

MA 412 Complex Analysis Summer, 2006 Practise Final

- (a) Show that the following functions is harmonic and find a harmonic conjugate: $u(z, y) = 2xy(1 - y)$
(b) What do the level sets look like?
- Compute $\int_{\Gamma} \operatorname{Re}(z)dz$, around the right half circle with radius 4.
(ie: $\Gamma = \{z : |z| = 4, \operatorname{Re}z \geq 0\}$)
- Write two Laurent series for each of the following functions

$$f(z) = \frac{1}{z(1+z^2)} \qquad f(z) = \frac{z+1}{z-1}$$

in certain domains, and specify those domains.

- State the maximum modulus principle.
- (a) Determine the zeros and singularities of the function,

$$f(z) = \frac{3z^2 + 2}{(z-1)(z^2-9)} dz$$

- (b) Find all the residues of $f(z)$
- (c) Let Γ be the positively oriented circle $|z-2| = 2$ and compute

$$\int_{\Gamma} f(z) dz$$

- Use residues to evaluate the following definite integrals

$$(a) \int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} \qquad (b) \int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} \quad (-1 < a < 1)$$

- Use Rouché's Theorem to show that the polynomial $z^6 + 4z^2 - 1$ has exactly two zeros in the disk $|z| < 1$

8. Find

(a)

$$\int_0^{\infty} \frac{x^2}{(x^2 + 9)(x^2 + 4)^2} dx$$

(b)

$$p.v. \int_{-\infty}^{\infty} \frac{x}{(x^2 + 1)(x^2 + 2x + 2)} dx$$

9. Compute

$$\int_0^{\infty} \frac{\cos ax}{x^2 + 1} dx \quad (a > 0)$$

10. Find

$$(a) \int_0^{\infty} \frac{\cos x - 1}{x} dx \quad (b) \int_0^{\infty} \frac{e^{ix}}{(x-1)(x-2)} dx$$

11. Find

$$(a) \int_0^{\infty} \frac{\log x}{x^2 + 1} dx \quad (b) \int_0^{\infty} \frac{\sqrt{x}}{x^2 + 1} dx$$

12. Evaluate

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$$

$$\text{where } f(z) = \frac{z^2(z-i)^3 e^z}{3(z+2)^4(3z-18)^5}$$

13. Prove that the equation $z^3 + 9z^2 + 27 = 0$ has no roots in the disk $|z| < 2$