Iteration of an Even-Odd Splitting Map Can Make Integration Easier

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September 12, 2002

Abstract

We study the dynamics of the map $\mathcal{F}(R(x)) = \frac{R(\sqrt{x})-R(-\sqrt{x})}{2\sqrt{x}}$ on the space of rational functions, in the context of a new method of integration. We give a recursive formula for the iterates of a model family of rational functions, which is closed under the action of $\mathcal{F}$. We give a class of rational functions that are mapped to zero by two iterations of $\mathcal{F}$.

We prove that all polynomials are eventually mapped to even functions by $\mathcal{F}$, and we determine the number of iterations required for a given polynomial. We use power series representation to determine which rational functions are eventually mapped to even functions by $\mathcal{F}$.

1 Introduction

The integration of rational functions is one of the central tasks in calculus. The classical method of partial fractions reduces the problem to that of solving an algebraic equation. If $P(x)$ and $Q(x)$ are polynomials, the evaluation of

$$I = \int_0^\infty \frac{P(x)}{Q(x)}\,dx \tag{1.1}$$

requires factorization of the denominator

$$\begin{aligned} Q(x) &= (x-x_1)^{n_1}(x-x_2)^{n_2}\times\cdots\times(x-x_j)^{n_j} \qquad (1.2)\\ &= (x-x_1)^{n_1}(x-x_2)^{n_2}\times\cdots\times(x-x_k)^{n_k}\times\\ &\quad (x^2+2a_1x+a_1^2+b_1^2)^{m_1}\times\cdots\times(x^2+2a_px+a_p^2+b_p^2)^{m_p}. \end{aligned}$$

where $x_1, \ldots, x_j$ are the roots of $Q(x)=0$, and the factorization is converted to a real form by combining any non-real roots in conjugate pairs.

The difficulty associated with this method is that, as Abel showed, it is impossible to solve the general equation of degree 5 or more by radicals. Exact formulas for the roots of a polynomial are not always available. Therefore an interesting question is to classify the rational functions R for which the integral (1.1) can be evaluated without factoring the polynomial Q.

The integration of *even rational functions* seems to be an easier problem. Two examples are the classical Wallis formula [5]

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}, \qquad m \in \mathbb{N}, \tag{1.3}$$

and the evaluation in [1] of

$$\begin{aligned} N_{0,4}(a;m) &= \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \\ &= \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a), \qquad m \in \mathbb{N}, \end{aligned} \tag{1.4}$$

where

$$P_m(a) = 2^{-2m}\sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k. \tag{1.5}$$

The special case

$$N_{0,4}(a;0) = \frac{\pi}{2\sqrt{2(a+1)}} \tag{1.6}$$

of (1.4) will be employed in Section 3. See [3] for many more examples.

Recall that the even and odd parts of a function are defined as, respectively,

$$R_e(x) = \frac{R(x)+R(-x)}{2} \qquad \text{and} \qquad R_o(x) = \frac{R(x)-R(-x)}{2}. \tag{1.7}$$

We can rewrite $\int_0^\infty R(x)\,dx$ as

$$\int_0^\infty R(x)\,dx = \int_0^\infty R_e(x)\,dx + \int_0^\infty R_o(x)\,dx. \tag{1.8}$$

The first integral on the right has an even integrand, and so is likely to be easier to evaluate than the original. The change of variables $t = x^2$ in the second integral yields the identity

$$\int_0^\infty R(x)\,dx = \int_0^\infty R_e(x)\,dx + \frac{1}{2}\int_0^\infty \mathcal{F}(R(x))\,dx, \tag{1.9}$$

where the map $\mathcal{F}$ is defined by

$$\mathcal{F}(R(x)) = \frac{R(\sqrt{x})-R(-\sqrt{x})}{2\sqrt{x}}. \tag{1.10}$$

If the n^{th} iterate $\mathcal{F}^{(n)}(R(x))$ is even for some n, then the integral $\int_0^\infty R(x)\,dx$ reduces to an integral of even functions. In this paper we give necessary and sufficient conditions on the rational function $R(x)$ for this to occur.

The paper is organized as follows. In Section 2 we show that the map $\mathcal{F}$ preserves rationality of the function $R(x)$. Sections 3 and 4 contain examples. In Section 3 we show that $\mathcal{F}$ preserves the family of rational functions

$$R_m(a,x) = \frac{G_m(a)}{x^2 + H_m(a)x + 1}\,, \qquad G_0(a) = 1, \;\; H_0(a) = 2a, \tag{1.11}$$

and we give recursive formulas for $G_m(a)$ and $H_m(a)$. We also discuss analogous results for the family of rational functions

$$R(x) = \frac{1}{x^3 + ax^2 + bx^3 + 1}\,, \qquad a, b \in \mathbb{R}\,, \tag{1.12}$$

where we now include a substitution $x \to -x$ in our mapping function $\mathcal{F}$.

In Section 4 we show that the rational functions of the form

$$R(x) = \frac{xP(x^4) + x^2Q(x^2)}{V(x^4)}\,, \tag{1.13}$$

where P, Q, and V are polynomials, are mapped to *even* functions by one iteration of $\mathcal{F}$.

In Section 5 we establish a necessary and sufficient condition for a rational function $R(x)$ to be mapped to an even function by n iterations of $\mathcal{F}$. The condition is that certain coefficients in the power series for $R(x)$ about zero must vanish. In Section 6 we prove that all polynomials are eventually mapped to even functions by $\mathcal{F}$, and we determine the number of iterations required.

2 $\mathcal{F}$ preserves rationality

In this section we prove that the map $\mathcal{F}$ preserves the class of rational functions.

Proposition 2.1 *If $R(x)$ is a rational function, then $\mathcal{F}(R(x))$ is also rational.*

PROOF: Write $R(x) = P(x)/Q(x)$. A direct calculation shows that

$$\mathcal{F}(R(x)) = \frac{P(\sqrt{x})Q(-\sqrt{x}) - P(-\sqrt{x})Q(\sqrt{x})}{Q(\sqrt{x})Q(-\sqrt{x})2\sqrt{x}}. \tag{2.1}$$

Now observe that $Q(t)Q(-t)$ is an even polynomial in $t = \sqrt{x}$, so it is a polynomial in $t^2 = x$. Similarly $P(t)Q(-t) - P(-t)Q(-t)$ is an odd polynomial in t, so the numerator in (2.1) is also a polynomial in x, after cancellation with the $\sqrt{x}$ in the denominator. ∎

3 Examples of the dynamics of $\mathcal{F}$

Consider the rational function

$$R(a,x) = \frac{1}{x^2+2ax+1}\,, \qquad a \in \mathbb{R}\,. \tag{3.1}$$

The even part of $R(a,x)$ is

$$R_e(a,x) = \frac{1+x^2}{x^4+(2-4a^2)x^2+1}\,. \tag{3.2}$$

Integrating the even part, we obtain

$$\int_0^\infty \frac{1+x^2}{x^4+(2-4a^2)x^2+1}\,dx \quad = \quad 2\int_0^\infty \frac{1}{x^4+(2-4a^2)x^2+1}\,dx, \tag{3.3}$$

using the change of variables $x \mapsto 1/x$. The resulting integral can be evaluated using (1.6) to produce

$$\int_0^\infty R_e(a,x)\,dx \quad = \quad 2N_{0,4}(1-2a^2,0) \quad = \quad \frac{\pi}{2\sqrt{1-a^2}}\,. \tag{3.4}$$

Turning to the odd part of $R(a,x)$, we evaluate the rational function $\mathcal{F}(R(x))$. Direct calculation suggests that the iterates of R under $\mathcal{F}$ have the form

$$\begin{aligned} \mathcal{F}^{(m)}(R(a,x)) &= \frac{G_m(a)}{x^2+H_m(a)x+1} \\ &= G_m(a)\times R\left(H_m(a)/2,x\right)\,, \end{aligned} \tag{3.5}$$

where $G_m(a)$ and $H_m(a)$ are polynomials in a. This is established in the next proposition.

Proposition 3.1 *The functions $H_m(a)$ and $G_m(a)$ satisfy the recursion formulas*

$$\begin{aligned} H_{m+1}(a) &= 2-H_m(a)^2\,, \\ G_{m+1}(a) &= -G_m(a)H_m(a)\,, \\ G_{m+2}(a) &= \frac{G_{m+1}^3(a)}{G_m^2(a)} - 2G_{m+1}(a)\,, \end{aligned} \tag{3.6}$$

with initial conditions $G_0(a)=1$ and $H_0(a)=2a$. In particular, $H_m(a)$ and $G_m(a)$ are polynomials in a.

PROOF: The proof is by induction.

Base Case: For $G_0(a) = 1$ and $H_0(a) = 2a$ a direct computation shows that

$$\mathcal{F}^{(1)}(R(x)) = \frac{-2a}{x^2 + (2-4a^2)x + 1}\,, \tag{3.7}$$

as desired.

We assume that

$$\mathcal{F}^{(m)}(R(x)) = \frac{G_m(a)}{x^2 + H_m(a)x + 1}\,, \tag{3.8}$$

and evaluate $\mathcal{F}^{(m+1)}(R(x))$ to obtain the indicated recursion.

Define $R_m(x) = \mathcal{F}^{(m)}(R(x))$ and compute the odd part

$$R_{m,odd}(x) = \frac{1}{2}\left(\frac{G_m(a)}{x^2 + H_m(a)x + 1} - \frac{G_m(a)}{x^2 - H_m(a)x + 1}\right). \tag{3.9}$$

Substitute $x \to \sqrt{x}$, divide by $\sqrt{x}$ and combine the two fractions to produce

$$\begin{aligned}\mathcal{F}^{(m+1)}(R(x)) &= \frac{-2G_m(a)H_m(a)\sqrt{x}}{(x^2 + (2 - H_m^2(a))x + 1)2\sqrt{x}} \\ &= \frac{-G_m(a)H_m(a)}{x^2 + (2 - H_m^2(a))x + 1}\,. \end{aligned} \tag{3.10}$$

It follows that $\mathcal{F}^{(m+1)}(R(x))$ has the required form and that the functions $H_m(a)$ and $G_m(a)$ satisfy the recursion stated above.

Since $H_m(a) = -G_{m+1}(a)/G_m(a)$, the second recursion for $G_m(a)$ now follows from the recursion for $H_m(a)$. The fact that $G_{m+2}(a)$ is necessarily a polynomial can be proved by induction. ■

We summarize our discussion in a theorem:

Theorem 3.1 *The family of functions*

$$R(a,x) := \frac{G_m(a)}{x^2 + H_m(a)x + 1}\,, \qquad a \in \mathbb{R}\,, \tag{3.11}$$

is closed under the action of $\mathcal{F}$*, and the following integral formula holds:*

$$\int_0^\infty \frac{G_m(a)\,dx}{x^2 + H_m(a)x + 1} = \frac{\pi G_m(a)}{\sqrt{1 - H_m^2(a)}} + \frac{1}{2}\int_0^\infty \frac{G_{m+1}(a)\,dx}{x^2 + H_{m+1}(a)x + 1}\,, \tag{3.12}$$

where $H_m(a)$ *and* $G_m(a)$ *are as defined in (3.6).*

For rational functions of the form $R(a,x) = 1/(x^2 + 2ax + 1)$ we are able to determine which rational functions are eventually mapped to even functions. Simply, the solution a to $H_m(a) = 0$ will give particular rational functions $R(a,x)$ that map to an even function after m applications of $\mathcal{F}$, because when $H_m(a) = 0$ the resulting rational function $\mathcal{F}^{(m)}(R(a,x))$ is even.

Turning our attention to rational functions of the form $R(x) = 1/(x^3+ax^2+bx^3+1)$ we can obtain similar, more complicated, recursive equations for the coefficients. There $\mathcal{F}$ must be slightly modified in order for iterations of $\mathcal{F}$ to preserve the structure of $R(x)$. If we add a second substitution $x \to -x$ to $\mathcal{F}$, then this modified map $\mathcal{G}$ allows similar results.

So far we have not considered the convergence of our integrals. It is shown in [4] that $\int_0^\infty 1/(x^3+ax^2+2bx+1)\,dx$ converges if a and b satisfy the condition $4a^3-18ab+4b^3+27 > 0$. It is an open question whether one can find conditions on a for convergence of $\int_0^\infty \mathcal{F}^{(m)}\left(1/(x^2+2ax+1)\right)\,dx$. Similarly, it is an open question whether one can find conditions on a and b for the convergence of $\int_0^\infty \mathcal{G}^{(m)}\left(1/(x^3+ax^2+2bx+1)\right)\,dx$.

4 Mapping rational to even functions by $\mathcal{F}^{(m)}$

If a rational function $R(x)$ is mapped to an even function by applying $\mathcal{F}$ finitely many times, then $R(x)$ can be integrated in finitely many steps, provided one has an algorithm for the integration of even functions. The beginnings of such an algorithm are described in [3].

Recall that R is an even function if and only if $\mathcal{F}(R(x)) = 0$. We now describe a family of rational functions $R(x)$ that become even after one application of $\mathcal{F}$. In other words, $\mathcal{F}^{(2)}(R(x)) = 0$.

Theorem 4.1 *Let P, Q, and V be polynomials in x, and consider the rational function*

$$R(x) = \frac{xP(x^4)+x^2Q(x^2)}{V(x^4)}. \tag{4.1}$$

Then $\mathcal{F}(R(x))$ is an even function.

PROOF: The odd part of $R(x)$ simplifies to

$$\left(\frac{xP(x^4)+x^2Q(x^2)}{V(x^4)}\right)_{odd}$$

$$\begin{aligned} &= \frac{xP(x^4)+x^2Q(x^2)}{V(x^4)} - \frac{(-x)P((-x)^4)+(-x)^2Q((-x)^2)}{V((-x)^4)} \\ &= \frac{2xP(x^4)}{V(x^4)}. \end{aligned} \tag{4.2}$$

The substitution $x \to \sqrt{x}$ and division by $2\sqrt{x}$ result in

$$\frac{2\sqrt{x}P((\sqrt{x})^4)}{2\sqrt{x}Q((\sqrt{x})^4)} = \frac{P(x^2)}{V(x^2)}, \tag{4.3}$$

and this function is even. ■

Observe that the resulting even function is independent of Q.

Example. Consider the case $P(x) = bx + c$, $Q(x) = dx + e$ and $V(x) = x^2 + 2ax + 1$. We wish to evaluate the integral

$$\int_0^\infty \frac{x(bx^4+c)+x^2(dx^2+e)}{x^8+2ax^4+1}\,dx \quad = \quad \int_0^\infty \frac{bx^5+dx^4+ex^2+cx}{x^8+2ax^4+1}\,dx\,. \tag{4.4}$$

The even part reduces to the integral

$$\int_0^\infty R_e(x)dx = \int_0^\infty \frac{dx^4+ex^2}{x^8+2ax^4+1}\,dx \;=\; \frac{(d+e)\pi}{2^{3/2}(1+a)^{1/2}(4+\sqrt{8(1+a)})^{1/2}}\,dx\;. \tag{4.5}$$

$\mathcal{F}$ converts the odd part to an even function, which can be integrated:

$$\int_0^\infty \mathcal{F}(R(x))dx = \int_0^\infty \frac{bx^2+c}{x^4+2ax^2+1}\,dx \;=\; \frac{(b+c)\pi}{2^{3/2}(1+a)^{1/2}}. \tag{4.6}$$

For both integrals we have used equations from [3]. The integral becomes

$$\begin{aligned}\int_0^\infty \frac{bx^5+dx^4+ex^2+cx}{x^8+2ax^4+1}\,dx \quad &= \quad \int_0^\infty R_e(x)\,dx + \frac{1}{2}\int_0^\infty \mathcal{F}(R(x))\,dx \qquad (4.7)\\ &= \quad \frac{2\pi(d+e)+\pi\left(4+\sqrt{8(1+a)}\right)^{1/2}(b+c)}{2^{3/2}(1+a)^{1/2}\left(4+\sqrt{8(1+a)}\right)^{1/2}}\,.\end{aligned}$$

5 The power series condition for $\mathcal{F}^{(n)}(R) = 0$

The power series expansion $R(x) = \sum_{j=-\infty}^{\infty} a_j x^j$ about $x = 0$ of a rational function has only a finite number of non-zero terms with negative powers of z [6, Section 5.6]. The coefficients satisfy a periodicity condition: there exists an $m \in \mathbb{N}$ such that $a_j = a_{j+m}$ for all j. This property of a rational function is key to finding a necessary and sufficient condition on $R(x)$ to ensure that some $\mathcal{F}^{(n)}(R(x))$ is even.

Theorem 5.1 *Consider the rational function*

$$R(x) = \sum_{j=-m}^{\infty} a_j x^j\,. \tag{5.1}$$

The n^{th} *iterate* $\mathcal{F}^{(n)}(R(x))$ *is even if and only if* $a_{2^{n+1}j+2^{n+1}-1} = 0$ *for all integers* $j \geq -m$.

First we state a simple Lemma.

Lemma 5.1 *The power series for the n^{th} iterate of $R(x)$ is given by*

$$\mathcal{F}^{(n)}(R(x)) = \frac{1}{2^n}\sum_{j=0}^{\infty} a_{2^n j+2^n-1}x^j \,. \tag{5.2}$$

PROOF: Apply $\mathcal{F}$ to equation (5.1) and use induction. ■

Now we can prove Theorem 5.1. We use induction on the number of iterations.

PROOF: *Base Case:* $n = 1$. First assume $\mathcal{F}^{(1)}(R(x))$ is even. Now

$$\begin{aligned}
\mathcal{F}^{(1)}(R(x)) &= \mathcal{F}^{(1)}\left(\sum_{j=-m}^{\infty} a_j x^j\right) \\
&= \frac{1}{2\sqrt{x}}\sum_{j=-m}^{\infty} a_{2j+1}(\sqrt{x})^{2j+1} \\
&= \frac{1}{2\sqrt{x}}\sum_{j=-m}^{\infty} a_{2j+1}x^{j+\frac{1}{2}} \\
&= \frac{1}{2}\sum_{j=-m}^{\infty} a_{2j+1}x^{j} \\
&= \frac{1}{2}\sum_{j=-m}^{\infty} a_{4j+3}x^{2j+1} + \frac{1}{2}\sum_{j=-m}^{\infty} a_{4j+1}x^{2j} \,.
\end{aligned} \tag{5.3}$$

But we know that $\mathcal{F}^{(1)}(R(x))$ is even, which can only occur if the odd part of $\mathcal{F}^{(1)}(R(x))$ vanishes. Therefore $a_{4j+3} = 0$ for all $j \in \mathbb{Z}$, as required.

Now suppose $a_{4j+3} = 0$ for all $j \in \mathbb{N}$. The function $R(x)$ can be expressed in the form

$$\begin{aligned}
R(x) &= \sum_{j=-m}^{\infty} a_j x^j \qquad (5.4) \\
&= \sum_{j=-m}^{\infty} a_{4j}x^{4j} + \sum_{j=-m}^{\infty} a_{4j+1}x^{4j+1} + \sum_{j=-m}^{\infty} a_{4j+2}x^{4j+2} + \sum_{j=-m}^{\infty} a_{4j+3}x^{4j+3} \,.
\end{aligned}$$

But we know that

$$\sum_{j=-m}^{\infty} a_{4j+3}x^{4j+3} = 0 \,, \tag{5.5}$$

and

$$\mathcal{F}^{(1)}\Big(\sum_{j=-m}^{\infty} a_{4j+2}x^{4j+2}\Big) = 0, \qquad \mathcal{F}^{(1)}\Big(\sum_{j=-m}^{\infty} a_{4j}x^{4j}\Big) = 0, \tag{5.6}$$

because the arguments are even functions. So the substitution $x \to \sqrt{x}$ and division by $2\sqrt{x}$ lead to

$$\begin{aligned} \mathcal{F}^{(1)}(R(x)) &= \mathcal{F}^{(1)}\Big(\sum_{j=-m}^{\infty} a_{4j+1}x^{4j+1}\Big) \\ &= \frac{1}{2\sqrt{x}}\sum_{j=-m}^{\infty} a_{4j+1}(\sqrt{x})^{4j+1} \\ &= \frac{1}{2}\sum_{j=-m}^{\infty} a_{4j+1}x^{2j}\,, \end{aligned} \tag{5.7}$$

which is even. This establishes the base case.

Assume Theorem 5.1 holds for some n, and suppose $\mathcal{F}^{(n+1)}(R(x))$ is even. Now by Lemma 5.1,

$$\mathcal{F}^{(n+1)}(R(x)) = \frac{1}{2^{n+1}}\sum_{j=-m}^{\infty} a_{2^{n+1}j+2^{n+1}-1}x^{j} \tag{5.8}$$

$$= \frac{1}{2^{n+1}}\Big(\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n+1}-1}x^{2j} + \sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n+2}-1}x^{2j+1}\Big)\,.$$

But $\mathcal{F}^{(n+1)}(R(x))$ is even, so each coefficient $a_{2^{n+2}j+2^{n+2}-1}$ must be 0.

Now suppose $a_{2^{n+2}j+2^{n+2}-1} = 0$ for all $j \geq -m$. By our lemma,

$$\mathcal{F}^{(n)}(R(x)) = \frac{1}{2^{n}}\sum_{j=-m}^{\infty} a_{2^{n}j+2^{n}-1}x^{j} \tag{5.9}$$

$$\begin{aligned} &= \frac{1}{2^{n}}\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n}-1}x^{4j} + \frac{1}{2^{n}}\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n+1}-1}x^{4j+1} \\ &+ \frac{1}{2^{n}}\sum_{j=-m}^{\infty} a_{2^{n+2}j+3(2^{n})-1}x^{4j+2} + \frac{1}{2^{n}}\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n+2}-1}x^{4j+3}\,. \end{aligned}$$

Also,

$$\mathcal{F}\Big(\frac{1}{2^{n}}\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n}-1}x^{4j}\Big) = 0\,, \quad \mathcal{F}\Big(\frac{1}{2^{n}}\sum_{j=-m}^{\infty} a_{2^{n+2}j+3(2^{n})-1}x^{4j+2}\Big) = 0, \tag{5.10}$$

because both sums in (5.10) are even. By assumption $a_{2^{n+2}j+2^{n+2}-1} = 0$ for all $j \geq 0$. Therefore

$$\begin{aligned} \mathcal{F}^{(n+1)}(R(x)) &= \mathcal{F}\Big(\frac{1}{2^{n}}\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n+1}-1}x^{4j+1}\Big) \\ &= \frac{1}{2^{n+1}}\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n+1}-1}x^{2j}\,, \end{aligned} \tag{5.11}$$

which is even, as required. ■

This result is useful because now we can take any rational function, express it as a power series and determine whether it will *ever* result in an even function. It also implies that $R(\sqrt{x}) - R(-\sqrt{x}) = iR(-i\sqrt{x}) - iR(i\sqrt{x})$ if and only if $\mathcal{F}$ maps $R(x)$ to an even function. Now we have a closed form test for whether a rational function will be mapped to an even function under $\mathcal{F}$ on the next iteration.

6 A special property of mapping polynomials

By definition, $\mathcal{F}$ maps every even function to 0. The converse is also true: if $\mathcal{F}(R(x)) = 0$, then $R(x)$ is even. An interesting open problem is to classify all functions R for which there exists an integer n such that $\mathcal{F}^{(n)}(R(x)) = 0$.

All polynomials are eventually mapped to 0 by repeated application of the map $\mathcal{F}$. Further, the number of iterations required can be exactly determined from the exponents present in the polynomial.

Theorem 6.1 *Let $P(x)$ be a polynomial. Then there exists a non-negative integer n such that*

$$\mathcal{F}^{(n)}(P(x)) = 0\,. \tag{6.1}$$

Proof: Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$. Any monomial of even degree is mapped to zero after one iteration of $\mathcal{F}$:

$$\mathcal{F}(x^{2k}) = \frac{x^{2k} - x^{2k}}{2\sqrt{x}} = 0\,. \tag{6.2}$$

So it suffices to look at powers of x of the form $x^{2^m p-1}$, where m is a positive integer and p is odd, and to show that there is a positive integer n such that $\mathcal{F}^{(n)}(x^{2^m p-1}) = 0$. Notice that

$$\mathcal{F}^{(1)}\left(x^{2^m p-1}\right) = x^{2^{m-1}p-1}\,, \qquad \mathcal{F}^{(2)}\left(x^{2^m p-1}\right) = x^{2^{m-2}p-1}\,, \tag{6.3}$$

and so on. Iterating this procedure yields

$$\mathcal{F}^{(m)}\left(x^{2^m p-1}\right) = x^{p-1}\,. \tag{6.4}$$

Notice that x^{p-1} is an even power of x. Therefore

$$\mathcal{F}^{(m+1)}\left(x^{2^m p-1}\right) = \mathcal{F}\left(x^{p-1}\right) = 0\,. \tag{6.5}$$

■

Corollary 6.1 *Let* $P(x) = a_1 x^{2^{k_1} p_1 - 1} + a_2 x^{2^{k_2} p_2 - 1} + \cdots + a_n x^{2^{k_n} p_n - 1}$*, where* p_i *is odd, and* $1 \leq i \leq n$*. Define*

$$k = \max(k_i)\,, \qquad 1 \leq i \leq n\,.$$

Then

$$\mathcal{F}^{(k+1)}\left(P(x)\right) = 0\,,$$

and this is the first iterate that vanishes.

PROOF: This is the situation of Theorem (6.1) with $m = k$. ■

7 Open Questions

Finding other particular classes of functions such as (4.1) for the second, third and n^{th} iterates of $\mathcal{F}$ would be very useful for recognizing which rational functions $R(x)$ eventually map to even functions. Currently, our general Theorem 5.1 can tell us this by looking at the power series expansion, but specific cases would also be interesting.

Unfortunately, many rational functions will never map to even functions. Describing the behavior of these rational functions under the map $\mathcal{F}$ becomes complicated. In the case of the reciprocal of the quadratic, applying our mapping function results in a standard formula (3.5) for the iterates. An idea related to the integrability over $[0, \infty)$ is to use these recursion formulas to classify the behavior of the zeroes of a rational function under iteration of $\mathcal{F}$; we have started to consider this.

We have used Mathematica and Maple to study the behavior of $\mathcal{F}$ on rational functions, to find fixed points of $\mathcal{F}$, to find periodic points of $\mathcal{F}$, and to measure the length of their orbits. We have also found functions that are pre-periodic under $\mathcal{F}$. One can evaluate the integral of $e^x \cos x$ by integrating by parts twice. Similarly, when $\mathcal{F}^{(n)}(R(x)) = R(x)$, the integral of $R(x)$ can be expressed as a sum of integrals of even rational functions, and hence evaluated (if the even integrals can be done).

The fixed points of $\mathcal{F}$ have recently been found [2], and they are of the form

$$R(x) = \frac{x^{m-1}}{x^m - 1}\,, \tag{7.1}$$

where $m \in \mathbb{N}$ and m is odd. Unfortunately, these fixed points are *not* integrable on $[0, \infty)$.

Finally, we continue to study the dynamics of $\mathcal{F}$ on the space of rational functions, following [2]. It is an open question whether the fixed points of $\mathcal{F}$ are attracting or repelling, and how one might define the multiplier of $\mathcal{F}$.

8 Acknowledgements

The work in this paper began as a student project at the 2000 Summer Institute in Math for Undergraduates (SIMU) REU program at the University of Puerto

Rico, Humacao. We thank Herbert Medina, Ivelisse Rubio, Olgamary, Monica and the entire SIMU staff for organizing SIMU and giving us the opportunity to participate in an undergraduate research project. In addition, we would like to thank Jean Carlos Cortissoz and Victor Moll for their guidance and contributions to our work. We thank Andrew Bernoff for the Maple algorithm. Finally, we give special thanks to Lesley Ward (Harvey Mudd College) for mentoring this project beyond SIMU.

References

[1] Boros, G. and Moll, V., *An integral hidden in Gradshteyn and Ryzhik*. Jour. Comp. Appl. Math. **106**, 1999, 361-368.

[2] Boros, G., Joyce, M., and Moll. V., *A map of rational functions*. Preprint available at `http://math.tulane.edu/∼vhm/pap.html`

[3] Boros, G. and Moll, V., *Landen transformations and the integration of rational functions*. Preprint available at `http://math.tulane.edu/∼vhm/pap.html`

[4] Cortissoz, J., Gutierrez, E., Moll, V., Rosario-Garcia, M., and Torres, M., *Convergence of a Landen transformation of degree 6*. Preprint available at `http://cuhwww.upr.clu.edu/∼simu/technical_reports2000.htm`

[5] Spivak, M., *Calculus, 2nd Ed.*, Publish or Perish Inc., Houston, 1980, 370ff.

[6] Saff, E.B., and Snider, A.D., *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering, 2nd Ed.*, Prentice Hall, 1976.

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