional very ample lattice polytope. Then:*

(1) For any vertex $m_i \in P \cap M$, we have $X_P \cap U_i = U_{\sigma_i}$, where $\sigma_i \subseteq N_{\mathbb{R}}$ is the strongly convex, rational polyhedral cone σ_{m_i} (dual to $C_{m_i} = \operatorname{Cone}(P \cap M - m_i)$).

(2) The torus of $X_{P\cap M}$ is T_N.

(3) $X_P = X_\Sigma$, where Σ is the normal fan of P.

Note II.5.9. The normal fan of P is equal to the normal fan of kP for all $k \geq 1$.

Corollary II.5.10. *If P is a full-dimensional very ample lattice polytope, then $X_P = X_{kP}$ for all $k \geq 1$.*

What if a full-dimensional lattice polytope P is not very ample? Recall:

- kP is normal for all $k \geq \dim P - 1$.
- normal $\implies$ very ample.

Definition II.5.11. Let P be a full-dimensional lattice polytope. Then:

- As an abstract variety, $X_P := X_{kP}$ for any $k \gg 0$.

- X_P also has a distinguished map $X_P \to \mathbb{P}^{\#(P\cap M)-1}$ whose image is $X_{P\cap M}$.

Later, we'll prove: For any full-dimensional lattice polytope P, we get a pair (X_P, D_P), where D_P is the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$. This naturally corresponds to pairs (X, A) with X a projective toric variety and A an ample (but not necessarily very ample) line bundle.

Theorem II.5.12. *Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional lattice polytope. Then:*

(a) X_P is normal.

(b) $X_{kP} \subseteq \mathbb{P}^{\#(kP)\cap M-1}$ is projectively normal if and only if kP is normal.

Proof. Note that $X_P = X_{\Sigma(P)}$, which is normal. For part (b), recall that $Y_{kP\cap M}$ is the affine cone over $X_{kP\cap M}$, and $X_{kP\cap M} \subseteq \mathbb{P}^m$ is projectively normal $\iff$ $Y_{(kP\cap M)\times\{1\}}$ is normal $\iff$ $\mathbb{N}(\{kP\cap M\}\times\{1\})$ is saturated, which follows if kP is normal. $\square$

Chapter III

Normal toric varieties

III.1 2014-10-09

III.1.1 Examples

Example III.1.1. The d-dimensional rectangular polytope with opposite diagonal vertices $(0,\dots,0)$ and $(a_1,\dots,a_d)$ corresponds to the embedding $(\mathbb{P}^1)^{\times m} \hookrightarrow \mathbb{P}^{\prod_i (a_i+1)-1}$, which comes from the line bundle $\mathcal{O}(a_1,\dots,a_d)$ on $(\mathbb{P}^1)^{\times m}$.

Example III.1.2. Consider the polytope $P = \mathrm{Conv}(\{(0,0),(2,0),(1,1),(1,0)\})$. This gives a toric surface

$$\begin{gathered} X_P \to \mathbb{P}^4, \\ (\mathbb{C}^*)^2 \ni (s,t) \mapsto [1 : s : s^2 : t : st], \end{gathered}$$

defined by the 2×2 minors of $\begin{bmatrix} x_0 & x_1 & y_0 \\ x_1 & x_2 & y_1 \end{bmatrix}$. This is a rational normal scroll.

III.1.2 Smoothness

Definition III.1.3. Let P be a lattice polytope.

(1) Given a vertex $v \in P$ and an edge $E \ni v$, let w_E be the first lattice on E after v.

(2) P is *smooth* if for all v, the vectors

$$\{w_E - v \mid E \text{ is an edge containing } v\}$$

form a subset of a lattice basis.

Theorem III.1.4. *Let P be a full-dimensional lattice polytope. Then X_P is smooth $\iff$ P is smooth.*

Proof. X_P is covered by affines U_v for vertices v of P. It suffices to check that $X_P \cap U_v$ is smooth for all vertices v.

Recall that $C_v = \operatorname{Cone}(P \cap M - v) = \operatorname{Cone}(\{m - v \mid m \in P \cap M\}) \subseteq M_{\mathbb{R}}$ and $\sigma_v = C_v^{\vee}$. The variety

$$X_P \cap U_v = \operatorname{Spec} \mathbb{C}[\mathbb{N} \cdot \{m - v \mid m \in P \cap M\}] = Spec\mathbb{C}[\sigma_v^{\vee} \cap M]$$

is smooth $\iff$ σ_v is smooth $\iff$ $\sigma_v^{\vee}$ is smooth $\iff$ C_v is smooth. □

Proposition III.1.5. *Let P be a full-dimensional smooth lattice polytope. Then P is very ample.*

Proof. Recall that P is very ample if $S_{P,v} = \mathbb{N}\{m - v \mid m \in P \cap M\}$ is saturated for all v. Since P is smooth, $S_{P,v}$ is generated by a lattice basis $m_1, \ldots, m_n$. Hence, if $km \in S_{P,v}$, then there are unique $a_i \in \mathbb{N}$ such that $km = \sum_i a_i m_i$. Also, $m = \sum_i b_i m_i$ for some $b_i \in \mathbb{Z}$. Thus, $km = \sum_i kb_i m_i$. By uniqueness, $a_i = kb_i$, so $b_i \geq 0$, whence $m \in S_{P,v}$. □

Conjecture III.1.6. *Let P be a full-dimensional smooth lattice polytope. Then P is normal.*

Note III.1.7. $X_P \subseteq \mathbb{P}^{\#P\cap M-1}$ is projectively normal $\iff$ P is normal.

Theorem III.1.8. *$X_{kP} \subseteq \mathbb{P}^m$ is projectively normal if $k \geq \dim P - 1$.*

III.1.3 Products

If $P \subseteq M_{\mathbb{R}}$ and $P' \subseteq M'_{\mathbb{R}}$, then $P \times P' \subseteq M_{\mathbb{R}} \times M'_{\mathbb{R}}$.

Theorem III.1.9. *Assume P, P' are very ample, $X_P \subseteq \mathbb{P}^{a-1}$, and $X_{P'} \subseteq \mathbb{P}^{b-1}$. Then*

$$X_{P\times P'} \cong X_P \times X_{P'} \subseteq \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \xrightarrow{\textit{Segre}} \mathbb{P}^{ab-1},$$

and the composition $X_{P\times P'} \hookrightarrow \mathbb{P}^{ab-1}$ is the embedding given by the polytope $P \times P'$.

Proof. Note that $(P \times P') \cap (M \times M') = (P \cap M) \times (P' \cap M')$. Let $T'' = T \times T'$ be the torus of $M \times M'$. The following diagram commutes:

$$\begin{tikzcd}
T'' \arrow[rr] \arrow[d, equal] & & \mathbb{P}^{ab-1} \\
T \times T' \arrow[r] & \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \arrow[ur] &
\end{tikzcd}$$

□

Theorem III.1.10. *For fans $\Sigma \subseteq N_{\mathbb{R}}$ and $\Sigma' \subseteq N'_{\mathbb{R}}$, define*

$$\Sigma \times \Sigma' := \{\sigma \times \sigma' \mid \sigma \in \Sigma,\ \sigma' \in \Sigma'\} \subseteq N_{\mathbb{R}} \times N'_{\mathbb{R}}.$$

Then $\Sigma \times \Sigma'$ is a fan, and $X_{\Sigma\times\Sigma'} = X_\Sigma \times X_{\Sigma'}$.

Example III.1.11. Let Δ_n be the standard n-simplex. Then $\Delta_n \times \Delta_m$ corresponds to the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$.

III.1.4 Orbit-cone correspondence

Example III.1.12. Let $P = \operatorname{Conv}\{(0,0),(1,0),(0,1)\}$ be the triangle, corresponding to the normal fan in $N_{\mathbb{R}}$ that corresponds to $\mathbb{P}^2$. Points of the lattice N correspond to one-parameter subgroups of $(\mathbb{C}^*)^2 \subseteq \mathbb{P}^2$. For example, $(3,4)$ corresponds to

$$\begin{aligned} \mathbb{C}^* &\xrightarrow{\lambda^{(3,4)}} (\mathbb{C}^*)^2 \to \mathbb{P}^2 \\ u &\mapsto (u^3, u^4) \\ &\qquad\qquad (s,t) \mapsto [1 : s : t]. \end{aligned}$$

Observe that

$$\lim_{u\to 0} \lambda(\mathbb{C}^*) = \lim_{u\to 0}[1 : u^3 : u^4] = [1 : 0 : 0].$$

Note: the same computation works for any $(a,b) \in N$ with $a > 0$ and $b > 0$.

III.2 2014-10-14: Orbit-cone correspondence

Recall: Let σ be a rational polyhedral convex cone in $N_{\mathbb{R}}$. Then:

- $U_\sigma = \operatorname{Spec}\mathbb{C}[S_\sigma] = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$. The set of closed points of U_σ are in bijection with semigroup morphisms $S_\sigma \to \mathbb{C}$.
- If σ is strongly convex, then there is a distinguished point $\gamma_\sigma \in U_\sigma$ corresponding to the semigroup morphism

$$m \mapsto \begin{cases} 1 & \text{if } m \in S_\sigma \cap \sigma^\perp = \sigma^\perp \cap M, \\ 0 & \text{otherwise.} \end{cases}$$

- γ_σ is torus-fixed $\iff$ σ is full-dimensional.

Definition III.2.1. Let $O(\sigma)$ be the torus orbit of $\gamma_\sigma \in U_\sigma$. We have

$$O(\sigma) = T_N \cdot \gamma_\sigma \subseteq U_\sigma.$$

When $\sigma \in \Sigma$ a fan, this definition always makes sense.

Lemma III.2.2. *Let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. Then*

$$O(\sigma) = \left\{\gamma : S_\sigma \to \mathbb{C} \;\middle|\; \gamma(m) \neq 0 \iff m \in \sigma^\perp \cap M\right\} \cong \operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*).$$

Proof. Recall how $t \in T$ acts on a semigroup morphism: For $\gamma : S_\sigma \to \mathbb{C}$, $t \cdot \gamma$ is defined by $m \mapsto \chi^m(t) \cdot \gamma(m)$. Let

$$O' := \left\{\gamma : S_\sigma \to \mathbb{C} \;\middle|\; \gamma(m) \neq 0 \iff m \in \sigma^\perp \cap M\right\}.$$

Note that $\gamma_\sigma \in O'$. Also, O' is closed under the action of the torus T, since $O(\sigma) = T \cdot \gamma_\sigma \subseteq O'$.

Observe that $\sigma^\perp$ is the largest vector subspace of $M_\mathbb{R}$ contained in $\sigma^\vee$. Hence, $\sigma^\perp \cap M$ is a subgroup of M and a subsemigroup of S_σ. Restricting $\gamma \in O'$ to $\sigma^\perp \cap M$ yields a group morphism

$$\hat{\gamma} = \gamma|_{\sigma^\perp \cap M} : \sigma^\perp \cap M \to \mathbb{C}^*.$$

Note that $\hat{\gamma} \in \operatorname{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$.

Conversely, given $\hat{\gamma} \in \operatorname{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$, we get a semigroup morphism $\gamma : S_\sigma \to \mathbb{C}$ by extending by zero, i.e., setting $\gamma(m) = 0$ for all $m \in S_\sigma \setminus (S^\perp \cap M)$.

We've shown that $O(\sigma) \subseteq O' \cong \operatorname{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$. It remains to show that T_N acts transitively on O'. Note that $T_N = N \otimes_\mathbb{Z} \mathbb{C}^* \cong \operatorname{Hom}_\mathbb{Z}(M, \mathbb{C}^*)$. The inclusion $\sigma^\perp \cap M \subseteq M$ induces a surjection

$$T_N = \operatorname{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \twoheadrightarrow \operatorname{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) = O',$$

proving the result. $\square$

Theorem III.2.3 (Orbit-cone correspondence). *Let $\Sigma \subseteq N_\mathbb{R}$ be a fan, and X_Σ the corresponding toric variety.*

(1) There is a bijective correspondence

$$\{\textit{cones } \sigma \in \Sigma\} \longleftrightarrow \{T_N\textit{-orbits in } X_\Sigma\},$$
$$\sigma \mapsto O(\sigma).$$

(2) The above correspondence is dimension-reversing, i.e.,

$$\dim O(\sigma) = \dim N_\mathbb{R} - \dim \sigma.$$

(3) If we take orbit closures, then the correspondence is inclusion-reversing, i.e.,

$$\tau \preceq \sigma \iff \overline{O(\sigma)} \subseteq \overline{O(\tau)}.$$

Even stronger,

$$\overline{O(\tau)} = \coprod_{\tau \preceq \sigma} O(\sigma).$$

Proof. See book. $\square$

Note III.2.4. We sometimes denote $V(\sigma) := \overline{O(\sigma)}$.

Remark III.2.5. Each orbit closure $V(\sigma)$ is itself a toric variety. Fix $\tau \in \Sigma$, and define $N_\tau := \mathbb{Z}\langle \tau \cap N \rangle \subseteq N$ and $N(\tau)_\mathbb{R} := N_\mathbb{R}/(N_\tau)_\mathbb{R}$. Write $\bar{\sigma}$ for the image of σ under the quotient map $N_\mathbb{R} \twoheadrightarrow N(\tau)_\mathbb{R}$. Then

$$\operatorname{Star}(\tau) := \left\{ \bar{\sigma} \subseteq N(\tau)_\mathbb{R} \;\middle|\; \sigma \preceq \tau \in \Sigma \right\} \subseteq N(\tau)_\mathbb{R}.$$

$\operatorname{Star}(\tau)$ is a fan, and $V(\tau) = X_{\operatorname{Star}(\tau)}$.

III.3 2014-10-16: Toric morphisms

III.3.1 Example

An example: the cone of a square corresponds to the affine cone of $\mathbb{P}^1 \times \mathbb{P}^1$. Let's classify $V(\sigma)$ for $\sigma \in \Sigma$:

$$\begin{aligned} V(\text{whole thing}) &= \text{origin in } \mathbb{A}^4 \\ V(p_i) &= \mathbb{A}^2 = \widehat{[0:1] \times \mathbb{P}^1} && \text{(2-dimensional)} \\ V(\sigma_{01}) &= [0:\widehat{1] \times [0}:1] && \text{(1-dimensional)} \\ V(\text{origin}) &= X_\Sigma \end{aligned}$$

For example, $V(p_0) \cong X_{\mathrm{Star}(p_0)}$, which comes from

$$N_{p_0} \xrightarrow{[0\ 0\ 1]} N \xrightarrow{\pi} N(p_0) \to 0.$$

III.3.2 Toric morphisms

Recall that a map $\phi : V_1 = \operatorname{Spec} \mathbb{C}[S_1] \to V_2 = \operatorname{Spec} \mathbb{C}[S_2]$ between affine toric varieties is *toric* if it's induced by a semigroup homomorphism $S_2 \to S_1$, or equivalently, if ϕ maps T_1 into T_2 and $\phi|_{T_1} : T_1 \to T_2$ is a group morphism. The latter definition is the one that generalizes nicely.

Definition III.3.1. Let $X_{\Sigma_1}, X_{\Sigma_2}$ be normal toric varieties with $\Sigma_i \subseteq (N_i)_\mathbb{R}$ a fan. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is *toric* if ϕ sends T_1 into T_2 and if $\phi|_{T_1} : T_1 \to T_2$ is also a group morphism.

Corollary III.3.2. *Any toric morphism is equivariant, i.e., the following diagram commutes:*

$$\begin{tikzcd} T_1 \times X_{\Sigma_1} \arrow[r, "\text{toric action}"] \arrow[d, "{(\phi,\phi)}"'] & X_{\Sigma_1} \arrow[d, "\phi"] \\ T_2 \times X_{\Sigma_2} \arrow[r, "\text{toric action}"'] & X_{\Sigma_2} \end{tikzcd}$$

Proof. Since $\phi|_{T_1} : T_1 \to T_2$ is a group homomorphism, the diagram commutes on a dense subset, hence everywhere. $\square$

Definition III.3.3. For $i = 1, 2$, let $\Sigma_i \subseteq (N_i)_\mathbb{R}$ be fans. Let $\Phi : N_1 \to N_2$ be a $\mathbb{Z}$-linear map, and let $\Phi_\mathbb{R} : (N_1)_\mathbb{R} \to (N_2)_\mathbb{R}$ be the induced map. We say Φ is *compatible with* Σ_1 *and* Σ_2 if for all $\sigma_1 \in \Sigma_1$, there exists $\sigma_2 \in \Sigma_2$ such that $\Phi_\mathbb{R}(\sigma_1) \subseteq \sigma_2$.

Theorem III.3.4. *Let $\Sigma_1 \subseteq (N_i)_\mathbb{R}$ be fans for $i = 1, 2$.*

(1) If $\Phi : N_1 \to N_2$ is a $\mathbb{Z}$-linear map that is compatible with Σ_1 and Σ_2, then there is a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ such that

$$\phi|_{T_1} = \Phi \otimes 1 : N_1 \otimes_\mathbb{Z} \mathbb{C}^* \to N_2 \otimes_\mathbb{Z} \mathbb{C}^*.$$

(2) *Conversely, if $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism, then ϕ induces a $\mathbb{Z}$-linear map $\Phi : N_1 \to N_2$ that is compatible with Σ_1 and Σ_2.*

Proof. (1) For all $\sigma_1 \in \Sigma_1$, there exists $\sigma_2 \in \Sigma_2$ where $\Phi_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$. Hence, $\Phi^*_{\mathbb{R}}(\sigma_2^\vee) \subseteq \sigma_1^\vee$, and so the map $\sigma_2^\vee \cap M_2 \to \sigma_1^\vee \to M_1$ induces a toric morphism $U_{\sigma_1} \to U_{\sigma_2}$. Then glue these.

(2) $\phi|_{T_1}$ is a group homomorphism, so $\Phi : N_1 \to N_2$ is $\mathbb{Z}$-linear. Take $\sigma_1 \in \Sigma_1$. By the orbit-cone correspondence, σ_1 corresponds to an orbit $O(\sigma_1) \subseteq \Sigma_1$. By equivariance, there is a T_2-orbit containing $\phi(O(\sigma_1))$, i.e., there exists $\sigma_2 \in \Sigma_2$ with $\phi(O(\sigma_1)) \subseteq O(\sigma_2)$.

By reversing the argument in (1), it suffices to show that $\phi|_{U_{\sigma_1}}(U_{\sigma_1}) \subseteq U_{\sigma_2}$. Note that $U_{\sigma_1} = \bigcup_{\tau_1 \preceq \sigma_1} O(\tau_1)$. Also, $\phi(O(\tau_1)) \subseteq O(\tau_2)$ for some $\tau_2 \in \Sigma_2$.

Since $\tau_1 \preceq \sigma_1$, $O(\sigma_1) \subseteq \overline{O(\tau_1)}$. So $O(\sigma_2) \supseteq \phi(O(\sigma_1)) \subseteq \overline{\phi(O(\tau_1))} \subseteq \overline{O(\tau_2)}$. Hence by equivariance, $O(\sigma_2) \subseteq \overline{O(\tau_2)} = \bigcup_{\tau_2 \preceq \rho} O(\rho)$. This implies $\tau_2 \preceq \sigma_2$. Hence, Φ is compatible with Σ_1, Σ_2. □

III.3.3 Blowups

Given a fan $\Sigma \subseteq N_{\mathbb{R}}$, we say another fan $\Sigma' \subseteq N_{\mathbb{R}}$ is a *refinement* of Σ if every cone in Σ' is contained in a cone of Σ and $|\Sigma| = |\Sigma'|$.

Note III.3.5. In this case, we get a map $X_{\Sigma'} \to X_{\Sigma}$.

Definition III.3.6. Let Σ be a fan in $N_{\mathbb{R}} = \mathbb{R}^n$. Let $\sigma = \mathrm{Cone}(u_1, \ldots, u_n)$ be a smooth cone in Σ. Let $u_0 := u_1 + \cdots + u_n$, and let

$$\Sigma'(\sigma) = \left\{ \begin{array}{c} \text{cones generated by subsets of} \\ \{u_0, \ldots, u_n\} \text{ not containing} \\ \{u_1, \ldots, u_n\} \end{array} \right\}.$$

Then $\Sigma^*(\sigma) := (\Sigma \setminus \{\sigma\}) \cup \Sigma'(\sigma)$ is a fan called the *star subdivision* of Σ along σ.

Theorem III.3.7. *The induced toric morphism $\phi : X_{\Sigma^*(\sigma)} \to X_{\Sigma}$ is the blowup of X_Σ at the distinguished point γ_σ (i.e., at the smooth point $V(\sigma)$).*

III.4 2014-10-21: Bundles

Recall: a fan $\Sigma' \subseteq N_{\mathbb{R}}$ is a *refinement* of a fan $\Sigma \subseteq N_{\mathbb{R}}$ if $|\Sigma'| = |\Sigma|$ and for all $\sigma' \in \Sigma'$, there exists $\sigma \in \Sigma$ with $\sigma' \subseteq \sigma$.

Idea: Refining Σ corresponds to blowing up X_Σ. (For example, we can even study blowups at non-reduced ideals this way, such as blowing up $\mathbb{A}^2$ at (x^a, y^b).)

Question: What does a vector bundle, projective bundle, or fibration of normal toric varieties look like at the level of fans? In particular, we look at a toric morphism $\phi : X_\Sigma \to X_{\Sigma'}$ whose fibers are all isomorphic to X_{Σ_0}.

Let $\Phi : N \twoheadrightarrow N'$ be $\mathbb{Z}$-linear. Let $\Sigma \subseteq N_{\mathbb{R}}$ and $\Sigma' \subseteq N'_{\mathbb{R}}$ be fans compatible with Φ, yielding a toric morphism $X_\Sigma \to X_{\Sigma'}$. Let $N_0 = \ker \Phi$, so

$$0 \to N_0 \to N \xrightarrow{\Phi} N' \to 0$$

is a short exact sequence, which splits. Define

$$\Sigma_0 := \left\{ \sigma \in \Sigma \;\middle|\; \sigma \subseteq (N_0)_{\mathbb{R}} \right\} \subseteq (N_0)_{\mathbb{R}}.$$

Let X_{Σ_0, N_0} be the toric variety associated to $\Sigma_0 \subseteq (N_0)_{\mathbb{R}}$.

Question: When is $\phi : X_\Sigma \to X_{\Sigma'}$ a X_{Σ_0,N_0}-bundle over $X_{\Sigma'}$? In other words, when is there an open cover $\{V_\alpha\}$ of $X_{\Sigma'}$ where $\phi^{-1}(U_\alpha) \cong U_\alpha \times X_{\Sigma_0,N_0}$ for all α?

Definition III.4.1. We say that Σ is *split* by Σ' if there is a subfan $\hat{\Sigma} \subseteq \Sigma$ where:

(1) Φ maps each cone $\hat{\sigma} \in \hat{\Sigma}$ bijectively to a cone $\sigma' \in \Sigma'$ such that $\Phi(\hat{\sigma} \cap N) = \sigma' \cap N'$ and $\hat{\sigma} \mapsto \sigma'$ is a bijection $\hat{\Sigma} \to \Sigma'$.

(2) Given $\hat{\sigma} \in \hat{\Sigma}$ and $\sigma_0 \in \Sigma_0$, we have that $\hat{\sigma} + \sigma_0 \in \Sigma$, and every cone in Σ arises in this way.

Remark III.4.2. This is like a "twisted" product. For example, $\Sigma_1 \times \Sigma_2$ is split by Σ_1.

Theorem III.4.3. *If Σ is split by Σ', then $\phi : X_\Sigma \to X_{\Sigma'}$ is an X_{Σ_0,N_0}-bundle.*

We'll prove this next time.

III.4.1 Hirzebruch surfaces

Definition III.4.4. A *Hirzebruch surface* is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, i.e., $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a)) \to \mathbb{P}^1$.

Remark III.4.5. Hirzebruch surfaces are always toric.

What are all smooth fans $\Sigma \subseteq \mathbb{R}^2$ with 3 rays and $|\Sigma| = \mathbb{R}^2$? Without loss of generality, $(1,0)$ and $(0,1)$ are two of the points defining rays. To meet the condition on the support, the third point must be (a,b) with $a, b < 0$. Also, for Σ to be smooth, $(1,0),(0,1),(a,b)$ must be a lattice basis, so $|a| = |b| = 1$.

What about smooth fans $\Sigma \subseteq \mathbb{R}^2$ with 4 rays and $|\Sigma| = \mathbb{R}^2$? Two points are $(1,0)$ and $(0,1)$, and the remaining two, by determinant considerations, are $\rho = (-1,b)$ and $\rho' = (a,-1)$. Also,

$$\det \begin{bmatrix} -1 & a \\ b & -1 \end{bmatrix} = 1 - ab = \pm 1,$$

so $ab = 0$ (whence $b = 0$ without loss of generality) or $ab = 2$ (whence (a,b) is one of the following: $(2,1),(1,2),(-2,-1),(-1,-2)$). This gives a complete list.

III.5 2014-10-23: Proper toric varieties

III.5.1 Split fans, continued

Recall from last time:

Theorem III.5.1. *If Σ is split by Σ' and Σ_0, then $\phi : X_\Sigma \to X_{\Sigma'}$ is an X_{Σ_0,N_0}-bundle, i.e., $\phi^{-1}(U_{\sigma'}) \cong U_{\sigma'} \times X_{\Sigma_0,N_0}$ for all $\sigma' \in \Sigma'$.*

Proof. Fix $\sigma' \in \Sigma'$, and let $\Sigma(\sigma') = \{\sigma \in \Sigma \mid \Phi(\sigma) \subseteq \sigma'\}$. Then $\phi^{-1}(U_{\sigma'}) = X_{\Sigma(\sigma')}$. We need to show that $X_{\Sigma(\sigma')} = U_{\sigma'} \times X_{\Sigma_0,N_0}$.

Note that $\Sigma(\sigma')$ is split by $\Sigma_0 \cap \Sigma(\sigma')$ and $\{\tau' \in \Sigma' \mid \tau' \preceq \sigma'\}$ (where $\hat{\Sigma} \rightsquigarrow \hat{\Sigma} \cap \Sigma(\sigma')$). We can now reduce to the case where $\Sigma' = \{\tau' \preceq \sigma'\}$, $\Sigma = \Sigma(\sigma')$, and so on.

Consider the short exact sequence of $\mathbb{Z}$-modules $0 \to N_0 \to N \to N' \to 0$. We have a splitting $\bar{\nu} : N' \to N$ that induces an isomorphism $N \cong N' \times N_0$, and we want a splitting $\bar{\nu}$ inducing $\Sigma \cong \Sigma' \times \Sigma_0$. For all $\hat{\tau} \in \hat{\Sigma}$, this maps bijectively to $\tau' \in \Sigma'$. Furthermore, the map is bijective on lattice points!

Let $\hat{\sigma} \mapsto \sigma'$. Let $\hat{N} \subseteq N$ be the sublattice spanned by $\hat{\sigma} \cap N$. Let $N'' \subseteq N'$ be the sublattice spanned by $\sigma' \cap N' \subseteq N'$.

Let $\bar{\nu}$ be any splitting that identifies N'' with $\hat{N}$. This works. $\square$

III.5.2 Proper varieties

Definition III.5.2. A variety X is *proper* (or *complete*) if for all Z, the induced map $X \times Z \to Z$ is closed in the Zariski topology.

Proposition III.5.3. *Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex, rational polyhedral cone, and let $u \in N$. Then $u \in \sigma$ if and only if $\lim_{t\to 0} \lambda^u(t)$ exists in U_σ.*

Proof. Given $u \in N$, $\lim_{t\to 0} \lambda^u(t)$ exists in $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma] \iff \lim_{t\to 0} \chi^m(\lambda^u(t))$ exists in $\mathbb{C}$ for all $m \in \mathcal{A} \iff \lim_{t\to 0} t^{\langle m,n\rangle}$ exists in $\mathbb{C}$ for all $m \in S_\sigma \iff \langle m, u\rangle \geq 0$ for all $m \in \sigma^\vee \cap M \iff u \in (\sigma^\vee)^\vee = \sigma$. $\square$

Theorem III.5.4. *Let X_Σ be a normal toric variety. The following are equivalent:*

(1) X_Σ is compact in the Euclidean topology.

(2) The limit $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ for all $u \in N$.

(3) $|\Sigma| = N_\mathbb{R}$.

(4) X_Σ is proper.

Remark III.5.5. Criterion (2) can be thought of as a vastly strengthened analogue of the valuative criterion: it says that we only need to check torus-equivariant copies of $\mathbb{C}^*$.

Proof. By Serre's GAGA theorem, (1) and (4) are equivalent.

We first show that (1) implies (2): Suppose X_Σ is compact, and fix $u \in N$. A sequence $t_k \in \mathbb{C}^*$ converging to 0 yields a sequence $\{\lambda^u(t_k)\}_{k\in\mathbb{N}}$ in X_Σ. Since X_Σ is compact, there is a convergent subsequence, so passing to this, $\lim_{k\to\infty} \lambda^u(t_k) = \gamma \in X_\Sigma = \bigcup_\sigma U_\sigma$. Assume $\gamma \in U_\sigma$. Take $m \in \sigma^\vee \cap M$. Then χ^m is a regular function on U_σ, so

$$\mathbb{C} \ni \chi^m(\gamma) = \lim_{k\to\infty} \chi^m(\lambda^u(t_k)) = \lim_{k\to\infty} t_k^{\langle m,u\rangle}.$$

Thus $\langle m, u\rangle \geq 0$ for all $m \in \sigma^\vee \cap M$, whence $u \in (\sigma^\vee)^\vee = \sigma$. By Proposition III.5.3, it follows that $\lim_{t\to 0} \lambda^u(t)$ exists in $U_\sigma \subseteq X_\Sigma$.

Now we show (2) implies (3): For all $u \in N$, the limit $\lim_{t\to 0} \lambda^u(t)$ exists in $U_\sigma \subseteq X_\Sigma$ for some σ, so $u \in \sigma$ by Proposition III.5.3. Thus, $|\Sigma| = N_\mathbb{R}$.

It remains to show that (3) implies (1). We'll prove this next time. □

III.6 2014-10-28: Proper morphisms

III.6.1 Proper toric varieties, continued

Last time, we began proving:

Theorem III.6.1. *Let X_Σ be a normal toric variety. The following are equivalent:*

(1) X_Σ is compact in the Euclidean topology.

(2) The limit $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ for all $u \in N$.

(3) $|\Sigma| = N_\mathbb{R}$.

(4) X_Σ is proper.

The equivalence of (1) and (2) is Serre's GAGA. Last time, we showed that (1) $\implies$ (3) $\implies$ (4).

We now show (4) $\implies$ (1): Induct on $n = \dim N_\mathbb{R}$. For $n = 1$, the only complete fan in $N_\mathbb{R}$ corresponds to $X_\Sigma = \mathbb{P}^1_\mathbb{C}$. Assume the statement for all lower-dimensional fans. Let $\gamma_k \in X_\Sigma$ be a sequence of cones. The fan X_Σ is the union of a finite number of torus orbits. We may assume $\gamma_k \in O(\tau)$ for some $\tau \in \Sigma$. If $\tau \neq \{0\}$ (so $O(\tau) \neq T_N$), then $\gamma_k \in O(\tau) \subseteq V(\tau) = X_{\mathrm{Star}(\tau)}$. After checking that $|\Sigma| = N_\mathbb{R} \implies |\mathrm{Star}(\tau)| = N(\tau)_\mathbb{R}$, the statement holds by induction (so $\tau = \{0\}$).

So we now have $\gamma_k \in T_N = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*)$. Define

$$\begin{aligned} L : T_N = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*) &\to N_\mathbb{R} = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{R}) \\ (m \mapsto \gamma(m)) &\mapsto (m \mapsto \log|\gamma(m)|). \end{aligned}$$

Given $\gamma \in T_N$, we get $L(\gamma) \subseteq N_\mathbb{R}$. If $L(\gamma) \in -\sigma$ for $\sigma \in \Sigma$, and if $m \in \sigma^\vee \cap M$, then $\log|\gamma(m)| = \langle m, L(\gamma)\rangle \leq 0$ and $|\gamma(m)| \leq 1$, so $\gamma(m) \in$ (unit disk).

Since $|\Sigma| = N_\mathbb{R}$, we can assume $L(\gamma_k)$ lies in $-\sigma$ for some $\sigma \in \Sigma$. Thus γ_k is a map $M \to$ (closed unit disk in $\mathbb{C}$) for all γ_k. Since the closed unit disk is compact, there is a subsequence converging to a map γ_∞ sending $\sigma^\vee \cap M$ to the closed unit disk, thus $\gamma_\infty \in U_\sigma$. □

III.6.2 Proper morphisms

A morphism of algebraic varieties $\phi : X \to Y$ is *proper* if it is universally closed, i.e., if for all $\psi : Z \to Y$, the mapping $\phi' : X \times_Y Z \to Z$ is a closed mapping in the Zariski topology.

Example III.6.2. The projection $\mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{A}^1$ is proper. On the other hand, the projection $\mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ isn't proper.

Remark III.6.3 (Proper implies proper fibers). A necessary condition for a morphism being proper is that all of its fibers are proper. However, this isn't sufficient — for example, consider an open embedding.

Remark III.6.4 (Topological analogue). A continuous map of topological spaces $f : X \to Y$ is proper if $f^{-1}(T)$ is compact for all compact $T \subseteq Y$.

Theorem III.6.5. *Let $\phi : X_\Sigma \to X'_\Sigma$ be a toric morphism of normal toric varieties induced by $\Phi : N \to N'$. The following are equivalent:*

(1) ϕ is proper as a continuous map of topological spaces.

(2) ϕ is proper as a morphism of algebraic varieties.

(3) $\bar{\Phi}_{\mathbb{R}}^{-1}(|\Sigma'|) = |\Sigma|$.

Chapter IV

Divisors on toric varieties

IV.1 2014-10-28: Weil and Cartier divisors

IV.1.1 Weil divisors

Let X be a normal toric variety. The *divisor group* of X is

$$\mathrm{Div}(X) = \bigoplus_{P \subseteq X} \mathbb{Z} \cdot [P],$$

where P ranges over a codimension 1 closed subvarieties of X. Let $K(X)$ denote the function field of X, given by $K(X) = \mathrm{Frac}(A) = A_{(0)}$, where $U = \mathrm{Spec}(A) \subseteq X$ is an affine open subset.

For $f \in K(X)^*$, we define an element $\mathrm{div}(f) \in \mathrm{Div}(X)$ by

$$\mathrm{div}(f) := \sum_{P \subseteq X \text{ codim } 1} v_P(f) \cdot [P],$$

where the valuation $v_P(f) \in \mathbb{Z}$ is defined as follows: If $U = \mathrm{Spec}(A)$ contains P, then $P \cap U \subseteq U$ is a closed subvariety of codimension 1, so $P \cap V = V(\mathfrak{p}) \subseteq U$ for some prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, for which $A_{\mathfrak{p}} = \mathcal{O}_{X,P}$. Since X is normal, X is smooth in codimension 1, so $\mathrm{Sing}(X) \subseteq X$ has codimension ≥ 2, hence $P \subseteq X$ is generically smooth. Thus, $\mathcal{O}_{X,P}$ is a 1-dimensional regular local ring, hence a DVR. Let v_P be the associated valuation

$$v_P : K(X)^* \to \mathbb{Z}.$$

A regular function $f = \frac{g}{h}$ can be restricted to U, and $v_P(f)$ is the multiplicity of $V(g|_U)$ along P, which can also be computed as $\min_{x \in P} \{\text{multiplicity of } g|_U \text{ at } P\}$.

We have $v_P(f) \neq 0$ for only finitely many P, so $\mathrm{div}(f)$ is indeed a well-defined element of $\mathrm{Div}(X)$. We define

$$\mathrm{Div}_0(X) = \left\{ \mathrm{div}(f) \;\middle|\; f \in K(X)^* \right\} \subseteq \mathrm{Div}(X).$$

A *Weil divisor* is an element of $\mathrm{Div}(X)$, and the *divisor class group* of X is the quotient group

$$\mathrm{Cl}(X) = \frac{\mathrm{Div}(X)}{\mathrm{Div}_0(X)}.$$

Example IV.1.1. Consider $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[t]$. The divisor group is

$$\operatorname{Div}(\mathbb{A}^1) = \bigoplus_{\alpha\in\mathbb{C}} \mathbb{Z}\cdot[\alpha].$$

For any nonzero $g, h \in \mathbb{C}[t]$,

$$\operatorname{div}\left(\frac{g(t)}{h(t)}\right) = \operatorname{div}\left(\frac{(t-\alpha_1)\cdots(t-\alpha_r)}{(t-\beta_1)\cdots(t-\beta_s)}\right) = \sum_{i=1}^{r}[\alpha_i] - \sum_{j=1}^{s}[\beta_j].$$

In particular, $\operatorname{div}(t-\alpha) = [\alpha]$, so $\operatorname{Cl}(\mathbb{A}^1) = 0$.

Remark IV.1.2. The divisor class group is difficult to compute in general, but very easy to compute for toric varieties.

IV.1.2 Cartier divisors

A divisor $Z \in \operatorname{Div}(X)$ is a *Cartier divisor* if Z is locally defined by a single function, i.e., there is an open cover $\{U_i\}$ of X such that $Z|_{U_i} = \operatorname{div}(f_i)$ for all i and some $f_i \in K(X)^*$.

Example IV.1.3. Consider the cone $X = V(xz - y^2) \subseteq \mathbb{A}^3$. The line $V(x, y)$ is a codimension 1 closed subvariety of X, but it's not locally principal (scheme-theoretically).

Let $\operatorname{CDiv}(X) \subseteq \operatorname{Div}(X)$ be the group of Cartier divisors, which contains $\operatorname{Div}_0(X)$ by construction. The *Cartier class group* is

$$\operatorname{CaCl}(X) = \frac{\operatorname{CDiv}(X)}{\operatorname{Div}_0(X)}.$$

In all reasonable cases (including everything we do in this class), $\operatorname{CaCl}(X) = \operatorname{Pic}(X)$ is equal to the *Picard group*.

IV.2 2014-10-30: Class groups

IV.2.1 The class group of $\mathbb{P}^2$

Let's compute $\operatorname{Cl}(\mathbb{P}^2) = \operatorname{Div}(\mathbb{P}^2)/\operatorname{Div}_0(\mathbb{P}^2)$. We have

$$\operatorname{Div}(\mathbb{P}^2) = \bigoplus_{\substack{P\subseteq\mathbb{P}^2 \\ \text{codim 1, irred.}}} \mathbb{Z}\cdot[P].$$

Recall that

$$\left\{\begin{matrix}\text{codim. 1} \\ \text{irreducible} \\ \text{subvarieties of } \mathbb{P}^2\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{homogeneous, codim} \\ \text{1 prime ideals in} \\ \mathbb{C}[x,y,z]\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{homog. irred.} \\ \text{polynomials in} \\ \mathbb{C}[x,y,z]\text{, up to} \\ \text{scalar multiples}\end{matrix}\right\}.$$

Also,

$$\begin{aligned} \mathrm{Div}_0(\mathbb{P}^2) &= \left\{\mathrm{div}(f) \;\middle|\; f \in K(\mathbb{P}^2)^*\right\} \subseteq \mathrm{Div}(\mathbb{P}^2), \\ K(\mathbb{P}^2) &= \left\{\frac{f}{g} \;\middle|\; f, g \in \mathbb{C}[x,y,z] \text{ homogeneous}, \deg f = \deg g\right\}. \end{aligned}$$

Write $f = f_1^{a_1} \cdots a_r^{a_r}$ and $g = g_1^{b_1} \cdots g_s^{b_s}$ with f_i, g_j irreducible. Note that

$$\sum_{i=1}^{r} a_i \deg(f_i) = \sum_{j=1}^{s} b_j \deg(g_j).$$

Write $V_i = V(f_i)$ and $Z_j = V(g_j)$. Then

$$\mathrm{div}\left(\frac{f}{g}\right) = \sum_{i=1}^{r} a_i[V_i] - \sum_{j=1}^{s} b_j[Z_j] \in \mathrm{Div}_0(\mathbb{P}^2).$$

Note IV.2.1. There is a $\mathbb{Z}$-linear map

$$\begin{aligned} \deg : \mathrm{Div}(\mathbb{P}^2) &\to \mathbb{Z}, \\ [P] &\mapsto \deg P = \deg f \qquad\qquad \text{where } P = V(f). \end{aligned}$$

Note that $\mathrm{Div}_0(\mathbb{P}^2) \subseteq \ker(\deg)$. We have an exact sequence

$$\mathrm{Div}_0(\mathbb{P}^2) \to \mathrm{Div}(\mathbb{P}^2) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Hence, if we show this is exact in the middle, then $\mathrm{Cl}(\mathbb{P}^2) = \mathbb{Z}$. This means showing that any divisor of degree zero is principal. (See Hartshorne for the proof.)

IV.2.2 Miscellany on class groups

- If $X = \operatorname{Spec} A$ and A is a UFD, then $\mathrm{Cl}(X) = 0$, e.g., $\mathrm{Cl}(\mathbb{A}^n) = 0$ and $\mathrm{Cl}(T_N) = 0$.
- If X is smooth, then $\mathrm{Cl}(X) = \mathrm{Pic}(X)$.
- If $D \subseteq X$ is a prime Weil divisor, then there is a right exact sequence

$$\begin{aligned} &\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus D) \to 0. \\ &1 \mapsto [D] \end{aligned}$$

- If $D_1, \ldots, D_n$ are distinct prime Weil divisors on X, then there is a right exact sequence

$$\begin{aligned} &\mathbb{Z}^n \to \mathrm{Cl}(X) \to \mathrm{Cl}\Big(X \setminus \bigcup_i D_i\Big) \to 0. \\ &e_i \mapsto D_i \end{aligned}$$

Example IV.2.2. Let D be an elliptic curve in $\mathbb{P}^2$. Then the corresponding map $\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^2) = \mathbb{Z}$ has degree 3, so $\mathrm{Cl}(\mathbb{P}^2 \setminus D) = \mathbb{Z}/3\mathbb{Z}$.

Definition IV.2.3. A divisor in $\mathrm{Div}(X)$ is *effective* if all coefficients are nonnegative. The class of divisor in $\mathrm{Cl}(X)$ is effective if it is the image of an effective divisor.

From a Weil divisor, we get a subsheaf of $K(X)$. A subsheaf of $K(X)$ assigns a subset of $K(X)$ to each open set $U \subseteq X$. In particular, we define the subsheaf $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D)(U) = \left\{ f \in K(X)^* \;\middle|\; \mathrm{div}(f) + D|_U \text{ is effective in } \mathrm{Cl}(U) \right\}.$$

Example IV.2.4. If D is a prime divisor (or more generally, an effective divisor) on $\mathbb{A}^n$, then if $D = V(f_D)$, we have

$$\mathcal{O}_X(-D)(\mathbb{A}^n) = \left\{ \mathrm{div}\, \frac{f}{g} - D \text{ effective} \right\} = \langle f_D \rangle \cdot \mathbb{C}[x_1, \ldots, x_n] \subseteq K(\mathbb{A}^n).$$

Remark IV.2.5. D is Cartier $\iff$ $\mathcal{O}_X(D)$ is a line bundle.

IV.2.3 Toric varieties and divisors

Let X_Σ be an n-dimensional normal toric variety. Let $\Sigma(1)$ be the rays of the fan Σ, i.e., the codimension 1 torus orbits. If $\rho \in \Sigma(1)$ is a ray, we get a Weil divisor $D_\rho = V(\rho)$.

As we argued before, $\mathcal{O}_{X_\Sigma, D_\rho}$ is a DVR with a valuation v_ρ. For example, if $X_\Sigma = \mathbb{A}^2$ and $D_\rho = V(x_1)$, then $\mathcal{O}_{\mathbb{A}^2, D_\rho} = \mathbb{C}[x_1, x_2]_{\langle x_1 \rangle}$ has valuation $v_\rho : K(\mathbb{A}^2)^* \to \mathbb{Z}$ given by $v_\rho(\frac{f}{g}) = m$, where $\frac{f}{g} = x_1^m \frac{f'}{g'}$ and f', g' aren't divisible by x_1.

Recall that $u_\rho \in N$ is the ray generator of ρ and $\chi^m : T_N \to \mathbb{C}^*$ is a rational function on X_Σ.

Proposition IV.2.6. $v_\rho(\chi^m) = \langle m, u_\rho \rangle$.

Proof. Since u_ρ is primitive, we extend to a lattice basis $e_1 = u_\rho, e_2, \ldots, e_n$ of N. So

$$U_\rho = \operatorname{Spec} \mathbb{C}[x_1, x_2^\pm, \ldots, x_n^\pm] = \mathbb{A}^1 \times (\mathbb{C}^*)^{n-1}.$$

So $U_\rho \cap D_\rho = V(x_1) \subseteq U_\rho$. It turns out that

$$\mathcal{O}_{X_\Sigma, D_\rho} = \mathbb{C}[x_1, x_2^\pm, \ldots, x_n^\pm]_{\langle x_1 \rangle} = \mathbb{C}[x_1, x_2, \ldots, x_n]_{\langle x_1 \rangle}.$$

If $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ and $\frac{f}{g} = x_1^\ell \frac{f'}{g'}$ with $x_1 \nmid f'$ and $x_1 \mid g'$, then $v_\ell(\frac{f}{g}) = \ell$.

Let $m_1, \ldots, m_n$ be the dual basis of M. Then $x_i = \chi^{m_i}$, and

$$\chi^m = x_1^{\langle m, e_1 \rangle} x_2^{\langle m, e_2 \rangle} \cdots x_n^{\langle m, e_n \rangle} = x_1^{\langle m, u_\rho \rangle} x_2^{\langle m, e_2 \rangle} \cdots x_n^{\langle m, e_n \rangle}.$$

Thus, $v_\rho(\chi^m) = \langle m, e_1 \rangle = \langle m, u_\rho \rangle$. $\square$

Proposition IV.2.7. *For all $m \in M$,* $\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle \cdot [D_\rho]$.

Proof. Note that $X \setminus \bigcup_{\rho \in \Sigma(1)} D_\rho = T_N$. If $P \subseteq X$ where $[P]$ has a nonzero coefficient in $\mathrm{div}(\chi^m)$, then the same holds in T_N if P is not D_ρ for some ρ. Identify $T_N = \operatorname{Spec} \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ and $\chi^m = x_1^{m_1} \cdots x_n^{m_n}$. Then

$$\mathrm{div}(\chi^m) = \sum_{i=1}^n m_i \cdot [V(x_i)] = 0$$

in T_N. $\square$

IV.3 2014-11-04: Toric class groups and Picard groups

IV.3.1 Class groups of toric varieties

Our goal is to compute $\mathrm{Cl}(X_\Sigma)$. Let u_ρ be the first lattice point on the ray ρ. Recall that, for $m \in M$, $\mathrm{div}(\chi^m) = \sum_\rho \langle m, u_\rho \rangle D_\rho$.

Write $\mathrm{Div}_T(X_\Sigma) := \bigoplus_\rho \mathbb{Z} \cdot D_\rho \subseteq \mathrm{Div}(X)$.

Theorem IV.3.1. *We have an exact sequence*

$$\begin{aligned} M &\to \mathrm{Div}_T(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to 0 \\ m &\mapsto \mathrm{div}(\chi^m) \end{aligned}$$

which is exact on the left if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ *spans* $N_\mathbb{R}$.

Proof. Note that $\mathrm{Cl}(T) = 0$ since $T = \operatorname{Spec} \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ is a UFD. Since $T = X_\Sigma \setminus \bigcup_\rho D_\rho$, we have an exact sequence

$$\mathrm{Div}_T = \bigoplus_\rho \mathbb{Z} \to \mathrm{Cl}(X_\Sigma) \to \mathrm{Cl}(T) = 0.$$

So we have a surjection $\mathrm{Div}_T(X_\Sigma) \twoheadrightarrow \mathrm{Cl}(X_\Sigma)$. The composition $M \to \mathrm{Div}_T(X_\Sigma) \to \mathrm{Cl}(X)$ is zero. Assume $[\sum_\rho a_\rho D_\rho] = 0$ in $\mathrm{Cl}(X_\Sigma)$ (where $a_\rho \in \mathbb{Z}$). Then $\mathrm{div}(f) = \sum_\rho a_\rho D_\rho$ for some $f \in K(X)^*$. Since $\mathrm{div}(f)|_T = 0$ in $\mathrm{Div}(T)$, f is a unit on the torus, so $f = c\chi^m$ for some $m \in M$ and $c \in \mathbb{C}^*$. $\square$

Let $\{e_1, \dots, e_n\}$ be a basis for $M \cong \mathbb{Z}^n$, and choose a dual basis for N. Label rays $\rho_1, \dots, \rho_r$. This realizes $M \to \mathrm{Div}_T(X_\Sigma)$ as an integer matrix

$$\begin{aligned} \Phi : \mathbb{Z}^n &\to \mathbb{Z}^r, \\ e_i &\mapsto \sum_{j=1}^r \langle e_i, u_{\rho_j} \rangle D_{\rho_j}. \end{aligned}$$

Example IV.3.2. The fan of $\mathbb{P}^2$ has rays generated by $(1,0), (0,1), (-1,-1)$. Its class group is

$$\mathrm{Cl}(\mathbb{P}^2) = \operatorname{coker}\left(\mathbb{Z}^2 \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{smallmatrix}\right]} \mathbb{Z}^3 \right) = \mathbb{Z}.$$

Example IV.3.3. The fan of a Hirzebruch surface has rays generated by $(1,0), (0,1), (0,-1), (-1,a)$. Its class group is

$$\mathrm{Cl}(\text{Hirzebruch surface}) = \operatorname{coker}\left(\mathbb{Z}^2 \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & a \end{smallmatrix}\right]} \mathbb{Z}^4 \right) = \mathbb{Z}^2.$$

Example IV.3.4. The fan with rays generated by $(0,1), (d,-1)$ corresponds to the cone over $\mathbb{P}^1 \xrightarrow{|\mathcal{O}(d)|} \mathbb{P}^d$. Its class group is $\operatorname{coker} \begin{bmatrix} d & -1 \\ 0 & 1 \end{bmatrix} = \mathbb{Z}/d\mathbb{Z}$.

IV.3.2 Picard groups of toric varieties

Write $\mathrm{CDiv}_T(X_\Sigma)$ for the subgroup of torus-invariant Cartier divisors.

Note IV.3.5. If D is a Cartier divisor, then D is also a Weil divisor, so $D \sim \sum_\rho a_\rho D_\rho$ for some $a_\rho \in \mathbb{Z}$.

Corollary IV.3.6. *We have an exact sequence*

$$\begin{aligned} M &\to \mathrm{CDiv}_T(X_\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0 \\ m &\mapsto \mathrm{div}(\chi^m) \end{aligned}$$

which is exact on the left if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ *spans* $N_\mathbb{R}$.

Remark IV.3.7. Locally principal doesn't necessarily imply principal on *every* open affine set. For example, let $C \subseteq \mathbb{A}^2$ be a plane curve of degree $d > 1$, and let $P \in C$ be a point. Then P is locally principal, but not principal. Indeed, let $\ell = V(f)$ be a line transverse to C at P. Then $\ell \cap C = \{P, Q_1, \ldots, Q_{d-1}\}$ contains points other than P, and $\mathrm{div}(f|_C) = [P]+[Q_1]+\cdots+[Q_{d-1}]$. However, $\mathrm{div}(f|_U) = [P]$, where $U = C - \{Q_1, \ldots, Q_{d-1}\}$.

However, on toric varieties, the situation is nice:

Theorem IV.3.8. *Let* $D = \sum_\rho a_\rho D_\rho$ *on* X_Σ. *The following are equivalent:*

(1) D *is Cartier.*

(2) D *is principal on* U_σ *for all* $\sigma \in \Sigma$.

(3) For each cone $\sigma \in \Sigma$, *there exists* $m_\sigma \in M$ *such that* $\langle m_\sigma, u_\rho \rangle = -a_\rho$ *for all* ρ.

(4) For each maximal *cone* $\sigma \in \Sigma$, *there exists* $m_\sigma \in M$ *such that* $\langle m_\sigma, u_\rho \rangle = -a_\rho$ *for all* ρ.

Proof. Trivially, (2) $\implies$ (1). Moreover, (2) is equivalent to the existence of $m_\sigma \in M$ such that

$$D|_{U_\sigma} = \mathrm{div}(\chi^{-m_\sigma}) = \sum_{\rho in \sigma(1)} \langle -m_\sigma, u_\rho \rangle \, D_\rho.$$

But also $D|_{U_\sigma} = \sum_{\rho \in \sigma(1)} a_\rho D_\rho$, so (2) $\iff$ (3).

Trivially, (3) $\implies$ (4). Also, (4) $\implies$ (3) because if m_σ works for σ, then it works for any face of σ. Finally, (1) $\implies$ (2) is implied by the following proposition. $\square$

Proposition IV.3.9. *Let* $\sigma \in \Sigma$. *Then:*

(1) Every torus-invariant Cartier divisor on U_σ *is trivial.*

(2) $\mathrm{Pic}(U_\sigma) = 0$.

Example IV.3.10. Consider a cone X_Σ, and let X_{Σ_0} be the cone with the singular point removed. Then $\mathrm{Cl}(X_\Sigma) = \mathrm{Cl}(X_{\Sigma_0}) = \mathrm{Pic}(X_{\Sigma_0}) = \mathbb{Z}/d\mathbb{Z}$, but $\mathrm{Pic}(X_\Sigma) = \mathrm{Pic}(U_\sigma) = 0$.

Example IV.3.11. Consider $X_\Sigma = \operatorname{Cone}(\mathbb{P}^1 \times \mathbb{P}^1) = V(xz - yw) \subseteq \mathbb{C}^4$, where Σ has rays $(e_1, e_2, e_1 + e_3, e_2 + e_3)$. Then

$$\operatorname{Cl}(X_\Sigma) = \operatorname{coker}\left(\mathbb{Z}^3 \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]} \mathbb{Z}^4 \right) = \mathbb{Z}$$

is generated by D_ρ for any ray ρ. However, $\operatorname{Pic}(X_\Sigma) = 0$; this is an exercise in the book.

Theorem IV.3.12. *The following are equivalent:*

(1) $\operatorname{Cl}(X_\Sigma) = \operatorname{Pic}(X_\Sigma)$.

(2) X_Σ *is smooth.*

Theorem IV.3.13. *The following are equivalent:*

(1) The index of $\operatorname{Pic}(X_\Sigma)$ *in* $\operatorname{Cl}(X_\Sigma)$ *is finite.*

(2) X_Σ *is simplicial.*

IV.4 2014-11-06: Cartier divisors, continued

IV.4.1 Smooth and simplicial toric varieties

Recall that a divisor $D = \sum_\rho a_\rho D_\rho$ on X_Σ is Cartier if for each maximal cone $\sigma \in \Sigma$, there is $m_\sigma \in M$ with $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$. We call $\{m_\sigma\}_\sigma$ the *Cartier data*.

Theorem IV.4.1. *Let* X_Σ *be a normal toric variety. The following are equivalent:*

(1) Every Weil divisor is Cartier.

(2) $\operatorname{Cl}(X_\Sigma) = \operatorname{Pic}(X_\Sigma)$.

(3) X_Σ *is smooth.*

The easy part is (1) $\iff$ (2). Let's show that (3) implies (2). Fix σ. Without loss of generality, $\sigma = \operatorname{Cone}(e_1, \ldots, e_k)$. Then $D|_{U_\sigma} = \sum_{i=1}^k a_i D_i$, where D_i is the divisor corresponding to the ray $\operatorname{Cone}(e_i) \preceq \sigma$. Extend $e_1, \ldots, e_k$ to a basis $e_1, \ldots, e_n$ and choose a dual basis for $M = \mathbb{Z}^n$. Choose $m_\sigma = (a_1, \ldots, a_k, 0, \ldots, 0)$. So D is Cartier.

Now we show (2) implies (3). Choose $\sigma \in \Sigma$ with rays $\rho_1, \ldots, \rho_s$. Consider the map

$$\begin{aligned} M &\to \mathbb{Z}^s = \bigoplus_{i=1}^s \mathbb{Z} \cdot [D_i] = \operatorname{Div}_T(U_\sigma) \\ m &\mapsto \operatorname{div}(\chi^{-m}). \end{aligned}$$

This map is an $n \times s$ matrix whose rows correspond to U_{ρ_i} for $i = 1, \ldots, s$. This map is surjective if and only if the U_{ρ_i} can be extended to a lattice basis of M. By (2), the map is surjective and hence σ is a smooth cone. $\square$

There's a variant of this for simplicial toric varieties:

Theorem IV.4.2. *Let X_Σ be a normal toric variety. The following are equivalent:*

(1) Every Weil divisor is $\mathbb{Q}$-Cartier.

(2) $\mathrm{Pic}(X_\Sigma)$ has finite index in $\mathrm{Cl}(X_\Sigma)$.

(3) X_Σ is simplicial.

IV.4.2 Polytope divisors

Recall that a full-dimensional lattice polytope $P \subseteq M_\mathbb{R}$ has a canonical presentation

$$P = \left\{ m \in M_\mathbb{R} \;\middle|\; \langle m, u_F \rangle \geq -a_F \ \forall \text{ facets } F \preceq P \right\}.$$

From P, we build a normal fan Σ_P to get a toric variety $X_P := X_{\Sigma_P}$. Since $|\Sigma_P| = N_\mathbb{R}$, X_P is always proper.

Ray generators of Σ_P correspond to facets normal to u_F; let D_F be the corresponding divisor on X_P. Let

$$D_P := \sum_{F \text{ facet of } P} a_F D_F.$$

This is a torus-invariant Weil divisor on X_P.

Proposition IV.4.3. *D_P is a Cartier divisor.*

Proof. A vertex $v \in P$ corresponds to a maximal cone $\sigma_v \in \Sigma_P$. Since v is a vertex, $\langle v, u_F \rangle = -a_F$ for all $v \in F$. Since $v \in M$, we choose $\{v\}_{\sigma_v}$ as our Cartier data. □

So each full-dimensional lattice polytope P corresponds to a pair (X_P, D_P) where X_P is a proper normal toric variety and D_P is a Cartier divisor on X_P.

IV.4.3 Support functions

Definition IV.4.4. Let $\Sigma \subseteq N_\mathbb{R}$ be a fan.

(1) A *support function* is a function $\phi : |\Sigma| \to \mathbb{R}$ that is linear on each $\sigma \in \Sigma$.

(2) ϕ is *integral* with respect to N if $\phi(\Sigma \cap N) \subseteq \mathbb{Z}$. The space of all integral support functions on Σ is denoted $\mathrm{SF}(\Sigma, N)$.

Theorem IV.4.5. *Let $\Sigma \subseteq N_\mathbb{R}$ be a fan.*

(1) Let $D = \sum_\rho a_\rho D_\rho$ with Cartier data $\{m_\sigma\}_\sigma$. Then the function

$$\begin{aligned} \phi_D : |\Sigma| &\to \mathbb{R} \\ u &\mapsto \langle m_\sigma, u \rangle \quad \forall u \in \sigma \end{aligned}$$

is a well-defined element $\mathrm{SF}(\Sigma, N)$.

(2) $\phi_D(u_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$.

(3) $D \mapsto \phi_D$ is a group isomorphism $\mathrm{CDiv}_T(X_\Sigma) \xrightarrow{\simeq} \mathrm{SF}(\Sigma, N)$.

Proof. (1) The map is well-defined because each m_σ in the Cartier data is unique modulo $\sigma^\perp \cap M$. Clearly ϕ_D is linear on each σ and is integral.

(2) By the definition of Cartier data, $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$.

(3) The map $D \mapsto \phi_D$ is a group homomorphism. Part (2) implies injectivity. For surjectivity, take $\phi \in \mathrm{SF}(\Sigma, N)$ and fix $\sigma \in \Sigma$. Since ϕ is integral, the map $\phi|_{\sigma \cap N} : \sigma \cap N \to \mathbb{Z}$ is $\mathbb{N}$-linear, and extends to a $\mathbb{Z}$-linear map $\phi|_{\mathrm{span}(\sigma)\cap N} : \mathrm{span}(\sigma) \cap N \to \mathbb{Z}$.

Since $\mathrm{Hom}_{\mathbb{Z}}(N_\sigma, \mathbb{Z}) = M/M(\sigma)$, there exists $m_\sigma \in M$ where $\phi(u) = \langle m_\sigma, u \rangle$ for all $u \in \Sigma \cap N$. Thus we recover the Cartier data $\{m_\sigma\}_\sigma$. ☐

Proposition IV.4.6. *Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional lattice polytope with normal fan Σ_P. Consider the function*

$$\begin{aligned} \phi_P &: |\Sigma| \to \mathbb{R} \\ \phi_P(u) &= \min \{\langle m, u \rangle : m \in P\}. \end{aligned}$$

Then $\phi_P \in \mathrm{SF}(\Sigma, N)$ corresponds to D_P, i.e., $\phi_P = \phi_{D_P}$.

IV.5 2014-11-11

Recall that a Weil divisor D induces a sheaf $\mathcal{O}_X(D)$, which is a line bundle iff D is Cartier. Let's consider the case where X_Σ is smooth, hence all divisors are Cartier. On X_Σ, let $D = \sum_\rho a_\rho D_\rho$. Let

$$\Gamma(X_\Sigma, \mathcal{O}_X(D)) = \{\text{global sections of } \mathcal{O}_X(1)\} \subseteq K(X)^*.$$

The set $P_D := \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\} \subseteq M_{\mathbb{R}}$ is an intersection of half-spaces, hence a *polyhedron*. However, it may fail to be a polytope, or it may be a non-lattice polytope.

Example IV.5.1. Let $X = \mathbb{A}^1$ and Σ the usual fan for $\mathbb{A}^1$. Let $D = 0$. Then

$$P_D = \left\{ m \in \mathbb{R} \,\middle|\, \langle m, 1 \rangle \geq 0 \right\} = \mathbb{R}_{\geq 0}.$$

Theorem IV.5.2. *The set $\{\chi^m \mid m \in P_D \cap M\}$ is a $\mathbb{C}$-basis for the global sections of $\mathcal{O}_{X_\Sigma}(D)$.*

Example IV.5.3. Consider $\mathbb{P}^2$ with the fan Σ with rays ρ_0, ρ_1, ρ_2 generated by points $(1,0), (0,1), (-1,-1)$. Let $D = D_0 + D_1 + D_2$ be the corresponding divisor. Then

$$P_D = \left\{ (m_0, m_1) \,\middle|\, m_0 \geq -1, m_1 \geq -1, -m_0 - m_1 \geq -1 \right\}.$$

A basis of $\Gamma(\mathbb{P}^2, \mathcal{O}_X(D))$ is

$$\frac{1}{xy} \left\{ 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3 \right\} \subseteq K(\mathbb{P}^2) = \mathbb{C}(x, y).$$

We have $\deg D = 3$ and $\mathcal{O}_{\mathbb{P}^2}(D) = \mathcal{O}_{\mathbb{P}^2}(3)$.

IV.5.1 Digression on projective space

What is $\mathbb{P}^n_{\mathbb{C}}$?

- The space of lines in $\mathbb{A}^{n+1}$ through the origin.
- Points in $\mathbb{A}^{n+1} \setminus \{0\}$ modulo $\mathbb{C}^*$.
- $\operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n]$, whose points correspond to homogeneous prime ideals not containing the irrelevant ideal.
- The CW-complex $\mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \cdots \cup \mathbb{A}^0$.
- Gluing together a bunch of copies of $\mathbb{A}^n$.
- X_Σ defined via gluing.
- Subvarieties of $\mathbb{P}^n$ correspond to homogeneous prime ideals in $\mathbb{C}[x_0, \ldots, x_n]$ that don't contain the irrelevant ideal.

Let's consider the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ with its usual fan. We have

$$\mathbb{P}^1 \times \mathbb{P}^1 = \left(\frac{\mathbb{A}^2 - \{0\}}{\mathbb{C}^*} \right) \times \left(\frac{\mathbb{A}^2 - \{0\}}{\mathbb{C}^*} \right).$$

Give the first $\mathbb{A}^2$ coordinates (x_0, x_2) and the second coordinates (x_1, x_3). Then

$$\mathbb{P}^1 \times \mathbb{P}^1 = \frac{\mathbb{A}^4 - V(x_1, x_3) - V(x_0, x_2)}{(\mathbb{C}^*)^2},$$

where the action of $(\lambda, \mu) \in (\mathbb{C}^*)^2$ on $(x_0, x_1, x_2, x_3) \in \mathbb{A}^4$ is $(\lambda x_0, \mu x_1, \lambda x_2, \mu x_3)$.

Aside IV.5.4. The following are equivalent:

(i) a $(\mathbb{C}^*)^2$-action on $S = \mathbb{C}[x_0, x_1, x_2, x_3]$ that respects multiplication and addition.

(ii) a $\mathbb{Z}^2$-grading of S, i.e., $S = \bigoplus_{\alpha \in \mathbb{Z}^2} S_\alpha$ respecting multiplication and addition.

If $\deg(x_i) = (a_i, b_i) \in \mathbb{Z}^2$, then we get a $(\mathbb{C}^*)^2$-action on S by $(\lambda, \mu) \cdot x_i = \lambda^{a_i} \mu^{b_i} x_i$. Since $\deg(x_i x_j) = \deg(x_i + \deg(x_j)$, this respects the ring structure.

Conversely, given $(\mathbb{C}^*)^2 \circlearrowleft S$, decompose into irreducible representations, and get $S = \bigoplus_{\alpha \in \mathbb{Z}^2} S_\alpha$, the $\mathbb{Z}^2$-grading.

IV.5.2 Hirzebruch surfaces as quotients

Consider the Hirzebruch surface X_Σ with rays generated by $(1, 0), (0, 1), (0, -1), (-1, 3)$. Let $S = \mathbb{C}[x_0, x_1, x_2, x_3]$. We have exact sequences

$$0 \to M \to \operatorname{Div}_T(X_\Sigma) = \mathbb{Z}^{\Sigma(1)} \to \operatorname{Cl}(X_\Sigma) \to 0,$$
$$0 \leftarrow N \leftarrow \mathbb{Z}^{\Sigma(1)} \leftarrow \operatorname{Cl}(X_\Sigma)^* \leftarrow 0.$$

In our example, the latter is

$$0 \longleftarrow \mathbb{Z}^2 \xleftarrow{\left[\begin{smallmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -1 \end{smallmatrix}\right]} \mathbb{Z}^4 \xleftarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 3 & 1 \end{smallmatrix}\right]} \text{kernel} = \mathrm{Cl}(X_\Sigma)^* \longleftarrow 0.$$

We have $\deg(x_1) = (1,0)$, $\deg(x_2) = (0,1)$, $\deg(x_3) = (1,0)$, and $\deg(x_4) = (3,1)$. Since $V(x_1, x_3)$ and $V(x_2, x_4)$ are empty in X_Σ, so $(x_1, x_3) \cap (x_2, x_4)$ is the "irrelevant ideal" for X_Σ. So

$$X_\Sigma = \frac{\{(x_1, x_2, x_3, x_4)\} - \{x_1 = x_3 = 0\} - \{x_2 = x_4 = 0\}}{\text{equivalence induced by } (\mathbb{C}^*)^2\text{-action}}.$$

Chapter V

Quotients

V.1 2014-11-13: Quotients

Let G be a group and $G \times X \to X$ an action. What is the quotient X/G?

V.1.1 Topological quotients

For X a topological space and $G \times X \to X$ continuous, X/G is defined in the category of topological spaces so that points of X/G correspond to G-orbits in X, there is a surjective map

$$\begin{aligned} \pi : X &\to X/G \\ x &\mapsto G \cdot x, \end{aligned}$$

and $U \subseteq X/G$ is open $\iff$ $\pi^{-1}(U) \subseteq X$ is open.

V.1.2 Affine quotients

For $X = \operatorname{Spec} R$ a variety and $G \times X \to X$ a morphism of varieties, we want to define this via regular functions. If $f \in R$, $g \in G$, and $x \in X$, we define

$$\begin{aligned} g \cdot f : X &\to \mathbb{C}, \\ x &\mapsto f(g^{-1}x). \end{aligned}$$

Let $R^G \subseteq R$ be the ring of G-invariant functions. For $f \in R^G$, there is a well-defined map

$$\begin{aligned} \bar{f} : X/G &\to \mathbb{C}, \\ G \cdot x &\mapsto f(x), \end{aligned}$$

where X/G is a suitable quotient object. In the affine case, we can define $X/G = \operatorname{Spec}(R^G)$.

Example V.1.1 (Fairly good quotient). Let $X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$. Let $G = \{\pm 1\}$ act by $-1 \cdot (x, y) = (-x, -y)$. Then

$$\mathbb{C}[x, y]^G = \mathbb{C}[x^2, xy, y^2] = \frac{\mathbb{C}[a, b, c]}{\langle ac - b^2 \rangle},$$

and we have a surjective map

$$\mathbb{A}^2 \to V(ac - b^2) \subseteq \mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[a, b, c],$$
$$(x, y) \mapsto (x^2, xy, y^2).$$

Point of $V(ac - b^2)$ are in bijective correspondence with G-orbits of $\mathbb{A}^2$.

However, this isn't as nice as it might seem — the quotient isn't smooth, and the quotient map isn't flat (there are fibers of different dimensions).

Example V.1.2 (Not as good). Consider $\mathbb{C}^*$ acting on $\mathbb{C}^4 = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_4]$ by

$$\lambda \cdot (x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^{-1} x_3, \lambda^{-1} x_4).$$

We have $\mathbb{C}[x_1, \dots, x_4] = \mathbb{C}[x_1x_3, x_1x_4, x_2x_3, x_2x_4]$. Our quotient map is

$$\pi : \mathbb{C}^4 \to \operatorname{Spec} \mathbb{C}[x_1x_3, x_1x_4, x_2x_3, x_2x_4] = V(ad - bc) \subseteq \mathbb{C}^4 = \operatorname{Spec} \mathbb{C}[a, b, c, d].$$

Again, π is surjective. Also, if $p \in V(ad - bc) \setminus \{0\}$, then $\pi^{-1}(p)$ corresponds to a single $\mathbb{C}^*$-orbit.

However, $\pi^{-1}(0)$ is the union of all $\mathbb{C}^*$-orbits contained in $\mathbb{C}^2 \times \{(0,0)\} \cup \{(0,0)\} \times \mathbb{C}^2 \subseteq \mathbb{C}^4$. So we lose the bijection between G-orbits of X and points of X/G. This is bad.

Example V.1.3 (Really bad). Let $\mathbb{C}^*$ act on $\mathbb{A}^{n+1}$ by $\lambda \cdot (x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n)$. Then $\mathbb{C}[x_0, \dots, x_n]^{\mathbb{C}^*} = \mathbb{C}$, so $\pi : \mathbb{A}^{n+1} \to \operatorname{Spec} \mathbb{C}$.

Remark V.1.4. Another problem: R^G can fail to be a finitely-generated $\mathbb{C}$-algebra. (Hilbert 14, answered by Nagata.)

Lemma V.1.5. *Let G act on $X = \operatorname{Spec}(R)$ so that R^G is a finitely-generated $\mathbb{C}$-algebra. Then:*

(1) The natural map $\pi : X \to \operatorname{Spec}(R^G)$ satisfies a universal property: For any affine scheme $Z = \operatorname{Spec}(S)$ and any morphism $\varphi : X \to Z$ such that $\varphi(g \cdot x) = \varphi(x)$ for all $g \in G$ and all $x \in G$, there exists a unique morphism $\psi : Y \to Z$ such that $\varphi = \psi \circ \pi$.

(2) If X is irreducible, then Y is irreducible.

(3) If X is normal, then Y is normal.

V.1.3 Good categorical quotients

Is there a better X/G for X not necessarily affine? For example, can we interpret $\mathbb{P}^n = (\mathbb{A}^{n+1} - \{0\})/\mathbb{C}^*$ as a statement about quotients of varieties?

We will define a *good categorical quotient* $X /\!\!/ G$ defined by two properties (if it exists):

- Given a G-equivariant morphism $\varphi : X \to Z$ (i.e., $\varphi(gx) = \varphi(x)$ for all $g \in G$, $x \in X$), φ factors uniquely through $\pi : X \to X /\!\!/ G$.
- The morphisms $G \times X \to X \xrightarrow{\pi} X /\!\!/ G$ are the same whether $G \times X \to X$ is the group action or the projection map.

Remark V.1.6. Good categorical quotients may not always exist. When they exist, though, they have nice properties.

Good properties of good categorical quotients:

- $U \subseteq X /\!\!/ G$ is open $\iff$ $\pi^{-1}(U) \subseteq X$ is open.
- $U \subseteq X /\!\!/ G$ is nonempty and open, then $\pi : \pi^{-1}(U) \to U$ is also a good categorical quotient.

V.1.4 Good geometric quotients

A *good geometric quotient* is a good geometric quotient with the additional property that points in $X /\!\!/ G$ bijectively correspond to G-orbits. The main cases are where GL_n, a finite group, a torus, or a reductive algebraic group act on X.

Here's a case of interest: Suppose $X = \bigcup_\alpha \operatorname{Spec}(R_\alpha)$. We can try to build a good geometric quotient by gluing $\operatorname{Spec}(R_\alpha^G)$.

Example V.1.7. Consider $\mathbb{C}^2 - \{0\} = \operatorname{Spec}\mathbb{C}[x_0^\pm, x_1] \cup \operatorname{Spec}\mathbb{C}[x_0, x_1^\pm]$, and let $\mathbb{C}^*$ act by dilations. Then $\mathbb{C}[x_0^\pm, x_1]^{\mathbb{C}^*} = \mathbb{C}[\frac{x_1}{x_0}]$ and $\mathbb{C}[x_0, x_1^\pm]^{\mathbb{C}^*} = \mathbb{C}[\frac{x_0}{x_1}]$, and gluing yields $(\mathbb{C}^2 - \{0\}) /\!\!/ \mathbb{C}^* \cong \mathbb{P}^1$, a good geometric quotient.

V.1.5 Stack quotients

There's another perspective on quotients (and more generally, on algebraic geometry) due to Grothendieck: We think of a geometric object via the category of sheaves on it. (This is closely tied to the philosophy of derived categories.) Rather than specifying the quotient as a geometric object, we can specify the category of sheaves on it.

Building from this, a sheaf on X/G should be a G-equivariant sheaf on X, i.e., a sheaf on X with a G-action. There typically isn't a scheme whose category of sheaves is this category; this motivates stacks, which are geometric objects which can have such categories of sheaves. We have a *stack quotient* $[X/G]$ such that the category of coherent sheaves on $[X/G]$ is the category of coherent sheaves on X with a G-action.

V.2 2014-11-25: Toric varieties as quotients

We want to consider toric varieties X_Σ as (some variety) $/\!\!/$ (some group). What's the group? Let $N_\mathbb{R}$ be a fan. If the rays of Σ span $N_\mathbb{R}$, we get a short exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X_\Sigma) \to 0.$$

Apply $\operatorname{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ to get

$$1 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 1.$$

Theorem V.2.1. *(a)* $\mathrm{Cl}(X_\Sigma)$ *is the character group of* G.

(b) G *is isomorphic to a product of a torus and a finite abelian group. (*G *is reductive.)*

(c) Given a basis $e_1, \ldots, e_m$ *of* M*, we have*

$$G = \left\{ (t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \prod_\rho t_\rho^{\langle m, u_\rho \rangle} = 1 \ \forall m \in M \right\}$$
$$= \left\{ (t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \prod_\rho t_\rho^{\langle e_i, u_\rho \rangle} = 1 \ \forall i = 1, \ldots, n \right\}.$$

Proof. Since $\mathrm{Cl}(X_\Sigma)$ is a finitely-generated abelian group, $\mathrm{Cl}(X_\Sigma) \cong \mathbb{Z}^\ell \times H$ for some finite abelian group H. Then

$$G = \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*) = (\mathbb{C}^*)^\ell \times \mathrm{Hom}(H, \mathbb{C}^*).$$

This proves (b). Note that $\alpha \in \mathrm{Cl}(X_\Sigma)$ yields a character on G by $g \mapsto g(\alpha) \in \mathbb{C}^*$, giving (a). For (c), note that $M \to \mathbb{Z}^{\Sigma(1)}$ is $m \mapsto (\langle m, u_\rho \rangle)_\rho$. □

The group G is determined entirely by $\Sigma(1)$.

V.2.1 Quotient construction

Let $S := \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$. This is the *Cox ring* of X_Σ. Given $\sigma \in \Sigma$, we write $x^{\hat{\sigma}} := \prod_{\rho \notin \sigma} x_\rho$, and define

$$B(\Sigma) := \left\langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \right\rangle \subseteq S,$$

the *irrelevant ideal*.

The group G acts on S by the action as a subgroup of $(\mathbb{C}^*)^{\Sigma(1)}$.

The toric variety of the fan we are going to build is $\mathbb{C}^{\Sigma(1)} \setminus V(B(\Sigma))$. Let $\{e_\rho \mid \rho \in \Sigma(1)\}$ bet he standard basis of $\mathbb{Z}^{\Sigma(1)}$. For each $\sigma \in \Sigma$, let $\tilde{\sigma} = \mathrm{Cone}(e_\rho \mid \rho \in \sigma)$.

Theorem V.2.2. *Let* $\tilde{\Sigma} = \{\tilde{\sigma}\} \subseteq \mathbb{R}^{\Sigma(1)}$*. This is a fan.*

(1) $X_{\tilde{\Sigma}} = \mathbb{C}^{\Sigma(1)} \setminus V(B(\Sigma))$.

(2) The map $e_\rho \mapsto u_\rho$ *defines a map of lattices* $\mathbb{Z}^{\Sigma(1)} \to N$ *that is compatible with* $\tilde{\Sigma}$ *and* Σ.

(3) The resulting toric morphism $\pi : \mathbb{C}^{\Sigma(1)} \setminus V(B(\Sigma)) \to X_\Sigma$ *is constant on* G*-orbits.*

Proof. (1) Start with the full fan for $\mathbb{C}^{\Sigma(1)}$ (all subsets of $\{e_\rho\}$). Removing faces from the fan corresponds to removing torus orbits from $\mathbb{C}^{\Sigma(1)}$, which corresponds to removing orbit closures of all minimal non-cones of $\tilde{\Sigma}$. One can check that we've removed *exactly* $V(B(\Sigma))$.

(2) This is a straightforward verification.

(3) On tori, we have the map $\pi : (\mathbb{C}^*)^{\Sigma(1)} \to T_N$ from earlier. For $g \in G$ and $x \in \mathbb{C}^{\Sigma(1)} \setminus V(B(\Sigma))$, since the morphism is toric,

$$\pi(g \cdot x) = \pi(g) \cdot \pi(x) = \pi(x)$$

since $\pi(G) = 1$.

□

V.2.2 Good geometric quotients

Recall: A good categorical quotient $X /\!\!/ G$ is a good geometric quotient if there is also a bijection between G-orbits in X and points of $X /\!\!/ G$.

Theorem V.2.3. *Suppose X_Σ has no torus factors.*

(1) $\pi : \mathbb{C}^{\Sigma(1)} \setminus V(B(\Sigma)) \to X_\Sigma$ is an almost geometric quotient *for the action of G (i.e., a good categorical quotient and there is a dense open set $U_0 \subseteq X$ where $G \cdot x$ is closed for all $x \in U_0$).*

(2) π is a good geometric quotient if and only if X_Σ is simplicial.

V.2.3 Global coordinates

For example, on $\mathbb{P}(1,1,2)$, we can view points as $[x_0 : x_1 : x_2]$ with the equivalence relation $[x_0 : x_1 : x_2] \sim [\lambda x_0 : \lambda x_1 : \lambda^2 x_2]$ for $\lambda \in \mathbb{C}^*$.

On a Hirzebruch surface, points are given by $[x_1 : x_2 : x_3 : x_4]$ with equivalence relation $[x_1 : x_2 : x_3 : x_4] \sim [\lambda x_1 : \mu x_2 : \lambda x_3 : \lambda^a \mu x_4]$ for all $\lambda, \mu \in \mathbb{C}^2$. The weights are the kernel of the ray matrix

$$\begin{bmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Another example: $\mathrm{Bl}_0(\mathbb{C}^2)$. We have an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]} \mathbb{Z}^3 \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]} N.$$

The coordinates are $[x : y : t] \sim [\lambda x : \lambda y : \lambda^{-1} t]$ for all $\lambda \in \mathbb{C}^*$, where $(x, y) \neq (0,0)$.

V.3 2014-12-02

Recall that the Cox ring has a $\mathrm{Cl}(X)$-grading. Let $\beta \in \mathrm{Cl}(X)$ and $f \in S_\beta$. Let $x \in \mathrm{Spec}(S) = \mathbb{C}^{\#\Sigma(1)}$. Then

$$f(g \cdot x) = \chi^\beta(g) \cdot f(x).$$

Note that $\{x \mid f(x) = 0\}$ is a union of G-orbits, so $\{\pi(x) \mid f(x) = 0\} \subseteq X_\Sigma$ is well-defined.

Theorem V.3.1. *(1) If $I \subseteq S$ is a homogeneous prime ideal, then $V_X(I) := \{\pi(x) \in X \mid f(x) = 0\}$ is a closed subvariety of X_Σ.*

(2) All closed subvarieties arise in this way.

(3) The analogous statement for ideals and subschemes is true.

V.4 2014-12-04 [missing]

Chapter VI

Various topics

VI.1 2014-12-09: Maps into projective space

How do we give a map $V \to \mathbb{P}^n$?

To give a map $V \to \mathbb{A}^1$ is the same as giving a regular function $f \in \Gamma(V, \mathcal{O}_V)$. Likewise, to give a map $V \to \mathbb{A}^n$ is the same as giving regular functions $(f_1, \ldots, f_n)$.

Giving a map $V \to \mathbb{A}^n - \{0\}$ is the same as giving regular functions $(f_1, \ldots, f_n)$ such that $(f_1(v), \ldots, f_n(v)) \neq (0, \ldots, 0)$ for all $v \in V$.

What about $V \to (\mathbb{A}^n - \{0\})/\mathbb{C}^* = \mathbb{P}^n$?

Fact VI.1.1. A map $\pi : V \to \mathbb{P}^n$ is equivalent to the following data:

(1) a Cartier divisor $D \in \mathrm{Pic}(X)$ where $\mathcal{O}_V(D) = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$;

(2) $n+1$ global sections $s_0, \ldots, s_n \in H^0(V, \mathcal{O}_V(D))$ where $s_i = \pi^* x_i$ and $\{s_0 = s_1 = \cdots = s_n = 0\} = \varnothing \subseteq V$. (A Cartier divisor that admits such a set of global sections is called *base-point-free*.)

Two follow-ups for X_Σ:

(1) Clarify $X_\Sigma \to \mathbb{P}^n$.

(2) Classify maps $V \to X_\Sigma$.

VI.1.1 Maps from toric varieties into projective space

Given $D \in \mathrm{Pic}(X)$, we get a polytope P_D and $\{m_\sigma\}_{\sigma \in \Sigma(n)}$.

Theorem VI.1.2. *Let X_Σ be n-dimensional such that all maximal cones of Σ have dimension n. Then D is base-point-free if and only if the Cartier data $\{m_\sigma\}$ satisfies $m_\sigma \in P_D$ for all $\sigma \in \Sigma(n)$.*

VI.1.2 Maps into toric varieties

A map $\pi : V \to X_\Sigma$ is equivalent to the following data:

(1) a group homomorphism $D_\rho \mapsto L_\rho : \mathrm{Pic}(X_\Sigma) \to \mathrm{Pic}(V)$;

(2) $s_\rho \in H^0(V, L_\rho)$ for all $\rho \in \Sigma(1)$, where $\bigcap_{\rho \notin \sigma} \{s_\rho = 0\} = \varnothing \subseteq V$ for all $\sigma \in \Sigma$.

VI.1.3 Ample and very ample divisors

A divisor $D \in \mathrm{Pic}(V)$ is *very ample* if D is base-point-free and there exist sections $s_0, \ldots, s_n$ such that $v \mapsto [s_0(v) : \cdots : s_n(v)] : V \to \mathbb{P}^n$ is a closed embedding. A divisor D is *ample* if kD is very ample for some $k \geq 1$.

On $\mathbb{P}^n$, $\mathcal{O}(d)$ is base-point-free $\iff d \geq 0$, and $\mathcal{O}(d)$ is ample $\iff \mathcal{O}(d)$ is very ample $\iff d > 0$.

Theorem VI.1.3. *If D is an ample divisor on a smooth projective toric variety X_Σ, then D is very ample.*

Remark VI.1.4. This is vacuously true for proper varieties that aren't projective (since such varieties have no ample divisors). The theorem can fail for singular projective toric varieties.

Theorem VI.1.5. *Let D be an ample (or nef) divisor on X_Σ. Then $H^i(X, \mathcal{O}_X(D)) = 0$ for all $i > 0$.*

Corollary VI.1.6. *A normal toric variety is locally Cohen–Macaulay.*

Theorem VI.1.7 (Toric Chow lemma and resolution of singularities)**.** *If X_Σ is complete, then there is a refinement Σ' of Σ such that $\pi : X_{\Sigma'} \to X_\Sigma$ is birational and $X_{\Sigma'}$ is smooth and projective.*

VI.1.4 Euler sequence and canonical divisors

If X_Σ is smooth and has no torus factors, then there is an exact sequence

$$0 \to \Omega^1_{X_\Sigma} \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(-D_\rho) \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \to 0.$$

Corollary VI.1.8. *The canonical bundle is $\omega_{X_\Sigma} = \mathcal{O}_X(\sum_{\rho \in \Sigma(1)} -D_\rho)$.*

Example VI.1.9. $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$.

Example VI.1.10. $\omega_{\mathbb{P}^a \times \mathbb{P}^b} = \mathcal{O}(-a-1, -b-1)$.

Example VI.1.11. $\omega_{\mathbb{P}(q_1, \ldots, q_n)} = \mathcal{O}(\sum_i -q_i)$.

Example VI.1.12. $\omega_{\mathcal{H}_a} = \mathcal{O}(a-2, -2)$.

VI.2 2014-12-11: Mori dream spaces and T-varieties

VI.2.1 Mori dream spaces

Let X be a smooth projective variety. We have

$$\operatorname{Cox}(X) := \bigoplus_{L \in \operatorname{Pic} X} H^0(X, L).$$

For toric varieties, this is nice, but in general (e.g., for an elliptic curve E, for which $\operatorname{Pic}(E) \cong E \oplus \mathbb{Z}$), it might be wildly non-Noetherian.

Definition VI.2.1. A smooth projective variety X is a *Mori dream space* if $\operatorname{Cox}(X)$ is a finitely-generated $\mathbb{C}$-algebra.

Example VI.2.2. Smooth projective toric varieties are Mori dream spaces.

Example VI.2.3. Projective vector bundles over smooth projective toric variety are Mori dream spaces.

Example VI.2.4. Del Pezzo surfaces are Mori dream spaces.

VI.2.2 T-varieties

Let X be a variety with a torus action $(\mathbb{C}^*)^k \times X \to X$, where $\dim X = n \geq k$. (Maybe we also require an embedding $(\mathbb{C}^*)^k \hookrightarrow X$.) The *complexity of a T-variety* is $n - k$.

The complexity of a T-variety measures the balance between combinatorics and algebraic geometry. For $n - k = 0$ (i.e., toric varieties), all the geometry is encoded in the fan. On the other end, the affine cone of any n-dimensional projective variety is a T-variety of complexity n (because affine cones have a $\mathbb{C}^*$-action).

Essentially, the study of T-varieties of complexity $n - k$ amounts to combinatorics plus $(n - k)$-dimensional algebraic geometry.

Bibliography

[CLS] Cox, Little, and Schenck, *Toric Varieties.*

[F] Fulton, *Introduction to Toric Varieties.*

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