MA 412 Complex Analysis Summer, 2006 Practise Exam

1. Let $P(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$. Show by induction on n that,

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n}.$$

2. Use the Cauchy-Riemann equations to show that the following funcitons are nowhere differntiable.

   (a) $w = \overline{z}$

   (b) $w =$Re z

   (c) $w = 2y - ix$

3. Show that if v is the harmonic conjugate of u in a domain D, then uv is harmonic in D

4. Find an analytic function $f(z) = u(x, y) + iv(x, y)$ for the following expressions.

   (a) $u(x, y) = y^3 - 3x^2y$

   (b) $u(x, y) = \sin y \cdot \sinh x$

   (c) $v(x, y) = e^y \sin x$

   (d) $v(x, y) = \sin x \cosh y$

5. let $F(z) =$ Log z. With $\phi(x, y) =$ Log $|x|$ and $\psi(x, y) =$ Arg z. Sketch the equipotentials $\phi = 0$, Log 2 Log 3, Log 4 and the streamlines $\psi = \frac{k\pi}{8}$ for $k = 0, 1, ..., 7$.

6. Pick an appropriate branch of the logarithm, find $\frac{dw}{dz}$, and state where the formula is valid for.

   (a) $w = \log_\alpha(z^2 - z + 2)$

   (b) $w = z \log_\alpha(z)$

7. Compute $\int_\Gamma \overline{z} dz$, first over the countout $\Gamma =$ the polygonal path from -4 to $-4 + 4i$ to $4 + 4i$ to 4 then around the upper half circle with radius 4.

8. Let γ be a directed smooth curve with initial point α and terminal point β. Show that, $\int_\gamma z dz = \frac{\beta^2 - \alpha^2}{2}$

9. Evaluate $\int_\Gamma \frac{2z^2 - z + 1}{(z-1)^2(z+1)} dz$, where Γ is the figure-eight contour traversed once as shown.

10. Find $\int_\Gamma \frac{3z-2}{z^2-z} dz$, where Γ is the simple closed contour shown.

11. Let C the circle $|z| = 2$ traversed once in the positive sense. Compute the following integrals:

    (a) $\int_C \frac{\sin 3z}{z - (\pi/2)} dz$

    (b) $\int_C \frac{ze^z}{z - 2z - 3} dz$

    (c) $\int_C \frac{\cos z}{z^3 + 9z} dz$

12. Using the maximum modulus principle show that if a function f is analytic in $D : |z| < R$. If also we know that $f(0) = i$ and $|f(z)| \leq 1$ then f is constant.

13. Find the radius of convergence of the following,

    (a) $g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}$

    (b) $f(z) = \sum_{n=0}^{\infty} (2 - (-1)^n)^n z^n$

    (c) $h(z) = \sum_{n=0}^{\infty} (\frac{3n+7}{4n+2})^n z^n$

14. Evaluate $\int_\Gamma y$ for $-i$ to i along the polygonal path Γ with vertices $-i, -1-i, -1$, and i.