Dynamics of $F_\lambda = \lambda(z + \frac{1}{z^d})$

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1 Introduction

In this section we discuss the situation in which the polynomial being perturbed is the identity. Such functions, where $n = 1$ in the original formulation, are of interest since they do not satisfy the McMullen condition (i.e. $\frac{1}{n} + \frac{1}{d} < 1$). For the time being we will focus on the case where $d = 1$. The analogy to the previous examples is not so complete in this case since the function $z + \frac{\lambda}{z}$ is conjugate to the parameter free function $z + \frac{1}{z}$. For this reason, we will concentrate on the following related function

$$F_\lambda(z) = \lambda(z + \frac{1}{z}).$$

This function can be viewed, after a Möbius transformation, as a perturbation of the involution function by a linear term. As a further contrast to the previous examples, these functions have a repelling fixed point at infinity whenever $|\lambda| < 1$. Consequently, zero lies in the Julia set and thus there is no trap door as in the previous examples.

Viewing these functions as perturbations, we are mainly concerned with the behavior of $F_\lambda(z)$ for small values of λ. It has been shown by Yongcheng [6] that for $0 < |\lambda| \leq 1$ the Julia set is connected (otherwise it is a Cantor set). The parameter space is plotted in Figure 1. Similar figures have been produced by Hawkins [2] and Milnor [4]. We will approach the study of this function by focusing on the dynamical behavior as we approach the origin on both the real and imaginary axis. For any non-zero parameter lying on the real axis in parameter space the corresponding Julia set is a circle consisting of either the entire real axis or entire imaginary axis connecting zero and infinity. In contrast the behavior along the imaginary axis is much more complicated.

Given non-zero λ, this function is a degree-two rational map with two critical points, ± 1. These critical points behave symmetrically under $F_\lambda(z)$. For purely imaginary parameter values, this function has the desirable property that in the dynamical plane the real axis is mapped to the imaginary axis and vice versa. Therefore, for such parameter values we will consider the second iterate map restricted to the real axis, $\tilde{F}_\lambda^2 : \mathbb{R} \to \mathbb{R}$ where $\tilde{F}_\lambda^2 = F_\lambda^2|_\mathbb{R}$. Understanding $\tilde{F}_\lambda^2$ will be sufficient to understand the overall dynamics since both of the critical values are real. We compute this map directly as

$$\tilde{F}_\lambda^2(x) = \lambda^2(x + \frac{1}{x}) + \frac{1}{x + \frac{1}{x}}.$$

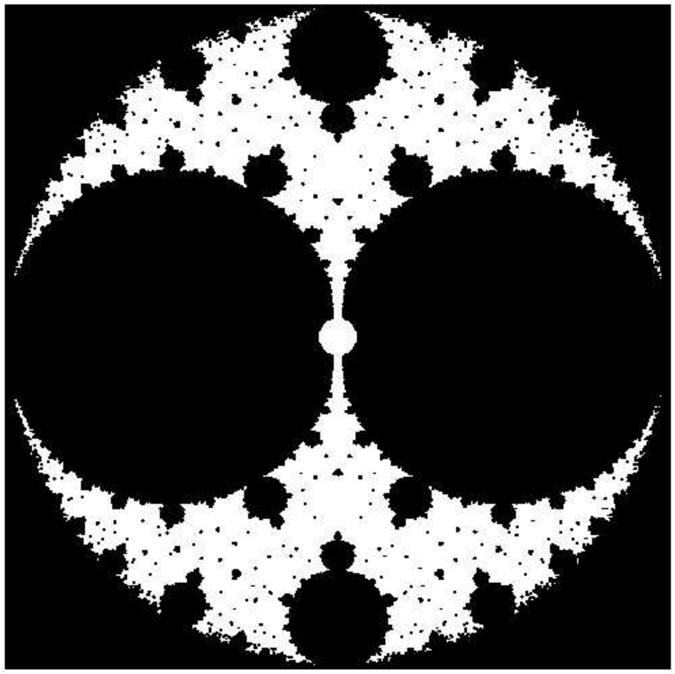

Figure 1: The λ parameter plane for the function $F_\lambda(z) = \lambda(z + \frac{1}{z})$

For small λ, the second iterate map can be viewed as a perturbation of the λ independent function $\frac{1}{x+\frac{1}{x}}$.

When a critical point lands on the repelling fixed point at infinity then the Julia set is the entire Riemann sphere [3]. Parameter values for which this occurs are known as m-ergodic rational maps (although m-ergodicity describes a larger set of maps than just those for which a critical point lands on infinity). Rees [5] has proved that m-ergodic maps comprise a set of positive Lebesgue measure in the parameter space of most rational maps. Hawkins [2] developed a computer algorithm for finding and plotting these parameter values. In that paper, it was shown numerically that for F_λ the m-ergodic maps accumulate on the origin along the imaginary axis in parameter space. We formalize this observation via the following theorem.

Theorem 1.1 *For the family of functions $F_\lambda(z) = \lambda(z + \frac{1}{z})$, in any neighborhood of $\lambda = 0$ we have*

1. *There exists a countably infinite set of parameter values lying on the imaginary axis for which the Julia set is the entire Riemann sphere (i.e. is m-ergodic);*

2. *There exists a countably infinite set of purely imaginary parameter values for which the critical point is part of a superattracting cycle.*

We will now briefly turn our attention to the cases when $d > 1$. The critical points are $c = d^{\frac{1}{d+1}}$. As in the $d = 1$ case the critical points do not depend on the parameter value.

Also there are lines, analogous to the imaginary axis for the case $d = 1$ and passing through the origin in parameter space for which F_λ^{d+1} is invariant over $\mathbb{R}$. The parameter space for several of these functions are plotted in Figure 2. For this class of rational function, the results of Rees [5] guarantee a set of positive Lebesgue measure in parameter space for which the Julia set is the whole Riemann sphere. However, it is unknown as to whether this behavior accumulates on the origin and hence whether a corollary to Theorem 1.1 is true for $d > 1$.

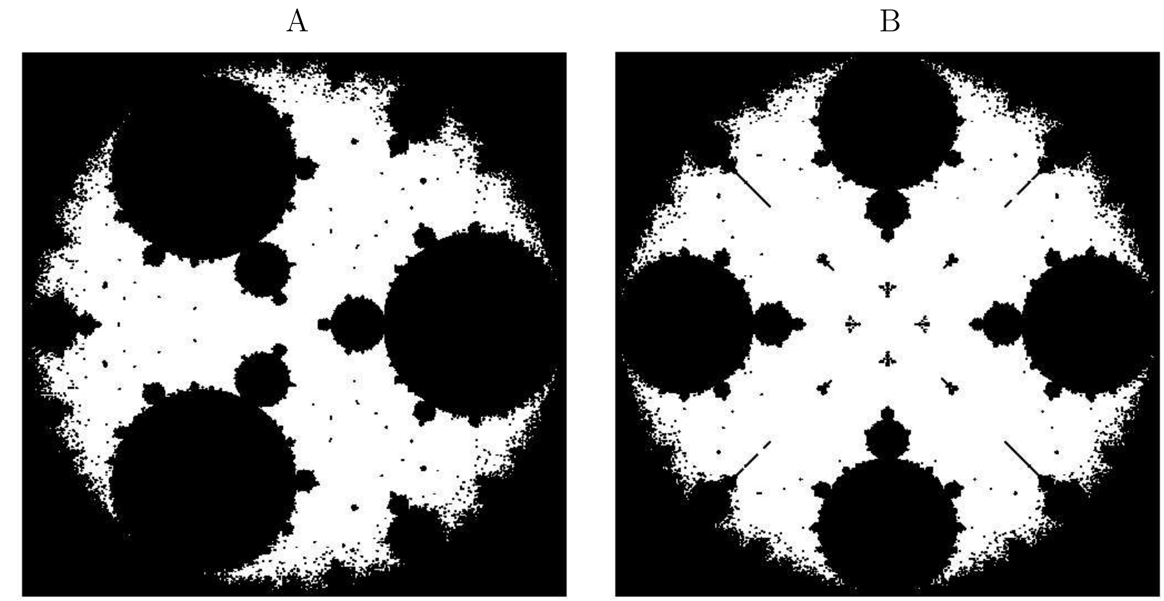

Figure 2: A) The λ parameter plane for the function $F_\lambda(z) = \lambda(z + \frac{1}{z^2})$ B) For $F_\lambda(z) = \lambda(z + \frac{1}{z^3})$

References

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