Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues

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The present paper is the beginning of an investigation into the congruence properties of systems of eigenvalues of Hecke operators acting on cohomology groups associated to automorphic forms over a reductive algebraic group G.

When $G = GL(2)$ this amounts to a study of the congruence properties of q-expansions of classical modular newforms of weight $\geqq 2$. This theory has been researched extensively and has found numerous applications (see for example [9], [16], [17], [22], [24], [28], [29], [31], [34]). The cohomological approach to the theory was initiated in 1968 by Shimura [33] (see Hida [17]). Other cohomological attacks on congruences can be found in [13] and [20]. In [20] forms over unit groups in quaternion algebras are studied as well.

In this paper we broaden the method in order to treat arithmetic subgroups of other reductive groups.

In section 1 we set up some general machinery for treating the cohomology of a group Γ (possibly with twisted coefficients) along with the action of a commutative Hecke algebra of double cosets on it. We consider pairs of groups and coefficient $\mathbf{Z}$-modules related in such a way that we obtain (1) a homomorphism ι between the Hecke algebras; (2) an ι-equivariant map between the cohomology groups $\bmod l$; and (3) a way of lifting systems of Hecke eigenvalues $\bmod l$ back to the integral cohomology. The net result is a pair of integral cohomology eigenclasses, one per group, and a congruence $\bmod l$ between the associated systems of Hecke eigenvalues.

Two remarks deserve special emphasis. First, despite the congruence between the eigenvalues associated to these eigenclasses, there may be no direct relationship between the eigenclasses themselves (compare the remark following proposition 1. 2. 3). Second, the cohomology groups under consideration may contain nontrivial torsion elements (cf. section 3). In the general situation, one of our eigenclasses may be a torsion class even if the other is not.

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Note that our Hecke algebras are assumed to be commutative, a condition which is satisfied in many important examples.

In section 2 we prove two results for arithmetic groups Γ generalizing known results for $GL(2)$. The first of these (theorem 2. 2) states that for fixed Γ and l, the set of systems of Hecke eigenvalues modulo l occurring in $\bigoplus H^*(\Gamma; E)$ is finite, where the direct sum is over all finite dimensional rational representations E of the ambient group which can be "reduced" modulo l. This may be viewed as a generalization of a theorem of Serre and Tate [31] (see also Jochnowitz [19]) which states that the set of systems of Hecke eigenvalues modulo l arising from modular forms of all weights and level 1 is finite. Our second result (theorem 2. 4) states that, given Γ, l sufficiently large, and E as above, there exists a subgroup Γ_1 of finite index in Γ and a trivial Γ_1-module F such that every system of Hecke eigenvalues occurring in $H^N(\Gamma; E)$ may be found modulo l in $H^N(\Gamma_1; F)$ where N is the virtual cohomological dimension of Γ. This generalizes results of Serre [28] and Serre-Fontaine [29] which state that modular eigenforms of arbitrary weight are congruent modulo l to weight two forms. In a later paper we will show how to get similar results in dimensions other than N. We close section 2 with an examination of the special case $G = GL(n)$.

One expects that our methods, when applied to specific groups, will yield stronger theorems than the general ones proved in § 2. In the special case $G = GL(2)$, for example, our methods can be refined to obtain more explicit statements about congruences among modular forms. We are also able to prove congruences between the algebraic parts of special values of associated L-functions. We will report on this in an upcoming paper.

In section 3, we use our methods along with some input from automorphic representation theory and l-adic Galois representation theory to prove (theorem 3. 5. 3) the existence of many l-torsion classes in the cohomology of certain arithmetic subgroups of $SL(3, \mathbf{Z})$. The torsion classes we construct are Hecke eigenclasses whose eigenvalues are congruent to the Hecke eigenvalues of an automorphic (hence nontorsion) cohomology class in a different cohomology group. We wonder whether or not *all* torsion eigenclasses in the cohomology of any arithmetic group have this property. We also ask the related question: Is there a correspondence à la Langlands between l-torsion eigenclasses and Galois representations on $\mathbf{F}_l$-vector spaces?

1. 1 Cohomology and Hecke operators

In this section we define the Hecke algebra and state the basic properties of this algebra acting on the group cohomology. We will use the notation of Andrianov [1].

Let G be a group. A *Hecke pair* consists of a subgroup Γ of G and a subsemigroup S of G such that

(1) $\Gamma \subseteq S$,

(2) Γ and $g^{-1}\Gamma g$ are commensurable for every $g \in S$.

We will write $L(\Gamma, S)$ for the free $\mathbf{Z}$-module on the right cosets Γg, $g \in S$, and $\mathscr{H} = \mathscr{H}(\Gamma, S)$ for the right Γ-invariant elements in $L(\Gamma, S)$. We define multiplication in $\mathscr{H}$ by the formula

$$\sum a_i(\Gamma g_i) \cdot \sum b_j(\Gamma h_j) = \sum a_i b_j(\Gamma g_i h_j).$$

Then $\mathscr{H}$ is an associative algebra. If $g \in S$ and $\Gamma g \Gamma$ is the disjoint union $\bigcup \Gamma g_i$ then we will write T_g for the element $\sum \Gamma g_i$ in $\mathscr{H}$. We will refer to $\mathscr{H}$ as the Hecke algebra of the pair (Γ, S).

Notation. If E is a right S-module, then we will denote the right action of $\sigma \in S$ on $e \in E$ by multiplication on the *left* by σ^{-1}:

$$E \times S \to E, \quad (e, \sigma) \mapsto \sigma^{-1} e.$$

It is well known that if E is a right $\mathbf{Z}S$-module, then there is a natural right action of the Hecke algebra $\mathscr{H}$ on the cohomology groups $H^r(\Gamma, E)$. For $g \in S$ the element T_g of $\mathscr{H}$ operates by the formula

$$(fT_g)(\gamma_0, \ldots, \gamma_r) = \sum g_i^{-1} f\big(t_i(\gamma_0), \ldots, t_i(\gamma_r)\big).$$

Here $f\colon \Gamma^{r+1} \to E$ is a homogeneous r-cocycle, $\gamma_0, \ldots, \gamma_r$ are in Γ, $\Gamma g \Gamma$ is the disjoint union $\bigcup \Gamma g_i$, and $t_i\colon \Gamma \to \Gamma$ is defined by the equations $\Gamma g_i \gamma = \Gamma g_j$ (for some j depending on i and $\gamma \in \Gamma$) and $g_i \gamma = t_i(\gamma) g_j$. The cohomology class of fT_g does not depend on the choice of the g_i.

Dually, if E is a left $\mathbf{Z}S$-module, then we can define a natural left action of $\mathscr{H}$ on the cohomology by formulas like the ones above. Alternatively, we set $S^{-1} = \{g^{-1} | g \in S\}$ and observe that since (Γ, S) is a Hecke pair, so also is (Γ, S^{-1}). As in the last paragraph we have a right action of $\mathscr{H}(\Gamma, S^{-1})$ on the cohomology. The left action of $\mathscr{H}$ is then given by the formula $T_g f = f T_{g^{-1}}$ for $g \in S$, and $f \in H^r(\Gamma, E)$. For more details see for instance [20].

It is fundamental to what follows that the action of the Hecke algebra on the cohomology groups respects the standard constructions of homological algebra. In the remainder of this section we record some instances of this principle.

Lemma 1.1.1. *Let* $0 \to D \to E \to F \to 0$ *be an exact sequence of (right or left)* $\mathbf{Z}S$*-modules. Then the long exact cohomology sequence*

$$\cdots \to H^r(\Gamma, D) \to H^r(\Gamma, E) \to H^r(\Gamma, F) \to H^{r+1}(\Gamma, D) \to \cdots$$

commutes with the action of $\mathscr{H}$.

Definition 1.1.2. A Hecke pair (Γ_0, S_0) is said to be compatible to the Hecke pair (Γ, S) if (a) $(\Gamma_0, S_0) \subseteqq (\Gamma, S)$, (b) $\Gamma S_0 = S$, and (c) $\Gamma \cap S_0 S_0^{-1} = \Gamma_0$.

If $(\Gamma_0, S_0) \subseteqq (\Gamma, S)$ are compatible, the cosets Γg with $g \in S_0$ span $L(\Gamma, S)$. So there is a unique linear map $L(\Gamma, S) \to L(\Gamma_0, S_0)$ sending Γg to $\Gamma_0 g$ for $g \in S_0$. The compatibility condition guarantees that the restriction to double cosets,

$$\iota\colon \mathscr{H} \to \mathscr{H}(\Gamma_0, S_0),$$

is an injective algebra homomorphism.

Thus if E is a $\mathbf{Z}S_0$-module, then we may (and will) view the cohomology groups $H^r(\Gamma_0, E)$ as $\mathcal{H}$-modules by composing the action of $\mathcal{H}(\Gamma_0, S_0)$ with ι.

The notion of compatibility (1. 1. 2) is not left-to-right symmetric. For this reason our next two results must treat right and left modules separately.

Lemma 1. 1. 3. (a) *If E is a right $\mathbf{Z}S$-module then the restriction map*

$$H^r(\Gamma; E) \xrightarrow{\text{res}} H^r(\Gamma_0, E)$$

commutes with the action of $\mathcal{H}$.

(b) *If the index $[\Gamma : \Gamma_0]$ is finite and E is a left $\mathbf{Z}S$-module then the corestriction map*

$$H^r(\Gamma, E) \xleftarrow{\text{cores}} H^r(\Gamma_0, E)$$

commutes with the action of $\mathcal{H}$.

Now suppose $[\Gamma : \Gamma_0] < \infty$ and let E be a right (resp. left) $\mathbf{Z}S_0$-module. Then in particular E is a $\mathbf{Z}\Gamma_0$-module and we may consider the induced $\mathbf{Z}\Gamma$-module $I = \operatorname{Ind}(\Gamma_0, \Gamma; E)$ of functions $f: \Gamma \to E$ such that $f(xy) = xf(y)$ for all x in Γ_0, y in Γ. We define a right (resp. left) action of S on I by the following formulas for $g \in S$, $f \in I$, and $x \in \Gamma$.

If E is a *right* S_0-module, choose $g_0 \in S_0$, $y \in \Gamma$ so that $xg^{-1} = g_0^{-1}y$ and set

$$(g^{-1}f)(x) = g_0^{-1}f(y).$$

If E is a *left* S_0-module, set

$$(gf)(x) = \sum xg\gamma^{-1}f(\gamma)$$

where the sum is over representatives, γ, of the cosets in $\Gamma_0 \backslash (\Gamma \cap S_0^{-1}xg)$. Using the compatibility of $(\Gamma_0, S_0) \subseteqq (\Gamma, S)$ it is not difficult to verify that these formulas define a right (resp. left) semigroup action of S on I extending the standard action of Γ.

Lemma 1. 1. 4. *Suppose $[\Gamma : \Gamma_0] < \infty$ and let E be a (right or left) $\mathbf{Z}S_0$-module. Then the Shapiro isomorphism*

$$\mathcal{S}: H^r(\Gamma, \operatorname{Ind}(\Gamma_0, \Gamma, E)) \xrightarrow{\sim} H^r(\Gamma_0, E)$$

commutes with the action of $\mathcal{H}$.

Proof. Let $I = \operatorname{Ind}(\Gamma_0, \Gamma; E)$. If E is a right $\mathbf{Z}S$-module then the map $\rho: I \to E$ which sends a function f to $f(1)$ is a morphism of right S_0-modules. The Shapiro isomorphism is the composition

$$H^r(\Gamma, I) \xrightarrow{\text{res}} H^r(\Gamma_0, I) \xrightarrow{\rho_*} H^r(\Gamma_0, E),$$

and hence commutes with the action of $\mathcal{H}$ by 1. 1. 1 and 1. 1. 3.

If E is a left $\mathbf{Z}S_0$-module, then the map $i: E \to I$ defined by

$$i(e)(x) = \begin{cases} xe, & \text{if } x \in \Gamma_0, \\ 0, & \text{otherwise} \end{cases}$$

is a morphism of left $\mathbf{Z}S_0$-modules. The inverse of the Shapiro isomorphism is the composition

$$H^r(\Gamma_0, E) \xrightarrow{i_*} H^r(\Gamma_0, I) \xrightarrow{\text{cores}} H^r(\Gamma, I).$$

Again we use 1.1.1 and 1.1.3 to conclude that this commutes with $\mathscr{H}$. □

We close this section with a discussion of nebentype operators. Suppose (Γ_1, S_1) is a Hecke pair compatible to (Γ_0, S_0) and which is normalized by Γ_0. Let E be an S_0-module. We have seen that the Hecke algebra $\mathscr{H}_0 = \mathscr{H}(\Gamma_0, S_0)$ acts on $H^*(\Gamma_1, E)$. But there is also a standard action of the quotient group Γ_0/Γ_1 on the cohomology ([27], VII §§ 5, 6). In our language this action can be described as follows. Since Γ_1 is normal in Γ_0 the pair (Γ_1, Γ_0) is a Hecke pair. The Hecke algebra $\mathscr{H}(\Gamma_1, \Gamma_0)$ is naturally isomorphic to the group ring $\mathbf{Z}[\Gamma_0/\Gamma_1]$. Thus the action of $\mathscr{H}(\Gamma_1, \Gamma_0)$ on the cohomology induces an action of Γ_0/Γ_1 on $H^*(\Gamma_1, E)$. For $a \in \Gamma_0$ we write $[a]$ for the associated operator on cohomology and refer to $[a]$ as the nebentype operator associated to a. One readily verifies that these operators commute with the action of $\mathscr{H}_0$. Since the nebentype operators are defined as Hecke operators, they enjoy all of the functorial properties attributed to Hecke operators in this chapter.

Let R be a commutative ring with identity and let $\varepsilon: \Gamma_0/\Gamma_1 \to R^*$ be a character. For a right (resp. left) RS_0-module E we define $H^*(\Gamma_1, E)(\varepsilon)$ to be the submodule of ξ in $H^*(\Gamma_1, E)$ on which the nebentype operators act via ε: $\xi[a] = \varepsilon(a)^{-1}\xi$ (resp. $[a]\xi = \varepsilon(a)\xi$).

The compatibility of $(\Gamma_1, S_1) \subseteq (\Gamma_0, S_0)$ guarantees that there is a unique extension of ε to a character $\varepsilon: S_0 \to R^*$ which is trivial on S_1. Let R_ε be the rank one R-module on which S_0 acts via ε.

Lemma 1.1.5. *Let (Γ_1, S_1) be a Hecke pair compatible to (Γ_0, S_0) and normalized by Γ_0. Suppose the index $[\Gamma_0 : \Gamma_1]$ is finite and invertible in R. Then for every (left or right) RS_0-module E and every character $\varepsilon: \Gamma_0/\Gamma_1 \to R^*$, the restriction map induces an isomorphism of $\mathscr{H}_0$-modules*

$$H^*(\Gamma_0, E \otimes R_\varepsilon) \cong H^*(\Gamma_1, E)(\varepsilon^{-1}).$$

Proof. The map $R \to R_\varepsilon$, $r \mapsto r$ is an isomorphism of RS_1-modules and thus induces an isomorphism of $\mathscr{H}_0$-modules $H^*(\Gamma_1, E) \to H^*(\Gamma_1, E \otimes R_\varepsilon)$. A simple calculation with cocycles shows that the space $H^*(\Gamma_1, E)(\varepsilon^{-1})$ is mapped onto $H^*(\Gamma_1, E \otimes R_\varepsilon)^{\Gamma_0}$. The invertibility of $[\Gamma_0 : \Gamma_1]$ together with the Hochschild-Serre spectral sequence now show that this is isomorphic to $H^*(\Gamma_0, E \otimes R_\varepsilon)$ via the restriction map. If the action of S on E is a right action then lemma 1.1.3(a) shows that restriction commutes with $\mathscr{H}_0$ and we are done. Otherwise we use 1.1.3(b) and the fact that cores ∘ res $= [\Gamma_0 : \Gamma_1]$ which is invertible in R to complete the proof. □

1.2 Systems of Hecke eigenvalues

In this section we discuss the functorial properties of systems of Hecke eigenvalues associated to eigenvectors in the cohomology groups $H^*(\Gamma, E)$. We will take a more abstract point of view and let $\mathscr{H}$ be an arbitrary *commutative* algebra.

Definition 1.2.1. (a) A system of eigenvalues of $\mathscr{H}$ with values in a commutative ring R is a set theoretic map $\Phi: \mathscr{H} \to R$.

(b) The system Φ is said to occur in the $R\mathscr{H}$-module A if there is a *nonzero* $a \in A$ such that $Ta = \Phi(T)a$ for all $T \in \mathscr{H}$. Such an a is called a Φ-eigenvector.

We will prove the following two propositions.

Proposition 1.2.2. *Suppose R is a discrete valuation ring (respectively field). Let A, B be $R\mathscr{H}$-modules, finitely generated over R and $f: A \to B$ be a surjective $R\mathscr{H}$-morphism. Let $\Phi: \mathscr{H} \to R$ be a system of eigenvalues occurring in B, and $v \in B$ be a Φ-eigenvector. Let $Q \subseteqq R$ be a prime ideal in the support of Rv. Then there is a discrete valuation ring (respectively field) R' of finite type over R and a system $\Psi: \mathscr{H} \to R'$ occurring in $A \otimes_R R'$ such that $\Psi(T) \equiv \Phi(T) \pmod{Q'}$ for all $T \in \mathscr{H}$ where Q' is the unique prime ideal of R' for which $Q' \cap R = Q$.*

This proposition generalizes a lemma of Deligne and Serre ([9], Lemma 6.11) which considers the special case where A is free over a discrete valuation ring R and B is the reduction of A modulo the maximal ideal.

If R is a local ring then we will use a bar to denote reduction modulo the maximal ideal. Thus, if $P \subseteqq R$ is the maximal ideal then $\bar{R} = R/P$; if M is an R-module then $\bar{M} = M \otimes \bar{R}$; if $m \in M$ then $\bar{m} = m \otimes 1 \in \bar{M}$; and if $\Phi: \mathscr{H} \to R$ is a system of eigenvalues then $\bar{\Phi}: \mathscr{H} \to \bar{R}$ is the composition of Φ with the canonical projection $R \to \bar{R}$.

Proposition 1.2.3. *Suppose R is a discrete valuation ring. Let A be an $R\mathscr{H}$-module finitely generated over R. If $\Phi: \mathscr{H} \to R$ occurs in A then $\bar{\Phi}$ occurs in $\bar{A}$.*

Remark. Proposition 1.2.2 states that a system of eigenvalues occurring in B may, after finite base extension, be "lifted" to a system occurring in A. Note, however, that an eigen*vector* in B need not lift to an eigenvector in A. Proposition 1.2.3 is also more subtle than it may seem at first. For example, a Φ-eigenvector $a \in A$ may reduce to zero modulo the maximal ideal. It is in general not even possible to solve the equation $rb = a$ for $r \in R$, $b \in A$ with $\bar{b}$ a $\bar{\Phi}$-eigenvector in $\bar{A}$.

In preparation for the proof of proposition 1.2.2 we state and prove two simple lemmas.

Lemma 1.2.4. *Let R be a discrete valuation ring (respectively field) and let A be an $R\mathscr{H}$-module, finitely generated over R. Then there is a discrete valuation ring (respectively field) R' finite over R such that $A' = A \otimes_R R'$ possesses an $R'\mathscr{H}$-stable filtration*

$$A' = A'_0 \supseteqq A'_1 \supseteqq \cdots \supseteqq A'_s = 0$$

in which the successive quotients are cyclic R'-modules.

Proof. It suffices to show that A has an $\mathscr{H}$-eigenvector after some finite base extension. If R is a field this is well known.

If A has no nonzero R-torsion then we may reduce to the case where R is a field by tensoring with the quotient field of R. Otherwise there is a nonzero $R\mathscr{H}$-submodule $A_0 \subseteqq A$ which is annihilated by the maximal ideal P of R. The action of $\mathscr{H}$ on A_0 factors through an action of $\mathscr{H} \otimes \bar{R}$. Thus we are again reduced to the case where the base ring is a field. □

We will refer to an $R\mathscr{H}$-filtration $A = A_0 \supseteqq \cdots \supseteqq A_s = 0$ as a *cyclic* $R\mathscr{H}$-filtration if the successive quotients are cyclic R-modules. If the integer s is minimal among all such filtrations of A we will call the filtration a *minimal* cyclic $R\mathscr{H}$-filtration.

Lemma 1. 2. 5. *Let R be a discrete valuation ring or a field. Let*

$$A = A_0 \supseteqq A_1 \supseteqq \cdots \supseteqq A_s = 0$$

be a minimal cyclic $R\mathscr{H}$-filtration of A, and suppose $T_0 \in \mathscr{H}$ annihilates the quotient A_i/A_{i+1} for some $i = 0, \ldots, s-1$. Then there is a nonzero $a \in A$ such that $T_0 a = 0$.

Proof. We have $T_0 A_i \subseteqq A_{i+1}$. For $j = 0, \ldots, s-1$ let

$$A'_j = \begin{cases} T_0 A_j & \text{if } \; 0 \leqq j \leqq i, \\ T_0 A_i \cap A_{j+1} & \text{if } \; i < j. \end{cases}$$

Then $T_0 A = A'_0 \supseteqq A'_1 \supseteqq \cdots \supseteqq A'_{s-1} = 0$ is a cyclic $R\mathscr{H}$-filtration of $T_0 A$ of length $s-1$. Because of the minimality of s, $T_0 A$ and A are not isomorphic as $R\mathscr{H}$-modules. Thus T_0 does not act injectively on A. □

Proof of Proposition 1. 2. 2. By lemma 1. 2. 4 it suffices to prove the following:

(∗) If A possesses a cyclic $R\mathscr{H}$-filtration then the proposition is true with

$$R' = R.$$

By replacing $\mathscr{H}$ by the R-algebra generated by the image of $\mathscr{H}$ in $\mathrm{End}_R(A)$ we may assume $\mathscr{H}$ is an R-subalgebra of $\mathrm{End}_R(A)$. Since A is a finitely generated R-module, so is $\mathscr{H}$. We will prove (∗) by induction on the minimal number of algebra generators of $\mathscr{H}$ over the image of the structure morphism $R \to \mathscr{H}$. If $R \to \mathscr{H}$ is surjective there is nothing to prove. So we let $\mathscr{H}_1$ be a subalgebra of $\mathrm{End}_R(A)$ and make the following inductive assumption.

(1) If $\mathscr{H} = \mathscr{H}_1$ then (∗) is valid.

Now suppose $\mathscr{H} = \mathscr{H}_1[T]$ for some $T \in \mathrm{End}_R(A)$. We will prove (∗) in this situation by induction on the length of a minimal cyclic $R\mathscr{H}$-filtration of A. If A is a cyclic R-module then (∗) is immediate. Thus we may make the following inductive assumption.

(2) $s > 1$ is an integer and (∗) holds for every $\mathscr{H}$-module A which possesses a cyclic $R\mathscr{H}$-filtration of length less than s.

Let A be an $R\mathscr{H}$-module which possesses a minimal cyclic $R\mathscr{H}$-filtration

$$A = A_0 \supseteqq A_1 \supseteqq \cdots \supseteqq A_s = 0$$

of length s. Without loss of generality we may also make the following assumption on B.

(3) B is the cyclic R-module generated by v.

Let $\lambda = \Phi(T)$ and let C be the submodule of A defined by

$$C = \bigcup \ker \prod_{j=1}^{n} (T - \lambda_j)$$

where the union is over all natural numbers n and n-tuples $(\lambda_1, \ldots, \lambda_n) \in R^n$ whose components satisfy the congruence $\lambda_j \equiv \lambda \pmod{Q}$. Then C is $\mathscr{H}$-stable since $\mathscr{H}$ is commutative.

Suppose $f(C) = 0$. Then $f\colon A \to B$ factors through an $R\mathscr{H}$-morphism

$$f'\colon A' = A/C \to B.$$

Let $A' = A'_0 \supseteqq A'_1 \supseteqq \cdots \supseteqq A'_s = 0$ be the $R\mathscr{H}$-filtration of A' induced by the given filtration of A. Let $0 \leqq i \leqq s-1$ be the least integer for which $f'(A'_{i+1}) = 0$. Let $\lambda' \in R$ satisfy $Tx = \lambda' x$ for all $x \in A'_i/A'_{i+1}$. Since f' induces a nonzero $R\mathscr{H}$-morphism $A'_i/A'_{i+1} \to B$ and B is cyclic (3), we have $\lambda' \equiv \lambda \pmod{Q}$. By the definition of C we know that $T - \lambda'$ acts injectively on A'. On the other hand $T - \lambda'$ annihilates A'_i/A'_{i+1}. Thus lemma 1.2.5 implies the filtration $A' = A'_0 \supseteqq \cdots \supseteqq 0$ is not minimal cyclic. Then by (2) there is a system of eigenvalues $\Psi'\colon \mathscr{H} \to R$ occurring in A' such that $\Psi' \equiv \Phi \pmod{Q}$. But then $\Psi'(T) \equiv \lambda \pmod{Q}$ and therefore, by the definition of C, $T - \Psi'(T)$ acts injectively on A', a contradiction.

This shows $f(C) \neq 0$. By applying (1) to the surjection $C \to f(C)$ we conclude that there is a system $\Psi_1\colon \mathscr{H}_1 \to R$ occurring in C such that $\Psi_1(t) \equiv \Phi(t) \pmod{Q}$ for all $t \in H_1$. Let $c \in C$ be a Ψ_1-eigenvector. By the definition of C there is a nonnegative integer n and $\lambda_1, \ldots, \lambda_n \in R$ with $\lambda_j \equiv \lambda \pmod{Q}$, $j = 1, \ldots, n$ such that $a = \prod_{j=2}^{n} (T - \lambda_j) c \neq 0$, and $(T - \lambda_1) a = 0$. Then a is a Ψ-eigenvector for a system of eigenvalues $\Psi\colon \mathscr{H} \to R$ satisfying $\Psi \equiv \Phi \pmod{Q}$. □

Proof of Proposition 1.2.3. As in the proof of proposition 1.2.2 we may suppose $\mathscr{H} \subseteqq \mathrm{End}_R(A)$ and $\mathscr{H} = \mathscr{H}_1[T]$ where $\Phi|\mathscr{H}_1$ occurs in $\bar{A}$. Without loss of generality we may replace A by the full inverse image of the $\Phi|\mathscr{H}_1$-eigenspace of $\bar{A}$, since Φ occurs already in this submodule. Thus it suffices to find a nonzero $x \in \bar{A}$ such that $Tx = rx$ where $r = \Phi(T)$.

Since A is a finite direct sum of cyclic R-modules, there is a free R-module F and a surjective R-morphism $\xi\colon F \to A$ whose reduction $\bar{\xi}\colon \bar{F} \to \bar{A}$ is an isomorphism. We may lift T to an R-endomorphism of F so that T commutes with ξ.

By proposition 1.2.2 there is a discrete valuation ring R' finite over R, and a nonzero $f \in F \otimes_R R'$ such that $Tf = sf$ for some $s \in R'$ with $\bar{s} = \bar{r}$. Since F is free we may *assume* $\bar{f} \neq 0$. Thus $(t - \bar{r})$ has a nonzero kernel in $\bar{F} \otimes_R \bar{R}' \cong \bar{A} \otimes_R \bar{R}'$, and hence also in $\bar{A}$. □

1.3 The main diagrams

We consider compatible Hecke pairs $(\Gamma_0, S_0) \subseteq (\Gamma, S)$ and let $\mathscr{H}$ be the associated Hecke algebra. We assume the following additional conditions:

(1.3.1) Γ is finitely presented and of type (WFL) ([26], section 1.8), $[\Gamma : \Gamma_0] < \infty$.

Let R be a discrete valuation ring with maximal ideal P generated by π, and let E be a right (left) RS-module and F be a right (left) RS_0-module. We assume that both E and F are finitely generated as R-modules. This assumption taken together with (1.3.1) implies that the cohomology groups of E and F are finitely generated over R as well, so that we can apply the results of the last section to them.

If E and F are right modules and $\phi: \bar{E} \to \bar{F}$ is an RS_0-morphism then the map $\alpha(\phi): \bar{E} \to \mathrm{Ind}(\Gamma_0, \Gamma; \bar{F})$ defined by $(\alpha(\phi)(e))(\gamma) = \phi(\gamma e)$ for $e \in \bar{E}$ and $\gamma \in \Gamma$ commutes with the right action of RS. The interesting cases of course occur when ϕ cannot be lifted to a morphism $E \to F$.

For a positive integer r we can now draw the main diagram for *right* modules.

(1.3.2)
$$\begin{array}{ccccc}
H^r(\Gamma, E) & \longrightarrow & H^r(\Gamma, \bar{E}) & \xrightarrow{\ \delta\ } & H^{r+1}(\Gamma, E) \\
& \swarrow^{\mathrm{res}} & & \searrow^{\alpha(\phi)_*} & \\
H^r(\Gamma_0, \bar{E}) & & \Big\downarrow A^r & & H^r(\Gamma, \mathrm{Ind}(\Gamma_0, \Gamma; \bar{F})) \\
& \searrow^{\phi_*} & & \swarrow^{\mathscr{S}} & \\
H^r(\Gamma_0, F) & \longrightarrow & H^r(\Gamma_0, \bar{F}) & \xrightarrow{\ \delta\ } & H^{r+1}(\Gamma_0, F)
\end{array}$$

In this diagram the horizontal arrows are extracted from the long exact cohomology sequences of $0 \to M \xrightarrow{\pi} M \to \bar{M} \to 0$ for $M = E, F$, the arrow res is the restriction morphism, and $\mathscr{S}$ is the Shapiro isomorphism. The arrow A^r is defined by the commutativity of the diagram. The results of 1.1 show that this diagram commutes with the action of $\mathscr{H}$.

Dually, if E and F are left modules and $\psi: \bar{F} \to \bar{E}$ is an RS_0-morphism then we define $\beta(\psi): \mathrm{Ind}(\Gamma_0, \Gamma; \bar{F}) \to \bar{E}$ by

$$\beta(\psi)(f) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \gamma^{-1} \psi(f(\gamma)).$$

A straightforward calculation shows that $\beta(\psi)$ is a morphism of left RS-modules.

Our main diagram for *left* modules is the following.

(1.3.3)
$$\begin{array}{ccccc}
H^r(\Gamma, E) & \longrightarrow & H^r(\Gamma, \bar{E}) & \xrightarrow{\ \delta\ } & H^{r+1}(\Gamma, E) \\
& \nearrow^{\mathrm{cores}} & & \nwarrow^{\beta(\psi)_*} & \\
H^r(\Gamma_0, \bar{E}) & & \Big\downarrow B^r & & H^r(\Gamma, \mathrm{Ind}(\Gamma_0, \Gamma; \bar{F})) \\
& \nwarrow^{\psi_*} & & \nearrow^{\mathscr{S}^{-1}} & \\
H^r(\Gamma_0, F) & \longrightarrow & H^r(\Gamma_0, \bar{F}) & \xrightarrow{\ \delta\ } & H^{r+1}(\Gamma_0, F)
\end{array}$$

Here cores is the corestriction morphism which commutes with the action of $\mathscr{H}$ by lemma 1.1.3, and B^r is defined by the commutativity of the diagram.

Theorem 1.3.4. *Suppose $\mathscr{H}$ is commutative and replace R by a finite extension if necessary so that R contains all of the eigenvalues of $\mathscr{H}$ acting on the cohomology groups in* (1.3.2) *and* (1.3.3). *Let $\Phi: \mathscr{H} \to R$ be a system of eigenvalues.*

(a) *Suppose A^r is injective or B^r is surjective. If Φ occurs in $H^r(\Gamma, E)$ then there is a system of eigenvalues $\Psi: \mathscr{H} \to R$ occurring in $H^r \oplus H^{r+1}(\Gamma_0, F)$ such that $\bar{\Phi} = \bar{\Psi}$.*

(b) *Suppose A^r is surjective or B^r is injective. If Φ occurs in $H^r(\Gamma_0, F)$ then there is a system of eigenvalues $\Psi: \mathscr{H} \to R$ occurring in $H^r \oplus H^{r+1}(\Gamma, E)$ such that $\bar{\Phi} = \bar{\Psi}$.*

Proof. We will give the proof of (a) only. The proof of (b) is similar. Suppose Φ occurs in $H^r(\Gamma, E)$. The long exact cohomology sequence of the sequence $0 \to E \xrightarrow{\pi} E \to \bar{E} \to 0$ provides a Hecke equivariant inclusion

$$H^r(\Gamma, E) \otimes R/P \hookrightarrow H^r(\Gamma, \bar{E}).$$

Thus proposition 1.2.3 shows that $\bar{\Phi}$ occurs in $H^r(\Gamma, \bar{E})$. If A^r is injective we conclude at once that $\bar{\Phi}$ occurs in $H^r(\Gamma_0, \bar{F})$, and if B^r is surjective we use proposition 1.2.2 to draw the same conclusion. In either case we can find a $\bar{\Phi}$-eigenvector $v \in H^r(\Gamma_0, F)$. If $\delta(v) \neq 0$ then Φ occurs in $H^{r+1}(\Gamma_0, F)$. Otherwise we can appeal to proposition 1.2.2 again to prove the existence of Ψ occurring in $H^r(\Gamma_0, F)$ with $\bar{\Phi} = \bar{\Psi}$. □

We close this section by giving a criterion for A^r or B^r to be surjective.

Theorem 1.3.5. *Let N be the virtual cohomological dimension of Γ and D be the greatest common divisor of the indices of the torsionfree subgroups of finite index in Γ. Assume D is invertible in R. Then we have the following implications:*

(i) *$\alpha(\phi)$ surjective $\Rightarrow$ A^N surjective;*

(ii) *$\beta(\psi)$ surjective $\Rightarrow$ B^N surjective.*

Proof. To prove (i) we consider the long exact cohomology sequence associated to

$$0 \to \ker(\alpha) \to \bar{E} \to \operatorname{Ind}(\Gamma_0, \Gamma; \bar{F}) \to 0.$$

By [6], p. 287 we have $D \cdot H^{N+1}(\Gamma, \ker(\alpha)) = 0$. Because D is invertible in R we have $H^{N+1} = 0$ and therefore $\alpha_*: H^N(\Gamma, \bar{E}) \to H^N(\Gamma, \operatorname{Ind})$ is surjective. Since $A^N = \mathscr{S} \circ \alpha_*$ and $\mathscr{S}$ is an isomorphism, (i) follows. A similar argument establishes (ii). □

1. 4 Arithmetic groups

In the applications Γ will be an arithmetic subgroup of a reductive algebraic group G over $\mathbf{Q}$, and S will be a subsemigroup of $G(\mathbf{Q})$. Then Γ satisfies (1. 3. 1) and the results of the last section are applicable.

The additional structure which we would like to utilize derives from the fact that Γ acts properly discontinuously on the symmetric space X of G. We define the integers d, N, m and D by

(1. 4. 1)
$d =$ the dimension of X;
$N =$ the virtual cohomological dimension of Γ;
$m =$ the least common multiple of the orders $|\Gamma_x|$ of the isotropy groups of all x in X;
$D =$ the greatest common divisor of the indices of the torsionfree subgroups of finite index in Γ.

Borel and Serre [4] have shown that N depends only on G and is equal to d minus the $\mathbf{Q}$-rank of G.

Let R be a ring in which m is invertible. For an $R\Gamma$-module E, let $\tilde{E}_\Gamma$ be the corresponding local coefficient system on the quotient $X_\Gamma = \Gamma \backslash X$, of X by Γ. The invertibility of m implies that the canonical map

$$H^r(X_\Gamma, \tilde{E}_\Gamma) \longrightarrow H^r(\Gamma, E)$$

is an isomorphism for every r. Letting H_c denote cohomology with compact supports and $H_!$ the image of H_c in H, we define

$$H_c^r(\Gamma, E) = H_c^r(X_\Gamma, \tilde{E}_\Gamma) \quad \text{and} \quad H_!^r(\Gamma, E) = H_!^r(X_\Gamma, \tilde{E}_\Gamma).$$

If E is a right (resp. left) RS-module, then the action of $\mathscr{H}$ can be described topologically. For g in S let

$$\Gamma(g) = \Gamma \cap g^{-1}\Gamma g \quad \text{and} \quad \Gamma(g^{-1}) = g\Gamma(g)g^{-1} = \Gamma \cap g\Gamma g^{-1},$$

and consider the diagram

(1. 4. 2)
$$\begin{array}{ccc} X_{\Gamma(g)} & \xrightarrow{L(g)} & X_{\Gamma(g^{-1})} \\ \downarrow{\scriptstyle \pi(g)} & & \downarrow{\scriptstyle \pi(g^{-1})} \\ X_\Gamma & & X_\Gamma \end{array}$$

where $\pi(g)$ and $\pi(g^{-1})$ are the natural projections and $L(g)$ is induced by left translation of X by g. Then T_g: $H_*^r(\Gamma, E) \longrightarrow H_*^r(\Gamma, E)$ is given by

$$T_g = \pi(g)_* \circ L(g)^* \circ \pi(g^{-1})^*$$

(resp. $T_g = \pi(g^{-1})_* \circ L(g)_* \circ \pi(g)^*$) where H_* denotes either H or H_c. The map $H_c^r \longrightarrow H^r$ commutes with the action of $\mathscr{H}$ so that $H_!^r(\Gamma, E)$ also inherits a structure of $\mathscr{H}$-module.

If E is a right (resp. left) RS-module finitely generated as an R-module, we define the contragredient left (resp. right) RS-module $E^* = \mathrm{Hom}_R(E, R)$ by $(gf)(e) = f(g^{-1}e)$ (resp. $(g^{-1}f)(e) = f(ge)$) for $f \in E^*$, $e \in E$, and $g \in S$.

The following lemma will not be used in the rest of the paper, though it seems appropriate to record it here for the sake of completeness.

Lemma 1.4.3. *Suppose Γ acts on X without reversing orientation, and let R be a field in which mD is invertible. Then cup product and the identification $H_c^d(\Gamma, R) = R$ induce $\mathscr{H}$-equivariant perfect pairings*

(i) $H_c^r(\Gamma, E) \times H^{d-r}(\Gamma, E^*) \to R$

and

(ii) $H_!^r(\Gamma, E) \times H_!^{d-r}(\Gamma, E^*) \to R$.

Proof. If Γ is torsion free then the Poincaré duality theorem ([7], p. 20—40) assures that the pairing (i) is nondegenerate. In the general case let Γ' be a torsion free subgroup of finite index in Γ such that $[\Gamma : \Gamma']$ is invertible in R. Let $\pi: X_{\Gamma'} \to X_\Gamma$ be the canonical projection. For each $R\Gamma$-module M and each integer $r \geqq 0$ we consider the maps

$$\pi^*: H_*^r(\Gamma, M) \to H_*^r(\Gamma', M), \quad \pi_*: H_*^r(\Gamma, M) \leftarrow H_*^r(\Gamma', M)$$

where H_* denotes either H or H_c. Since $\pi_* \circ \pi^*$ is multiplication by $[\Gamma : \Gamma']$ which is invertible in R we see that π_* is surjective and π^* is injective. If $x \in H_c^r(\Gamma, E)$ satisfies $\langle x, y \rangle = 0$ for every $y \in H^{d-r}(\Gamma, E^*)$ then $0 = \langle x, \pi_*(z) \rangle = \langle \pi^*(x), z \rangle$ for all $z \in H^{d-r}(\Gamma', E^*)$. Since Γ' is torsion free we see that $\pi^*(x) = 0$. But π^* is injective, so $x = 0$. Thus the pairing is nondegenerate on the left. The right nondegeneracy is proved similarly.

Next we check that (i) is $\mathscr{H}$-equivariant. To fix ideas we suppose E is a *right* RS-module. Let $x \in H_c^r(\Gamma, E)$ and $y \in H^{d-r}(\Gamma, E^*)$. For $g \in S$ we have

$$\langle x T_g, y \rangle = \langle \pi(g)_* \circ L(g)^* \circ \pi(g^{-1})^* x, y \rangle = \langle x, \pi(g^{-1})_* \circ L(g)_* \circ \pi(g)^* y \rangle = \langle x, T_g y \rangle.$$

The canonical maps $H_c^*(\Gamma, E) \to H^*(\Gamma, E)$ and $H_c^*(\Gamma, E^*) \to H^*(\Gamma, E^*)$ are dual under the pairing (i) and commute with $\mathscr{H}$. Thus (ii) is a consequence of (i). □

2. Systems of Hecke eigenvalues mod l

In this section, we prove two general theorems (2.2 and 2.4) about systems of Hecke eigenvalues (mod l) occurring in the cohomology groups of a fixed arithmetic group Γ. As mentioned in the introduction these results generalize known statements about classical modular forms [19], [29], [31].

We begin by defining precisely the objects to concern us for the rest of this paper. Let l be a rational prime and let $\mathbf{Z}_{(l)}$ be the ring of rational numbers with denominators prime to l. We make the following assignments.

G = a reductive linear algebraic group scheme defined over $\mathbf{Z}_{(l)}$.

Γ = an arithmetic group in $G(\mathbf{Z}_{(l)})$.

(Γ, S) = a Hecke pair.

$\mathscr{H}$ = the Hecke algebra $\mathscr{H}(\Gamma, S)$.

$\mathscr{E}$ = a finite dimensional representation of G defined over $\mathbf{Z}_{(l)}$.

E = a Γ-stable finitely generated $\mathbf{Z}$-module in $\mathscr{E}(\mathbf{Z}_{(l)})$ such that $\mathbf{Z}_{(l)}E = \mathscr{E}(\mathbf{Z}_{(l)})$.

We will call such an E an l-rational $\mathbf{Z}\Gamma$-module. We also let

$\mathcal{O}$ = the ring of algebraic integers in the algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$.

λ = a prime ideal in $\mathcal{O}$ lying over l.

We will identify $\mathcal{O}/\lambda$ with the algebraic closure of the finite field with l elements via a *fixed* isomorphism $\mathcal{O}/\lambda \cong \bar{F}_l$. If we reduce an integral object mod l or λ, we use a bar to denote the result. The completion of $\mathcal{O}$ at λ will be denoted by $\mathcal{O}_\lambda$.

Lemma 2. 1. *Suppose $\mathscr{H}$ is commutative. Let k be a field, and V be a (left or right) kS-module, finite dimensional over k. If Φ: $\mathscr{H} \to k$ occurs as a system of eigenvalues in $H^r(\Gamma, V)$, then Φ occurs in $H^r(\Gamma, W)$ for some irreducible kS-subquotient W of V.*

Proof. If V is not already irreducible, we have an exact sequence of kS-modules $0 \to V_1 \to V \to V_2 \to 0$ with nonzero V_1 and V_2. We get an exact sequence of $\mathscr{H}$-modules $H^r(\Gamma, V_1) \to H^r(\Gamma, V) \to H^r(\Gamma, V_2)$. If a Φ-eigenvector maps nonzero into $H^r(\Gamma, V_2)$, Φ occurs in the latter. Otherwise, this eigenvector is in the image of $H^r(\Gamma, V_1)$ and Φ occurs in $H^r(\Gamma, V_1)$ by proposition 1. 2. 2. The result follows from induction on $\dim_k V$. □

Theorem 2. 2. *Suppose the Hecke algebra $\mathscr{H}$ is commutative. Given an integer r, form the set $Z = \{\bar{\Phi}: \mathscr{H} \to \bar{F}_l\}$ with Φ running through all systems of eigenvalues Φ: $\mathscr{H} \to \mathcal{O}_\lambda$ occurring in $\oplus H^r(\Gamma, E \otimes \mathcal{O}_\lambda)$, where E runs over all l-rational $\mathbf{Z}\Gamma$-modules. Then Z is finite.*

Proof. Let $J \subseteqq G(\mathbf{F}_l)$ be the reduction of S mod l. Then J is a finite semigroup with cancellation laws and is therefore a finite group. Let $\{V\}$ be a set of representatives of isomorphism classes of irreducible finitely generated $\bar{F}_l J$-modules. This is a finite set (see, e.g. Part III of [30]). Define the set $Z_1 = \{\Psi: \mathscr{H} \to \bar{F}_l\}$ where Ψ runs through all systems of eigenvalues occurring in $\bigoplus H^r(\Gamma, V)$, V running over $\{V\}$. Each $H^r(\Gamma, V)$ is finite-dimensional over $\bar{F}_l$, so Z_1 is a finite set. We will show that Z is contained in Z_1.

Let E and Φ be as in the statement of the theorem. By lemma 1. 2. 4 there is a discrete valuation ring $R \subseteqq \bar{\mathbf{Q}}$ of finite type over $\mathbf{Z}_{(l)}$ and with maximal ideal $R \cap \lambda$, such that Φ takes values in R and occurs in $H^r(\Gamma, E \otimes R)$. Then from proposition 1. 2. 3 it follows that $\bar{\Phi}$ occurs in $H^r(\Gamma, E \otimes R) \otimes R/(\lambda \cap R)$. From the Bockstein exact sequence we know this last injects into $H^r(\Gamma, E \otimes R/(\lambda \cap R))$, which injects into $H^r(\Gamma, E \otimes \bar{\mathbf{F}}_l)$. So $\bar{\Phi}$ occurs in the latter.

Now S acts on $E \otimes \bar{\mathbf{F}}_l$ via the reduction map $S \to J$. By lemma 2.1, we see that Φ occurs in $H^r(\Gamma, V)$ for some V in $\{V\}$. So $\bar{\Phi}$ is in Z_1. □

Let E be an l-rational $\mathbf{Z}\Gamma$-module and e be a nonzero element of $\bar{E}$. Set

$$S(e) = \{g \in G(\mathbf{Z}_{(l)}) \mid ge \in \Gamma e\},$$
$$S_1(e) = \{g \in S(e) \mid ge = e\},$$
$$\Gamma_0 = \{g \in \Gamma \mid ge \in \mathbf{F}_l^* e\}.$$

Proposition 2.3. *Let S be a subsemigroup of $S(e)$ which contains Γ and set $\Gamma_1 = \Gamma \cap S_1(e)$, $S_1 = S \cap S_1(e)$ and $S_0 = \Gamma_0 S_1$. Then $(\Gamma_1, S_1) \subseteqq (\Gamma_0, S_0) \subseteqq (\Gamma, S)$ are compatible Hecke pairs and Γ_0 normalizes (Γ_1, S_1).*

Proof. Clearly Γ_1 and Γ_0 have finite index in Γ, so Γ_1, Γ_0 are arithmetic, and $G(\mathbf{Z}_{(l)})$ commensurates them. Thus (Γ_1, S_1), (Γ_0, S_0) and (Γ, S) are Hecke pairs.

We will prove only the compatibility of (Γ_1, S_1) to (Γ, S). The other two compatibility statements are easy consequences of this one. We must verify conditions (b) and (c) of definition 1.1.2. Clearly $\Gamma S_1 \subseteqq S$. If $g \in S$ then $ge = \gamma e$ for some $\gamma \in \Gamma$. Thus $\gamma^{-1} g \in S_1$ and it follows that $g \in \Gamma S_1$ proving (b). If $\gamma = gh^{-1} \in \Gamma$ with $g, h \in S_1$ then

$$\gamma e = (gh^{-1})e = (gh^{-1})he = ge = e,$$

so $\gamma \in \Gamma_1$. This shows $\Gamma \cap S_1 S_1^{-1} = \Gamma_1$ which is (c). □

Theorem 2.4. *Let N, D be as in (1.4.1) and suppose l does not divide D. Let S, Γ_1, Γ_0 be as in proposition 2.3 and $\varepsilon: \Gamma_0/\Gamma_1 \to \mathbf{Z}_l^*$ be the unique character satisfying $ae = \varepsilon(a)e$ for all a in Γ_0.*

If $\mathscr{H}$ is commutative and $\bar{E}$ is irreducible as an $\mathbf{F}_l\Gamma$-module, then for each system of eigenvalues $\Phi: \mathscr{H} \to \mathcal{O}_\lambda$ occurring in $H^N(\Gamma, E \otimes \mathcal{O}_\lambda)$ there is a system $\Phi_1: \mathscr{H} \to \mathcal{O}_\lambda$ occurring in $H^N(\Gamma_1, \mathcal{O}_\lambda)(\varepsilon^{-1})$ such that $\bar{\Phi} = \bar{\Phi}_1$.

Proof. Let S_1, S_0 be as in proposition 2.3 and let $\varepsilon: S_0 \to \mathbf{Z}_l^*$ be the unique extension of ε which is trivial on S_1. Then the map $\psi: (\mathbf{F}_l) \to \bar{E}$ defined by $\psi(r) = re$ is S_0-equivariant by the definition of ε. We can therefore draw diagram 1.3.3 with $F = (\mathbf{Z}_l)_\varepsilon$ and $\bar{F} = (\mathbf{F}_l)_\varepsilon$.

The image of $\beta(\psi)$: $\mathrm{Ind}(\Gamma_0, \Gamma; (\mathbf{F}_l)_\varepsilon) \to \bar{E}$ contains e and is therefore a nonzero $\mathbf{F}_l\Gamma$-submodule of $\bar{E}$. Since $\bar{E}$ is irreducible we see that $\beta(\psi)$ is surjective. Theorem 1.3.5 (ii) tells us that B is surjective. Since $H^{N+1}(\Gamma_0, (\mathbf{Z}_l)_\varepsilon)$ vanishes, theorem 1.3.4 (a) shows that there is a system of eigenvalues Φ_1 occurring in $H^N(\Gamma_0, (\mathbf{Z}_l)_\varepsilon)$ for which $\bar{\Phi}_1 = \bar{\Phi}$. The theorem is now a consequence of lemma 1.1.5. □

Remark. For the sake of simplicity this result has been stated only for the case when $\bar{E}$ is irreducible. However a similar result is true for reducible $\bar{E}$ if we replace e by an element of some irreducible subquotient of $\bar{E}$ and define S, Γ_1, Γ_0 as before.

We close this section with an investigation of the irreducible representations of $G = GL(n)$. We first give a few preliminary definitions and remarks.

Let T be a Young tableau with $r=r(T)$ rows of lengths $g_1 \geqq \cdots \geqq g_r > 0$. Let $g=g(T)=g_1+\cdots+g_r$ and number the positions of T from 1 to g in lexicographic order. Let $\mathscr{S}_g$ be the symmetric group on $\{1,\ldots,g\}$. As in [36], Chapter IV, there is a unique idempotent $C_T \in \mathbf{Q}\mathscr{S}_g$ such that

$$pC_T = C_T, \quad C_T q = \operatorname{sgn}(q) C_T$$

for all $p, q \in \mathscr{S}_g$ such that p preserves the rows of T and q preserves the columns of T. In fact

$$C_T = \frac{1}{\mu_T} \sum_{p,q} \operatorname{sgn}(q) pq$$

where the sum is over all p, q as above and μ_T is a positive integer which divides $g!$.

Now let $G=GL(n)$ and $W=\mathbf{Q}^n$ be the standard representation of G over $\mathbf{Q}$. The symmetric group $\mathscr{S}_g$ acts on $\otimes^g W$ by permuting the factors. Let $W_T = C_T \cdot \otimes^g W$. As in [36], theorem 4.4F, it follows that a finite dimensional rational representation V of G is irreducible if and only if there is an integer v and a Young tableau T with $r(T)<n$ such that $V \cong W_T \otimes \det(\;)^v$.

We fix a tableau T with $r=r(T)<n$. We assume that the prime l is greater than $g=g(T)$, and as usual denote reduction modulo l by bar. Let $M=\mathbf{Z}^n \subseteqq W$ be the standard lattice and set $E_T = C_T \cdot \otimes^g M \subseteqq W_T$. Then M and E_T are l-rational $\mathbf{Z}\Gamma$-modules for every arithmetic group $\Gamma \subseteqq SL(n,\mathbf{Z})$.

Next we fix Γ to be $SL(n,\mathbf{Z})$. Then the reduction map $\Gamma \to SL(n,\mathbf{F}_l)$ is surjective and our assumption $l>g$ implies $\bar{E}_T$ is irreducible as an $\mathbf{F}_l\Gamma$-module. Define $\Gamma_1(l)$ to be the group of all $\gamma \in \Gamma$ which are congruent modulo l to an upper triangular matrix with ones on the diagonal.

We will identify $\bar{M}$ with the space of n-dimensional column vectors over $\mathbf{F}_l$ with the usual left action of $G(\mathbf{Z}_{(l)})$. Let $e_1={}^t(1,0,\ldots,0),\ldots,e_n={}^t(0,\ldots,0,1)$ be the standard basis of $\bar{M}$ and set

$$e_T = C_T \cdot \big((\otimes^{g_1} e_1) \otimes \cdots \otimes (\otimes^{g_r} e_r)\big) \in \bar{E}_T.$$

One can check that $\Gamma_1(l)$ fixes e_T.

We take for S the set of all $\sigma \in M_n(\mathbf{Z})$ with $l \nmid \det(\sigma)$. Then $S \subseteqq S(e)$. Transposition defines an anti-isomorphism of S which leaves invariant the double cosets $\Gamma\sigma\Gamma$. Thus $\mathscr{H}=\mathscr{H}(\Gamma,S)$ is *commutative* ([32], proposition 3.8).

Proposition 2.5. *Let V be an irreducible finite dimensional rational representation of $GL(n,\mathbf{Q})$, and let $\Phi\colon \mathscr{H} \to \mathcal{O}$ be a system of eigenvalues occurring in $H^N(\Gamma, V)$ with $N=\frac{n(n-1)}{2}$. Then for sufficiently large l and every $\lambda \subseteqq \mathcal{O}$ extending l, there is a group Γ_1 intermediate to $\Gamma_1(l) \subseteqq \Gamma$ and a system of eigenvalues Φ_1 occurring in $H^N(\Gamma_1, \mathcal{O}_\lambda)$ such that $\bar{\Phi}=\bar{\Phi}_1$.*

Proof. The integer N in the proposition is the virtual cohomological dimension of Γ. Let m and D be as in (1.4.1). Let T be a Young tableau and $v \geqq 0$ be such that $V \cong W_T \otimes \det(\;)^v$. The conclusion of the proposition then follows from theorem 2.4 if we let l be greater than $g(T)$ and relatively prime to mD, and take Γ_1 to be the stabilizer of e_T. □

3. An example

In this section we combine the results of the first section with the theory of automorphic representations to prove the existence of torsion classes in the cohomology of certain subgroups of $SL(3, \mathbf{Z})$ (see theorem 3. 5. 3). To prove this theorem we need three lemmas which in principle are "well known". The first (proposition 3. 2. 1) states that any system of Hecke eigenvalues occurring in the cohomology of the boundary of the Borel-Serre compactification corresponds to a *reducible* Galois representation. The second (lemma 3. 3. 2) is a vanishing statement for interior cohomology with coefficients in a non self-dual representation. The third (lemma 3. 4. 3) concerns the existence in cohomology of the symmetric squares lift of a classical modular eigenform.

All of this is most easily understood in an adelic setting. Thus we begin by adelizing the cohomology groups and their associated Hecke algebras.

Notation. We will write $\mathbf{A}$ for the adeles of $\mathbf{Q}$ and $\mathbf{A}_f$ for the finite adeles. If B is an adelic object then we let B_∞, B_f, B_p be the infinite, respectively finite, respectively p-adic components of B. For example, if $g \in GL(3, \mathbf{A})$ then $g_\infty \in GL(3, \mathbf{R})$, $g_f \in GL(3, \mathbf{A}_f)$, and $g_p \in GL(3, \mathbf{Q}_p)$. Similarly, if $\chi: \mathbf{A}^* \to \mathbf{C}^*$ then $\chi_\infty: \mathbf{R}^* \to \mathbf{C}^*$, $\chi_f: \mathbf{A}_f^* \to \mathbf{C}^*$, and $\chi_p: \mathbf{Q}_p^* \to \mathbf{C}^*$.

3. 1 Adelization of the cohomology

Let $G_3 = GL(3)$, Z be the center of G_3, $Z_\infty = Z(\mathbf{R})$, and $K_\infty = SO(3)$. Then $X = G_3(\mathbf{R})/Z_\infty K_\infty$ is the symmetric space of G_3. To each compact open subgroup $K_f \subseteqq G_3(\mathbf{A}_f)$ we associate the topological space

$$X_{K_f} = G_3(\mathbf{Q}) \backslash G_3(\mathbf{A})/Z_\infty K_\infty K_f.$$

This space has finitely many connected components and moreover, there are arithmetic subgroups $\Gamma_i \subseteqq SL(3, \mathbf{R})$, $i = 1, \ldots, h$, such that X_{K_f} is the disconnected union

$$X_{K_f} \cong \bigcup_{i=1}^{h} \Gamma_i \backslash X.$$

Let E be a finite dimensional irreducible representation over $\mathbf{C}$ of the group $G_3(\mathbf{R})$. The action will be a *left* action. We describe a sheaf $\tilde{E}_{K_f}$ over X_{K_f} by describing its local sections. Let $\pi: G_3(\mathbf{A})/Z_\infty K_\infty K_f \to X_{K_f}$ be the natural projection and let U be an open subset of X_{K_f}. Then

$$\tilde{E}_{K_f}(U) = \left\{ s: \pi^{-1}(U) \to E \;\middle|\; \begin{array}{l} s \text{ is locally constant and} \\ \text{for all } g \in G_3(\mathbf{Q}) \text{ and } x \in \pi^{-1}(U) \\ s(gx) = g_\infty \cdot s(x) \end{array} \right\}.$$

We can now form the cohomology groups $H^*(X_{K_f}, \tilde{E}_{K_f})$. If K_f' is a compact open subgroup of K_f then we have the pullback map

(3. 1. 1) $$H^*(X_{K_f}, \tilde{E}_{K_f}) \to H^*(X_{K_f'}, \tilde{E}_{K_f'}).$$

We adopt Harder's notation [15] and write

$$H^*(\tilde{X}, \tilde{E}) \overset{\text{def}}{=} \varinjlim_{K_f} H^*(X_{K_f}, \tilde{E}_{K_f})$$

even though the symbols $\tilde{X}$, $\tilde{E}$ are given no independent meaning.

Right translation by $g \in G_3(\mathbf{A}_f)$ induces maps

$$X_{K_f} \xrightarrow{R(g)} X_{g^{-1}K_f g}, \quad G_3(\mathbf{Q})xZ_\infty K_\infty K_f \longmapsto G_3(\mathbf{Q})xgZ_\infty K_\infty(g^{-1}K_f g)$$

which in turn induce isomorphisms in cohomology

$$H^*(X_{K_f}, \tilde{E}_{K_f}) \xleftarrow{R(g)^*} H^*(X_{g^{-1}K_f g}, \tilde{E}_{g^{-1}K_f g}) \tag{3.1.2}$$

by pullback. The maps $R(g)$ commute with the inclusions (3. 1. 1) in the obvious way. Thus the maps (3. 1. 2) induce an automorphism of $H^*(\tilde{X}, \tilde{E})$. The resulting map $G_3(\mathbf{A}_f) \to \operatorname{Aut}(H^*(\tilde{X}, \tilde{E}))$ is an admissible left action. Moreover, if K_f is a compact open subgroup of $G_3(\mathbf{A}_f)$ then

$$H^*(\tilde{X}, \tilde{E})^{K_f} \cong H^*(X_{K_f}, \tilde{E}_{K_f}). \tag{3.1.3}$$

For a positive integer N we define $K(N) = \{g \in \prod_p G_3(\mathbf{Z}_p) \mid g \equiv 1 \pmod N\}$ and say that a compact open subgroup $K_f \subseteqq G_3(\mathbf{A}_f)$ has level N if N is the least positive integer for which $K(N) \subseteqq K_f$.

For K_f of level N we set

$$S_{K_f} = K_f \cdot \prod_{p \nmid N} (G_3(\mathbf{Q}_p) \cap M_3(\mathbf{Z}_p)). \tag{3.1.4}$$

where $M_3(\mathbf{Z}_p)$ is the set of 3×3 matrices with entries in $\mathbf{Z}_p$. Then (K_f, S_{K_f}) is a Hecke pair and we may form the Hecke algebra $\mathscr{H} = \mathscr{H}(K_f, S_{K_f})$. For each prime p not dividing N let $T_{p,1}, T_{p,2}, T_{p,3} \in \mathscr{H}$ be the elements associated to the following double cosets:

$$\begin{aligned} T_{p,1} &\longleftrightarrow K_f \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_p K_f, \\ T_{p,2} &\longleftrightarrow K_f \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix}_p K_f, \\ T_{p,3} &\longleftrightarrow K_f \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}_p K_f. \end{aligned} \tag{3.1.5}$$

Then $\mathscr{H}$ is commutative and is generated by the set $\{T_{p,1}, T_{p,2}, T_{p,3} : p \nmid N\}$. As in § 1. 1 $\mathscr{H}$ acts on $H^*(\tilde{X}, \tilde{E})^{K_f}$ and thus also on $H^*(X_{K_f}, \tilde{E}_{K_f})$ by transport of structure.

The center $Z(\mathbf{A}_f)$ acts on $H^*(X_{K_f}, \tilde{E}_{K_f})$ as a group of nebentype operators. For $a \in \mathbf{A}_f^*$ we write $[a]_{K_f}$ for the nebentype operator associated to the central element aI. Let $\chi_\infty: Z_\infty \to \mathbf{C}^*$ be the central character of E. Then a simple calculation shows that for $a \in \mathbf{Q}^*$ we have

$$[a_f]_{K_f} = \chi_\infty(a_\infty). \tag{3.1.6}$$

Thus the characters of $Z(\mathbf{A}_f)$ occurring in $H^*(X_{K_f}, \tilde{E}_{K_f})$ are Hecke characters of type χ_∞ and conductor a divisor of N.

We would like to relate this adelic Hecke algebra to the Hecke algebras of sections 1 and 2. To do this it is convenient to impose the following condition on K_f:

$$\det(K_f) = \prod_p \mathbf{Z}_p^*. \tag{3.1.7}$$

Under this hypothesis, the strong approximation theorem for $SL(3)$ implies

$$K_f G_3^+(\mathbf{Q}) = G_3(\mathbf{A}_f)$$

where $G_3^+(\mathbf{Q})$ is the subgroup of $G_3(\mathbf{Q})$ of matrices with positive determinant. In particular if we set

$$\Gamma = K_f \cap G_3^+(\mathbf{Q}), \quad S = S_{K_f} \cap G_3^+(\mathbf{Q}) \tag{3.1.8}$$

then the Hecke pairs $(\Gamma, S) \subseteq (K_f, S_{K_f})$ are compatible and the natural map

$$\iota: \mathscr{H}(K_f, S_{K_f}) \to \mathscr{H}(\Gamma, S) \tag{3.1.9}$$

is an isomorphism. Moreover the map

$$X_\Gamma \to X_{K_f}, \quad \Gamma g_\infty K_\infty \mapsto G_3(\mathbf{Q})(g_\infty, 1) K_\infty K_f$$

is a homeomorphism, and the induced isomorphism in cohomology

$$H^*(X_\Gamma, \tilde{E}_\Gamma) \to H^*(X_{K_f}, \tilde{E}_{K_f}) \tag{3.1.10}$$

commutes with the action of $\mathscr{H}(K_f, S_{K_f})$.

Of special interest to us will be the groups

$$K_1(N) = \left\{ g \in \prod_p G_3(\mathbf{Z}_p) \,\middle|\, g \equiv \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \pmod N \right\},$$

$$K_0(N) = \left\{ g \in \prod_p G_3(\mathbf{Z}_p) \,\middle|\, g \equiv \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \pmod N \right\}, \tag{3.1.11}$$

$$\Gamma_1(3, N) = K_1(N) \cap G_3^+(\mathbf{Q}),$$
$$\Gamma_0(3, N) = K_0(N) \cap G_3^+(\mathbf{Q}).$$

Then $K_1(N)$, $K_0(N)$ satisfy (3. 1. 7) and we can apply the above remarks. The group $\Gamma_0(3, N)$ normalizes $\Gamma_1(3, N)$ so that the quotient group acts on the cohomology of $\Gamma_1(3, N)$ as a group of nebentype operators. For $a \in \prod_p \mathbf{Z}_p^*$ let $[a]$ denote the nebentype operator associated to an element of $\Gamma_0(3, N)$ whose lower right hand corner is congruent to a modulo N.

The following dictionary is easily established.

Proposition 3. 1. 12. *Under the identification*

$$H^*(X_{K_1(N)}, \tilde{E}_{K_1(N)}) = H^*(X_{\Gamma_1(3,N)}, \tilde{E}_{\Gamma_1(3,N)})$$

of (3. 1. 10) *we have the following identification of adelic Hecke operators and* $\Gamma_1(3, N)$ *double coset operators.*

For $p \nmid N$

(i) $T_{p,1} \longleftrightarrow \Gamma_1(3, N) \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Gamma_1(3, N)$,

(ii) $T_{p,2} \longleftrightarrow \Gamma_1(3, N) \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \Gamma_1(3, N)$.

For $a \in \prod_p \mathbf{Z}_p^*$

(iii) $[a]_{K_f} \longleftrightarrow [a]$.

3. 2 Cohomology at infinity

Let $K_f \subseteqq G_3(\mathbf{A}_f)$ be a compact open subgroup of level N satisfying (3. 1. 7) and let $\mathscr{H} = \mathscr{H}(K_f, S_{K_f})$ be the associated Hecke algebra. Let $\bar{X}_{K_f}$ be the Borel-Serre compactification [4] of X_{K_f}. We will write ∂X_{K_f} for the boundary of $\bar{X}_{K_f}$. If E is a finite dimensional irreducible representation of $G_3(\mathbf{R})$, then the sheaf $\tilde{E}_{K_f}$ over X_{K_f} extends to a sheaf on $\bar{X}_{K_f}$. In this section we examine the systems of Hecke eigenvalues occurring in the cohomology groups

$$H^*(\partial X_{K_f}, \tilde{E}_{K_f}).$$

As in the last section this is closely related to the problem of determining the structure of the $G_3(\mathbf{A}_f)$-module

$$H^*(\partial \tilde{X}, \tilde{E}) \stackrel{\text{def}}{=} \varinjlim_{K_f} H^*(\partial X_{K_f}, \tilde{E}_{K_f}).$$

We will prove the following result.

Proposition 3. 2. 1. *Let E be a finite dimensional irreducible complex representation of $G_3(\mathbf{R})$, let $K_f \subseteqq G_3(\mathbf{A}_f)$ be a compact open subgroup of level N, and let*

$$\Phi\colon \mathscr{H}(K_f) \to \mathbf{C}$$

be a system of eigenvalues occurring in

$$H^r(\partial X_{K_f}, \tilde{E}_{K_f}).$$

Set $b_{p,1}=\Phi(T_{p,1})$, $b_{p,2}=\Phi(T_{p,2})$ and $b_{p,3}=\Phi(T_{p,3})$ for each prime p not dividing N. Let F be a number field which contains all of these eigenvalues and $\mathcal{O}_\lambda$ be the completion of the integers of F at a prime λ with residue characteristic l.

Then there is a **reducible** *Galois representation*

$$\rho_\lambda\colon \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL(3, \mathcal{O}_\lambda)$$

unramified outside Nl such that for every $p \nmid Nl$ we have

$$\det(1-\rho_\lambda(\mathrm{Frob}_p)T) = 1 - b_{p,1}T + pb_{p,2}T^2 - p^3 b_{p,3}T^3.$$

The structure of the boundary ∂X_{K_f} is described in [21]. It is a union of three subspaces, W_{P_i,K_f}, $i=0,1,2$ associated to the three standard parabolic subgroups

$$P_0=\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}; \quad P_1=\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}; \quad P_2=\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

The space W_{P,K_f} is homotopically equivalent to the space $P(\mathbf{Q})\backslash G_3(\mathbf{A})/K_\infty K_f$. Moreover, we have

$$W_{P_0,K_f} = W_{P_1,K_f} \cap W_{P_2,K_f}.$$

Thus we may form the Mayer-Vietoris sequence as in [21] but with twisted coefficients.

$$\text{(3. 2. 2)} \qquad \cdots \to H^{r-1}(W_{P_0,K_f}, \tilde{E}_{K_f}) \to H^r(\partial X_{K_f}, \tilde{E}_{K_f}) \to H^r(W_{P_1,K_f}, \tilde{E}_{K_f}) \oplus H^r(W_{P_2,K_f}, \tilde{E}_{K_f}) \to \cdots$$

Passing to the inductive limit over K_f we obtain an exact sequence

$$\text{(3. 2. 3)} \qquad \cdots \to H^{r-1}(\tilde{W}_{P_0}, \tilde{E}) \to H^r(\partial\tilde{X}, \tilde{E}) \to H^r(\tilde{W}_{P_1}, \tilde{E}) \oplus H^r(\tilde{W}_{P_2}, \tilde{E}) \to \cdots$$

It is easy to check that the action of $G_3(\mathbf{A}_f)$ commutes with this sequence. It follows that (3. 2. 2) is a sequence of $\mathscr{H}$-modules.

We introduce a bit of notation. For an algebraic subgroup H of G_3 we will write X_{H,K_f} for the space

$$H(\mathbf{Q})\backslash H(\mathbf{A})/K_\infty^H K_f^H$$

where $K_\infty^H = K_\infty \cap H(\mathbf{R})$ and $K_f^H = K_f \cap H(\mathbf{A}_f)$. We will use the same symbol $\tilde{E}_{K_f}$ for the restriction of $\tilde{E}_{K_f}$ to X_{H,K_f} and set

$$H^r(\tilde{X}_H, \tilde{E}) \overset{\text{def}}{=} \varinjlim_{K_f} H^r(X_{H,K_f}, \tilde{E}_{K_f}).$$

Clearly, $H^r(\tilde{X}_H, \tilde{E})$ is a left $H(\mathbf{A}_f)$-module.

Lemma 3.2.4. *Let P be one of the groups P_0, P_1, P_2. Then there is a natural isomorphism of $G_3(\mathbf{A}_f)$-modules*

$$H^r(\tilde{W}_P, \tilde{E}) \cong \mathrm{Ind}_{P(\mathbf{A}_f)}^{G_3(\mathbf{A}_f)} H^r(\tilde{X}_P, \tilde{E}),$$

where Ind *is defined as in* 1.1 *except that functions are now required to be locally constant.*

Proof. This is a consequence of the structure of the boundary (see [21], esp. 3.6 (2), and compare [15], p. 117). □

Now write

$$P = L \cdot U$$

where L is the Levi component and U is the unipotent radical of P. Let $\mathcal{U}$ be the Lie algebra of U.

Lemma 3.2.5. *There is a natural isomorphism of $P(\mathbf{A}_f)$-modules*

$$H^r(\tilde{X}_P, \tilde{E}) \cong \bigoplus_{s+t=r} H^s(\tilde{X}_L, \tilde{H}^t(\mathcal{U}, E)).$$

Remark. Note in particular that the action of $U(\mathbf{A}_f)$ on these groups is trivial.

Proof. For each K_f consider the fibration

$$\begin{array}{ccc} X_{U,K_f} & \longrightarrow & X_{P,K_f} \\ & & \downarrow \\ & & X_{L,K_f}. \end{array}$$

Associated to this fibration there is a spectral sequence

$$H^s(X_{L,K_f}, \tilde{H}^t(X_{U,K_f}, \tilde{E}_{K_f})) \Rightarrow H^{s+t}(X_{P,K_f}, \tilde{E}_{K_f}).$$

By Van Est's theorem [35] we have an isomorphism $H^t(X_{U,K_f}, \tilde{E}) \cong H^t(\mathcal{U}, E)$. Thus after passing to the inductive limit over K_f our spectral sequence takes the form

$$E_2^{s,t} = H^s(\tilde{X}_L, \tilde{H}^t(\mathcal{U}, E)) \Rightarrow H^{s+t}(\tilde{X}_P, \tilde{E}).$$

Since $E_2^{s,t} = 0$ for $s > 1$ the spectral sequence degenerates at the E_2 term and the lemma follows. □

Proof of Proposition 3.2.1. If Φ is a system of eigenvalues occurring in $H^r(\partial X_{K_f}, \tilde{E}_{K_f})$ then (3.2.2) implies Φ occurs in $H^*(W_{P_i,K_f}, \tilde{E}_{K_f})$ for some $i = 0, 1,$ or 2. Let $P = P_i$ and $P = L \cdot U$ be its Levi decomposition. Lemmas 3.2.4 and 3.2.5 show that there is a choice of s and t such that Φ occurs in

$$(\mathrm{Ind}_{P(\mathbf{A}_f)}^{G_3(\mathbf{A}_f)} H^s(\tilde{X}_L, \tilde{H}^t(\mathcal{U}, E)))^{K_f}.$$

Consider first the case $P=P_1$. Then $L=L'\times T$ where $L'\cong GL(2)$ and $T\cong GL(1)$. By ([14], Theorem 2.6.1) we know that $H^i(\tilde{X}_T,\cdot)$ vanishes for $i>0$. Thus a simple spectral sequence argument yields an isomorphism

$$H^s(\tilde{X}_L, \tilde{H}^t(\mathcal{U}, E))\cong H^s(\tilde{X}_{L'}, \tilde{H}^0(\tilde{X}_T, \tilde{H}^t(\mathcal{U}, E))).$$

To simplify the notation we write F for the $L'(\mathbf{R})\times T(\mathbf{A}_f)$-module $H^0(\tilde{X}_T, \tilde{H}^t(\mathcal{U}, E))$. Then F decomposes into a sum of character spaces under the action of $T(\mathbf{A}_f)$:

$$F=\bigoplus_{\chi} F_\chi$$

where χ runs through Hecke characters of $T(\mathbf{A}_f)$. Thus our system of eigenvalues Φ occurs in one of the spaces

$$(\mathrm{Ind}_{P(\mathbf{A}_f)}^{G_3(\mathbf{A}_f)} H^s(\tilde{X}_{L'}, \tilde{F}_\chi))^{K_f}.$$

Now let φ be a Φ-eigenvector in this space. Write π for the representation of $P(\mathbf{A}_f)$ on $H^s(\tilde{X}_{L'}, \tilde{F}_\chi)$, and π' for the restriction of π to $L'(\mathbf{A}_f)$. Then φ may be viewed as a function

$$\varphi\colon G_3(\mathbf{A}_f)/K_f \longrightarrow H^s(\tilde{X}_{L'}, \tilde{F}_\chi)$$

satisfying $\varphi(bg)=\pi(b)\,\varphi(g)$ for all b in $P(\mathbf{A}_f)$. Since $P(\mathbf{A}_f)\cdot\prod_p G_3(\mathbf{Z}_p)=G_3(\mathbf{A}_f)$ there is a γ in $\prod_p G_3(\mathbf{Z}_p)$ for which $\varphi(\gamma)$ is nonzero. Fix a prime p not dividing the level of K_f and let T_p denote the Hecke operator associated to the double coset

$$L'(\mathbf{Z}_p)\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}_p L'(\mathbf{Z}_p)$$

in the usual way. We now show that $\varphi_0=\varphi(\gamma)$ is a T_p-eigenvector and express $b_{p,1}$ in terms of the eigenvalue. In this calculation we will use the fact that φ_0 is $L'(\mathbf{Z}_p)$-invariant and also the fact that φ is invariant under left translation by $U(\mathbf{Q}_p)$.

$$\begin{aligned}
b_{p,1}\varphi_0 &= (T_{p,1}\varphi)(\gamma)\\
&= \sum_{a,b=0}^{p-1}\varphi\left(\gamma\begin{pmatrix} p & a & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}_p\right)+\sum_{c=0}^{p-1}\varphi\left(\gamma\begin{pmatrix} 1 & 0 & 0\\ 0 & p & c\\ 0 & 0 & 1\end{pmatrix}_p\right)+\varphi\left(\gamma\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & p\end{pmatrix}_p\right)\\
&= \sum_{a,b=0}^{p-1}\varphi\left(\begin{pmatrix} p & a & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}_p\gamma\right)+\sum_{c=0}^{p-1}\varphi\left(\begin{pmatrix} 1 & 0 & 0\\ 0 & p & c\\ 0 & 0 & 1\end{pmatrix}_p\gamma\right)+\varphi\left(\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & p\end{pmatrix}_p\gamma\right)\\
&= \sum_{a,b=0}^{p-1}\pi'\begin{pmatrix} p & a\\ 0 & 1\end{pmatrix}_p\varphi_0+\sum_{c=0}^{p-1}\pi'\begin{pmatrix} 1 & 0\\ 0 & p\end{pmatrix}_p\varphi_0+\chi_p(p)\varphi_0\\
&= pT_p\varphi_0+\chi_p(p)\varphi_0.
\end{aligned}$$

Thus we see that φ_0 is a T_p-eigenvector. Let a_p be the eigenvalue. A similar calculation can be carried out for the operators $T_{p,2}$, $T_{p,3}$. If we let ψ be the central character of $L'(\mathbf{A}_f)$ acting on φ_0 then the result of these calculations is summarized by

$$b_{p,1} = pa_p + \chi_p(p),$$
$$b_{p,2} = \chi_p(p)a_p + p^2\psi_p(p),$$
$$b_{p,3} = \chi_p(p)\,\psi_p(p).$$

We know from $GL(2)$-theory [8] and from class-field theory respectively, that there are Galois representations

$$\rho_0\colon \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL(2, \mathcal{O}_\lambda),$$
$$\rho_\chi\colon \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL(1, \mathcal{O}_\lambda)$$

unramified outside Nl and satisfying

$$\det(1 - \rho_0(\mathrm{Frob}_p)T) = 1 - a_pT + p\psi_p(p)T^2,$$
$$\det(1 - \rho_\chi(\mathrm{Frob}_p)T) = 1 - \chi_p(p)T$$

for all $p \nmid Nl$. The action of the Galois group on l-power roots of unity gives us a character ω_{cycl} satisfying $\omega_{\mathrm{cycl}}(\mathrm{Frob}_p) = p$. Set $\rho_\lambda = (\rho_0 \otimes \omega_{\mathrm{cycl}}) \oplus \rho_\chi$. A simple calculation shows that ρ_λ satisfies the conclusion of the proposition.

The proofs for the cases $P = P_0$, P_2 are similar. □

3.3 Relative Lie algebra cohomology

Suppose E is a finite dimensional irreducible rational representation over $\mathbf{C}$ of $G_3(\mathbf{R})$. Then for compact open subgroups K_f of $G_3(\mathbf{A}_f)$ we have isomorphisms ([5], VII 2.5)

$$H^*(sl_3, so_3; C^\infty(G_3(\mathbf{Q})\backslash G_3(\mathbf{A})/Z_\infty K_f) \otimes E) \cong H^*(X_{K_f}, \tilde{E}_{K_f})$$

where sl_3, so_3 are the Lie algebras of $SL(3, \mathbf{R})$, $SO(3)$ respectively, and C^∞ denotes smooth functions. These isomorphisms commute with the maps (3.1.2). Thus, passing to the inductive limit over K_f we obtain an isomorphism of $G_3(\mathbf{A}_f)$-spaces

$$(3.3.1) \qquad H^*(sl_3, so_3; C^\infty(G_3(\mathbf{Q})\backslash G_3(\mathbf{A})/Z_\infty)^o \otimes E) \cong H^*(\tilde{X}, \tilde{E})$$

where $C^\infty(\ )^o$ is the space of $K(1)$-finite smooth functions. We will use this isomorphism in lemma 3.4.3 to compute eigenvalues of Hecke operators.

In 3.5 we will need the following vanishing result. Recall the definition of the interior cohomology $H_!$ from section 1.4.

Lemma 3.3.2. *Suppose E is not isomorphic to its own contragredient. Let Γ be an arithmetic subgroup of $SL(3, \mathbf{Q})$. Then*

$$H_!^*(\Gamma, E) = 0.$$

Proof. Any interior cohomology class $\varphi \in H_!^*(\Gamma, E)$ can be represented by a smooth compactly supported $\tilde{E}_\Gamma$-valued differential form on X_Γ. We choose an admissible scalar product on E as in ([3], § 5.1; [5], II § 2). Then φ can be represented by an L^2 harmonic form ([23], p. 165). Propositions 5.5 and 5.6 of [3] now show that φ is in the image of the canonical map

$$(3.3.3) \qquad H^*(sl_3, so_3; L^2_{\mathrm{dis}}(\Gamma \backslash SL(3, \mathbf{R}))^\infty \otimes E) \to H^*(\Gamma, E)$$

where $L^2_{\mathrm{dis}}(\ )^\infty$ is the space of smooth vectors in the discrete spectrum of L^2. The left hand side of (3.3.3) decomposes into a finite direct sum

$$(3.3.4) \qquad \bigoplus H^*(sl_3, so_3; H_i^\infty \otimes E)$$

where for each i, H_i^∞ is the space of smooth vectors in a complete irreducible unitary representation of $SL(3, \mathbf{R})$. But proposition 6.12 II of [5] and our assumption that E is not self dual imply that the space (3.3.4) vanishes. This proves $\varphi = 0$. □

3.4 Symmetric squares

For the rest of the paper g denotes a nonnegative integer and R is a ring in which $g!$ is invertible. If M is a free R-module of finite rank there is a canonical splitting

$$\overset{g}{\otimes} M = \mathrm{Sym}^g(M) \oplus W$$

where $\mathrm{Sym}^g(M)$ is the module of symmetric tensors. The natural isomorphism

$$(\overset{g}{\otimes} M)^* = \overset{g}{\otimes}(M^*)$$

induces an isomorphism $\mathrm{Sym}^g(M)^* = \mathrm{Sym}^g(M^*)$.

Definition 3.4.1. For $n \geqq 1$ let M_n be the left $GL(n, R)$-module of column vectors R^n and set $S_g^n(R) = \mathrm{Sym}^g(M_n)$.

If $X_1, \ldots, X_n$ is the standard basis for M_n we may identify $S_g^n(R)$ with the module of degree g polynomials over R in $(X_1, \ldots, X_n)$. The action of $\sigma \in GL(n, R)$ is given by $(\sigma F)(X_1, \ldots, X_n) = F((X_1, \ldots, X_n)\sigma)$. Similarly if $(\xi_1, \ldots, \xi_n)$ is the dual basis in M_n^* to $(X_1, \ldots, X_n)$ then $S_g^n(R)^*$ may be identified with degree g polynomials in $(\xi_1, \ldots, \xi_n)$. The $GL(n, R)$-action is given by

$$(\sigma G)(\xi_1, \ldots, \xi_n) = G((\xi_1, \ldots, \xi_n)\, {}^t\sigma^{-1})$$

where ${}^t\sigma$ is the transpose of σ.

Multiplication by $\Delta^n = \sum_{i=1}^{n} X_i \otimes \xi_i$ induces a $GL(n, R)$-morphism

$$S^n_{g-1}(R) \otimes S^n_{g-1}(R)^* \xrightarrow{\Delta^n_g} S^n_g(R) \otimes S^n_g(R)^*.$$

Definition 3. 4. 2. (a) $V_g(R)$ is the $GL(2, R)$-module $S^2_g(R)$.

(b) $W_g(R)$ is the $GL(3, R)$-module $\operatorname{coker}(\Delta^3_g)$.

Elements of $W_g(R)$ will be denoted by representatives in $S^n_g(R) \otimes S^n_g(R)^*$ when no confusion will arise.

Our assumption that $g!$ is invertible in R assures that $V_g(R)$ is an irreducible $GL(2, R)$-module and also that $W_g(R)$ is an irreducible $GL(3, R)$-module.

Note that $V_g(\mathbf{C})$ runs through all irreducible rational representations of $SL(2, \mathbf{C})$ and $W_g(\mathbf{C})$ through all self-dual irreducible representations of $SL(3, \mathbf{C})$ as $g = 0, 1, 2, \dots$.

Recall the definition of $H_!$ from section 1. 4 and the definition of $T_{p,1}$, $T_{p,2}$ from the end of 3. 1 where we take $N = 1$.

Lemma 3. 4. 3. *If θ in $H^1_!(SL(2, \mathbf{Z}), V_g(\mathbf{C}))$ is an eigenclass for all the Hecke operators T_p with eigenvalues a_p then there exists Θ in $H^3_!(SL(3, \mathbf{Z}), W_g(\mathbf{C}))$ which is an eigenclass for all the Hecke operators $T_{p,1}$ and $T_{p,2}$ with eigenvalues respectively $p^{-g}(a_p^2 - p^{g+1})$ and $p^{-g}(a_p^2 - p^{g+1})$* (sic).

Remark. Note that "eigenclass" implies by definition that θ and Θ are nonzero.

Proof. Given θ, there exists a holomorphic cusp form in the classical sense of weight $g+2$ for $SL(2, \mathbf{Z})$ with the same Hecke eigenvalues as θ ([32], Chapter 8). Corresponding to the latter we have an irreducible cuspidal automorphic representation $\pi = \bigotimes \pi_v$ of $GL(2, \mathbf{A})$ with trivial central character ([11], 5. 19). The local representation π_∞ is the discrete series representation with lowest weight $g+2$ and trivial central character; for each prime p, π_p is a principal series representation $\pi(\mu_p, \mu_p^{-1})$ where μ_p is an unramified unitary character satisfying $a_p = p^{\frac{g+1}{2}}(\mu_p(p) + \mu_p^{-1}(p))$ ([11], 5. 21). By [25] we know that π is not a monomial representation.

Gelbart and Jacquet ([10], theorem 3; [12], theorem 9. 3) have shown that given a nonmonomial π as in the last paragraph there exists an irreducible cuspidal automorphic representation $\Pi = \bigotimes \Pi_v$ of $GL(3, \mathbf{A})$ which is a "symmetric square lift" of π. This means that for each prime p, Π_p is the principal series representation $\Pi(\mu_p^2, 1, \mu_p^{-2})$ ([12], § 3). The representation Π_∞ is described in terms of the associated representation of the Weil group. This can be translated to an explicit description of Π_∞ using theorem 4. 4. 1 of [18]. In this way one finds that Π_∞ is induced from the standard parabolic subgroup $P \subsetneqq GL(3)$ of type (2, 1) as follows. Let $P(\mathbf{R}) = {}^0M \cdot A \cdot N$ be the Langlands decomposition of $P(\mathbf{R})$ ([5], III 3. 2). Then ${}^0M$ is isomorphic to

$$SL^{\pm}(2, \mathbf{R}) = \{g \in GL(2, \mathbf{R}) \mid \det(g) = \pm 1\}.$$

Let σ_{2g+3} be the discrete series representation of ${}^0M$ with lowest weight $2g+3$ and χ_0 be the trivial character of AN. Then Π_∞ is the unitarily induced representation $I_{P,\sigma_{2g+3},\chi_0}$ ([5], III 3. 2).

Since Π is a direct summand of the space $L_0^2(G_3(\mathbf{Q})\backslash G_3(\mathbf{A})/Z_\infty)^\infty$ of smooth L^2 cuspidal functions we have an inclusion of $G_3(\mathbf{A}_f)$-modules

$$H^3(sl_3, so_3; \Pi \otimes W_g(\mathbf{C})) \subseteq H^3(sl_3, so_3; L_0^2(\quad)^\infty \otimes W_g(\mathbf{C})).$$

Taking $K(1)$-invariants we obtain an inclusion of $\mathscr{H}(K(1), S_{K(1)})$-modules

$$(3.4.4) \qquad H^3(sl_3, so_3; \Pi_\infty \otimes W_g(\mathbf{C})) \otimes \bigotimes_p \Pi_p^{G_3(\mathbf{Z}_p)}$$
$$\hookrightarrow H^3(sl_3, so_3; L_0^2(G_3(\mathbf{Q})\backslash G_3(\mathbf{A})/Z_\infty K(1))^\infty \otimes W_g(\mathbf{C})).$$

By ([2], cor. 5.5) the natural map from this latter group to $H^3(SL(3, \mathbf{Z}, W_g(\mathbf{C}))$ is injective and has image contained in $H_!^3(SL(3, \mathbf{Z}), W_g(\mathbf{C}))$.

On the other hand a calculation based on ([5], III 3.3) proves

$$H^3(sl_3, so_3; \Pi_\infty \otimes W_g(\mathbf{C})) \cong \mathbf{C}.$$

Clearly $\Pi_p^{G_3(\mathbf{Z}_p)}$ is one dimensional for each prime p. Hence the left hand side of (3.4.4) is isomorphic to $\mathbf{C}$. The eigenvalues of $T_{p,1}$ and $T_{p,2}$ acting on this space are easily calculated and seen to be those given in the statement of the proposition. □

3.5 Torsion

If l is a rational prime, the Teichmüller character is the unique character $\omega: \mathbf{F}_l^* \to \mathbf{Z}_l^*$ satisfying the congruence $\omega(a) \equiv a \pmod{l}$. Evaluation of ω on the lower right entry of a matrix in $\Gamma_0(3, l)$ induces a character $\Gamma_0(3, l)/\Gamma_1(3, l) \to \mathbf{Z}_l^*$ which we will also denote by ω.

Proposition 3.5.1. *Let θ be as in Lemma 3.4.3, $l > g$ a rational prime, $l \neq 2, 3$, and K a finite extension of $\mathbf{Q}$. Let λ be a prime of K lying over l and $\mathcal{O}_\lambda$ the ring of integers in K_λ. For any $p \neq l$, let $T_{p,1}$ and $T_{p,2}$ denote the Hecke operators associated to $\Gamma_1(3, l)$ as in 3.1. Set $b_p = p^{-g}(a_p^2 - p^{g+1})$.*

(1) *Then for a suitable choice of K there is a nonzero*

$$\Theta^0 \in H^3(\Gamma_1(3, l), S_g^3(\mathcal{O}_\lambda))(\omega^g)$$

such that Θ^0 is an eigenvector for all $T_{p,1}$ and $T_{p,2}$, $p \neq l$, with eigenvalues $b_{p,1}$ and $b_{p,2}$ respectively, satisfying the congruence $b_{p,1} \equiv b_{p,2} \equiv b_p \pmod{\lambda}$.

(2) *If we fix g and θ, then for sufficiently large l the class Θ^0 is a torsion class.*

Proof. We make the following assignments:

$$\Gamma = SL(3, \mathbf{Z}), \quad \Gamma_0 = \Gamma_0(3, l),$$
$$S = S_{K(1)} \cap G_3^+(\mathbf{Q}),$$
$$S_0 = S_{K_0(l)} \cap G_3^+(\mathbf{Q}).$$

As in Lemma 1. 2. 4, we know that there is a finite extension K of $\mathbf{Q}$ which will contain all the Hecke-eigenvalues appearing in cohomology groups we need to deal with. We fix such a K and set $R=\mathcal{O}_\lambda$, $P=(\lambda)$.

By lemma 3. 4. 3 there is a class Θ in $H^3(SL(3,\mathbf{Z}), W_g(R))$ such that for every prime $p \neq l$, Θ is an eigenvector for $T_{p,1}$ and $T_{p,2}$ with eigenvalues $b_p = p^{-g}(a_p^2 - p^{g+1})$.

We define $\psi\colon S_g^3(\bar{R}) \otimes R_{\omega^{-g}} \to W_g(\bar{R})$ by $\psi(F \otimes 1) = F \otimes \xi_3^g$ (notation of 3. 4). This is an S_0-morphism.

As in 1. 3 we have an RS-morphism $\beta(\psi)\colon \mathrm{Ind}(\Gamma_0, \Gamma, S_g^3(\bar{R}) \otimes \bar{R}_{\omega^{-g}}) \to W_g(\bar{R})$ and we may draw diagram 1. 3. 3. Clearly, $X_1^g \otimes \xi_3^g$ is a nonzero element of the image of $\beta(\psi)$. Since $\bar{E}$ is irreducible we must have $\beta(\psi)$ is surjective. By theorem 1. 3. 5, B^N is surjective. Assertion (1) follows from theorem 1. 3. 4 (a) and lemma 1. 1. 5.

A theorem of Deligne [8] proves that there is a two dimensional irreducible λ-adic representation σ_λ of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ which is unramified outside l and such that the characteristic polynomial of $\sigma_\lambda(\mathrm{Frob}_p)$ is $1 - a_p T + pT^2$. Let σ_λ^2 be the symmetric square of this representation and let ω_{cycl} be the character of the Galois group acting on l-power roots of unity. Then one easily verifies the identity

$$\det(1 - (\sigma_\lambda^2 \otimes \omega^{-g})(\mathrm{Frob}_p)) = 1 - b_p T + p b_p T^2 - p^3 T^3.$$

By a theorem of Ribet [25], $\sigma_\lambda(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})) \bmod \lambda$ contains $SL(2, \mathcal{O}/\lambda)$ for almost all l. For such l, $\sigma_\lambda^2(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})) \bmod \lambda$ contains $SL(3, \mathcal{O}/\lambda)$. Hence there is a $B>0$ such that whenever $l > B$, $\sigma_\lambda^2 \bmod \lambda$ is irreducible.

Now fix $l > B$. We claim that Θ^0 is a torsion class. For suppose Θ^0 were not torsion. Then by lemma 3. 3. 2 we know that the restriction of Θ^0 to the boundary does not vanish. Let $b_{p,i}$ be the eigenvalues of $T_{p,i}$, $i=1,2,3$, acting on Θ^0. Then theorem 3. 2. 1 says that there is a reducible three dimensional λ-adic Galois representation ρ_λ such that

$$\det(1 - \rho_\lambda(\mathrm{Frob}_p)T) = 1 - b_{p,1}T + p b_{p,2} T^2 - p^3 b_{p,3} T^3.$$

By (1) we have congruences $b_{p,1} \equiv b_{p,2} \equiv b_p \pmod{\lambda}$. Using (3. 1. 12 (iii)) and (3. 1. 6) we find $b_{p,3} = p^g \omega(p)^{-g}$. In particular $b_{p,3} \equiv 1 \pmod{\lambda}$. Thus we have a congruence

$$\det(1 - \rho_\lambda(\mathrm{Frob}_p)T) \equiv \det(1 - (\sigma_\lambda^2 \otimes \omega^{-g})(\mathrm{Frob}_p)T) \pmod{\lambda}$$

for every prime $p \neq l$. But ρ_λ is reducible and $\sigma_\lambda^2 \otimes \omega^{-g}$ is irreducible, so by the Čebotarev density theorem this contradicts the Brauer-Nesbitt theorem. □

Lemma 3. 5. 2. *Let f and f' be two classical holomorphic cusp forms of the same weight k for the full modular group $SL(2,\mathbf{Z})$. Assume each is an eigenform for all the Hecke operators T_p with eigenvalues a_p and a'_p respectively, for all primes p. Suppose $a_p = \pm a'_p$ for every p. Then f and f' are proportional.*

Proof. We prove the lemma by using the principle of [25] that the l-adic representations of f and f' are "as independent as possible."

Let $E = \mathbf{Q}[a_p | p \text{ prime}] = \mathbf{Q}[a'_p | p \text{ prime}]$ and $\mathcal{O}$ be the ring of integers of E. For a prime l let $\mathcal{O}_l = \mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_l$.

We first show that there is a $\sigma \in \mathrm{Aut}(\mathbf{C})$ such that $\sigma(a_p) = a'_p$ for all primes p. For suppose there is no such σ. Let $(\alpha, \beta) \in \mathcal{O} \times \mathcal{O}$ such that $E \times E = \mathbf{Q}[(\alpha^2, \beta^2)]$. For sufficiently large l we have $\mathbf{Z}_l[(\alpha^2, \beta^2)] = \mathcal{O}_l \times \mathcal{O}_l$ and by theorem 6.1 of [25] there is a prime p such that $(a_p, a'_p) \equiv (\alpha, \beta) \pmod{l}$. Then $(a_p^2, a_p'^2)$ generates $E \times E$ over $\mathbf{Q}$. This contradicts our hypothesis $a_p = \pm a'_p$.

By lemma 4.8 of [25] there is a prime p such that $\mathbf{Q}[a_p^2] = E$. Since

$$\sigma(a_p) = a'_p = \pm a_p$$

we have $\sigma|E$ is the identity map on E. In particular $\sigma(a_p) = a_p = a'_p$, proving the lemma. □

Theorem 3.5.3. *Let g be a fixed positive integer and set $d(g)$ = the dimension of the space of holomorphic cusp forms for $SL(2, \mathbf{Z})$ of weight $g+2$. Then if l is sufficiently large,*

$$\dim_{\mathbf{F}_l} H^3(\Gamma_1(3, l), S_g^3(\mathbf{Z}))_{l\text{-torsion}} \geqq d(g).$$

Proof. The space of cusp forms mentioned above is naturally a sub-Hecke module of $H_!^1(SL(2, \mathbf{Z}), V_g(\mathbf{C}))$ by Eichler-Shimura. Set $d = d(g)$. By lemmas 3.4.3 and 3.5.2 there are d linearly independent Hecke eigenclasses $\Theta_1, \ldots, \Theta_d$ in $H_!^3(SL(3, \mathbf{Z}), W_g(\mathbf{C}))$ with eigenvalues $p^{-g}(a_p(i)^2 - p^{g+1})$, $i = 1, \ldots, d$.

Let l be greater than g, large enough so that the d infinite-dimensional vectors

$$(\ldots, \bar{p}^{-g}(\bar{a}_p(i)^2 - \bar{p}^{g+1}), \ldots)$$

are distinct (bar denotes reduction mod λ), and large enough so that the conclusion of (2) of Proposition 3.5.1 is valid. Then corresponding to these systems of eigenvalues there are d linearly independent eigenclasses $\Theta_1^0, \ldots, \Theta_d^0$ in

$$H^3(\Gamma_1(3, l), S_g^3(\mathcal{O}_\lambda))_{\lambda\text{-torsion}}.$$

The theorem now follows immediately. □

Remark. In particular, the dimension of the l-torsion in $H^3(\Gamma_1(3, l), S_g^3(\mathbf{Z}))$ becomes arbitrarily large as $g \to \infty$ and l is sufficiently large. Compare this with the fact that there is *no* l-torsion in $H^1(\Gamma_1(2, l), S_g^2(\mathbf{Z}))$ whenever $l > g$.

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