The “missing” p-adic L-function – DRAFT

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Overconvergent Eigensymbols

1 Introduction

2 Modular Symbols

We begin by recalling the basic theory of modular symbols and how these symbols relate to modular forms.

2.1 General setup

Let

$$S_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) \text{ such that } p \nmid a, p \mid c \text{ and } ad - bc \neq 0 \right\}.$$

Fix N some integer prime to p and let $\Gamma := \Gamma_1(Np)$ be the standard congruence subgroup of level Np. If V is some $\mathbf{Q}_p$-vector space with an $S_0(p)$-action then there is an associated locally constant sheaf $\widetilde{V}$ on $Y_1(Np)$ and we define the space of V-valued modular symbols to be

$$H^1_c(\Gamma, V) := H^1_c(Y_1(Np), \widetilde{V})$$

the space of one-dimensional compactly supported cohomology classes of $\widetilde{V}$.

This space naturally has an action of $S_0(p)$ and hence is acted on by the Hecke operators T_l for $l \nmid Np$ and by U_q for $q|Np$. For example, the U_p operator is given by

$$\phi|U_p = \sum_{a=0}^{p-1} \phi \mid \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right).$$

There also is a natural involution ι on this space given by $\phi|\iota = \phi \mid \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This involution gives a decomposition

$$H^1_c(\Gamma, V) = H^1_c(\Gamma, V)^+ \oplus H^1_c(\Gamma, V)^-$$

of the space of modular symbols into its plus and minus parts.

If V is a Banach space and if $\Gamma_1(Np)$ acts by unitary operators on V then $H^1_c(\Gamma, V)$ is also a Banach space under the norm

$$||\phi|| = \sup_{D \in \Delta_0} ||\phi(D)||.$$

2.2 A concrete description of $H^1_c(\Gamma, V)$

We now give an explicit description of these space of modular symbols in terms of certain maps from degree 0 divisors of $\mathbf{P}^1(\mathbf{Q})$ into V. In section 9, we will use this description to perform computations in these spaces.

Let $\Delta_0 = \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ be the set of degree 0 divisors on $\mathbf{P}^1(\mathbf{Q})$. Then Δ_0 is a left $GL_2(\mathbf{Q})$-module by linear fractional transformations. If we view V as a right $S_0(p)$-module, the space $\mathrm{Hom}(\Delta_0, V)$ becomes a right $S_0(p)$-module by

$$(\phi|\gamma)(D) = \phi(\gamma D)|\gamma$$

where $\phi : \Delta_0 \longrightarrow V$, $D \in \Delta_0$ and $\gamma \in S_0(p)$. Let

$$\mathrm{Hom}_\Gamma(\Delta_0, V) = \{\phi \in \mathrm{Hom}(\Delta_0, V) \text{ such that } \phi|\gamma = \phi \text{ for all } \gamma \in S_0(p)\}$$

the subspace of Γ-invariant maps.

Proposition 2.1. *There is a canonical isomorphism*

$$H^1_c(\Gamma, V) \cong \mathrm{Hom}_\Gamma(\Delta_0, V).$$

Proof. ? □

In what follows we will implicitly identify these two spaces.

2.3 Eichler-Shimura theory

With the appropriate choice of V one can recover any space of classical modular forms of weight greater than one inside the space of V-valued modular symbols. For $k \geq 0$ an integer, consider

$$L_k = \{F(Z) \in \mathbf{Q}_p[Z] \text{ such that } \deg(F) \leq k\}$$

as a right $S_0(p)$-module by

$$(F\big|_k \gamma)(z) = (d - cZ)^k \cdot F\left(\frac{-b + aZ}{d - cZ}\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S_0(p)$ and $F \in L_k$.

Let $\mathcal{S}_k(\Gamma_1(Np))$ (resp. $\mathcal{E}_k(\Gamma_1(Np))$) denote the space of cusp forms (resp. Eisenstein series) on $\Gamma_1(Np)$. The following theorem describes $H^1_c(\Gamma, L_k)$ in terms of these spaces of modular forms.

Theorem 2.2 (Eichler-Shimura). *There is an isomorphism of $S_0(p)$-modules*

$$H^1_c(\Gamma, L_k \otimes \overline{\mathbf{Q}}_p) \cong \mathcal{S}_k(\Gamma_1(Np)) \oplus \mathcal{S}_k^{anti}(\Gamma_1(Np)) \oplus \mathcal{E}_k(\Gamma_1(Np))$$

where $\mathcal{S}_k^{anti}(\Gamma_1(Np))$ is the space of antiholomorphic cusp forms.

Proof. See [3, Chapter 8]. □

3 Distributions

We will ultimately want to study overconvergent modular symbols which are modular symbols whose values lie in certain spaces of p-adic distributions. In this section, we will review the basic theory of these distributions.

3.1 Definitions

For each $r \in |\mathbf{C}_p^\times|$, let

$$B[\mathbf{Z}_p, r] = \{x \in \mathbf{C}_p \mid \text{there exists some } a \in \mathbf{Z}_p \text{ with } |x - a| \leq r\}.$$

Then $B[\mathbf{Z}_p, r]$ is the $\mathbf{C}_p$-points of a $\mathbf{Q}_p$-affinoid variety. For example, if $r \geq 1$ then $B[\mathbf{Z}_p, r]$ is the closed disc in $\mathbf{C}_p$ of radius r around 0. If $r = \frac{1}{p}$ then $B[\mathbf{Z}_p, r]$ is the disjoint union of the p discs of radius $\frac{1}{p}$ around the points $0, 1, \ldots, p-1$.

Let $A[\mathbf{Z}_p, r]$ denote the $\mathbf{Q}_p$-Banach algebra of $\mathbf{Q}_p$-affinoid functions on $B[\mathbf{Z}_p, r]$. For example, if $r \geq 1$

$$A[\mathbf{Z}_p, r] = \left\{ f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbf{Q}_p[[x]] \text{ such that } \{|a_n| \cdot r^n\} \longrightarrow 0 \right\}.$$

The norm on $A[\mathbf{Z}_p, r]$ is given by the supremum norm. That is, if $f \in A[\mathbf{Z}_p, r]$ then

$$||f||_r = \sup_{x \in B[\mathbf{Z}_p, r]} |f(x)|_p.$$

For $r_1 > r_2$, there is a natural restriction map $A[\mathbf{Z}_p, r_1] \longrightarrow A[\mathbf{Z}_p, r_2]$ that is injective, completely continuous and has dense image. We define

$$\mathcal{A}(\mathbf{Z}_p) = \varinjlim_{s>0} A[\mathbf{Z}_p, s] \text{ and } \mathcal{A}^\dagger(\mathbf{Z}_p, r) = \varinjlim_{s>r} A[\mathbf{Z}_p, s].$$

(It should be pointed out that these direct limits are taken over sets with no smallest element and therefore are not vacuous.) We endow each of these spaces with the inductive limit topology. Then $\mathcal{A}(\mathbf{Z}_p)$ is naturally identified with the space of locally analytic $\mathbf{Q}_p$-valued functions on $\mathbf{Z}_p$ while $\mathcal{A}^\dagger(\mathbf{Z}_p, r)$ is identified with the space of $\mathbf{Q}_p$-overconvergent functions on $B[\mathbf{Z}_p, r]$. Note that there are natural continuous inclusions

$$\mathcal{A}^\dagger(\mathbf{Z}_p, r) \hookrightarrow A[\mathbf{Z}_p, r] \hookrightarrow \mathcal{A}(\mathbf{Z}_p).$$

Moreover, the image of each of these maps is dense in its target space.

We now define our distributions modules as dual to these topological vector spaces. Namely, set $\mathcal{D}(\mathbf{Z}_p)$, $D[\mathbf{Z}_p, r]$ and $\mathcal{D}^\dagger(\mathbf{Z}_p, r)$ to be the space of continuous $\mathbf{Q}_p$-linear functionals on $\mathcal{A}(\mathbf{Z}_p)$, $A[\mathbf{Z}_p, r]$, and $\mathcal{A}^\dagger(\mathbf{Z}_p, r)$ respectively, each endowed with the strong topology. Equivalently,

$$\mathcal{D}(\mathbf{Z}_p) = \varprojlim_{s>0} D[\mathbf{Z}_p, s] \text{ and } \mathcal{D}^\dagger(\mathbf{Z}_p, r) = \varprojlim_{s>r} D[\mathbf{Z}_p, s],$$

each endowed with the projective limit topology.

Note that $D[\mathbf{Z}_p, r]$ is a Banach space under the norm

$$||\mu||_r = \sup_{\substack{f \in A[\mathbf{Z}_p, r] \\ f \neq 0}} \frac{|\mu(f)|}{||f||_r}.$$

for $\mu \in D[\mathbf{Z}_p, r]$. On the other hand, $\mathcal{D}(\mathbf{Z}_p)$ (resp. $\mathcal{D}^\dagger(\mathbf{Z}_p, r)$) has its topology defined by the family of norms $\{|| \cdot ||_s\}$ for $s \in |\mathbf{C}_p^\times|$ with $s > 0$ (resp. $s > r$). By duality, we have continuous linear injective maps

$$\mathcal{D}(\mathbf{Z}_p) \hookrightarrow D[\mathbf{Z}_p, r] \hookrightarrow \mathcal{D}^\dagger(\mathbf{Z}_p, r).$$

3.2 The action of $\Sigma_0(p)$

Let

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p) \text{ such that } p \nmid a, p \mid c \text{ and } ad - bc \neq 0 \right\}$$

be the p-adic version of $S_0(p)$. We now define an action of $\Sigma_0(p)$ on the spaces defined in the previous section. As in the classical case, we will incorporate a weight into this action. Fix k a non-negative integer. Let $\Sigma_0(p)$ act on $A[\mathbf{Z}_p, r]$ on the left by

$$(\gamma \cdot_k f)(x) = (a+cx)^k \cdot f\left(\frac{b+dx}{a+cx}\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$ and $f \in A[\mathbf{Z}_p, r]$. Then $\Sigma_0(p)$ acts on $D[\mathbf{Z}_p, r]$ on the right by

$$(\mu\big|_k\gamma)(f) = \mu(\gamma \cdot_k f).$$

where $\mu \in D[\mathbf{Z}_p, r]$.

These two actions then induce actions on $\mathcal{A}(\mathbf{Z}_p)$,$\mathcal{A}^\dagger(\mathbf{Z}_p, r)$,$\mathcal{D}(\mathbf{Z}_p)$ and $\mathcal{D}^\dagger(\mathbf{Z}_p, r)$. To emphasis the role of k in this action, we will sometimes write k in the subscript, i.e. $\mathcal{A}_k(\mathbf{Z}_p)$, $\mathcal{D}_k(\mathbf{Z}_p)$, etc.

3.3 An explicit description of $D[\mathbf{Z}_p, 1]$ and $\mathcal{D}^\dagger(\mathbf{Z}_p, 1)$

In what follows, we will primarily be interested in the distribution modules $\mathcal{D}(\mathbf{Z}_p)$, $D[\mathbf{Z}_p, 1]$ and $\mathcal{D}^\dagger(\mathbf{Z}_p, 1)$. We are obliged to study $\mathcal{D}(\mathbf{Z}_p)$ since this is the natural space where p-adic L-functions live. We also study the larger spaces of distribution $D[\mathbf{Z}_p, 1]$ and $\mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ as one has a bit more freedom in these spaces to perform certain constructions. In this subsection, we give a concrete description of the latter two spaces in terms of the sequence of moments attached to a distribution.

First note $\{x^j\}_{j\in\mathbf{N}}$ is dense in $A[\mathbf{Z}_p, r]$ for $r \geq 1$. Thus, any element of $D[\mathbf{Z}_p, 1]$ and $\mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ is determined by its values on powers of x (i.e. by its moments). Thus, we have an injective map

$$\mathcal{D}^\dagger(\mathbf{Z}_p, 1) \xrightarrow{M} \prod_{j=0}^{\infty} \mathbf{Q}_p$$

that sends μ to the sequence $(c_j)_{j\in\mathbf{N}}$ where $c_j = \mu(x^j)$. The following proposition says that the only obstruction to constructing a distribution by specifying its moments is a growth condition on the sequence of moments.

Proposition 3.1.

1. *The image of M is precisely*

$$\left\{(c_j) \in \prod_{j=0}^{\infty} \mathbf{Q}_p \text{ where } |c_j| \text{ is } o(r^j) \text{ as } j \longrightarrow \infty \text{ for each } r > 1\right\}.$$

2. *The map M restricts to give an isomorphism of Banach spaces between $D[\mathbf{Z}_p, 1]$ and the space of bounded sequences in $\mathbf{Q}_p$ (under the sup norm).*

Proof. For the first part, let $\mu \in D^\dagger(\mathbf{Z}_p, 1)$ and $r, s \in |\mathbf{C}_p^\times|$ with $1 < s < r$. Then

$$|\mu(x^j)| \leq ||x^j||_s \cdot ||\mu||_s = s^j \cdot ||\mu||_s$$

and hence $|\mu(x^j)|$ is $\mathrm{O}(s^j)$ as $j \longrightarrow \infty$ and in particular is $\mathrm{o}(r^j)$.

Conversely, let (c_j) be some $\mathbf{Q}_p$-sequence that is $\mathrm{o}(r^j)$ for every $r > 1$. Take $f \in \mathcal{A}^\dagger(\mathbf{Z}_p, 1)$ with $f = \sum_j a_j x^j$. For some $r > 1$, $f \in A[\mathbf{Z}_p, r]$ and hence $\{|a_j| \cdot r^j\} \longrightarrow 0$ as $j \longrightarrow \infty$. Set $\mu(f) = \sum_j a_j c_j$ which converges since $|c_j|$ is $\mathrm{o}(r^j)$. Then $M(\mu) = (c_j)$ and the first claim is proven.

The second claim follow similarly. Note that

$$||\mu||_1 = \sup_{\substack{f \in A[\mathbf{Z}_p,1] \\ f \neq 0}} \frac{|\mu(f)|}{||f||_1} = \sup_{j \geq 0} \frac{|\mu(x^j)|}{||x^j||_1} = \sup_{j \geq 0} |c_j| = ||(c_j)||.$$

$\square$

4 Log-differentials on Wide Open Subspaces

In the previous section, we saw a concrete description of an overconvergent distributions in terms of its sequence of moments. In this section, we will give another description of these spaces of distributions in terms of log-differentials. This later description will be very convenient both theorectically and computationally.

4.1 Differentials on Wide Open Subspaces

For each $r \in |\mathbf{C}_p^\times|$, let

$$W_r = W(\mathbf{Z}_p, r) = \mathbf{P}^1(\mathbf{C}_p) - B[\mathbf{Z}_p, r].$$

The space W_r is the standard example of a wide open subspace of $\mathbf{P}^1(\mathbf{C}_p)$. The ring of $\mathbf{Q}_p$-rigid analytic functions $A(W_r)$ on W_r is a topological $\mathbf{Q}_p$-algebra and the space $\Omega(W_r)$ of Kahler differentials on W_r is an $A(W_r)$-module.

Note that $1/z \in A(W_r)$ and thus $dz/z^2 \in \Omega(W_r)$. However, $dz/z \notin \Omega(W_r)$ as it has a pole at infinity.

Proposition 4.1. *Let $r \in |\mathbf{C}_p^\times|$ be greater than or equal to 1. Then we have the following descriptions of $A(W_r)$ and $\Omega(W_r)$:*

1. *Every function $f \in A(W_r)$ has a unique representation in the form*

$$f = \sum_{j=0}^{\infty} a_j z^{-j}$$

with each $a_j \in \mathbf{Q}_p$.

2. *Every $\omega \in \Omega(W_r)$ has a unique representation in the form*

$$\omega = \sum_{n=1}^{\infty} a_j z^{-j} \frac{dz}{z}$$

with each $a_j \in \mathbf{Q}_p$.

3. *Conversely, an expression of the form (1) (resp. (2)) represents an element of $A(W_r)$ (resp. $\Omega(W_r)$) if and only if for every real number $t > r$ the coefficient a_j satisfy*

$$|a_j|_p \text{ is } o(t^n) \text{ as } n \longrightarrow \infty.$$

Proof. Reference... □

Note that $\Sigma_0(p)$ preserves W_r and thus acts on $A(W_r)$ and $\Omega(W_r)$. Explicitly, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$ and $f \in A(W_r)$ then

$$(\gamma \cdot f)(z) = (\gamma \cdot_0 f)(z) = f\left(\frac{b+dz}{a+cz}\right)$$

by a weight zero action. This action naturally induces a left action on $\Omega(W_r)$ which we will express as a right action

$$\omega\big|_0 \gamma := \gamma^{-1}(\omega)$$

for $\omega \in \Omega(W_r)$. (We do this because $\Sigma_0(p)$ naturally acts on distribution spaces on the right.)

The weight k action of $\Sigma_0(p)$ on $\Omega(W_r)$ is a little more complicated and we will postpone discussion of it until section 4.3.

4.2 Log-differentials

Glenn's notes.

4.3 Weight k action

We will now discribe the weight k action of $\Sigma_0(p)$ on $\Omega_{\log}(W_r)$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$. We will define the action by describing it on δ_0 and on $z^{-j}dz/z$ for $j > 0$. Namely set

$$\delta_0\big|_k \gamma = a^k \delta_{b/a}$$

and

$$z^{-j}\frac{dz}{z}\bigg|_k \gamma = \left(\sum_{r=\max(j-k,0)}^{j} \binom{k}{j-r} a^{k-j+r} c^{j-r} z^{-r} \frac{dz}{z}\right)\bigg|_0 \gamma.$$

One can equivalently describe this action as follows: for $\omega \in \Omega(W_r)$, consider ω multiplied by $(a+cz)^k$. This product is no longer a log-differential, but one can make it into one by throwing away the new terms that appear that are holomorphic at zero. Then to form $\omega\big|_k \gamma$ simply apply the weight zero action of γ on this truncated product.

It is not a priori clear that these formulas lead to a well-defined action. However, this will follow from Theorem 4.2.

4.4 Relation between log-differentials and distributions

In this section, we will construct an $\Sigma_0(p)$-equivariant isomorphism between spaces of log-differentials and spaces of overconvergent distributions.

Set $W := W_1 = \Omega_{\log}(\mathbf{Z}_p, 1)$. For $\omega \in \Omega_{\log}(W)$, define a distribution $\mu_\omega \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ by

$$\int_{\mathbf{Z}_p} f d\mu_\omega := \rho_{\partial W}(fw)$$

for each $f \in A^\dagger(\mathbf{Z}_p, 1)$. Here $\rho_{\partial W}$ is the residue around the unit disc of $\mathbf{C}_p$. Taking the residue of $f\omega$ is a valid operation because f is overconvergent and hence defined on some disc of radius strictly larger than 1.

Theorem 4.2. *The map*

$$\mu : \Omega_{log}(W(\mathbf{Z}_p, 1)) \longrightarrow \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$$
$$\omega \mapsto \mu_\omega$$

is an isomorphism. Moreover, for each integer $k \geq 0$ we have that

$$(\mu_\omega)\big|_k \gamma = \mu_{\omega|_k \gamma}$$

for $\gamma \in \Sigma_0(p)$. (That is, μ is a $\Sigma_0(p)$-equivariant map under its weight k action.)

Proof. In the notes you say that the first part of this theorem is a simple extension of results of Vishik and Teitelbaum. The second part is essentially just a computation, but in fact, I found it quite complicated. I typed up several pages of notes where I nearly checked this, but I ran out of steam at some point. ☐

Under this isomorphism between distributions and log-differentials, the moments of a distribution correspond to the coefficients of the associated log-differential. Namely we have the following corollary.

Corollary 4.3. *If $\omega = a_0\delta_0 + \sum_{j=1}^{\infty} a_j z^{-j} \frac{dz}{z} \in \Omega_{log}(W)$ then the j-th moment of μ_ω is a_j.*

Proof. We have that

$$\mu_\omega(x^j) = \int_{\mathbf{Z}_p} x^j d\mu_\omega = \rho_{\partial W}(x^j\omega) = a_j$$

since the residue function returns the coefficient of dz/z. ☐

With this equivalence between distributions and log-differentials in hand, we will tacitly identify these two spaces.

5 Filtrations on spaces of distributions and log-differentials

5.1 A $\Sigma_0(p)$-stable filtration of $D[\mathbf{Z}_p, 1]$

Let

$$\Omega_0 = \left\{\mu \in D[\mathbf{Z}_p, 1] \text{ with } \mu(x^j) \in \mathbf{Z}_p \text{ for all } j \geq 0\right\},$$

the set of distributions with integral moments. By Proposition 3.1, this subspace is the unit ball under $||\cdot||_1$.

The natural filtration $\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_r \supset \cdots$ defined by

$$\Omega_r = \left\{\mu \in \Omega_0 \text{ such that } \mu(x^j) = 0 \text{ for } 0 \leq j \leq r-1\right\}$$

is unfortunately not stable under the weight k action of $\Sigma_0(p)$. Instead we need to replace Ω_r with a larger subspace to produce a filtration compatible with the action of $\Sigma_0(p)$. This is done in the following proposition.

Proposition 5.1. *The subspace*

$$\widetilde{\Omega}_r = \left\{\mu \in \Omega_0 \textit{ scuh that } \mu(x^j) \in p^{r-j}\mathbf{Z}_p\right\}$$

is a $\Sigma_0(p)$-module via its weight k action for any $k \geq 0$.

Proof. Let μ_j be the distribution defined by $\mu_j(x^s) = \delta_{js}$. To prove this proposition, it suffices to check that

$$\mu_j\big|_k\gamma \in \widetilde{\Omega}_r \quad \text{for} \quad j \geq r$$

and

$$p^{r-j}\mu_j\big|_k\gamma \in \widetilde{\Omega}_r \quad \text{for} \quad 0 \leq j < r.$$

In the first case, by definition it suffices to check that

$$(\mu_j\big|_k\gamma)(x^s) \in p^{r-s}\mathbf{Z}_p \quad \text{for} \quad 0 \leq s \leq r.$$

In the second case, it suffices to see that

$$p^{r-j}(\mu_j\big|_k\gamma)(x^s) \in p^{r-s}\mathbf{Z}_p \quad \text{for} \quad 0 \leq s \leq r$$

for which it is enough to see that

$$(\mu_j\big|_k\gamma)(x^s) \in p^{j-s}\mathbf{Z}_p \quad \text{for} \quad 0 \leq s \leq j.$$

In fact, it suffices to check this last condition in both cases since in the first case $j \geq r$. For $0 \leq s \leq j$ we have

$$\begin{aligned}
(\mu_j\big|_k\gamma)(x^s) &= \mu_j\left((a+cx)^{k-s}(b+dx)^s\right) \\
&= \mu_j\left(\left(\sum_{m=0}^{\infty}\binom{k-s}{m}a^{k-s-m}c^m x^m\right)\left(\sum_{m=0}^{s}\binom{s}{m}b^{s-m}d^m x^m\right)\right) \\
&= \sum_{m=0}^{s}\binom{k-s}{j-m}\binom{s}{m}a^{k-s-j+m}b^{s-m}c^{j-m}d^m
\end{aligned}$$

which is divisible by c^{j-s} and thus divisible by p^{j-s}. (Note that the potentially negative exponent on the a term is irrelevant since $a \in \mathbf{Z}_p^\times$.) $\square$

Remark 5.2. Arguing as in the above proof, we have in general that

$$(\mu_j\big|_k\gamma)(x^s) = \sum_{m=0}^{\min(j,s)} \binom{k-s}{j-m}\binom{s}{m} a^{k-s-j+m} b^{s-m} c^{j-m} d^m.$$

Although Ω_r is not stable under the weight k action of $\Sigma_0(p)$ for general r, Ω_{k+1} is in fact a $\Sigma_0(p)$-module. This is true since Ω_{k+1} is the kernel of a natural $\Sigma_0(p)$-equivariant map (defined later) from Ω_0 to L_k. Nonetheless, we check this claim directly below.

Proposition 5.3. *The subspace Ω_{k+1} is a $\Sigma_0(p)$-module under its weight k action.*

Proof. It suffices to see that

$$\mu_j\big|_k\gamma \in \Omega_{k+1} \quad \text{for} \quad j \geq k+1$$

and thus we must check that $(\mu_j\big|_k\gamma)(x^s) = 0$ for $0 \leq s \leq k$. We have

$$(\mu_j\big|_k\gamma)(x^s) = \mu_j\left((a+cx)^{k-s}(b+dx)^s\right)$$

which is indeed zero since it equals the j-th coefficient of a polynomial of degree k with $k < j$. $\square$

The subspace Ω_{k+1} thus inherits an induced filtration which will be of interest to us in section 8. For $r \geq k+1$ set

$$\widetilde{\Omega}_r^0 = \Omega_{k+1} \cap \widetilde{\Omega}_r$$

which is a $\Sigma_0(p)$-module.

Lemma 5.4. *If $\mu \in \widetilde{\Omega}_r^0$ and $\gamma \in \Sigma_0(p)$ with $\det(\gamma) = p$ then $\mu\big|_k\gamma \in p^r\Omega_{k+1}$.*

Proof. Arguing as in Proposition 5.5, we need to check that

$$(\mu_j\big|_k\gamma)(x^s) \in p^j\mathbf{Z}_p \quad \text{for all } s.$$

From Remark 5.2, we have that

$$(\mu_j\big|_k\gamma)(x^s) = \sum_{m=0}^{\min(j,s)} \binom{k-s}{j-m}\binom{s}{m} a^{k-s-j+m} b^{s-m} c^{j-m} d^m.$$

Since $\det(\gamma) = p$, $p|c$ and $a \in \mathbf{Z}_p^\times$ we have that $p|d$. Thus $p^j|c^{j-m}d^m$ and therefore $(\mu_j\big|_k\gamma)(x^s) \in p^j\mathbf{Z}_p$ proving the claim. $\square$

5.2 A $\Sigma_0(p)$-stable filtration of $\Omega_{\mathbf{log}}(W)$

In this section we repeat the results of the previous section but now in the language of log-distributions. Let

$$\Omega_0 = \left\{ \omega = a_0\delta_0 + \sum_{j=0}^{\infty} a_j z^{-j}\frac{dz}{z} \in \Omega_{\log}(W) \text{ with } a_j \in \mathbf{Z}_p \text{ for all } j \geq 0 \right\},$$

the set of log-differentials with integral coefficients.

The natural filtration $\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_r \supset \cdots$ defined by

$$\Omega_r = \left\{ \omega = a_0\delta_0 + \sum_{j=0}^{\infty} a_j z^{-j}\frac{dz}{z} \in \Omega_0 \text{ such that } a_j = 0 \text{ for } 0 \leq j \leq r-1 \right\}$$

is also not stable under the weight k action of $\Sigma_0(p)$. Instead we need to replace Ω_r with a larger subspace to produce a filtration compatible with the action of $\Sigma_0(p)$. This is done in the following proposition.

Proposition 5.5. *The subspace*

$$\widetilde{\Omega}_r = \left\{ \omega = a_0\delta_0 + \sum_{j=0}^{\infty} a_j z^{-j}\frac{dz}{z} \in \Omega_0 \textit{ with } a_j \in p^{r-j}\mathbf{Z}_p \right\}$$

is a $\Sigma_0(p)$-module via its weight k action for any $k \geq 0$.

Before proving this propostion, we begin with a few lemmas.

Lemma 5.6. *For $j \geq 0$ and $\gamma \in \Sigma_0(p)$ we have that*

$$z^{-j}\frac{dz}{z}\Big|_0 \gamma = \sum_{r=0}^{\infty} b_s z^{-s}\frac{dz}{z}$$

where

$$b_s = \begin{cases} \left(\dfrac{b}{a}\right)^{s+1} & j = 0 \\ \det(\gamma)\left(\dfrac{-1}{a}\right)^{j+s} \displaystyle\sum_{m=0}^{\min(j,s)-1} \binom{j-1}{m}\binom{-j-1}{s-1-m} a^m b^{s-1-m} c^{j-1-m} d^m & j > 0 \end{cases}.$$

Proof. For $j > 0$ we have

$$\begin{aligned}
z^{-j}\frac{dz}{z}\bigg|_0 \gamma = \gamma^{-1}\cdot_0\left(z^{-j}\frac{dz}{z}\right) &= \left(\frac{-b+az}{d-cz}\right)^{-(j+1)} d(\gamma^{-1}) \\
&= (\det\gamma)(d-cz)^{j-1}(-b+az)^{-(j+1)}dz \\
&= (\det\gamma)\left(\frac{d}{z}-c\right)^{j-1}\left(\frac{-b}{z}+a\right)^{-(j+1)}\frac{dz}{z^2} \\
&= (\det\gamma)\left(\sum_{m=0}^{j-1}(-1)^{j-m-1}\binom{j-1}{m}c^{j-m-1}d^m z^{-m}\right)\cdot \\
&\qquad \left(\sum_{m=0}^{\infty}(-1)^m\binom{-j-1}{m}a^{-j-m-1}b^m z^{-m}\right)\frac{dz}{z^2}.
\end{aligned}$$

Computing the coefficient of $z^{-r}\frac{dz}{z}$ in this product yields the formula of the lemma.

For $j = 0$ we have

$$\begin{aligned}
\delta_0\big|_0\gamma = \delta_{\frac{b}{a}} = \delta_0 - (\delta_{\frac{b}{a}} - \delta_0) &= \delta_0 + \det\begin{pmatrix}1 & 0\\ a & b\end{pmatrix}\frac{dz}{z(az-b)} \\
&= \delta_0 + \frac{b}{a}(1-\frac{b}{a}z^{-1})^{-1}\frac{dz}{z^2} = \delta_0 + \sum_{m=1}^{\infty}\left(\frac{b}{a}\right)^{m+1} z^{-m}\frac{dz}{z}
\end{aligned}$$

which has the correct coefficient of $z^{-r}dz/z$. ∎

Lemma 5.7. *The subspace $\widetilde{\Omega}_r$ is a $\Sigma_0(p)$-module via its weight 0 action.*

Proof. We need to check that

$$p^{\min(r-j,0)}z^{-j}\frac{dz}{z}\bigg|_0 \gamma \in \widetilde{\Omega}_r$$

for all $j \geq 0$. Let $z^{-j}\frac{dz}{z}\big|_0\gamma = \sum_{s=0}^{\infty} b_s z^{-s}\frac{dz}{z}$. Looking at the formulas of Lemma 5.6, we see that for $0 \leq s \leq j$, we have that b_s is divisible by c^{j-s} and thus by p^{j-s}. (Note that we are using that $a \in \mathbf{Z}_p^\times$.)

If $j > r$ we are done since then p^{r-s} divides b_s. For $j \leq r$, we have that $p^{r-j}b_s$ is divisible by $p^{r-j}p^{j-s} = p^{r-s}$ and again we are done. ∎

Proof of Prop 5.5. We need to check that

$$p^{\min(r-j,0)}z^{-j}\frac{dz}{z}\bigg|_k \gamma \in \widetilde{\Omega}_r$$

for all $j \geq 0$. By definition

$$z^{-j}\frac{dz}{z}\bigg|_k \gamma = \left(\sum_{m=\max(j-k,0)}^{j}\binom{k}{j-m}a^{k-j+m}c^{j-m}z^{-m}\frac{dz}{z}\right)\bigg|_0 \gamma.$$

By Lemma 5.7, we know that

$$p^{\min(r-m,0)} z^{-m} \frac{dz}{z}\bigg|_0 \gamma \in \widetilde{\Omega}_r.$$

So for $j > r$ we have that $c^{j-m} z^{-m} \frac{dz}{z}\big|_0 \gamma \in \widetilde{\Omega}_r$ and thus $z^{-j} \frac{dz}{z}\big|_k \gamma \in \widetilde{\Omega}_r$. For $j \leq r$ we have that $p^{r-j} c^{j-m} z^{-m} \frac{dz}{z}\big|_0 \gamma \in \widetilde{\Omega}_r$ and thus $p^{r-j} z^{-j} \frac{dz}{z}\big|_k \gamma \in \widetilde{\Omega}_r$ which proves the proposition. □

Although Ω_r is not stable under the weight k action of $\Sigma_0(p)$ for general r, Ω_{k+1} is in fact a $\Sigma_0(p)$-module. This is true since Ω_{k+1} is the kernel of a natural $\Sigma_0(p)$-equivariant map (defined later) from Ω_0 to L_k. Nonetheless, we check this claim directly below.

Proposition 5.8. *The subspace Ω_{k+1} is a $\Sigma_0(p)$-module under its weight k action.*

Proof. I was surprised that this claim is not obvious from direct computations. I did work this out, but it took lots of computations and some funny identities with binomial coefficients. □

The subspace Ω_{k+1} thus inherits an induced filtration which will be of interest to us in section 8. For $r \geq k+1$ set

$$\widetilde{\Omega}_r^0 = \Omega_{k+1} \cap \widetilde{\Omega}_r$$

which is a $\Sigma_0(p)$-module.

Lemma 5.9. *If $\mu \in \widetilde{\Omega}_r^0$ and $\gamma \in \Sigma_0(p)$ with $\det(\gamma) = p$ then $\mu\big|_0 \gamma \in p^r \Omega_1$.*

Proof. We need that for all $j \geq 1$

$$p^{\min(r-j,0)} z^{-j} \frac{dz}{z}\bigg|_0 \gamma \in p^r \Omega_1.$$

From Lemma 5.6, we can see that p^j divides each coefficient of $z^{-j} \frac{dz}{z}\big|_0 \gamma$. Each coefficient of $z^{-j} \frac{dz}{z}\big|_0 \gamma$ is given by a sum where in each term of the sum there appears $c^{j-m-1} d^m$. Since p divides both c and d this accounts for $j-1$ powers of p. The last factor of p comes from the $\det(\gamma)$ term appearing in the formula outside of the sum.

Thus if $j > r$ we are done. If $j \leq r$ we then have that $p^{r-j} p^j$ divides $p^{r-j} z^{-j} \frac{dz}{z}\big|_0 \gamma$ and we are again done. □

Lemma 5.10. *If $\mu \in \widetilde{\Omega}_r^0$ and $\gamma \in \Sigma_0(p)$ with $\det(\gamma) = p$ then $\mu\big|_k \gamma \in p^r \Omega_{k+1}$.*

Proof. We need that for all $j \geq k+1$

$$p^{\min(r-j,0)} z^{-j} \frac{dz}{z}\bigg|_0 \gamma \in p^r \Omega_{k+1}.$$

Note that from Proposition 5.8, it is enough to check that this log-differential is in $p^r\Omega_1$. We have that

$$z^{-j}\frac{dz}{z}\bigg|_k \gamma = \left(\sum_{m=j-k}^{j}\binom{k}{j-m}a^{k-j+m}c^{j-m}z^{-m}\frac{dz}{z}\right)\bigg|_0 \gamma.$$

From Lemma 5.6,

$$z^{-m}\frac{dz}{z}\bigg|_0 \gamma \in p^m\Omega_1.$$

Thus

$$c^{j-m}z^{-m}\frac{dz}{z}\bigg|_0 \gamma \in p^j\Omega_1.$$

Then, as always, considering either $j > r$ or $j \leq r$ yields the result. □

6 Overconvergent modular symbols

In this section, we consider $\mathbf{D}_k$-valued modular symbols where $\mathbf{D}_k = \mathcal{D}_k(\mathbf{Z}_p)$, $D_k[\mathbf{Z}_p, 1]$ or $\mathcal{D}_k^\dagger(\mathbf{Z}_p, 1)$. Such modular symbols will be referred to as overconvergent modular symbols.

6.1 Changing the distribution module

In this subsection, we will see that when seeking U_p-eigenvectors in $H^1_c(\Gamma, \mathbf{D}_k)$, it does not matter which one of the above three distribution modules is used. First note that U_p is well behaved on these possibly infinite dimensional spaces.

Proposition 6.1. *For $\mathbf{D}_k = \mathcal{D}_k(\mathbf{Z}_p)$, $D_k[\mathbf{Z}_p, 1]$ and $\mathcal{D}_k^\dagger(\mathbf{Z}_p, 1)$, U_p is a completely continuous operator on $H^1_c(\Gamma, \mathbf{D}_k)$.*

Proof. See [**?**]. □

As U_p is a completely continuous operator, its finite slope subspaces are finite dimensional. The following proposition states that these finite slope subspaces are the same for each choice of $\mathbf{D}_k$.

Proposition 6.2. *The natural maps*

$$H^1_c(\Gamma, \mathcal{D}_k(\mathbf{Z}_p)) \longrightarrow H^1_c(\Gamma, D_k[\mathbf{Z}_p, 1]) \longrightarrow H^1_c(\Gamma, \mathcal{D}_k^\dagger(\mathbf{Z}_p, 1))$$

are isomorphisms when restricted to any finite slope subspace of U_p.

Proof. See [**?**]. □

6.2 Comparison theorem

There is a natural map from overconvergent modular symbols to classical modular symbols. Namely, we have a map

$$\mathbf{D}_k \xrightarrow{\rho_k} L_k$$

given by

$$\mu \mapsto \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \mu(x^{k-j}) Z^j.$$

This map is equivariant for the action of $\Sigma_0(p)$ and hence induces an equivariant map

$$H^1_c(\Gamma, \mathbf{D}_k) \xrightarrow{\rho_k} H^1_c(\Gamma, L_k)$$

which will again be denoted by ρ_k and called the *specialization* map. Note that the kernel of specialization is $H^1_c(\Gamma, \Omega_{k+1}) \otimes \mathbf{Q}_p$ when $\mathbf{D}_k = D_k[\mathbf{Z}_p, 1]$.

While $H^1_c(\Gamma, \mathbf{D}_k)$ appears to be much larger than the classical space of modular symbols, the specialization maps becomes an isomorphism when restricted to the subspace where U_p acts with slope less than $k+1$. Let $\mathbf{D}_k^{(<h)}$ denote the subspace of $\mathbf{D}_k$ where U_p acts with slope less than h.

Theorem 6.3. *For $\mathbf{D}_k = \mathcal{D}_k(\mathbf{Z}_p)$, $D_k[\mathbf{Z}_p, 1]$ or $\mathcal{D}^\dagger_k(\mathbf{Z}_p, 1)$,*

$$H^1_c(\Gamma, \mathbf{D}_k)^{(<h)} \xrightarrow{\rho} H^1_c(\Gamma, \mathbf{Q}_p)^{(<h)}$$

is an isomorphism of $\Sigma_0(p)$-modules for $h < k+1$.

Proof. See [**?**]. ◻

6.3 p-adic L-functions arising from Hecke-eigensymbols

Theorem 6.3 can be viewed as an equivariant construction of p-adic L-functions. Namely, let f be an eigenform of weight $k+2$ and level N ($p \nmid N$) such that $f|T_l = a_l f$ for $l \nmid N$ and $f|U_q = a_q f$ for $q|N$. Let f_α be a p-stabilized form of f of level Np so that $f_\alpha|U_p = \alpha f_\alpha$ with α a root of $x^2 - a_p x + p^{k+1}$.

To view f_α as a p-adic object, we must fix an embedding of $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Then let K be the finite extension of $\mathbf{Q}_p$ generated by the images of the a_l and α under this map. Let $\mathcal{O}_K$ be the ring of integers of K.

By Theorem 2.2, there is a two dimensional subspace of $H^1_c(\Gamma, L_k \otimes K)$ with the same Hecke-eigenvalues as f_α. Fix $\phi_{f,\alpha}$ with these eigenvalues such that $\phi_{f,\alpha}|\iota = \phi_{f,\alpha}$ and $||\phi_{f,\alpha}|| = 1$ (so in particular $\phi_{f,\alpha} \in H^1_c(\Gamma, L_k \otimes \mathcal{O}_K)^+$).

If $\mathrm{ord}_p(\alpha) < k+1$ then by Theorem 6.3 there is a unique Hecke-eigensymbol

$$\Phi_{f,\alpha} \in H^1_c(\Gamma, \mathcal{D}(\mathbf{Z}_p) \otimes \mathcal{O}_K)^{(<k+1)}$$

with the same Hecke-eigenvalues as $\phi_{f,\alpha}$ such that $\rho(\Phi_{f,\alpha}) = \phi_{f,\alpha}$. Consider the $\mathcal{O}_K$-valued distribution

$$\mu_{f,\alpha} = \Phi_{f,\alpha}\left(\{0\} - \{\infty\}\right)$$

obtained by "integrating" from $\{0\}$ to $\{\infty\}$. Then since $\Phi_{f,\alpha}|U_p = \alpha\Phi_{f,\alpha}$ and since $\Phi_{f,\alpha}$ lifts $\phi_{f,\alpha}$, it is a straightforward computation to verify that $\mu_{f,\alpha}$ is precisely the Mazur-Tate-Teitelbaum p-adic L-function attached to f and α.

This construction fails precisely when both a_p is a unit and when we consider the root β of $x^2 - a_p x + p^{k+1}$ with slope $k+1$. In this case, Theorem 6.3 does not apply and we do not know a priori that there is any eigensymbol with the same eigenvalues as f_β. We also do not know the uniqueness of such a symbol if it existed. The remainder of this part of the paper will focus upon this highest slope case.

7 Forming a β Hecke-eigensymbol

7.1 Lifting $\phi_{f,\beta}$

Assume that a_p is a unit and let β be the root of slope $k+1$ of $x^2 - a_p x + p^{k+1}$. Let $\phi_{f,\beta} \in H^1_c(\Gamma, L_k \otimes \mathcal{O}_K)^\varepsilon$ be the modular symbol corresponding to f_β with $\varepsilon = \pm 1$. The following proposition guarentees that we can lift $\phi_{f,\beta}$ to some overconvergent symbol.

Proposition 7.1. *There exists $\Phi \in H^1_c(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1) \otimes K)^\varepsilon$ such that $\rho(\Phi) = \phi_{f,\beta}$.*

Proof. I don't know the proof of this. I think you said it was something like the H^2 term is small (maybe just composed of Eisenstein elements). $\square$

Remark 7.2. In section 9.3, we will give an alternative proof of this fact which is completely explicit.

The space $H^1_c(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1))$ is not a Banach space and can be a bit unwieldy to work in. However, we can use the U_p operator to force our modular symbols to take values in $D[\mathbf{Z}_p, 1]$.

Lemma 7.3. *If $\mu \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ then $\mu\big|_k \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right) \in D[\mathbf{Z}_p, 1]$.*

Proof. Let $\gamma_a = \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)$. We have

$$(\mu\big|_k \gamma_a)(x^j) = \mu\left((a+px)^j\right) = \sum_{k=0}^{j} \binom{j}{k} a^{j-k} p^k \mu(x^k).$$

But $\left\{p^k \mu(x^k)\right\}$ is bounded by Proposition 3.1 and thus (again by Proposition 3.1) $\mu\big|_k \gamma_a \in D[\mathbf{Z}_p, 1]$. $\square$

Corollary 7.4. *There exists $\Phi \in H^1_c(\Gamma, D[\mathbf{Z}_p, 1] \otimes K)^\varepsilon$ such that $\rho(\Phi) = \phi_{f,\beta}$.*

Proof. By Proposition 7.1, let $\Psi \in H^1_c(\Gamma, \mathcal{D}(\mathbf{Z}_p, 1) \otimes K)^\varepsilon$ such that $\rho(\Psi) = \phi_{f,\beta}$. Now set $\Phi = \frac{1}{\beta}\Psi|U_p$. Then

$$\rho(\Phi) = \frac{1}{\beta}\rho(\Psi)|U_p = \frac{1}{\beta}\phi_{f,\beta}|U_p = \phi_{f,\beta}$$

since $\phi_{f,\beta}$ is β-eigenvector for U_p. Now, for $D \in \Delta_0$,

$$\Phi(D) = \frac{1}{\beta}(\Psi|U_p)(D) = \frac{1}{\beta}\sum_{a=0}^{p-1} \Psi(\gamma_a D)\big|_k \gamma_a \in D[\mathbf{Z}_p, 1] \otimes K$$

by Lemma 7.3. Therefore, $\Phi \in H^1_c(\Gamma, D[\mathbf{Z}_p, 1] \otimes K)^\varepsilon$. □

Remark 7.5. It will not in general be possible to lift $\phi_{f,\beta}$ to an element of $H^1_c(\Gamma, \Omega_0 \otimes \mathcal{O}_K)$.

7.2 Slope $k+1$ subspace of lifts of $\phi_{f,\beta}$

Our task now is to refine our lift of $\phi_{f,\beta}$ to a Hecke-eigensymbol. Let

$$X = \{\Phi \in H^1_c(\Gamma, D[\mathbf{Z}_p, 1] \otimes K)^\varepsilon \mid \rho(\Phi) = c \cdot \phi_{f,\beta} \text{ for some } c \in K\}.$$

Note that X contains the ε-part of the kernel of specialization. However, by Corollary 7.4, there is some modular symbol in X not in the kernel of specialization.

One must take some care in trying to simultaneously diagonalize the action of the Hecke algebra on X since it is most likely an infinite dimensional space. We will accomplish this diagonalization by restricting to the "slope $k+1$" subspace of X for the operator U_p. This subspace is finite dimensional since U_p is a completely continuous operator.

Lemma 7.6. *We have that $H^1_c(\Gamma, \Omega_{k+1})|U_p \subseteq p^{k+1} H^1_c(\Gamma, \Omega_{k+1})$.*

Proof. Take $\Phi \in H^1_c(\Gamma, \Omega_{k+1})$ with $||\Phi|| = 1$. Then

$$(\Phi|U_p)(D) = \sum_{a=0}^{p-1} \Phi(\gamma_a D)\big|_k \gamma_a \in p^{k+1}\Omega_{k+1}$$

by Lemma 5.4. □

Proposition 7.7. *There is no eigenvalue of U_p in X with slope less than $k+1$.*

Proof. If some U_p-eigenspace intersects the kernel of specialization then by Lemma 7.6, its eigenvalue has slope at least $k+1$. If some eigenspace specializes to $c \cdot \phi_{f,\beta}$ with $c \neq 0$ then necessarily its eigenvalue is β which has slope $k+1$. □

The following is a general proposition about completely continuous operators on Banach spaces.

Proposition 7.8. *If U is a completely continuous operator on a Banach space H then there is a decomposition*

$$H = H^{(\le h)} \oplus H^{(>h)}$$

such that $p^{-h}U$ is topologically nilpotent on $H^{(>h)}$. Moreover, $H^{(\le h)}$ is a finite dimensional space on which U acts invertibly and where $\{p^{nh}U^{-n}\}$ is a bounded sequence of operators.

Proof. See [**?**]. ☐

Applying the above proposition with $H = X$ and $h = k+1$ yields

$$X = X^{(\leq k+1)} \oplus X^{(>k+1)}$$

with $X^{(\leq k+1)}$ finite dimensional. Note that by Proposition 7.7, $X^{(\leq k+1)}$ is composed entirely of pseudo-eigenspaces of slope exactly $k+1$.

Remark 7.9. I had hoped that from this I could conclude that $||U_p||_{X^{(\leq k+1)}} \leq \frac{1}{p^{k+1}}$ and hence $u = U_p/p^{k+1}$ preserves its unit ball. Then one knows explicitly how to project onto this space; namely map v to $\lim_n v|u^{n!}$. But this now seems not to work. For example, take $V = \mathbf{Q}_p^2$ and $U = \left(\begin{smallmatrix} p & 1 \\ 0 & p \end{smallmatrix}\right)$. On this space $||U|| = 1$ but it is a generalized eigenspace of slope 1.

Lemma 7.10. *We have that* $\rho(X^{(>k+1)}) = 0$.

Proof. Since U_p/p^{k+1} is topologically nilpotent on $X^{(>k+1)}$, it must also be topologically nilpotent on $\rho(X^{(>k+1)})$. But since β/p^{k+1} is a unit, U_p/p^{k+1} is not topologically nilpotent on $\phi_{f,\beta}$. Hence, $\rho(X^{(>k+1)})$ must vanish. ☐

7.3 Forming the β-Hecke eigensymbol

We begin with a general lemma from linear algebra.

Lemma 7.11. *Let V and W be finite dimensional vector spaces over a field K. Let $\{A_i\}$ be a countable family of commuting operators on these spaces and let $f : V \longrightarrow W$ be a linear map equivariant for each A_i. If there is some $w \in W$ such that $w|A_i = \lambda_i w$ for each i with $\lambda_i \in K$ then there is some $v \in V$ such that $v|A_i = \lambda_i v$ for each i.*

Proof. Replacing W with $K \cdot w$ and V with $f^{-1}(K \cdot w)$, we may assume that W is one dimensional and f is surjective. Then write $V = \oplus_j V_j$ with each V_j a simultaneous eigenspace for all the A_i. Since f is non-zero and W is itself a simultaneous eigenspace for the family $\{A_i\}$, one of the V_j must be a λ_i-eigenspace for each A_i. Since all of the λ_i are in K, V_j has some bonafide eigenvector v which proves the lemma. ☐

Remark 7.12. One cannot in general find such an $\{A_i\}$-eigenvector which maps to w. The set of all such eigenvectors with the same eigenvalues as w might lie entirely within the kernel of f.

Theorem 7.13. *There exists some $\Phi_{f,\beta} \in H^1_c(\Gamma, \mathcal{D}(\mathbf{Z}_p) \otimes K)^\varepsilon$ with the same Hecke-eigenvalues as f_β.*

Proof. First apply Corolllary 7.4 to find $\Psi \in H^1_c(\Gamma, D[\mathbf{Z}_p, 1] \otimes K)^\varepsilon$ such that $\rho(\Psi) = \phi_{f,\beta}$. Then write

$$\Psi = \Psi^0 + \Psi^{\text{nil}}$$

with $\Psi^0 \in X^{(\leq k+1)}$ and $\Psi^{\rm nil} \in X^{(>k+1)}$. By Lemma 7.10, $\rho(\Psi^{\rm nil}) = 0$ and hence, $\rho(\Psi^0) = \phi_{f,\beta}$. Then applying Lemma 7.11 with $V = X^{(\leq k+1)}$, $W = H^1_c(\Gamma, K)$ and $\{A_i\} = \{U_q\} \cup \{T_l\} \cup \{\iota\}$ yields $\Phi_{f,\beta} \in H^1_c(\Gamma, D[\mathbf{Z}_p, 1] \otimes K)$ with the correct eigenvalues. Then Proposition 6.2 implies that $\Phi_{f,\beta}$ actually lies in $H^1_c(\Gamma, \mathcal{D}(\mathbf{Z}_p))^\varepsilon$ since it is a U_p-eigensymbol. $\square$

8 Can $\Phi_{f,\beta}$ be in the kernel of specialization?

In this section, we will form a sufficient condition for $\Phi_{f,\beta}$ not to be in the kernel of specialization.

Lemma 8.1. *If $\Phi \in H^1_c(\Gamma, \Omega_0)$ such that $||\Phi|| = 1$ and $\Phi|U_p = \lambda\Phi$ with $\mathrm{ord}_p(\lambda) = r \geq k+1$ then $\Phi \notin H^1_c(\Gamma, \widetilde{\Omega}^0_{r+1})$.*

Proof. Assume that $\Phi \in H^1_c(\Gamma, \widetilde{\Omega}^0_{r+1})$. Since $||\Phi|| = 1$ there is some $D \in \Delta_0$ such that $||\Phi(D)|| = 1$. Then

$$\Phi(D) = \frac{1}{\lambda}(\Phi|U_p)(D) = \frac{1}{\lambda}\sum_{a=0}^{p-1}\Phi(\gamma_a D)|\gamma_a.$$

By Lemma 5.4, $\Phi(\gamma_a D)|\gamma_a \in p^{r+1}\Omega_0$ since $\Phi(\gamma_a D) \in \widetilde{\Omega}_{r+1}$ and hence $\Phi(D) \in p\Omega_0$. This contradicts the fact that $||\Phi(D)|| = 1$. $\square$

Let π be some uniformizer in $\mathcal{O}_K$. Then for some r, $||\pi^r \cdot \Phi_\beta|| = 1$. Hence, if Φ_β is in the kernel of specialization, by the above lemma, the image of $\pi^r \cdot \Phi_\beta$ is non-zero in $H^1_c(\Gamma, (\Omega_{k+1}/\widetilde{\Omega}^0_{k+2}) \otimes \mathcal{O}_K)$. We will now examine this latter module.

Lemma 8.2. *Let ψ be the character on $\Sigma_0(p)$ that sends $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $d^{k+2} \cdot \det^{-1}$ $= d^{k+2} \cdot (ad-bc)^{-1}$. We have that*

$$\Omega_{k+1}/\widetilde{\Omega}^0_{k+2} \cong \mathbf{Z}/p\mathbf{Z}(\psi)$$

as $\Sigma_0(p)$-module where $\Sigma_0(p)$ acts on $\mathbf{Z}/p\mathbf{Z}(\psi)$ via the character ψ.

Proof. As a set it is clear that $\Omega_{k+1}/\widetilde{\Omega}^0_{k+2} \cong \mathbf{Z}/p\mathbf{Z}$. To see the $\Sigma_0(p)$-action, note that the distribution μ defined by $\mu(x^j) = \begin{cases} 1 & j = k+1 \\ 0 & j \neq k+1 \end{cases}$ generates $\Omega_{k+1}/\widetilde{\Omega}^0_{k+2}$. Now, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then

$$\begin{aligned}
(\mu\big|_k\gamma)(x^{k+1}) &= \mu\left((a+cx)^{-1}(b+dx)^{k+1}\right) \\
&= a^{-1}\mu\left(\left(1 - \frac{c}{a}x + \frac{c^2}{a^2}x^2 - \dots\right)(b+dx)^{k+1}\right) \\
&= \frac{1}{a}\sum_{j=0}^{k+1}\binom{k+1}{j}\left(\frac{c}{a}\right)^j d^{k+1-j} \\
&\equiv \frac{d^{k+1}}{a} \equiv \frac{d^{k+2}}{ad} \equiv d^{k+2}(ad-bc)^{-1} \pmod{p}
\end{aligned}$$

which proves the claim. $\square$

Proposition 8.3. *Let Φ be some overconvergent Hecke-eigensymbol with $||\Phi|| = 1$ and with the same Hecke-eigenvalues as f_β. If Φ is in the kernel of specialization then there is a system of eigenvalues occuring in $\mathcal{M}_{k+2}(\Gamma_1(Np), \omega^{k+2}, \mathbf{C})$ that is congruent to $\{la_l\}$ mod π. (Here ω is the Teichmuller character mod p.)*

Proof. By Lemma 8.1, the image of Φ in $H^1_c(\Gamma, (\Omega_{k+1}/\widetilde{\Omega}^0_{k+2}) \otimes \mathcal{O}_K)$ is non-zero and by Lemma 8.2 this later space is isomorphic to $H^1_c(\Gamma, \mathcal{O}_K/p\mathcal{O}_K(\psi))$ with $\psi = d^{k+2} \cdot \det^{-1}$. Then, twisting away the determinant and reducing mod π, we see that the system of eigenvalues $\{la_l\}$ appears in $H^1_c(\Gamma, \mathcal{O}_K/\pi\mathcal{O}_K(d^{k+2}))$. Then by ?? this system of eigenvalues occurs in $\mathcal{M}_{k+2}(\Gamma_1(Np), \omega^{k+2}, \mathcal{O}_K/\pi\mathcal{O}_K)$. Finally, by [1, Corollary 1.2] this system of eigenvalues appears in $\mathcal{M}_{k+2}(\Gamma_1(Np), \omega^{k+2}, \mathbf{C})$. $\square$

Corollary 8.4. *Let $\phi_{f,\beta}$ be the modular symbol corresponding to f_β in $H^1_c(\Gamma, L_k)^\varepsilon$. If there is no system of eigenvalues in $\mathcal{M}_{k+2}(\Gamma_1(Np), \omega^{k+2}, \mathbf{C})$ congruent to $\{la_l\}$ mod π then there is a unique Hecke-eigensymblol $\Phi_{f,\beta} \in H^1_c(\Gamma, \mathcal{D}(\mathbf{Z}_p) \otimes K)^\varepsilon$ such that $\rho(\Phi_{f,\beta}) = \phi_{f,\beta}$.*

Proof. By Theorem 7.13, there is some eigensymbol $\Phi_{f,\beta} \in H^1_c(\Gamma, \mathcal{D}(\mathbf{Z}_p) \otimes K)^\varepsilon$ with the same Hecke-eigenvalues as $\phi_{f,\beta}$. Then by our assumptions on $\mathcal{M}_{k+2}(\Gamma_1(Np), \omega^{k+2}, \mathbf{C})$ and by Proposition 8.3, $\rho(\Phi_{f,\beta}) \neq 0$.

By Theorem 2.2, any element of $H^1_c(\Gamma, L_k \otimes \mathcal{O}_K)^\varepsilon$ with the same Hecke-eigenvalues as f_β is a scalar multiple of $\phi_{f,\beta}$. So by rescaling $\Phi_{f,\beta}$, we have that $\rho(\Phi_{f,\beta}) = \phi_{f,\beta}$.

For the uniqueness, if there were two such lifts of $\phi_{f,\beta}$ then their difference would be a Hecke-eigensymbol in the kernel of specialization which is not possible again by Proposition 8.3. $\square$

We now prove the uniqueness and existence of a Hecke-eigensymbol $\Phi_{f,\beta}$ lifting $\phi_{f,\beta}$ for $X_0(11)$ and $p = 3$.

Proposition 8.5. *The hypotheses of Corollary 8.4 are satisfied for $X_0(11)$ and $p = 3$.*

Proof. First note that $\mathcal{M}_2(\Gamma_1(33), \omega^2) = \mathcal{M}_2(\Gamma_0(33))$ since ω has order 2. Let f be the modular form corresponding to $X_0(11)$ and let a_l be the l-th Fourier coefficient of f. Since $\{a_l\}$ is not congruent to an Eisenstein series, it suffices to look at $\mathcal{S}_2(\Gamma_0(33))$.

By standard formulas, $\mathcal{S}_2(\Gamma_0(33))$ is 3 dimensional. Let f_α and f_β be the two 3-stabilizations of f to level 33. This accounts for two of the three dimensions. The third dimension comes from a new form of level 33. This form has rational coefficients and is actually congruent to f mod 3 (see [2]). Hence, away from 3 and 11, the only system of eigenvalues that occurs mod 3 in this space is $\{a_l\}$. $\square$

Part II
Computations

9 Explicitly computing with modular symbols

9.1 The Steinberg module as a Γ-module

Recall that in Proposition 2.1, we had that

$$H^1_c(\Gamma, V) \cong \mathrm{Hom}_\Gamma(\Delta_0, V)$$

where $\Gamma = \Gamma_0(Np)$. Thus in order to write down a modular symbol we need to understand the structure of $\Delta_0 = \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ (the Steinberg module) as a Γ-module.

If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbf{Q})$, let us denote by $[\gamma]$ the singular 1-chain in the extended upper half-plane $\mathcal{H}^*$ represented by the geodesic path joining $\frac{a}{c}$ to $\frac{b}{d}$. We will call any such 1-chain a modular path and any finite formal sum of such modular paths, a modular 1-chain. The $\mathbf{Z}$-module of all such modular chians will be denoted by

$$Z_1 = Z_1(\mathcal{H}^*, \mathbf{P}^1(\mathbf{Q})),$$

which we regard as a module of 1-cycles in $\mathcal{H}^*$ relative to the boundary of $\mathbf{P}^1(\mathbf{Q})$ of $\mathcal{H}^*$.

The group $PGL_2^+(\mathbf{Q})$ acts on Z_1 via standard fractional linear transformation on $\mathcal{H}^*$, hence Z_1 is naturally a $PGL_2^+(\mathbf{Q})$-module. If $\beta, \gamma \in GL_2^+(\mathbf{Q})$ then we have

$$\beta \cdot [\gamma] = [\beta\gamma].$$

The boundary map gives us a surjective $GL_2^+(\mathbf{Q})$-morphism

$$\partial : Z_1 \longrightarrow \Delta_0.$$

We say two modular chains c,c' are homologous if $\partial c = \partial c'$. Thus ∂ induces a $PGL_2^+(\mathbf{Q})$-isomorphism from the one-dimensional relative homology of the pair $(\mathcal{H}^*, \mathbf{P}^1(\mathbf{Q}))$ to the Steinberg module Δ_0:

$$\partial : H^1(\mathcal{H}^*, \mathbf{P}^1(\mathbf{Q}); \mathbf{Z}) \xrightarrow{\cong} \Delta_0.$$

Let $G = PSL_2(Z)$. A modular path of the form $[\gamma]$ with $\gamma \in G$ is called a "unimodular path" and any finite formal sum of such unimodular paths is called a unimodular 1-chain. Using continued fractions it is easy to see (and is a well-known result of Manin [**?**]) that every modular chain is homologous to a unimodular chain. Moreover, G acts transitively on the unimodular paths. Indeed, the map

$$G \longrightarrow Z_1$$
$$\gamma \mapsto [\gamma]$$

is a bijection from G to the set of unimodular pahts in Z_1. Extending by linearity, we obtain a G-morphism $\mathbf{Z}[G] \longrightarrow Z_1$, and composing with the boundary map ∂ we obtain a surjective G-morphism

$$e : \mathbf{Z}[G] \longrightarrow \Delta_0.$$

We know from a result of Manin that the kernel of e is the right ideal

$$I = \mathbf{Z}[G](1 + \tau + \tau^2) + \mathbf{Z}[G](1 + \sigma)$$

where $\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. These are the well-known Manin relations.

The Manin relations allow us to describe the structure of Δ_0 as a Γ-module in terms of generators and relations. Note that the map

$$G \longrightarrow P^1(\mathbf{Z}/Np\mathbf{Z})$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{d}{c}$$

is surjective and its fibers are the right Γ-cosets. We fix a choice of a section

$$g : \mathbf{P}^1(\mathbf{Z}/Np\mathbf{Z}) \longrightarrow G$$

of this map such that $g(\infty)$ equals the identity matrix. In this way, the set $\{g(v)\}$ as v varies over $\mathbf{P}^1(\mathbf{Z}/Np\mathbf{Z})$ gives a complete set of right coset representatives of $\Gamma\backslash G$.

We have that $\mathbf{Z}[G]$ is a free Γ-module generated by the $g(v)$. Of course though, the Steinberg module is not free as we have to consider the Manin relations given by the right ideal I. Namely, for each $v \in P^1(\mathbf{Z}/Np\mathbf{Z})$ we have that

$$e(g(v) + g(v)\tau + g(v)\tau^2) = e(g(v) + g(v)\sigma) = 0.$$

For any $\eta \in G$, we can write $g(v)\eta = \eta' g(v')$ for some $\eta' \in \Gamma$ and $v' \in P^1(\mathbf{Z}/Np\mathbf{Z})$. Hence, we see that the images of the $g(v)$ under e in Δ_0 satisfy many two and three term relations with coefficients in Γ.

In practice it is probably possible to solve these relations when Γ is torsion free. We consider the case when $N = 11$ and $p = 3$. Then among the $48 = (3+1)(11+1)$ values of $g(v)$ that generate Δ_0, there is a subset of 9 elements that span and satisfy a single relation. Namely, one can find $v_1, \ldots, v_t, \infty \in \mathbf{P}^1(\mathbf{Z}/Np\mathbf{Z})$ such that

$$e(g(v_1)), \ldots, e(g(v_t)), e(g(\infty)) \ \text{ generate } \ \Delta_0$$

and satisfy a unique $\mathbf{Z}[\Gamma]$-relation. (Here $t = 8$.) Let $D_i = e(g(v_i))$ and $D_\infty = e(g(\infty)) = \{0\} - \{\infty\}$. Then the single $\mathbf{Z}[\Gamma]$-relation is of the form

$$\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - 1 \right) D_\infty + \sum_{i=0}^{t} (\gamma_i - \delta_i)\, D_i = 0$$

with $\gamma_i, \delta_i \in \Gamma$ for each i.

The fact that Δ_0 is generated over Γ by certain elements that satisfy one relation as above seems to occur more generally than just when $N = 11$ and $p = 3$. This seems to happend whenever Γ is torsion free. We state this as a hypothesis that we will assume for the remainder of the paper.

Condition (TF): There exist $v_1, \ldots, v_t, infty \in P^1(\mathbf{Z}/Np\mathbf{Z})$ such that

$$e(g(v_1)), \ldots, e(g(v_t)), e(g(\infty)) \ \text{ generate } \ \Delta_0$$

and a satisfy a unique $\mathbf{Z}[\Gamma]$-relation:

$$\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - 1 \right) D_\infty + \sum_{i=0}^{t} (\gamma_i - \delta_i) D_i = 0$$

where $D_i = e(g(v_i))$ and $\gamma_i, \delta_i \in \Gamma$ for each i.

Thus, assuming (TF), a modular symbol $\phi \in H^1_c(\Gamma, M)$ (viewed as a Γ-invariant homomorphism from the Steinberg module to M) is uniquely determined by its values on D_i for $i = 0, \ldots, r, \infty$. Here M is any right $S_0(p)$-module. Conversely, consider a collection of $m + 1$ elements of M, say $m_1, \ldots, m_t, m_\infty$ satisfying

$$m_\infty | \Delta = \sum_{i=0}^{t} m_i | \left(\delta_i^{-1} - \gamma_i^{-1} \right)$$

where Δ is the difference operator $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1$. Then there exists a unique modular symbol $\phi \in H^1_c(\Gamma, M)$ such that

$$\phi(m_i) = D_i$$

for $i = 1, \ldots, r, \infty$. (Recall that $\phi(\gamma D) = \phi(D)|\gamma^{-1}$.)

Note that once the values m_i are known, the continued fraction algorithm of Manin allows one to compute the associated ϕ on any $D \in \Delta_0$.

Remark 9.1. With the description of $H^1_c(\Gamma, M)$ given in this section, one could easily represent an M-valued modular symbol on a computer if one is able to represent elements of M on a computer. Namely, for $\phi \in H^1_c(\Gamma, M)$, one stores ϕ as the list of $t + 1$ elements: $\phi(D_1), \ldots, \phi(D_t), \phi(D_\infty)$.

9.2 Solving the difference equation

From the results in the previous section, we see that in order to build modular symbols in $H^1_c(\Gamma, M)$ one must be able the solve the difference equation

$$u | \Delta = v$$

for a given $v \in M$. We begin by discussing the case where $M = L_k$.

Proposition 9.2. *For each non-zero $g \in L_k$ with $\deg(g) < k$ there exists an $f \in L_k$ such that*
$$f|\Delta = g.$$
Moreover, f is unique up to the addition of a constant.

Proof. First note that if $h \in L_k$ and $h|\Delta = 0$ then $h(x-1) = h(x)$ and thus h is a constant. Therefore, we have

$$0 \longrightarrow \mathbf{Q}_p \longrightarrow L_k \xrightarrow{\Delta} L_k \longrightarrow \operatorname{coker}(\Delta) \longrightarrow 0$$

where $\operatorname{coker}(\Delta)$ is one dimensional over $\mathbf{Q}_p$. Moreover, directly from the definition of acting by Δ, one sees that $\operatorname{im}(\Delta) \subseteq L_{k-1}$. Since L_{k-1} is of codimension 1 in L_k, we must have that $\operatorname{im}(\Delta) = L_{k-1}$ which is the content of the first part of the proposition. The second follows since the kernel of Δ is exactly the set of constants. $\square$

We next consider the case of $M = \mathcal{D}^\dagger(\mathbf{Z}_p, 1) \cong \Omega_{\log}(W)$.

Lemma 9.3. *We have that*

$$\ker(\Delta : \mathcal{D}^\dagger(\mathbf{Z}_p, 1) \longrightarrow \mathcal{D}^\dagger(\mathbf{Z}_p, 1)) = 0.$$

Proof. Let $\mu \in \ker(\Delta)$ and let n be the smallest non-negative integer such that $\mu(x^n) \neq 0$. Then by assumption $\mu((x-1)^{n+1}) = \mu(x^{n+1})$. (Note that the weight k action of Δ is the same as its weight 0 action). We then have that

$$\mu(x^{n+1}) = \mu(x^{n+1}) + (-1)^{n+1}(n+1)\mu(x^n)$$

and thus $\mu(x^n) = 0$. This contradiction implies that μ is identically zero. $\square$

Lemma 9.4. *We have that*

$$\ker(\Delta : \Omega_{log}(W) \longrightarrow \Omega_{log}(W)) = 0.$$

Proof. Let
$$\omega = a_0\delta_0 + \sum_{r=0}^{\infty} a_r z^{-r}\frac{dz}{z} \in \ker(\Delta)$$
and thus
$$\omega \,\Bigg|\, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = a_0\delta_0 + \sum_{r=0}^{\infty} a_r z^{-r}\frac{dz}{z}.$$
Assume that n is the smallest non-negative integer such that $a_n \neq 0$. Then since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ preserves Ω_n, from the formulas of Lemma 5.6, we see that

$$a_{n+1} = a_{n+1} + (n+1)a_n.$$

Thus $a_n = 0$ and ω is identically zero. $\square$

We begin by solving the difference equation $\mu\big|_k\Delta = \nu$ for ν of the form $z^{-j}dz/z$. (Since the weight k action of Δ is the same as the weight 0 action of Δ, we will simply write $\mu\big|\Delta = \nu$.)

Lemma 9.5. *Let*

$$\eta_j = \begin{cases} \displaystyle\sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} z^{-r} \frac{dz}{z} & j \neq 0 \\ \displaystyle\delta_0 + \sum_{r=1}^{\infty} b_r z^{-r} \frac{dz}{z} & j = 0 \end{cases}$$

where b_r is the r-th Bernoulli number. Then $\eta_j \in \Omega_{log}(W)$ and

$$\eta_j|\Delta = \frac{j+1}{z^{j+1}} \frac{dz}{z}.$$

Proof. By the von Staudt-Clausen theorem, $pb_n \in \mathbf{Z}_p$ for each n. Thus $p\eta_j \in \Omega_0$ and η_j is in $\Omega_{\log}(W)$.

As for the second part, for $j > 0$ we compute:

$$\begin{aligned}
\eta_j \Bigg| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} z^{-r} \frac{dz}{z} \Bigg| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} (z-1)^{-r} \frac{dz}{z-1} \\
&= \sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} z^{-r} (1-z^{-1})^{-r-1} \frac{dz}{z} \\
&= \sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} z^{-r} \left(\sum_{m=0}^{\infty} \binom{-r-1}{m} (-1)^m z^{-m} \right) \frac{dz}{z} \\
&= \sum_{r=j}^{\infty} \sum_{m=0}^{\infty} \binom{r}{j} \binom{r+m}{m} b_{r-j} z^{-r-m} \frac{dz}{z} \\
&= \sum_{s=j}^{\infty} \left(\sum_{r=j}^{s} \binom{r}{j} \binom{s}{r} b_{r-j} \right) z^{-s} \frac{dz}{z}.
\end{aligned}$$

But we have the following identity of Bernoulli numbers:

$$\sum_{r=j}^{n} \binom{n}{r} \binom{r}{j} b_{r-j} = \begin{cases} \binom{n}{j} b_{n-j} & n \neq j+1 \\ \binom{n}{j} b_{n-j} + (j+1) & n = j+1 \end{cases}.$$

Thus

$$\eta_j|\Delta = \frac{j+1}{z^{j+1}} \frac{dz}{z}$$

as claimed. $\square$

Theorem 9.6. *For any $\nu \in \Omega_{log}(W)$ of total measure zero, there exists a unique $\mu \in \Omega_{log}(W)$ such that*

$$\mu|\Delta = \nu.$$

Proof. Let

$$\nu = \sum_{m=1}^{\infty} a_m z^{-m} \frac{dz}{z}.$$

Then consider

$$\mu = \sum_{m=1}^{\infty} \frac{a_m}{m} \eta_{m-1}.$$

Since $\eta_j \in \Omega_{\log}(W)$, by Proposition 4.1, the coefficients of μ grow slowly enough so that $\mu \in \Omega_{\log}(W)$. Now from Lemma 9.5, it is clear that $\mu|\Delta = \nu$. □

Remark 9.7. Note that the proof of Theorem 9.6 is completely explicit.

9.3 Explicitly lifting modular symbols

We will now use Theorem 9.6 to give an explicit proof that

$$H^1_c(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1)) \xrightarrow{\rho_k} H^1_c(\Gamma, L_k)$$

is surjective. Take $\phi \in H^1_c(\Gamma, L_k)$. By the results of section 9.1, we know that ϕ is determined by its values on $D_1, \ldots, D_t, D_\infty$ and moreover if $m_i = \phi(D_i)$ and $m_\infty = \phi(D_\infty)$ then

$$m_\infty|\Delta = \sum_{i=0}^{t} m_i| \left(\delta_i^{-1} - \gamma_i^{-1}\right).$$

We begin by lifting each $m_i \in L_k$ to a distribution. We do this in the simplest possible way; namely, if

$$m_i = \sum_{j=0}^{k} a_j Z^j$$

set

$$\nu_i = \sum_{j=0}^{k} (-1)^j \binom{k}{j}^{-1} a_{k-j} z^{-j} \frac{dz}{z}.$$

Then $\rho_k(\nu_i) = m_i$ and it is the unique such distribution $\sum_j c_j z^{-j} dz/z$ with this property such that $c_j = 0$ for $j > k$.

Now let

$$\nu = \sum_{i=1}^{t} \nu_i| \left(\delta_i^{-1} - \gamma_i^{-1}\right).$$

Clearly, ν has total measure zero and thus Theorem 9.6 applies. Namely, there exists a unique $\mu \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ such that

$$\mu|\Delta = \nu.$$

Again using the results of section 9.1, we can form a modular symbol $\Phi \in H^1_c(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1))$ such that $\Phi(D_i) = \nu_i$ and $\Phi(D_\infty) = \mu$. We would like to say then that Φ is a lifting of ϕ under ρ. Unfortuntately, we do not yet have enough control on the value of $\rho(\Phi(D_\infty))$. We know that

$$\begin{aligned}\rho(\Phi(D_\infty))|\Delta = \rho(\mu)|\Delta = \rho(\mu|\Delta) = \rho(\nu) = \rho\left(\sum_{i=1}^t \nu_i| \left(\delta_i^{-1} - \gamma_i^{-1}\right)\right) \\ = \sum_{i=1}^t \rho(\nu_i)| \left(\delta_i^{-1} - \gamma_i^{-1}\right) = \sum_{i=1}^t m_i| \left(\delta_i^{-1} - \gamma_i^{-1}\right) = m_\infty|\Delta.\end{aligned}$$

However, Δ has a kernel on L_k and since $(\rho(\Phi(D_\infty)) - m_\infty)|\Delta = 0$ we can only conclude that $\rho(\Phi(D_\infty))$ and m_∞ differ by a constant in $\mathbf{Q}_p$. In fact, there is no reason to expect that this constant is zero. To succeed in lifting ϕ, we need to make a different choice of ν_i lifting m_i.

The μ that we constructed has all the correct coefficients of $z^{-j}dz/z$ for j between 0 and $k-1$ to be a lift of m_∞. It is only the coefficient of $z^{-k}dz/z$ that is a problem. The following lemma describes this $z^{-k}dz/z$ coefficient of μ in terms of the coefficients of ν.

Lemma 9.8. *If $\nu = \sum_j a_j z^{-j} dz/z$ and $\mu|\Delta = \nu$ then the coefficient of $z^{-k}dz/z$ in μ is given by*

$$\sum_{j=0}^k \frac{a_{j+1}}{j+1}\binom{k}{j} b_{k-j}.$$

In particular, it depends only on $a_1, \ldots, a_{k+1}$.

Proof. This lemma is immediate from the explicit nature of the construction of μ given in Theorem 9.6. $\square$

Let us now consider what happens to the coefficients of ν if we change one of our chosen lifts ν_i. Note that we can only change each ν_i in such a way that their specialization to L_k is unaffected. The simplest way to do this is to replace ν_1 with $\nu_1 + cz^{-(k+1)}dz/z$ with c some element of $\mathbf{Q}_p$. So let $\nu_1' = \nu_1 + cz^{-(k+1)}dz/z$ and for i between 2 and r, let $\nu_i' = \nu_i$. Let $\nu' = \sum_{i=1}^t \nu_i'| \left(\delta_i^{-1} - \gamma_i^{-1}\right)$ and μ' be the unique solution of $\mu'|\Delta = \nu'$. The following lemma will be useful in computing the coefficients of ν' in terms of the coefficient of ν.

Lemma 9.9. *If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$ and if*

$$z^{-(k+1)} \frac{dz}{z}\bigg|_k \gamma = \sum_{j=0}^\infty a_j z^{-j} \frac{dz}{z}$$

then

$$a_j = \begin{cases} 0 & 0 \le j \le k \\ \displaystyle\sum_{m=0}^{k+1} (-1)^m a^{-(m+1)} b^m c^m d^{k+1-m} & j = k+1 \end{cases}.$$

Proof. Since Ω_{k+1} is a $\Sigma_0(p)$-module, it is clear that $a_j = 0$ for $j \leq k$. The coefficient a_{k+1} can be computed directly from the definitions as in Proposition 5.5. $\square$

Thus, from Lemma 9.9, we see that if $\nu = \sum_j a_j z^{-j} dz/z$ and $\nu' = \sum_j a'_j z^{-j} dz/z$ then for j between 1 and k we have $a'_j = a_j$. Furthermore, $a'_{k+1} = a_{k+1} + c\delta$ where δ is some constant that depends only on γ_1 and η_1. Therefore, by judicious choice of c, we can force a'_{k+1} to take on any possible value.

By Lemma 9.8, the coefficient of $z^{-k} dz/z$ in μ' is

$$\sum_{j=0}^{k} \frac{a'_{j+1}}{j+1} \binom{k}{j} b_{k-j} = \frac{a'_{k+1}}{k+1} + \sum_{j=0}^{k-1} \frac{a_{j+1}}{j+1} \binom{k}{j} b_{k-j}.$$

Thus by altering the constant c appropriately, we can alter the value of a'_{k+1} appropriately and thus force the coefficient of $z^{-k} dz/z$ in μ' to equal the constant term of m_∞. This fact together with the above argument about the kernel of Δ, proves that we can choose c so that $\rho(\mu') = m_\infty$. Then the modular symbol Φ defined by $\Phi(D_i) = \nu'_i$ for i between 1 and r and $\Phi(D_\infty) = \mu'$ is indeed a lift of ϕ.

10 Finite approximation modules

10.1 Approximating distributions

We now need to describe a method of representing distributions on a computer so that we can perform explicit computations in $H^1_c(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1))$. Of course though, any given distribution contains an infinite amount of data. One first guess on how to store an approximation to $\omega \in \Omega_{\log}(W)$ on a computer would be to fix two integers M and N and stored the first M coefficients of ω mod p^N. Unfortunately, these approximations are not stable under the action of $\Sigma_0(p)$ and so we have to proceed a differently.

Let

$$\mathcal{K}_0 = \left\{ \omega = \sum_{j=0}^{\infty} a_j z^{-j} \frac{dz}{z} \mid p^j a_j \in \mathbf{Z}_p \right\} \subseteq \Omega_{\log}(W(\mathbf{Z}_p, p)).$$

Proposition 10.1. *We have that $\mathcal{K}_0$ is a $\Sigma_0(p)$-module.*

Proof. We need to check that $p^{-j}\omega_j|_k\gamma \in \mathcal{K}_0$ for $\gamma \in \Sigma_0(p)$ and $\omega_j = z^{-j} dz/z$. Knowing this is equivalent to knowing that the coeffcient of $z^{-n} dz/z$ of $p^{n-j}\omega_j|_k\gamma$ is in $\mathbf{Z}_p$. For $n \geq j$ there is nothing to check since all the coefficients of $\omega_j|_k\gamma$ are integral. For $n < j$ this is true since $\omega_j \in \Omega_j$ and thus $\omega_j|_k\gamma \in \Omega_j$. $\square$

Definition 10.2. For $M > 0$, define the M-th *finite approximation module* of Ω_0 to be

$$\mathcal{F}(M) := (\Omega_0 + p^M \mathcal{K}_0)/\Omega_0 \cong \Omega_0/(\Omega_0 \cap p^M \mathcal{K}_0).$$

Proposition 10.3. *We have that $\mathcal{F}(M)$ is a $\Sigma_0(p)$-module and that*

$$\mathcal{F}(M) \cong (\mathbf{Z}/p^M\mathbf{Z}) \times (\mathbf{Z}/p^{M-1}\mathbf{Z}) \cdots \times (\mathbf{Z}/p\mathbf{Z})$$

where the map is given by

$$\overline{\omega} \mapsto (a_0 + p^n\mathbf{Z}_p, a_1 + p^{M-1}\mathbf{Z}_p, \dots, a_{M-1} + p\mathbf{Z}_p)$$

where $\omega = \sum_j a_j z^{-j} dz/z \in \Omega_0 + p^M \mathcal{K}_0$.

Proof. By Proposition 10.1, we know that $\mathcal{F}(M)$ is a $\Sigma_0(p)$-module. As for the isomorphism, first note that the above map makes sense since if $\omega \in \Omega_0 + p^M\mathcal{K}_0$ then $a_j \in \mathbf{Z}_p$ for j between 0 and M. With this said, checking that the map is an isomorphism is straightforward. $\square$

Proposition 10.3 tells us that $\mathcal{F}(M)$ is actually a finite set which is easily represented on a computer. So for a given element $\mu \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ we can project μ onto $\mathcal{F}(M)$ and then store its image in $\mathcal{F}(M)$ as a sequence of integers mod various powers of p.

With this description of $\mathcal{F}(M)$ in hand, we can now represent the space $H^1_c(\Gamma, \mathcal{F}(M))$ on a computer and elements of this space can be considered as approximations to overconvergent modular symbols.

10.2 Solving the difference equation in $H^1_c(\Gamma, \mathcal{F}(M))$

To construct modular symbols in $\mathcal{F}(M)$, we must be able to solve the difference equation for elemnts in $\mathcal{F}(M)$. Some new difficulties appear in this case because in the construction of μ troublesome denominators appear. For example, even if ν is in Ω_1 the associated μ will not even be an element of $\Omega_0 + p^M\mathcal{K}_0$. To fix this, we must scale by a power of p that is small relative to M.

Lemma 10.4. *Let $\mu \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ such that $\mu|\Delta \in \Omega_0$. If $m, M \geq 0$ are integers for which $p^m > M + 1$ then*

$$p^m\mu \in \Omega_0 + p^M\mathcal{K}_0.$$

Proof. It suffices to prove that if $p^m > M + 1$ then for all $j \geq 0$

$$p^m \cdot \frac{1}{j+1} \cdot \eta_j \in \Omega^0 + p^M\mathcal{K}_0.$$

For this, it suffices to show that for all integers n, j with $n \geq j \geq 0$,

$$\frac{p^m}{j+1}\binom{n}{j} b_{n-j} \in \mathbf{Z}_p \quad \text{if} \quad n < M$$
$$p^n \cdot \frac{p^m}{j+1}\binom{n}{j} b_{n-j} \in p^M\mathbf{Z}_p \quad \text{if} \quad n \geq M$$

If $n < M$ then also $j < M$, hence $p^m > j + 1$ and it follows that $\mathrm{ord}_p(p^m/(j + 1)) \geq 1$, so the first assertion follows from the Clausen-von Staudt theorem. On the other hand, if $n \geq M$, then $n = M + r$ with $r \geq 0$, so $p^{m+r} > M + 1 + r = n + 1$. Thus $\mathrm{ord}_p(p^{m+r}/(j+1)) \geq 1$ for every $j \geq 0$ with $j \leq n$. Again, from the Clausen-von Staudt theorem it follows that

$$\frac{p^{m+r}}{j+1}\binom{n}{j} b_{n-j} \in \mathbf{Z}_p$$

and consequently that

$$p^n \cdot \frac{p^m}{j+1}\binom{n}{j} b_{n-j} = p^M \cdot \frac{p^{m+r}}{j+1}\binom{n}{j} b_{n-j} \in p^M \mathbf{Z}_p.$$

$\square$

Corollary 10.5. *If $\widetilde{\nu} \in p^m \mathcal{F}(M)$ of total measure zero such that $p^m > M + 1$ then there exists $\widetilde{\mu} \in \mathcal{F}(M)$ such that $\widetilde{\mu}\big|_k \Delta = \widetilde{\nu}$.*

Proof. First lift $p^{-m}\widetilde{\nu}$ to some element ν of Ω_0. Then solving the difference equation for ν yields some μ such that $\mu|_k\Delta = \nu$. By Lemma 10.4, we have that $p^m\mu \in \Omega_0 + p^M\mathcal{K}_0$. Let $\widetilde{\mu}$ be the image of $p^m\mu$ in $\mathcal{F}(M)$. Then $\widetilde{\mu}\big|_k\Delta = \widetilde{\nu}$ as desired. $\square$

10.3 Lifting classical modular symbols to $H^1_c(\Gamma, \mathcal{F}(M))$

We begin by writting down a way of representing L_k on a computer (compatible with our representation of $\mathcal{D}^\dagger(\mathbf{Z}_p, 1)$). For $M > 0$ define,

$$L_k(M) = \left\{ f = \sum_{j=0}^{k} a_j Z^k \mid a_j \in \mathbf{Z}/p^{M-j+e_j}\mathbf{Z} \right\}$$

where $e_j = \mathrm{ord}_p\left(\binom{k}{j}\right)$. The reason for the e_j terms is that under the specialization map ρ_k the coefficients of the given distribution are scaled by certain binomial coefficients and the extra powers of p that appear are being accounted for.

The specialization map reduces to give a map $\mathcal{F}(M) \xrightarrow{\rho_k} L_k(M)$ defined by

$$\rho_k\left(\sum_j a_j z^{-j} dz/z\right) = \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}(a_{k-j} + p^{M-j+e_j}\mathbf{Z}_p)Z^j$$

and it is with respect to this map that we will be lifting our classical modular symbols.

Theorem 10.6. *Let $\widetilde{\phi} \in p^m H^1_c(\Gamma, L_k(M))$ for some integers $m \geq 0$. Then if $p^m > M + 1$ there exists a modular symbol $\widetilde{\Phi} \in H^1_c(\Gamma, \mathcal{F}(M))$ such that $\rho_k(\widetilde{\Phi}) = \widetilde{\phi}$.*

Proof. To prove this we can repeat the arguments of section 9.3 verbatim making use of Corollary 10.5. Note that the assumptions of Corollary 10.5 are satisfied since $\widetilde{\phi}$ is divisible by p^m. $\square$

10.4 Defining $p^{-(k+1)}U_p$ exactly

In what follows, we are going to need to apply the operator $u := U_p/p^{k+1}$ many times to project our lifted symbol to the "slope k+1" subspace of $H^1_c(\Gamma, \mathcal{F}(M))$. In this section, we will describe how one can exactly compute this operator in $H^1_c(\Gamma, \mathcal{F}(M))$ when applied to certain symbols that lift a constant multiple of a β-eigensymbol for U_p. (Note that the trouble in defining this operator is to make sense of dividing by p^{k+1}.)

Let Ψ be some symbol in $H^1_c(\Gamma, \mathcal{F}(M))$ that lifts a β-eigensymbol in $H^1_c(\Gamma, L_k(M))$ and that satisfies the following condition:

$$\text{For every } D \in \Delta_0, \text{ if } \Psi(D) = \sum_j a_j z^{-j} \frac{dz}{z} \text{ then } a_j \in p^{k+1-j}\mathbf{Z}_p \text{ for } j \leq k.$$

We will be able to precisely compute the u-operator on symbols of this type.

For $D \in \Delta_0$ we have

$$(\Psi|U_p)(D) = \sum_{a=0}^{p-1} \Psi\left(\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right) D\right) |\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right) =: \sum_{j=0}^{M-1} c_j z^{-j} \frac{dz}{z}.$$

From the data of Ψ, for j between 0 and k we can compute $p^{-(k+1)}a_j \bmod p^{M-j}$ since

$$p^{-(k+1)}\rho(\Psi|U_p)(D) = p^{-(k+1)}(\rho(\Psi)|U_p)(D) = p^{-(k+1)}\beta\rho(\Psi)(D)$$

since $p^{-(k+1)}\beta \in \mathbf{Z}_p$.

For $j > k$, to deterine c_j we need to consider the contributions from each term of $\Psi\left(\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right) D\right) |\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right) =: \sum_r d_r z^{-r} dz/z$. We have

$$d_r z^{-r} \frac{dz}{z} |\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right) = d_r p^r (1 - az^{-1})^{-(r+1)} z^{-r} \frac{dz}{z}.$$

For $r > k$ we can compute this expression exactly by simply replacing the p^r term in the front of the formula to $p^{r-(k+1)}$. For $r \leq k$, we know by assumption that d_r is divisible by p^{k+1-r}. Thus, we can replace $d_r p^r$ at the start of the formula with d_r/p^{k+1-r} to successfully divide by p^{k+1}. Note that d_r is determined mod p^{M-r} and thus d_r/p^{k+1-r} is only determined mod $p^{M-(k+1)}$. However, since $j > k$ and we are only looking to compute a_j mod p^{M-j}, the above computation is accurate enough.

11 Explicitly forming a β Hecke-eigensymbol

The goal of this section is to give an algorithm that produces a Hecke-eigensymbol in $H^1_c(\Gamma, \mathcal{F}(M))$ with the same eigenvalues as $\phi_{f,\beta}$. As long as the hypotheses of Theorem 8.4 are satisfied then we know that up to scaling there is a unique such symbol in $H^1_c(\Gamma, \mathcal{F}(M))$ and any such symbol is the projection of some bonafide eigensymbol in $H^1_c(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1))$.

Let us begin by recalling the steps of constructing this β eigensymbol in $H_c^1(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1))$. First, we form a lift of $\phi_{f,\beta}$, say Φ. Then we project Φ to Φ^0, it's image in the "slope $k+1$" subspace. Since this is a finite dimensional space, linear algebra techiniques can then be used to form the desired eigensymbol.

For $\mathcal{F}(M)$, the first step of lifting $\phi_{f,\beta}$ requires some care because denominators are introduced. We need to be certain that we satisfy the hypothesis of Corollary 10.5. Namely, instead of lifting $\phi_{f,\beta}$ we will lift $p^m \phi_{f,\beta}$ for m satisfying $p^{m-||\phi_{f,\beta}||} > M+1$. Denote the lifted symbol in $H_c^1(\Gamma, \mathcal{F}(M))$ by $\widetilde{\Phi}$.

To project onto the slope $k+1$ subspace, we will iterate the u-operator on $\widetilde{\Phi}$. Since, u is topologically nilpotent on Φ^{nil} and p^M kills $\mathcal{F}(\mathcal{M})$, we have that for n large enough, the image of $\Phi^0|u^n$ in $\mathcal{F}(M)$ equals $\widetilde{\Phi}|u^n$. (In practice, it suffices to apply u only M times. However, if there was some piece of Φ with eigenvalue of slope bigger than $k+1$ and smaller than $k+2$ it is plausible that more applications of u would be necessary.)

Let $\widetilde{\Phi}' = \widetilde{\Phi}|u^n$ for n large enough. Then since the slope $k+1$ subspace of $H_c^1(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1))$ is finite dimensional, we can view the Hecke-stable subspace generated by $\widetilde{\Phi}'$ as having dimension small compared to M. At this point, basic methods of linear algebra can be used to produce the desired eigensymbol. In practice, the dimension of the slope $k+1$ subspace has been very small (no bigger than 4) and so in particular it is smaller than $t+1$ (the number of divisors needed to determine a modular symbol). The following procedure has been quite effective in producing our eigensymbol.

Let $\widetilde{\Psi} = \widetilde{\Phi}'|U_p - \beta\widetilde{\Phi}'$ which is in the kernel of specialization. We want to kill off $\widetilde{\Psi}$ using the full Hecke algebra. Take T_l for some prime l and apply it many times to $\widetilde{\Psi}$. Since $\widetilde{\Psi}$ lives in a small dimensional Hecke-stable subspace, the elements $\{\widetilde{\Psi}|T_l^j\}$ should posses a linear relation. To find this relation, look at the coefficient of $z^{-(k+1)}dz/z$ in these symbols evaluated at each of the divisors $D_1, \ldots, D_t, D_\infty$. For each $\widetilde{\Psi}|T_l^j$, this list of coefficients gives us an element of $(\mathbf{Z}/p^{M-k-1}\mathbf{Z})^{r+1}$. Call this element v_j. The elements $v_0, v_1, \ldots, v_d$ should then posses a relation $\sum_{j=0}^d c_j v_j = 0$ for some small value of d.

Let $f(T) = \sum_{j=0}^d c_j T^j$ and consider the symbol $\widetilde{\Psi}' := \widetilde{\Psi}|f(T_l)$. By construction, $\widetilde{\Psi}'$ has the property that $\widetilde{\Psi}'(D)$ has its coefficient of $z^{-(k+1)}dz/z$ equal to 0 for all D. If Ψ' is some lift of $\widetilde{\Psi}'$ in $H_c^1(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1))$, then we have that $\Psi' \in H_c^1(\Gamma, \Omega_{k+2})$. Thus by Lemma 7.6, Ψ' has slope at least $k+2$. However, since Ψ' is in the slope $k+1$ subspace it must be the zero symbol. So in fact $\widetilde{\Psi}' = 0$.

Therefore, by construction, $\widetilde{\Phi}_\beta := \widetilde{\Phi}'|f(T_l)$ is a β-eigensymbol for U_p. Note that $\widetilde{\Phi}_\beta$ still specializes to a multiple of $\phi_{f,\beta}$. However, it is now possible that this multiple is zero, if for instance $f(a_l) = 0$. Also, it is not always possible to choose l such $f(a_l) \neq 0$. (There will be no such possible choice if there is an element of the kernel of specialization with the same eigenvalues as f.)

Now, it is plausible at this point that $\widetilde{\Phi}_\beta$ is a β-eigensymbol for U_p, but not a eigensymbol for the whole Hecke algebra. To remedy this, consider $\widetilde{\Psi}_q :=$

$\widetilde{\Phi}_\beta | T_q - a_q \widetilde{\Phi}_\beta$ for some prime $q \neq p$. Repeat, the process described above to kill off $\widetilde{\Psi}_q$. Repeat this for as many q's as necessary until one has a eigensymbol for the full Hecke-algebra. (This is a finite process since we are working in a finite dimensional subspace.)

12 Computing p-adic L-functions

12.1 p-adic L-functions of overconvergent modular symbols

If $\Phi \in H^1_c(\Gamma, \mathcal{D}^\dagger(\mathbf{Z}_p, 1))$ is a β-eigensymbol for U_p, then by Theorem 6.3, Φ descends to an element fo $H^1_c(\Gamma, \mathcal{D}_k(\mathbf{Z}_p))$. We then define the p-adic L-function of Φ to be $\mu_\Phi := \Phi(\{0\} - \{\infty\}) \in \mathcal{D}_k(\mathbf{Z}_p)$.

By choosing a generator of γ of $1 + p\mathbf{Z}_p$, we can express μ_Φ as a power series in $\mathcal{A}(\mathbf{Z}_p, 1)$. Namely, for u in the open unit disc of $\mathbf{C}_p$ around 1, let χ_u be a character of $\mathbf{Z}_p^\times$ defined by first projecting to the 1-units and then mapping γ to u. Then the function

$$L(\mu_\Phi, u) := \int_{\mathbf{Z}_p^\times} \chi_u d\mu_\Phi$$

is analytic as a function of u and its Taylor series expansion around $u = 1$ gives us a power series representation of μ_Φ which we will denote as $L(\mu_\Phi, T) \in \mathcal{A}(\mathbf{Z}_p, 1)$.

Very explicitly, for $m \geq 1$ let

$$R_m(u) = \sum_{a=1}^{p-1} \sum_{j=1}^{p^{m-1}-1} \sum_{i=0}^{k+1} \frac{\chi_u^{(i)}(\{a\}\gamma^j)}{i!} \int_{\{a\}\gamma_j + p^n \mathbf{Z}_p} (x - \{a\}\gamma^j)^i d\mu_\Phi$$

where $\{a\}$ is the Teichmuller lift of a to $\mathbf{Z}_p^\times$ and $\chi_u^{(i)}$ is the i-th derivative of the character χ_u viewed as a locally analytic function on $\mathbf{Z}_p^\times$. Thus $R_m(u)$ is the m-th (enhanced) Riemann sum approximation of $\int_{\mathbf{Z}_p^\times} \chi_u d\mu_\Phi$. Computing the derivatives of χ and replacing u by $1 + T$ yields

$$\begin{aligned} R_m(1+T) &= \sum_{a=1}^{p-1} \sum_{j=1}^{p^{m-1}-1} \sum_{i=0}^{k+1} \frac{(L)(L-1)\dots(L-i+1)}{i!\{a\}^i \gamma^{ij}} (1+T)^j \cdot \\ &\qquad \left(\int_{\{a\}\gamma_j + p^n \mathbf{Z}_p} \sum_{r=0}^{i} \binom{i}{r} (-1)^{i-r} \{a\}^{i-r} \gamma^{j(i-r)} x^r d\mu_\Phi \right) \\ &= \sum_{a=1}^{p-1} \sum_{j=1}^{p^{m-1}-1} \sum_{i=0}^{k+1} \frac{(L)(L-1)\dots(L-i+1)}{i!} (1+T)^j \cdot \\ &\qquad \left(\sum_{r=0}^{i} \binom{i}{r} (-1)^{i-r} (\{a\}\gamma^j)^{-r} \left(\int_{\{a\}\gamma_j + p^n \mathbf{Z}_p} x^r d\mu_\Phi \right) \right) \end{aligned}$$

where $L = \log_\gamma(1+T) = \log_p(1+T)/\log_p(\gamma)$.

We then have that

$$L(\mu_\Phi) = \lim_{m\to\infty} R_m(1+T).$$

Hence, to obtain approximations to the p-adic L-function, we need to be able to compute the values of

$$\int_{\{a\}\gamma_j + p^n\mathbf{Z}_p} x^r d\mu_\Phi$$

in terms of Φ. The following lemmas describe how to do this.

Lemma 12.1. *Let $\mu \in \mathcal{D}_k(\mathbf{Z}_p)$. Then the support of $\mu\big|\left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right)$ is contained in $a + p^n\mathbf{Z}_p$.*

Proof. Let f_b be the characteristic function of $b + p^n\mathbf{Z}_p$. Then

$$(\mu|\left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right))(f_b(x)) = \mu(f_b(a+p^n x)) = \begin{cases}\mu(\mathbf{Z}_p) & a \equiv b \pmod{p^n}\\ 0 & a \not\equiv b \pmod{p^n}\end{cases}$$

. $\square$

Lemma 12.2. *If Φ is a β-eigensymbol for U_p then*

$$\int_{a+p^n\mathbf{Z}_p} x^r d\mu_\Phi = \beta^{-n}(\mu_{\Phi,a}|\left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right))(x^r)$$

where $\mu_{\Phi,a} = \Phi(\{\frac{a}{p^n}\} - \{\infty\})$.

Proof. Since Φ is a β-eigensymbol, we have that

$$\begin{aligned}\beta^n\Phi(\{0\}-\{\infty\})(x^r) &= (\Phi|U_p^n)(\{0\}-\{\infty\})(x^r)\\ &= \sum_{a=0}^{p-1}(\Phi\left(\left\{\frac{a}{p^n}\right\} - \{\infty\}\right)|\left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right))(x^r).\end{aligned}$$

By the previous lemma, the support of $\Phi\left(\left\{\frac{a}{p^n}\right\} - \{\infty\}\right)|\left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right)$ is contained in $a + p^n\mathbf{Z}_p$. Therefore, since

$$\Phi(\{0\}-\{\infty\})(x^r) = \sum_{a=0}^{p-1}\int_{a+p^n\mathbf{Z}_p} x^r d\mu_\Phi,$$

matching the two equal sums up term-by-term yields the lemma. $\square$

Lemma 12.3. *We have that*

$$(\mu_{\Phi,a}\big|\left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right))(x^r) = \sum_{j=0}^{r}\binom{r}{j}a^{r-j}p^{nj}\mu_{\Phi,a}(x^j).$$

Proof. This is a simple computation. We have

$$(\mu_{\Phi,a}\big|\left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right))(x^r) = \mu_{\Phi,a}((a+p^nx)^r) = \mu_{\Phi,a}\left(\sum_{j=0}^{r}\binom{r}{j}a^{r-j}p^{nj}x^j\right)$$
$$= \sum_{j=0}^{r}\binom{r}{j}a^{r-j}p^{nj}\mu_{\Phi,a}(x^j).$$

□

12.2 Twists

12.3 Some data

Using the modular symbol $\widetilde{\Phi}_\beta \in H^1_c(\Gamma, \mathcal{F}(M))$ contructed in section 11, we can compute approximations to $R_m(1+T)$ since the results of the previous section show that to compute R_m one only needs the data of the first $k+1$ moments of $\Phi(\{\frac{a}{p^n}\} - \{\infty\})$ for a between 0 and p^{n-1}. A good enough approximations to R_m should yield a good approximation of the newton polygon of $L(\mu_{\Phi_\beta}, T)$. In this section, we display the data arising from taking $m =$? and computing such newton polygons for various curves and twists of those curves. (The formulas for twists are given in the previous section.)

12.4 Persistence of zeros of small slope

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