

## Poster #45

### Abstract:

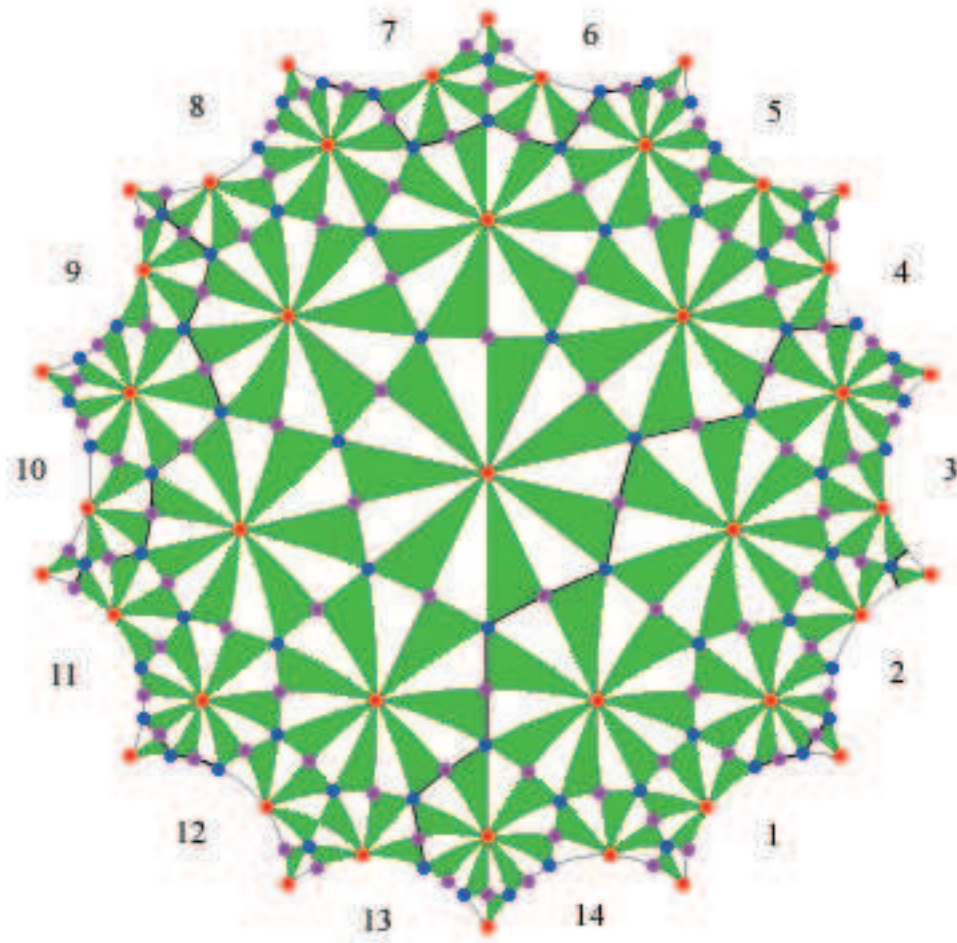
Topologically, surfaces are classified by genus. Within the moduli space of compact Riemann surfaces, we consider surfaces of genus  $g \geq 2$ . An ordinary Weierstrass point is a point where such a surface has more functions than expected. Higher-order Weierstrass points are a natural generalization of these. There are many connections between the automorphisms of compact Riemann surfaces and their Weierstrass points, both ordinary and higher-order, specifically with regards to the fixed points of the automorphisms. Klein's quartic curve,  $\mathfrak{X}$ , is the genus 3 surface with 168 automorphisms, the maximum for that genus. We will show the extent to which the fixed points of the automorphisms of  $\mathfrak{X}$  are higher-order Weierstrass points for  $\mathfrak{X}$ .

## Klein's quartic curve and its Automorphisms

Klein's quartic,  $\mathfrak{X}$ , is the genus 3 curve canonically modelled by the equation

$$XY^3 + YZ^3 + ZX^3 = 0.$$

$\mathfrak{X}$  has 168 automorphisms, the maximum for a curve of genus 3. The group of automorphisms is generated by three automorphisms:  $\sigma$  which is of order 7 and fixes three points,  $\tau$  which is of order 3 and fixes two points, and  $\lambda$  which is of order 2 and fixes four points. Among the orbits of points generated by the group of automorphisms there are three exceptional cases: 24 points of multiplicity 7, 56 points of multiplicity 3, and 84 points of multiplicity 2. After Klein [9], we will call these *a-points*, *b-points* and *c-points* respectively. The fixed points of  $\sigma$  and its conjugates are *a-points*, for  $\tau$  and its conjugates they are *b-points* and for  $\lambda$  and its conjugates they are *c-points*. See Figure 1 (modified) from [9].



**Figure 1**

After identifying the edges ( $1 \leftrightarrow 6$ ,  $3 \leftrightarrow 8$ ,  $5 \leftrightarrow 10$ ,  $7 \leftrightarrow 12$ ,  $9 \leftrightarrow 14$ ,  $11 \leftrightarrow 2$ ,  $13 \leftrightarrow 4$ ) this is  $\mathfrak{X}$ . The  $a$ -points are marked in red, the  $b$ -points in blue, and the  $c$ -points in purple. We can see the generating automorphisms for  $\mathfrak{X}$  ( $\sigma$ ,  $\tau$  and  $\lambda$ ) as rotations of angle  $2\pi/7$  (respectively  $2\pi/3$  or  $2\pi/2$ ) radians about their fixed points, which are  $a$ -points,  $b$ -points and  $c$ -points, respectively.

## Divisors—Definitions

Let  $M$  be a compact Riemann surface of genus  $g \geq 2$  and  $\mathcal{M}(M)$  the set of meromorphic functions on  $M$ . A **divisor** on  $M$  is a singular one-chain  $\sum n_p \cdot p$  where  $p \in M$  and  $n_p \in \mathbb{Z}$ . The **degree of a divisor**  $D$  is the sum of the coefficients,  $\deg(D) = \sum n_p$ . A divisor  $D$  is called **integral**, denoted  $D \geq 0$ , if all of the coefficients satisfy  $n_p \geq 0$ .

The **order function**,  $\text{ord}_p(f) : M \mapsto \mathbb{Z}$ , counts the multiplicity of the zeros and poles of  $f \in \mathcal{M}(M)$ . The divisor of a meromorphic function  $f$  is  $\text{div}(f) = \sum \text{ord}_p(f) \cdot p$  and is called **principal**. Two divisors  $D$  and  $D'$  are **linearly equivalent**, denoted  $D \equiv D'$  if there exists  $f \in \mathcal{M}(M)$  such that  $D - D' = \text{div}(f)$ .

Meromorphic differentials on  $M$ ,  $\omega$ , can locally be written as  $\omega = f(z)dz$  where  $f \in \mathcal{M}(M)$  and hence we can define divisors for them,  $\text{div}(\omega) = \sum \text{ord}_p(f) \cdot p$ . Since meromorphic differentials differ by multiplication by a meromorphic function, the divisors of meromorphic differentials are linearly equivalent. For this reason we call the divisor of a meromorphic function **canonical**.

## Linear Systems of Divisors—Definitions

The **linear system of a divisor**  $D$  is the complex vector space of meromorphic functions with poles bounded by  $D$ ,

$$\mathcal{L}(D) = \{f \in \mathcal{M}(X) \mid \operatorname{div}(f) + D \geq 0\}.$$

Let  $r(D)$  denote  $\dim(\mathcal{L}(D)) - 1$ . Thus if  $D \geq 0$  then  $r(D) \geq 0$ . Let  $i(D)$  denote the dimension of the space of meromorphic differentials  $\omega$  whose divisors satisfy  $\operatorname{div}(\omega) - D \geq 0$ . If  $i(D) > 0$  then  $D$  is called **special**.

The **complete linear system of a divisor**  $D$  is

$$|D| = \{E \in \operatorname{Div}(M) \mid E \equiv D \text{ and } E \geq 0\}.$$

We sometimes also write  $|D| = g_d^r$ , where  $r = r(D)$  and  $d = \operatorname{deg}(D)$ . If  $K$  is a canonical divisor, then  $r(K)$  and  $\operatorname{deg}(K)$  are known, in fact  $|K| = g_{2g-2}^{g-1}$  where  $g$  is the genus of  $M$ .

Let  $g_d^r$  be a linear system on  $M$ . A point  $x \in M$  is a **base point of**  $g_d^r$  if every divisor  $E \in g_d^r$  contains  $x$ , which means  $E - x \geq 0$ . A linear system with no base points is called base-point-free.

## Weierstrass points—Definitions

Let  $g_d^r$  be a base-point free complete linear system on  $M$ . Let  $x \in M$ . An integer  $n$ ,  $0 \leq n \leq d$ , is called a  $g_d^r$ -**nongap** at  $x$  if there is a divisor  $D \in g_d^r$  with  $D \equiv nx + E$ , where  $E$  is integral and doesn't contain  $x$ . Let  $n_i$  denote the  $i^{\text{th}}$  nongap. There are  $r + 1$   $g_d^r$ -nongaps at  $x$ ,  $0 = n_0 < n_1 < \dots < n_r \leq d$ . The complement of this set in the integers between 0 and  $d$  is called the  $g_d^r$ -**gaps** at  $x$  and is denoted  $g_1, \dots, g_{d-r}$ .

A **generalized Weierstrass point** at  $g_d^r$  is a point  $x \in M$  where the  $r^{\text{th}}$   $g_d^r$ -nongap,  $n_r$  is greater than  $r$ . A **Weierstrass point** is the special case where  $g_d^r = |K|$  and a **higher-order Weierstrass point** is where  $g_d^r = |nK|$  and  $n \geq 2$ .

The **weight** of a **generalized Weierstrass point** for  $g_d^r = |G|$  at  $x \in M$ ,  $w(G, x)$ , is defined to be the sum

$$\sum_{j=0}^r (n_j - j).$$

## Connections with Automorphisms

Being a curve of genus 3,  $\mathfrak{X}$  has at most  $(g - 1)g(g + 1) = 24$  ordinary Weierstrass points, counted with weight. It is not hard to show that  $\mathfrak{X}$  achieves this upper bound (see [12] or [13] for details). Thus the 24  $a$ -points of  $\mathfrak{X}$  are the ordinary Weierstrass points, each with weight 1.

**Proposition 1.** [1] *Let  $T \in \text{Aut}(M)$  be of prime order  $n$ . If  $T$  has 3 or more fixed points, then every fixed point of  $T$  is an  $n$ -Weierstrass point.*

**Theorem 1.** [7] *Let  $T \in \text{Aut}(M)$  of prime order  $n$  fixing two points,  $p_1$  and  $p_2$ . Assume  $M/\langle T \rangle$  has genus 1. Let  $\varphi$  be the projection  $M \rightarrow M/\langle T \rangle$  and  $\varphi(p_i) = a_i$ ,  $i = 1, 2$ . Then  $p_1$  and  $p_2$  are  $q$ -Weierstrass points for  $q \geq 2$  if and only if  $a_1 - a_2$  is a nonzero rational point on  $M/\langle T \rangle$ .*

**Proposition 2.** [4] *Let  $T \in \text{Aut}(M)$  be of order 2. If  $\nu(T) \geq 3$ , then every fixed point of  $T$  is a  $q$ -Weierstrass point for  $q \geq 2$ .*

## Implications

Since the  $a$  – *points* of  $\mathfrak{X}$  are individually fixed in sets of 3 by  $\sigma$  and its conjugates, which have order 7, Proposition 1 tells us that the  $a$  – *points* are 7-Weierstrass points.

Since the  $b$  – *points* of  $\mathfrak{X}$  are individually fixed in pairs by  $\tau$  and its conjugates, and it is not hard to show that their differences are not rational points on  $M/\langle\tau\rangle$ . Thus Theorem 1 tells us that the  $b$  – *points* are never higher-order Weierstrass points for  $\mathfrak{X}$ .

Proposition 2 implies that all of the  $c$  – *points* of  $\mathfrak{X}$  are  $q$ -Weierstrass points for every  $q \geq 2$ .

## A-points

**Proposition 3.** [1] *Let  $T$  be an automorphism of  $M$  of order  $n$ ,  $W$  the orbit space,  $\varphi : M \rightarrow W$  the natural map and  $x \in M$  a fixed point of  $T$ . If  $W$  has genus 0, then  $nK \equiv n(2g - 2)x$ .*

Recall the situation for the  $a$ -points of  $\mathfrak{X}$ : they are individually fixed in sets of three by the automorphisms  $\sigma$  and its conjugates, each of which is of order 7. If we apply the Riemann-Hurwitz formula, ( $T : M \rightarrow N$  a map of degree  $n$ )

$$2g_M - 2 = n(2g_N - 2) + \sum_{x \in M} (\text{mult}(x) - 1),$$

we find that the orbit space of  $\mathfrak{X}$  under each of these automorphisms has genus 0. So the proposition above tells us that  $7K \equiv 28x$  for  $x$  any  $a$ -point, and of course we already know that each  $a$ -point is a 7-Weierstrass point.

## Some Weights of A-Points

**Proposition 4.** [1] Suppose for  $x \in M$  we have  $mK \equiv m(2g - 2)x$ . Then

(1) If  $m = 1, l = 2, 3, \dots$ , we have  $w(lK, x) = g + w(K, x)$ .

(2) If  $m \geq 2$ , and  $l = 1, 2, \dots$  we have  $w(lmK, x) = w((lm + 1)K, x) = g + w(K, x)$ .

(3) If  $m \geq 3$ ,  $l = 1, 2, \dots$  and  $k = 2, 3, \dots, m - 1$  we have  $w((lm + k)K, x) = w(kK, x)$ .

(4) If  $m \geq 4$ ,  $l = 1, 2, \dots$  and  $k = 2, 3, \dots, m - 1$  we have  $w((lm - k + 1)K, x) = w(kK, x)$ .

We find, for instance, that for  $x$  an ordinary Weierstrass point, an  $a$  - point,

$$w(7K, x) = w(8K, x) = 3 + w(K, x) = 4$$

using (2), and

$$w(9K, x) = w(2K, x) = w(6K, x)$$

using (3) with  $l = 1$  and  $k = 2$ , and (4) with  $l = 1$  and  $k = 2$ .

## Weights of Higher-Order Weierstrass points

Let  $w_a(n)$  be the weight of the  $a$  – *points* as  $n$ -Weierstrass points,  $w_b(n)$  the weight of the  $b$  – *points* and  $w_c(n)$  the weight of the  $c$  – *points*. Also, let  $w_0(n)$  be the combined weight as  $n$ -Weierstrass points of the general points of  $\mathfrak{X}$  that are on separate orbits for the automorphisms group, so that  $168w_0(n)$  is the total weight of all the general points of  $\mathfrak{X}$  as  $n$ -Weierstrass points. Then (b)

$$168w_0(n) + 84w_c(n) + 56w_b(n) + 24w_a(n) = (2n - 1)^2 \cdot 12$$

using the formula  $(2n - 1)^2(g - 1)^2g$  for the total weight of all  $n$ -Weierstrass points in [2] and [3]. Note of course, that we know already that  $w_b(n) = 0$  for all  $n \geq 2$ .

Thus for  $k = 2$  we have

$$168w_0(2) + 84w_c(2) + 56w_b(2) + 24w_a(2) = 108,$$

the only integral solution to which is  $w_0(2) = w_b(2) = 0$ ,  $w_c(2) = w_a(2) = 1$ .

## Bounds on Weights

**Theorem 2.** [1] Let  $|D| = g_d^r$  be a complete linear series without fixed points of index  $i(D)$ . If  $x \in M$  then

$$w(D, x) \leq (g - i(D))(g - i(D) + 1)/2.$$

If we have equality, then  $M$  is hyperelliptic.

So  $w(nK, x) < 6$  for all  $n \geq 2$  and  $x \in \mathfrak{X}$ , since  $\mathfrak{X}$  is not hyperelliptic and  $i(nK) = 0$ . Considering equation (b) modulo 7, we can then determine the remaining values of  $w_a(n)$ :

$$w_a(3) = w_a(5) = 2, \quad w_a(4) = 0.$$

For  $x$  a  $c$ -point we can do even better at bounding  $w(nK, x)$ :

**Theorem 3.** [1] Let  $|G| = g_d^r$  be complete and base-point free with  $i(G) = 0$ . Suppose that  $x \in M$  is not an ordinary Weierstrass point. Then

$$w(G, x) \leq [(g + 1)^2/4].$$

If we have equality then the surface is hyperelliptic.

## Other Higher-Order Weierstrass Points

Since the  $c$ -points of  $\mathfrak{X}$  are not ordinary Weierstrass points and  $i(nK) = 0$  for  $n \geq 2$ , we can apply the theorem to find that

$$w_c(n) \leq 3.$$

Finally, considering the equation (b) and the values we have for  $w_a(n)$ , we find that  $w_c(n)$  is odd, hence  $w_c(n) = 1$  or  $3$ . This also means for  $n \geq 4$  there must be  $n$ -Weierstrass points of  $\mathfrak{X}$  which are not fixed points for any of the 168 automorphisms.

## Uses of Higher-Order Weierstrass points

Since the beginning of the field in the 19<sup>th</sup> century, curves with automorphisms (a rarity among all curves) have played a central role in algebraic geometry. The automorphisms reveal much about the geometric properties of the curve, as we saw here looking at higher-order Weierstrass points. Weierstrass points can also be used to reveal information about automorphisms. Most famously, Hurwitz [8] proved using Weierstrass points that a curve of genus  $g \geq 2$  can only have finitely many automorphisms. Kuribayashi and Komiya [10] use the Weierstrass points of genus 3 curves to determine their automorphism groups. In modern times curves with automorphisms have become important in cryptography and coding theory.

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