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p-adic deformations of cohomology classes of subgroups of $GL(n, \mathbb{Z})$

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Abstract

We construct p-adic analytic families of p-ordinary cohomology classes in the cohomology of arithmetic subgroups of $GL(n)$ with coefficients in a family of representation spaces for $GL(n)$. These analytic families are parametrized by the highest weights of the coefficient modules. More precisely, we consider the cohomology of a compact $\mathbb{Z}_p$-module $\mathbb{D}$ of p-adic measures on a certain homogeneous space of $GL(n, \mathbb{Z}_p)$. For any dominant weight λ with respect to a fixed choice (B, T) of a Borel subgroup B and a maximal split torus $T \subseteq B$ and for any finite "nebentype" character $\epsilon : T(\mathbb{Z}_p) \longrightarrow \mathbb{Z}_p^\times$ we construct a $\mathbb{Z}_p$-map from $\mathbb{D}$ to $V_{\lambda,\epsilon}$. These maps are equivariant for commuting actions of $T(\mathbb{Z}_p)$ and Γ_ν where $\Gamma_\nu \subseteq GL(n, \mathbb{Z})$ is a congruence subgroup analogous to $\Gamma_0(p^\nu)$ where p^ν is the conductor of ϵ. We also make the matrix $\pi := \mathrm{diag}(1, p, p^2, \ldots, p^{n-1})$ act equivariantly on all these modules. We obtain a $\Lambda := \mathbb{Z}_p[[T(\mathbb{Z}_p)]]$-module structure on $H^*(\Gamma, \mathbb{D})$ and Hecke actions on $H^*(\Gamma, \mathbb{D})$ and $H^*(\Gamma_\nu, V_{\lambda,\epsilon})$ with Hecke equivariant maps $\phi_{\lambda,\epsilon} : H^*(\Gamma, \mathbb{D}) \longrightarrow H^*(\Gamma_\nu, V_{\lambda,\epsilon})$, where Γ is a congruence subgroup of $GL(n, \mathbb{Z})$ of level prime to p and Γ_ν is one of a certain family of congruence subgroups of Γ with p in their level. Let $\phi^0_{\lambda,\epsilon}$ denote the map induced by $\phi_{\lambda,\epsilon}$ on the $\Gamma\pi\Gamma$-ordinary part of $H^*(\Gamma, \mathbb{D})$. Our main theorem states that the kernel of $\phi^0_{\lambda,\epsilon}$ is $I_{\lambda,\epsilon} H^*(\Gamma, \mathbb{D})^0$ where $I_{\lambda,\epsilon}$ is the kernel of the ring homomorphism induced on Λ by the character $\lambda\epsilon$.

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In the conference the first author's talk had two parts. The first was a survey of computational results concerning the cohomology of $GL(3,\mathbb{Z})$ and its congruence subgroups. The content of this part can be found in the references [1–6] and [10].

The second part presented a control theorem on the ordinary part of a p-adic deformation of the cohomology of congruence subgroups of $GL(3,\mathbb{Z})$, following the lines of a reinterpretation of some of H. Hida's work for $GL(2)$ due to R. Greenberg and the second author [11]. These ideas work just as well for $GL(N)/\mathbb{Z}$, in fact for any Chevalley group, and perhaps for any arithmetic subgroup of any reductive $\mathbb{Q}$-group. One should mention here a recent announcement of J. Tilouine and E. Urban [18] in which they get similar and more complete results for the group GSp(4) of symplectic similitudes.

We have chosen in this paper to restrict ourselves to $GL(n)/\mathbb{Z}$ in order to make the exposition as clear as possible.

The principal innovations compared with the $GL(2)$ case include the following: (1) Finding the right way to make the Hecke pair act on a suitable coset space since now more than just the top row of a matrix must be considered. In particular, it doesn't seem possible to make the whole semigroup of matrices of p-power determinant to act simultaneously. (2) Therefore, the definition of ordinary may be weaker than asking for all the Hecke operators at p to act invertibly. (Hida uses the same notion of ordinary as we do, and calls it "p-nearly ordinary".) (3) Working with cohomology classes becomes trickier because now the cohomology is a subquotient of the cochains, whereas in the $GL(2)$ case the cohomology could be viewed as a subset of modular symbols.

We consider a congruence subgroup Γ of $GL(n,\mathbb{Z})$ of level prime to p and a coefficient module L_V which is a lattice in $V(\mathbb{Q}_p)$, where V is an irreducible rational representation of $GL(n)$. We consider $H^*(\Gamma, L_V)$ as a module for a ring H of Hecke operators. To construct a p-adic deformation of this cohomology group, we form a large $\mathbb{Z}_p$-module $\mathbb{D}$ of measures on a certain coset space and maps from $\mathbb{D}$ to L_V, for varying V's. These induce H-maps on the cohomology and we determine the kernels on the ordinary parts.

More precisely, we let Γ_ν denote the intersection of Γ with a certain subgroup of $GL(n,\mathbb{Z})$ of level $p^{(n-1)\nu}$. Then we construct

$$\phi_V^0\colon H^*(\Gamma,\mathbb{D})^0 \to H^*(\Gamma_\nu, L_V)^0$$

where the superscript 0 denotes the ordinary part (defined below).

To specify the kernel, let Λ denote the completed group ring $\mathbb{Z}_p[[T(\mathbb{Z}_p)]]$, where T is the torus of diagonal matrices in $GL(n)$. Then $\mathbb{D}$ has a Λ-module structure commuting with the Γ-action which induces a Λ-action on $H^*(\Gamma,\mathbb{D})$ commuting with the H-action. Let λ be the highest weight of V (with respect to an appropriately chosen Borel subgroup containing T) and let I_λ denote the kernel of the $\mathbb{Z}_p$-algebra homomorphism $\lambda^\dagger$ from Λ to $\mathbb{Z}_p$ induced by λ. Then our main theorem states that the kernel of ϕ_V^0 equals $I_\lambda H^*(\Gamma,\mathbb{D})^0$. In the actual theorem (5.1) we include a nebentype character.

We also can assert the surjectivity of ϕ_V^0 under certain hypotheses.

The theorem should be thought of as giving p-adic deformations as follows: If α is a Hecke-eigenclass in $H^*(\Gamma,\mathbb{D})^0$, then its images in $H^*(\Gamma_\nu,L_V)^0$ as V varies will form a family of cohomology classes whose Hecke eigenvalues will be congruent modulo powers of p that depend on what congruences obtain among the λ's modulo powers of p. Of course you must maintain control on when the images are nonzero, and that is what the theorem does.

Since non-torsion Hecke eigenclasses in $H^*(\Gamma_\nu,L_V)^0$ give rise to irreducible automorphic representations on $GL(n)$, we are also getting p-adic deformations of certain automorphic forms.

In Hida's work, such a theorem is called a control theorem. In fact, Hida [12] has results similar to ours and in some ways stronger. However, our approaches and results differ significantly in the details. For example, our approach is dual to Hida's: our map $\phi_{\lambda,\varepsilon}$ in section 4 is the Pontryagin dual to a map considered by Hida. We also think that our measure-theoretical methods may more easily be applied to studying the non-ordinary case. In this connection, see the recent work of Coleman [9] and the second author [16], [17].

The first author thanks J. Tilouine for an illuminating conversation at the conference, and for bringing Hida's paper, of which we had been unaware, to our attention. We also thank Steve Rallis for reminding us of the "ord" function, and we thank the referee for helpful comments.

Outline:

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Section 1: The space of measures. The ring Λ. Koszul complexes

We introduce some notation. All the algebraic groups below should be thought of as group schemes over $\mathbb{Z}$:

$G = GL(n)$

$N =$ lower triangular unipotent matrices in G

$T =$ diagonal matrices in G

$B = TN = NT$

We denote by a raised "o" the transposed group. For example $B^o = TN^o$ is the opposite Borel subgroup to B.

$I =$ Iwahori subgroup of $G = \{g \in G(\mathbb{Z}_p) \mid g \bmod p \in B \bmod p\}$

$I = N(\mathbb{Z}_p)T(\mathbb{Z}_p)U$ where $U = (N^o \cap I)$

$W =$ permutation matrices in $G =$ Weyl group in G

$G(\mathbb{Z}_p) = IWI = \cup IC(w)$ (disjoint union of open and closed subsets)

$X' = N(\mathbb{Z}_p)\backslash N(\mathbb{Z}_p)B^o(\mathbb{Z}_p) \subset X = N(\mathbb{Z}_p)\backslash G(\mathbb{Z}_p) \subseteq Y = N(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$

Therefore, we have that $X \cong \cup T(\mathbb{Z}_p)UC(w)$ (disjoint union of open and closed "cells"), and the "big cell" in this decomposition is $X' = T(\mathbb{Z}_p)UC(w') = N(\mathbb{Z}_p)\backslash N(\mathbb{Z}_p)I^o$, where w' is the long Weyl element.

We have $T(\mathbb{Z}_p)$ acting on Y on the left, leaving X stable, and $G(\mathbb{Z}_p)$ acting on Y on the right, leaving X stable.

For any space Z, let

Step $(Z) = \{$loc. const. $f: Z \to \mathbb{Z}_p$ with compact supp.$\}$

$\mathbb{D}_Z =$ Hom$_{\mathbb{Z}_p}$ (Step (Z), $\mathbb{O}$) $= \mathbb{O}$-valued measures on Z.

In particular we set

$\mathbb{D} = \mathbb{D}_X$.

The action of $g \in G(\mathbb{Q}_p)$ on $\mu \in \mathbb{D}_Y$ is given by the formula $\int_Y f(x)d(\mu g)(x) = \int_Y f(xg)d\mu(x)$. If Z is a measurable subset of Y we will use the notation $\int_Z f(x)d\mu(x)$ for $\int_Y ch_Z(x)f(x)d\mu(x)$ where ch_Z is the characteristic function of Z. We view $\mathbb{D}_Z$ as a subset of $\mathbb{D}_X$ via the extension of measures by 0.

We define the analog of the Iwasawa algebra:

$\Lambda = \mathbb{Z}_p[[T(\mathbb{Z}_p)]]$ (completed group ring).

Then Λ acts on $\mathbb{D}$ via the left action of $T(\mathbb{Z}_p)$ on X.

We fix a topological generator γ of $\mathbb{Z}_p^\times$, and set γ_i to be the diagonal matrix with eigenvalues 1 except for γ at the i^{th} place. For any element t in $T(\mathbb{Z}_p)$, we let $[t]$ denote its natural image in Λ.

If $\chi: T(\mathbb{Z}_p) \to \mathbb{O}^\times$ is a character, where $\mathbb{O}$ is the ring of integers in a finite extension of $\mathbb{Q}_p$, we let $\chi^\dagger$ denote the unique extension of χ to a continuous $\mathbb{Z}_p$-algebra homomorphism from Λ to $\mathbb{O}$. We then set $I_\chi = \ker(\chi^\dagger)$.

We adopt the following convention: whenever we are implicitly viewing $\mathbb{D}$ as an $\mathbb{O}$-module, as in the lemma below, or in the notation $I_\chi\mathbb{D}$, we will still write $\mathbb{D}$ for the extension of scalars $\mathbb{D}\otimes\mathbb{O}$. We trust this will cause no confusion.

Lemma 1.1

The sequence $(\ldots,[\gamma_i]-\chi(\gamma_i),\ldots)$ is a $\mathbb{D}$-regular sequence in $\Lambda\otimes\mathbb{O}$.

Proof. The space X is a disjoint union of open and closed subsets of the form $T(\mathbb{Z}_p)\times Z$, so that $\mathbb{D}$ is a corresponding direct product of modules. Hence without loss of generality, we can replace $\mathbb{D}$ by $\mathbb{D}_{T(\mathbb{Z}_p)\times Z}$. Also, Λ acts through the first factor, leaving Z alone.

For this paragraph let T be any profinite group, with a basis $\{U\}$ of open subgroups of finite index, and let A be any abelian group. Then $A[[T]]=\lim A[T/U]=$ {coherent sequences of elements $\alpha=\{\sum a_g(gU)\}$} is isomorphic to the space of A-valued measures on T by the map that sends α to the measure μ such that $\mu(gU)=a_g$. If A is an $\mathbb{O}$-module, this isomorphism is equivariant for $\mathbb{O}[[T]]$.

Now let $T=T(\mathbb{Z}_p)$ and $A=\mathbb{D}_Z=\lim A_m$, where $A_m=$ Functions $(Z_m,\mathbb{O})\approx\mathbb{O}^{N(m)}$, where $Z=\lim Z_m$ and $N(m)=\operatorname{card}(Z_m)$. Then the A-valued measures on T in this case become $\lim\operatorname{Hom}_{\mathbb{O}}(\operatorname{Step}(T)\otimes\mathbb{O},A_m)=\lim\operatorname{Hom}_{\mathbb{Z}_p}(\operatorname{Step}(T),\mathbb{Z}_p)\otimes A_m=\lim\Lambda\otimes A_m$ (where Λ acts through the first factor in the tensor product).

Moreover, we have a Λ-equivariant isomorphism from $\mathbb{D}$ to {A-valued measures on T} sending μ to $(F\to\nu(F,\mu))$ where $F\in\operatorname{Step}(T)$ and $\int_Z h\,d\nu(F,\mu)=\int_X F(t)h(z)d\mu(tz)$ for any $h\in\operatorname{Step}(Z)$.

Therefore, it suffices to show that the given sequence is $\Lambda\otimes\mathbb{O}$-regular. Letting ψ run through the characters of $T(\mu_{p-1}(\mathbb{Z}_p))$ and letting e_ψ denote projection onto the ψ-eigenspace of Λ, we need to show that

$$(\ldots,e_\psi([\gamma_i]-\chi(\gamma_i)),\ldots)$$

is a regular sequence in $\Lambda_\psi\otimes\mathbb{O}\approx\mathbb{O}[[t_1,\ldots,t_n]]$ where t_i is the image of the diagonal matrix with eigenvalues 1 except for $[1+p]-1$ at the i^{th} place. Then under this isomorphism our sequence has the form $(\ldots,a_it_i+b_i,\ldots)$ with a_i, b_i in $\mathbb{O}$, $a_i\neq 0$. This is certainly regular. □

For later use we prove here a cohomological lemma.

Lemma 1.2

Let R be a commutative ring, G a group, M a right RG-module, I an ideal of R. Suppose I is generated by an M-regular sequence $(x_1,\ldots,x_r)$. Then the image of the map

$$i_*\colon H^*(G,IM)\to H^*(G,M)\,,$$

induced by the inclusion $i\colon IM\to M$, equals $IH^*(G,M)$.

Proof. Define the Koszul complex $K(X)$ as follows:
$K_0(X) =$ R;
$K_p(X) =$ free R-module with basis $\{e_{i(1)} \wedge \ldots \wedge e_{i(p)}\}$, $i(1) < \cdots < i(p)$;
with boundary maps defined by

$$d(e_{i(1)} \wedge \ldots \wedge e_{i(p)}) = \sum (-1)^{j-1} x_{i(j)} e_{i(1)} \wedge \ldots \wedge e^*_{i(j)} \wedge \cdots \wedge e_{i(p)}\,,$$

where the $*$ denotes the omission of a term.

Set $K(X, M) = K(X) \otimes M$. We have the exactness of the following sequence (e.g. p. 596 of Lang[15]):

$$0 \to K_r(X, M) \to \cdots \to K_1(X, M) \to M \to M/IM \to 0\,,$$

where the penultimate nontrivial map on the right is

$$\phi\colon K_1(X, M) \approx M^r \to M\,; \text{ where } \phi(m_i, \ldots, m_r) = \sum x_i m_i\,.$$

We turn this into a cochain complex by defining $C^{-i} = K_{i+1}(X, M)$, $i = 0, \ldots, r$. Then

$$0 \to C^{-r} \to \cdots \to C^{-1} \to C^0 \to 0$$

is a complex whose cohomology $H^*(C)$ is IM if $* = 0$ and 0 otherwise, and the isomorphism $H^0(C) \approx IM$ is induced by ϕ. Another way to say this is that $C \to IM^\#$ is a weak equivalence of RG-complexes, where for any RG-module N, we let $N^\#$ denote the complex concentrated in degree 0 and equal to N there.

It then follows from the cohomology version of Proposition 5.2, p. 169 of K. Brown's book [8], that ϕ induces an isomorphism on cohomology:

$$\phi_*\colon H^*(G, C) \to H^*(G, IM^\#) = H^*(G, IM).$$

We have that ϕ induces a map of RG-complexes $C \to M^\#$, call it p, so that $p = i\phi$:

$$\begin{array}{ccc} & & IM^\# \\ & \overset{\phi}{\nearrow} & \\ C & & \downarrow i \\ & \underset{p}{\searrow} & \\ & & M^\# \end{array}\,.$$

Therefore, looking at the induced maps on cohomology, we see that $Im(i_*) = Im(p_*)$. Since plainly $Im(i_*) \supset IH^*(G, M)$, what we must do is show that $Im(p_*) \subseteq IH^*(G, M)$.

Recall that $C^{-i} = K_{i+1}(X) \otimes M$ and G acts only through M.

Let $F \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. Then by definition, $H^*(G,C)$ is the cohomology of

$$\mathrm{Hom}_G^*(F,C) = \bigoplus_{p+q=*} \mathrm{Hom}_G(F_q, C^p) = \bigoplus_{p+q=*} K_{1-p}(X) \otimes \mathrm{Hom}_G(F_q, M) = \bigoplus_{p+q=*} J^{p,q}.$$

Now the double complex $J^{p,q}$ leads to two spectral sequences (cf. pp. 168-70 of [8]) of which one is

$$E_1^{p,q} = K_{1-p}(X) \otimes H^q(G,M) \Rightarrow H^{p+q}(G,C).$$

Similarly we obtain a spectral sequence

$$E_1^{p,q} = H^q(G,(M^\#)^p) \Rightarrow H^{p+q}(G,M)$$

and the map $p\colon C \to M^\#$ induces a map on the spectral sequences, hence on their abutments, which are the associated gradeds of $H^*(G,C)$ and $H^*(G,M)$ respectively. We are using Brown's conventions, so the filtrations are all increasing, and the filtration degree p runs from $-r$ to 0, 0 being the top degree.

So we have $p_*\colon H^*(G,C) \to H^*(G,M)$ preserving filtrations, so that $p_*(\mathrm{Fil}_{-1}H^*(G,C)) = 0$. Therefore, $Im(p_*) = Im(Gr^0 p_* \mid E_\infty)$. But E_∞ is a subquotient of E_1, so that $Im(p_*)$ is contained in $Im(Gr^0 p_* \mid E_1) =$

$$Im\, p_*\colon \bigoplus_q K_1(X) \otimes H^q(G,M) \to H^q(G,M)$$

under the map $\sum e_i \otimes \beta_i \to \sum x_i \beta_i$. Thus $Im(p_*) \subseteq IH^*(G,M)$. $\square$

Remark. It is true that the spectral sequence $E_1^{p,q} = K_{1-p}(X) \otimes H^q(G,M) \Rightarrow H^{p+q}(G,C)$ need not degenerate at E_1 unless (x_i) is also $H^*(G,M)$-regular. Thus $H^*(G,C) = H^*(G,IM)$ may in fact be "larger" than $IH^*(G,M)$. But taking its image in $H^*(G,M)$ kills the "extra part".

Section 2: Highest weights and representations

Now we let λ denote the highest weight of an irreducible right rational G-module V with respect to (B,T). From the theory of the highest weight (e.g., Howe's Schur lecture [13] or Jantzen's book [14]), we know that V can be realized in the space of N-invariant regular functions on G. Let us so embed $V(\mathbb{Q}_p)$ as a module of $\mathbb{Q}_p$-valued regular functions on $N(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$, where $g \in G(\mathbb{Q}_p)$ acts on the right of a function f by $(fg)(x) = f(xg^{-1})$.

Now let v be a highest weight vector of $V(\mathbb{Q}_p)$, so that $vn = v$ for $n \in N(\mathbb{Q}_p)$ and $vt = \lambda(t)v$ for $t \in T(\mathbb{Q}_p)$. Since $G(\mathbb{Z}_p)$ is compact, without loss of generality we may assume that the fractional ideal generated by the values of v on $G(\mathbb{Z}_p)$ is exactly $\mathbb{Z}_p$. We make this normalization so that we can reduce the functions modulo p, although we will not need to do so in this paper.

Since $N(\mathbb{Z}_p)B^o(\mathbb{Z}_p)$ is Zariski dense in $G(\mathbb{Q}_p)$, it follows that the $\mathbb{Q}_p$-linear span of the translates of v by $B^o(\mathbb{Z}_p)$ is all of $V(\mathbb{Q}_p)$. Then the $\mathbb{Z}_p$-span of the translates of v by $B^o(\mathbb{Z}_p)$ is a $B^o(\mathbb{Z}_p)$-stable $\mathbb{Z}_p$-lattice L in $V(\mathbb{Q}_p)$ consisting of functions whose values on $G(\mathbb{Z}_p)$ lie in $\mathbb{Z}_p$. We see no reason that L should be $G(\mathbb{Z}_p)$-invariant, although it will be so for sufficiently large p. What we need is for L to be I^o-invariant, where I^o is the Iwahori subgroup of $G(\mathbb{Z}_p)$ defined in Section 1. Because $I^o = (N(\mathbb{Z}_p) \cap I^o)T(\mathbb{Z}_p)B^o(\mathbb{Z}_p)$, L can also be described as the $\mathbb{Z}_p$-span of the translates of v by I^o, and therefore is I^o-stable.

We shall also write $L = L_V = L_\lambda$, which is thus a right I^o-module, finitely generated and free as $\mathbb{Z}_p$-module.

We need to extend this I^o-action on L to the semigroup generated by I^o and the diagonal matrix $\pi = \operatorname{diag}(1, p, p^2, \ldots, p^{n-1})$. Let the usual action of $G(\mathbb{Z}_p)$ on $V(\mathbb{Q}_p)$ be denoted by juxtaposition, and define the twisted action of π by

$$w * \pi = \lambda^{-1}(\pi)(w\pi)\,.$$

If $w = vb$, for some b in $B^o(\mathbb{Z}_p)$, then we have that

$$w * \pi = \lambda^{-1}(\pi)(vb\pi) = v\pi^{-1}b\pi \in L\,,$$

since π normalizes $B^o(\mathbb{Z}_p)$. Thus $L * \pi \subset L$.

It is easy to see that this $*$ action together with the usual action of I^o extends to an action of the whole semigroup S^* generated by I^o and π. The reason is that the determinant can be used to keep track of the π's in a given element of S^*. Specifically, if $s \in S^*$, set $w * s = \lambda^{-d}(\pi)(ws)$, where $\det(s) = \det(\pi^d)$.

It seems that the impossibility of extending this further to an action of the whole p-part of the semigroup S is the obstruction to incorporating the whole Hecke algebra at p into our construction.

Section 3: Arithmetic groups, Hecke actions, ord

Now we fix a congruence subgroup Γ of $G(\mathbb{Z})$ of level prime to p, i.e., Γ contains the principal congruence subgroup of level N for some N prime to p. We also fix a Hecke pair (Γ, S) where S is some semigroup in $G^+(\mathbb{Q}) \cap M_n(\mathbb{Z})$ where the plus denotes positive determinant. We assume that $SS^{-1} \cap G(\mathbb{Z}) = \Gamma$, which is a harmless condition satisfied by all Hecke pairs in practical use.

Let H^* denote the Hecke algebra $H(\Gamma, S)$. We assume that away from the level N, H^* is standard, i.e., generated as a polynomial algebra by $T(s) = \Gamma s \Gamma$, where s runs over the diagonal matrices

$$d(l,k) = \operatorname{diag}(l, \ldots, l, 1, \ldots, 1) \quad (k\ l's)$$

for primes l not dividing N. If $s = d(l,k)$, we denote $T(s)$ by $T(l,k)$. We let S' denote the subsemigroup of S consisting of those matrices with determinant prime to p. Note that $S' \subset G(\mathbb{Z}_p)$. We let $H' = H(\Gamma, S')$ and H be the subalgebra of H^* generated by H' and $T(\pi)$.

For each integer $\nu \geq 0$ we set

$$\Gamma_\nu = \{g \in \Gamma \mid g \text{ is upper triangular modulo } p \text{ and } p^{(i-j)\nu} \text{ divides } g_{ij} \text{ for all } i > j\}.$$

We assume given a semigroup S'_ν so that (Γ_ν, S'_ν) and (Γ, S') are compatible Hecke pairs. Note that S'_ν is contained in the Iwahori subgroup I^o. We then set

$$S_\nu = \text{semigroup generated by } S'_\nu \text{ and } \pi.$$

We let $H'_\nu = (\Gamma_\nu, S'_\nu)$ and $H_\nu = (\Gamma_\nu, S_\nu)$. We have an isomorphism from H to H_ν which we will use to identify these two Hecke algebras.

Now $G(\mathbb{Z}_p)$ and hence S' act on the right of $Y = N(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$ in the natural way. We make π act on Y as follows:

$$N(\mathbb{Q}_p)g * \pi = N(\mathbb{Q}_p)\pi^{-1}g\pi.$$

This makes sense since π normalizes $N(\mathbb{Q}_p)$. As before, this extends to an action of the semigroup S^π generated by S' and π , so $H = H(\Gamma, S^\pi)$. Note that X is stable under S' but not under π. On the other hand, X', being the image of I^o or alternatively of $B^o(\mathbb{Z}_p)$ in X, is stable under all of S_ν, including π. Most importantly, we have:

Lemma 3.1

Let G^π denote the semigroup generated by $G(\mathbb{Z}_p)$ and π, and let Y' denote the smallest subset of Y containing X and stable under G^π. Then $Y' - X$ is also stable under G^π. Moreover, if $x \in X$ and $x * \pi \in X$, then $x \in X'$.

Proof. First we have to define and discuss the function ord from $G(\mathbb{Q}_p)$ to $\mathbb{Z}^n$. For any element $a \in \mathbb{Q}_p^\times$, let $v(a)$ denote its p-adic valuation, so $v(a) = 0$ if and only if a is a unit, and $v(p) = 1$. The Iwasawa decomposition tells us that $G(\mathbb{Q}_p) = B(\mathbb{Q}_p)G(\mathbb{Z}_p)$, and if $g \in G(\mathbb{Q}_p)$ is written as bk, with $b \in B(\mathbb{Q}_p)$ and $k \in G(\mathbb{Z}_p)$, then the valuations of the diagonal entries of b are uniquely determined. Hence we define:

DEFINITION 3.2. $\operatorname{ord} g = (v(b_{11}), \ldots, v(b_{nn}))$.

Clearly ord has the following properties:

(1) ord descends to a function on $Y = N(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$;

(2) $X = \operatorname{ord}^{-1}((0, \ldots, 0))$;

(3) $\operatorname{ord}(tg) = \operatorname{ord} t + \operatorname{ord} g$, if $t \in T(\mathbb{Q}_p)$;

(4) $\operatorname{ord}(gk) = \operatorname{ord} g$, if $k \in G(\mathbb{Z}_p)$.

Given g as above, we can compute its ord as follows: Let Λ^i denote the i-th fundamental representation of $GL(n)$ and e^i the highest weight vector in it (with respect to (B, T)). Set, for any $w \in \mathbb{Q}_p^N$, $|w| = \min(v(w_1), \ldots, v(w_n))$. Then if $(d_1, \ldots, d_n) = (|e^1\Lambda^1 g|, \ldots, |e^n\Lambda^n g|)$, we have $\operatorname{ord} g = (d_1, d_2 - d_1, \ldots, d_n - d_{n-1})$. In more pedestrian terms, one computes $d_1 = |\text{first row of } g|$, and in general,

$$d_i = |\ i\text{-by-}i \text{ minors of the first } i \text{ rows of } g|.$$

We put a partial ordering on $\mathbb{Z}^n$ by $(d_1, \ldots, d_n) \geq (c_1, \ldots, c_n)$ if and only if $d_i \geq c_i$ for all i.

Now we compute $\operatorname{ord}(g\pi)$. Suppose $\operatorname{ord} g = (d_1, \ldots, d_n)$. Consider

$$|e^i\Lambda^i(g\pi)| = |e^i(\Lambda^i g)(\Lambda^i \pi)|.$$

Now $\Lambda^i\pi$ is diagonal with first entry $p^{1+2+\cdots+(i-1)}$ and all other entries strictly greater powers of p. It follows that $|e^i\Lambda^i(g\pi)| \geq |e^i\Lambda^i(g)| + 1 + 2 \cdots + (i-1)$, with equality if and only if the principal (left-most) i-by-i minor gives the min in $|e^i\Lambda^i(g)|$. Therefore we have

(5) $\operatorname{ord}(g\pi) \geq \operatorname{ord} g + (0, 1, \ldots, n-1)$, with equality if and only if the principal i-th minor of g has minimum valuation among all the i-by-i minors of the top i rows of g.

(6) $\operatorname{ord}(\pi^{-1}g\pi) \geq \operatorname{ord} g$.

Now to show that $Y' - X$ is G^π-stable. Since both Y' and X are G-stable, we need only worry about π. From (6) we see that if g represents an element y of Y', then $\operatorname{ord} g \geq \operatorname{ord} 1 = 0$. Now if $y * \pi = \pi^{-1}g\pi$ lies in X, then we have $0 = \operatorname{ord}(\pi^{-1}g\pi) \geq \operatorname{ord} g \geq 0$, and therefore y lies in X already.

To prove the last statement of the lemma, suppose x is represented by $g \in G(\mathbb{Z}_p)$. Then the hypothesis that $x * \pi \in X$ implies that $\operatorname{ord}(\pi^{-1}g\pi) = 0 = \operatorname{ord} g$. Therefore all the principal minors of g are units. Then premultiplying g by an element of $N(\mathbb{Z}_p)$ can make g into an element of $B^o(\mathbb{Z}_p)$, i.e., we have that $x \in X'$. □

We make S^π act on $\mathbb{D} = \mathbb{D}_X$ as follows: We have maps

$$\mathbb{D} \xrightarrow{i} \mathbb{D}_{Y'} \xrightarrow{p} \mathbb{D}$$

where the first map is extension by zero of a measure and the second map is induced by projection of a measure from Y' to X. We let $s \in S^\pi$ act on $\mu \in \mathbb{D}$ by setting $\mu * s = p((i\mu)s)$ back to $\mathbb{D}$, where juxtaposition is induced by the usual action of $G(\mathbb{Q}_p)$ on $\mathbb{D}_Y$. This defines an action, since by Lemma 3.1 the kernel of p is stable under S^π.

We let ε denote a nebentype character (which may be trivial), i.e., a character $\varepsilon: T(\mathbb{Z}/p^\nu) \to \mathbb{O}^\times$. We can also then denote by ε the induced characters $\varepsilon: S'_\nu \to \mathbb{O}^\times$ and $\varepsilon: B^o(\mathbb{Z}_p) \to \mathbb{O}^\times$ where $\varepsilon(x)$ depends only on the values of the diagonal entries of x modulo p^ν, and also the induced function $\varepsilon: X' \to \mathbb{O}^\times$ where $\varepsilon(N(\mathbb{Z}_p)g) = \varepsilon(g)$ for $g \in B^o(\mathbb{Z}_p)$. We extend ε from S'_ν to S_ν by setting $\varepsilon(\pi) = 1$.

Given a representation V as above containing the lattice L, we can twist the action of S_ν on L by ε, which we denote by $L(\varepsilon) = L \otimes \mathbb{O}_\varepsilon$.

Finally, we obtain actions of the Hecke algebra H on $H^*(\Gamma, \mathbb{D})$ and $H^*(\Gamma_\nu, L_\lambda(\varepsilon))$, which we will compare in the next section.

Section 4: Construction of the map $\phi_{\lambda,\varepsilon}$. Verification that $\phi_{\lambda,\varepsilon}$ is Hecke equivariant

Given a highest weight λ, we define

$$\phi_{\lambda,\varepsilon}: \mathbb{D} \to L_\lambda(\varepsilon)$$

by

$$\phi_{\lambda,\varepsilon}(\mu) = \int_{X'} \varepsilon(x) v x \, d\mu(x).$$

It's easy to check that the integrand makes sense since $vN(\mathbb{Z}_p) = v$. The integral makes sense since the integrand is a continuous function of x. Since we may take x running over $B^o(\mathbb{Z}_p)$, the image of $\phi_{\lambda,\varepsilon}$ does lie in $L_\lambda(\varepsilon)$.

Lemma 4.1

The map $\phi_{\lambda,\varepsilon}$ is equivariant with respect to S_ν.

Proof. First, if $s \in S_\nu$ has determinant prime to p, we have

$$\begin{aligned}\phi_{\lambda,\varepsilon}(\mu s) &= \int_{X'} \varepsilon(x) vx \, d(\mu s)(x) = \int_{X'} \varepsilon(xs)\, d\mu(x) \\ &= \varepsilon(s) \left[\int_{X'} \varepsilon(x) vx \, d\mu(x) \right] s = \phi_{\lambda,\varepsilon}(\mu) s.\end{aligned}$$

We now check the equivariance with respect to π:

$$\begin{aligned}\phi_{\lambda,\varepsilon}(\mu\pi) &= \int_{X'} \varepsilon(x) vx \, d(\mu\pi)(x) = \int_{X'} \varepsilon(x * \pi) v(x * \pi) d\mu(x) \\ &= \int_{B^o(\mathbb{Z}_p)} \varepsilon(\pi^{-1} x \pi) v(\pi^{-1} x \pi)\, d\mu(x) \\ &= \lambda^{-1}(\pi) \left[\int_{X'} \varepsilon(x) vx \, d\mu(x) \right] \pi = \phi_{\lambda,\varepsilon}(\mu) * \pi. \ \square\end{aligned}$$

Proposition 4.2

The map $\phi_{\lambda,\varepsilon}$ induces an H-equivariant map on cohomology:

$$\phi_{\lambda,\varepsilon} \colon H^*(\Gamma, \mathbb{D}) \to H^*\big(\Gamma_\nu, L_\lambda(\varepsilon)\big)\,.$$

Proof. This follows in the usual way from Lemma 4.1 for Hecke operators away from p. To prove it for $T(\pi)$ we make an explicit calculation with single cosets.

We use the following notation here and hereafter in connection with the computation of Hecke operators.

Fix a positive integer m and write $\Gamma\pi^m\Gamma$ as the disjoint union of single cosets ΓE_t. We may and shall assume that each E_t is upper triangular, integral, with determinant equal to $\det(\pi^m)$ and with diagonal entries positive powers of p. We denote those E_t that have diagonal part $= \operatorname{diag}(1, p^m, p^{2m}, \ldots, p^{(n-1)m})$ by A_i, and those which do not by B_j. One checks then that $\Gamma_\nu \pi^m \Gamma_\nu$ is the disjoint union of single cosets $\Gamma_\nu A_i$.

We fix a resolution F_* of $\mathbb{Z}_p$ by free, finitely generated $\mathbb{Z}_p\Gamma$-modules. We use F to compute the cohomology of Γ and $\pi^{-m}\Gamma\pi^m \cap \Gamma$ in terms of cochains. For the group $\pi^{-m}\Gamma\pi^m$ we use $F^\bullet$ where the underlying groups of $F^\bullet$ are the same as in F, and the group action is defined by $f^\bullet \pi^{-m}\gamma\pi^m = f\gamma$. We also fix a homotopy equivalence τ between the two $\pi^{-m}\Gamma\pi^m \cap \Gamma$-resolutions F and $F^\bullet$.

By definition, the Hecke operator T_{π^m} on the cohomology of Γ with coefficients in an S_ν-module M equals $tr \circ res \circ \pi^m$, where $\pi^m \colon H^*(\Gamma, M) \to H^*(\pi^{-m}\Gamma\pi^m, M)$ is induced by conjugation $\pi^{-m}\Gamma\pi^m \to \Gamma$ $(\gamma \mapsto \pi^m \gamma \pi^{-m})$ and the right action on M

$(x \mapsto x\pi^m)$, $res: H^*(\pi^{-m}\Gamma\pi^m, M) \to H^*(\pi^{-m}\Gamma\pi^m \cap \Gamma, M)$ is the restriction map and $tr: H^*(\pi^{-m}\Gamma\pi^m \cap \Gamma, M) \to H^*(\Gamma M)$ is the transfer map. The definition of T_{π^m} on the cohomology of Γ_ν is similar.

It's not hard to translate this definition into an operator on cochains z in $\mathrm{Hom}_\Gamma(F_k, M)$. We compute the transfer as in (B) p. 81 of [8], except that we are using right modules and Brown is using left modules. We choose coset representatives for $\pi^{-m}\Gamma\pi^m \cap \Gamma\backslash\Gamma$ to be $\{\gamma_t = \pi^{-m}E_t\}$ and obtain:

Formulae 4.3

For a cocycle z in $\mathrm{Hom}_\Gamma(F_k, M)$, $T_{\pi^m} z$ is represented by the cochain

$$f \mapsto \Sigma_t z\big(\tau(f\gamma_t^{-1})\big)E_t.$$

Similarly, if z is a cocycle for Γ_ν in $\mathrm{Hom}_{\Gamma_\nu}(F_k, M)$, $T_{\pi^m} z$ is represented by the cochain

$$f \mapsto \Sigma_i z\big(\tau(f\gamma_t^{-1})\big)A_i.$$

Using these formulae when $m = 1$, we see that for the proof of the proposition in the case of $T(\pi)$, it suffices to show that for any $\mu \in \mathbb{D}$

(1) for every i, $X' * A_i \subset X'$;

(2) for every j, the integral $\int_{X'} \varepsilon(x)vx\,d\mu * B_j(x)$ vanishes.

For (2) it suffices to show that $X'B_j$ is disjoint from X, considering our definition of the $*$ action on measures.

So let $C \in B^o(\mathbb{Z})$ be of the form $k\pi k'$, with $k, k' \in G(\mathbb{Z}_p)$ and with all diagonal entries nonnegative powers of p. Then if $b \in B^o(\mathbb{Z}_p)$ represents $x \in X'$, we have $x * C$ represented by $\pi^{-1}bC$. This is in $B^o(\mathbb{Q}_p)$ and its diagonal entries are $p^{e(1)}b_1, p^{e(2)-1}b_2, \ldots, p^{e(n)-(n-1)}b_n$, where $p^{e(1)}, p^{e(2)}, \ldots, p^{e(n)}$ are the diagonal entries of C.

If C is one of the A_i, obviously $\pi^{-1}bC \in B^o(\mathbb{Z}_p)$ again and $x^*C \in X'$. If C is one of the B_j's, then for some i, $e(i) - (i-1) \neq 0$, so $\mathrm{ord}\,(\pi^{-1}bC) \neq 0$ and $x^*C \notin X$. □

Section 5: The ordinary part. Statement of the control theorem

Recall the definition of Γ_ν from Section 3. We now fix $\nu \geq 1$. We also have chosen a representation V with highest weight λ with respect to (B, T), a lattice L_λ, and a nebentype character ε.

Let A be a compact $\mathbb{Z}_p$-module with an operator U acting continuously on it. We define the ordinary part of A to be $A^0 = \cap U^i A$, where i runs over all positive integers. Then A^0 is the largest submodule of A on which U acts invertibly, and $A \to A^0$ is an exact functor.

We apply this to the map $\phi_{\lambda,\varepsilon}$ to obtain an induced map on ordinary cohomology:

$$\phi^0_{\lambda,\varepsilon}\colon H^*(\Gamma,\mathbb{D})^0 \to H^*\big(\Gamma_\nu, L_\lambda(\varepsilon)\big)^0,$$

where we take for U the operator $T_\pi = \Gamma\pi\Gamma$. Now $H^*(\Gamma_\nu, L_\lambda(\varepsilon))$ is finitely generated over $\mathbb{Z}_p$, so compact. To see that $H^*(\Gamma,\mathbb{D})$ is compact, compute the cohomology using a $\mathbb{Z}_p\Gamma$-resolution F of $\mathbb{Z}_p$ which is free and finitely generated over $\mathbb{Z}_p$ in each dimension. Such exist (see for example Brown's book [8]). Since $\mathbb{D}$ is compact, so are the cochains in each dimension, and hence so is the cohomology. Incidentally, compactness of the cochains also implies that the coboundary operators have closed images as well as kernels. It is easy to see that the Hecke operators act continuously.

Recall that Λ acts on $H^*(\Gamma,\mathbb{D})$ commuting with the Hecke action, so that $H^*(\Gamma,\mathbb{D})^0$ is a compact Λ-module.

Theorem 5.1

The kernel of the map

$$\phi^0_{\lambda,\varepsilon}\colon H^*(\Gamma,\mathbb{D})^0 \to H^*\big(\Gamma_\nu, L_\lambda(\varepsilon)\big)^0$$

is $I_{\lambda\varepsilon}H^*(\Gamma,\mathbb{D})^0$ *where* $I_{\lambda\varepsilon}$ *denotes the kernel of the* $\mathbb{Z}_p$*-algebra homomorphism* $(\lambda\varepsilon)^\dagger$ *from* Λ *to* $\mathbb{Z}_p$ *induced by* $\lambda\varepsilon$.

We will prove this in the following sections.

Corollary 5.2

$H^*(\Gamma,\mathbb{D})^0$ *is finitely generated as* Λ*-module.*

Proof. The space $\mathbb{D}$ is endowed with a natural topology making $\mathbb{D}$ a compact Λ-module. This topology induces topologies on the spaces of $\mathbb{D}$-valued Γ-cocycles and Γ-coboundaries and both are compact spaces. In particular, the space of coboundaries is a closed subspace of the space of cocycles, hence the cohomology group $H^*(\Gamma,\mathbb{D})$ naturally inherits a structure of compact topological Λ-module and consequently $H^*(\Gamma,\mathbb{D})^0$ is also a compact Λ-module. Now by Theorem 5.1, $H^*(\Gamma,\mathbb{D})^0 \otimes \Lambda/I_\chi$ is finitely generated as $\mathbb{O} = \Lambda/I_\chi$-module. Using a simple compactness argument, we see that $H^*(\Gamma,\mathbb{D})^0$ is generated by any finite subset that generates modulo I. □

Suplement 5.3

(i) If $\alpha \in H^*(\Gamma_\nu, L_\lambda(\varepsilon))^0$ is in the image of $\phi^0_{\lambda,\varepsilon}$, then the package of Hecke eigenvalues attached to α can be lifted to $H^*(\Gamma,\mathbb{D})^0$, at least after localization.

(ii) if $* = n(n-1)/2$, and if p doesn't divide the torsion in Γ, then $\phi^0_{\lambda,\varepsilon}$ is surjective.

Proof. (i) this purely algebraic fact will be proved in section 8.

(ii) Since $v = n(n-1)/2$ is the virtual cohomological dimension of Γ, then $H^{v+1}(\Gamma, M) = 0$ for any Γ-module M on which the torsion primes of Γ are invertible. Letting I denote the induced module $I(\Gamma_\nu, \Gamma, L_\lambda(\varepsilon))$ and invoking Shapiro's lemma and the long exact sequence of cohomology we get that

$$H^v(\Gamma,\mathbb{D}) \to H^v\big(\Gamma_\nu, L_\lambda(\varepsilon)\big) \to H^{v+1}(\Gamma, M) = 0$$

is exact. □

Section 6: Proof of Theorem 5.1

From Lemma 1.1 and 1.2 we have that $I_{\lambda\varepsilon}H^*(\Gamma,\mathbb{D})^0$ is the image of $H^*(\Gamma, I_{\lambda\varepsilon}\mathbb{D})^0$ in $H^*(\Gamma,\mathbb{D})^0$.

Let's compute the cohomology using the resolution F chosen in the previous section. Then an element $\beta \in H^*(\Gamma,\mathbb{D})^0$ will be represented by a cocycle b in $\mathrm{Hom}_{\mathbb{Z}_p\Gamma}(F_*,\mathbb{D})$. That element is in the image of $H^*(\Gamma, I_{\lambda\varepsilon}\mathbb{D})^0$ if and only if b can be chosen so that it takes values in $I_{\lambda\varepsilon}\mathbb{D}$.

First we prove that $H^*(\Gamma, I_{\lambda\varepsilon}\mathbb{D})$ is contained in the kernel of $\phi_{\lambda,\varepsilon}$. Suppose that β is represented by a cocycle b which takes values in $I_{\lambda\varepsilon}\mathbb{D}$. Then for any $f \in F_*$, $b(f)$ is a measure and

$$\phi_{\lambda,\varepsilon}\big(\beta(f)\big) = \text{coh. class of} \int_{X'} \varepsilon(x) vx \, db(f)(x) \, .$$

By assumption $b(f)$ is a linear combination of measures of the form $([\gamma_i] - \lambda\varepsilon(\gamma_i))\mu_i$, where the γ_i are generators of $T(\mathbb{Z}_p)$ chosen in Section 1. So it will be enough to show that if $\gamma \in T(\mathbb{Z}_p)$ and $\mu \in \mathbb{D}$, then

$$\int_{X'} \varepsilon(x) vx \, d\big([\gamma] - \lambda\varepsilon(\gamma)\big)\mu(x) = 0 \, .$$

But the left hand integral equals

$$\int_{X'} \varepsilon(x) vx d[\gamma]\mu(x) - \lambda\varepsilon(\gamma) \int_{X'} \varepsilon(x) vx d\mu(x) \, .$$

On the other hand,

$$\int_{X'} \varepsilon(x) vxd[\gamma]\mu(x) = \int_{X'} \varepsilon(\gamma x) v\gamma xd\mu(x) = \lambda\varepsilon(\gamma)\int_{X'} \varepsilon(x) vxd\mu(x).$$

The harder part of the proof is to show that $H^*(\Gamma, I_{\lambda\varepsilon}\mathbb{D})$ contains the kernel of $\phi_{\lambda,\varepsilon}$.

Recall that we have chosen a resolution F_* of $\mathbb{Z}_p$ by free, finitely generated $\mathbb{Z}_p\Gamma$-modules.

Lemma 6.1

Let $\{E_t\}$ denote the coset representatives of $\Gamma\pi^m\Gamma$ chosen in the proof of Lemma 4.2, where for later reference we further suppose $A_1 = E_1 = \pi^m$. Let $\zeta \in H^k(\Gamma, \mathbb{D})$ be represented by a cochain z in $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$ that takes values in a given subset $\mathbb{D}^*$ of $\mathbb{D}$. Then for any $m \geq 1$, $T_{\pi^m}(\zeta)$ can be represented by a cochain $z_m = \sum c_{m,t}$, where $c_{m,t}$ is a function (not necessarily a cochain) in $Fns(F_*, \mathbb{D})$ that takes values in $\mathbb{D}^* E_t$.

Proof. This follows immediately from Formulae 4.3. □

DEFINITION 6.2. Given the positive integer m and the character $\chi: T(\mathbb{Z}_p) \to \mathbb{O}^\times$, define $\psi_{m,\chi}: X \to \mathbb{O}$ by

$$\psi_{m,\chi}(x) = \begin{cases} \chi(x) & \text{if } x \in X_m \\ 0 & \text{otherwise} \end{cases}$$

where X_m is the image in X of

$$U_m = \{g \in B^o(\mathbb{Z}_p) \mid g_{ij} = 0 \mod p^{m(j-i)} \text{ for } 1 \leq i < j \leq n\}.$$

Note that $X_m \subset X'$. If $x = N(\mathbb{Z}_p)g$, with g in U_m, write $\chi(x) = \chi($diagonal part of $g)$.

Lemma 6.3

Let $\chi^\dagger$ denote the natural extension of χ to a $\mathbb{Z}_p$-algebra homomorphism from Λ to $\mathbb{O}$ and set $I_\chi = \ker(\chi^\dagger)$. Let $\mu \in \mathbb{D}$. Then the following are equivalent:

(1) $\mu \in I_\chi\mathbb{D}$,

(2) $\int_X f(x)d\mu(x) = 0$ for every continuous function $f: X \to \mathbb{O}$ such that $f(tx) = \chi(t)f(x)$ for all $t \in T(\mathbb{Z}_p)$.

Proof. First note that if we consider an element $y \in \Lambda = \mathbb{Z}_p[[T(\mathbb{Z}_p)]]$ as a measure on $T(\mathbb{Z}_p)$, then $\chi^\dagger$ is nothing other than integration of χ with respect to dy. This is because $\int_{T(\mathbb{Z}_p)} \chi(t)dy(t)$ is continuous and $\mathbb{Z}_p$-linear in y and agrees with $\chi^\dagger$ on $\mathbb{Z}_p[T(\mathbb{Z}_p)]$, which is dense in Λ. (For $y \in T(\mathbb{Z}_p)$, note that dy is the Dirac measure at y, so that $\chi^\dagger(y) = \chi(y) = \int_{T(\mathbb{Z}_p)} \chi(t)dy(t)$.)

We now revert to the notation in the proof of Lemma 1.1. Then $X = T \times Z$ and $\mathbb{D} \approx \lim \Lambda \otimes A_m$, where $A_m = Fns(Z_m, \mathbb{O})$. Suppose that $\mu \in \mathbb{D}$ corresponds to $\sum y_{i,m} \otimes \rho_{i,m}$ under this isomorphism, where $y_{i,m} \in \Lambda$ and the $\rho_{i,m}$ form an $\mathbb{O}$-basis for A_m.

Then (2) holds if and only if $\int_X \chi(t)h(z)d\mu(tz) = 0$ for every $h \in A_m$, for every m. But this is equivalent to $\sum y_{i,m}(\chi) \otimes \rho_{i,m} = 0$ for every m and hence to $y_{i,m}(\chi) = 0$ for every i and m. But by the remark in the first paragraph of this proof, $y_{i,m}(\chi) = 0$ if and only if $y_{i,m} \in \mathrm{Ker}(\chi^\dagger) = I_\chi$. So (2) holds if and only if $\mu \in \lim I_\chi \otimes A_m = I_\chi \mathbb{D}$. □

Lemma 6.4

Hypotheses as in Lemma 6.3. Then $\mu \in I_\chi\mathbb{D}$ if and only if $\int_X \psi_{m,\chi}(x\gamma)d\mu(x) = 0$ for every $m \geq 1$ and every $\gamma \in \Gamma$.

Proof. Use that the level of Γ is prime to p and thus any f as in Lemma 6.3 is the uniform limit of functions of the form $\psi_{m,\chi}(x\gamma)$. □

Lemma 6.5

Suppose z is a cocycle in $\mathrm{Hom}_{\mathbb{Z}_p\Gamma}(F_k, \mathbb{D})$. Then z takes values in $I_\chi\mathbb{D}$ if and only if $\int_X \psi_{m,\chi}(x)dz(f)(x) = 0$ for every $m \geq 1$ and every $f \in F_k$.

Proof. By Lemma 6.4, we must show $\int_X \psi_{m,\chi}(x\gamma)dz(f)(z) = 0$ for every m, f, and γ. But this integral equals $\int_X \psi_{m,\chi}(x)d[z(f)\gamma](x) = \int_X \psi_{m,\chi}(x)dz(f\gamma)(x) = 0$, and $f\gamma$ is just another f. □

Lemma 6.6

With notation as in Lemma 6.1 and $X_m = \mathrm{supp}\,(\psi_{m,\chi})$ (cf. Definition 6.2), then for all $m \geq 1$,

(1) $X * B_j \cap X_m = \emptyset$ for all j.
(2) $X * A_i \cap X_m = \emptyset$ for all $i \neq 1$.
(3) $(X - X') * A_1 \cap X_m = \emptyset$.
(4) $x \to x * A_1$ gives a bijection from X' to X_m .

Proof. Since $A_1 = \pi^m$, (4) is an immediate computation. Assertion (3) follows from Lemma 3.1, since $X_m \subset X$. To prove (1) and (2) it is enough to show the following:

Claim. Let $C = k\pi^m k'$ for some $k, k' \in K = G(\mathbb{Z}_p)$. If $x \in X$ and $x * C \in X_m$, then $KC = K\pi^m$.

Indeed, let the given A_i or B_j be denoted C, since they have the form of C in the claim. If (1) or (2) failed then we would conclude from the claim that $KC = K\pi^m$. Hence $\Gamma C \subset K\pi^m \cap \Gamma C$. If $\gamma \in \Gamma$ and $k \in K$ and $\gamma C = k\pi^m$, then k (resp. k^{-1}) is a matrix in $G(\mathbb{Z}_p)$ whose entries are rational numbers with denominators powers of p, and so $k \in G(\mathbb{Z})$. But also $k \in SS^{-1}$. Then, recalling that our Hecke pair (Γ, S) has the property that $\Gamma = G(\mathbb{Z}) \cap SS^{-1}$, we obtain that $k \in \Gamma$. Thus $\Gamma C \subset \Gamma\pi^m$, and $C = \pi^m = A_1$.

To prove the claim, we define $R = \{x \in G(\mathbb{Z}_p) \mid x_{11}, \dots, x_{nn}$ are p-adic units and $x_{ij} \in p^{(j-i)m}\mathbb{Z}_p$ for $i < j\}$.

Sublemma

(i) *If $n \in N(\mathbb{Z}_p)$, $nR = R$;*
(ii) *X_m = image of $R \cap B^o(\mathbb{Z}_p)$ = image of R in X;*
(iii) *if $k \in R \cap G(\mathbb{Z}_p)$, then $\pi^m k \pi^{-m} \in G(\mathbb{Z}_p)$;*
(iv) *if $b \in R \cap B^o(\mathbb{Z}_p)$ and $k \in G(\mathbb{Z}_p)$ and $bk \in R$, then $k \in R$;*
(v) *if $k \in G(\mathbb{Z}_p)$ and $X_m * k$ meets X_m, then $\pi^m k \pi^{-m} \in G(\mathbb{Z}_p)$.*

Proof of Sublemma. The first three statements are obvious.

For (iv), it is easiest to work directly with the matrices. Using the hypotheses, since the diagonal entries of b and bk are units and the upper triangular entries of b are all divisible by p, one gets that the diagonal entries of k are units. Next one uses that information to show that the superdiagonal entries of k are all divisible by p^m. Then the next superdiagonal entries of k are all divisible by p^{2m}, and so on.

For (v), if $x \in X_m * k \cap X_m$, then x is represented by some $b \in R \cap B^o(\mathbb{Z}_p)$ and $bk \in R$. So (v) follows from (iv). □

Proof of Claim. Let $C = k\pi^m k'$ for some $k, k' \in K = G(\mathbb{Z}_p)$. If $x \in X$ and $x * C \in X_m$, then $y = x * k * \pi^m \in X_m * k'^{-1} \subset X$. Of course $x * k \in X$, so that by Lemma 3.1 we have that $x * k \in X'$. Hence y is in $X' * \pi^m = X_m$. But $y * k' = x * C$ is also in X_m. It follows from (v) of the sublemma that $\pi^m k' \pi^{-m} \in G(\mathbb{Z}_p) = K$. Thus $KC = Kk\pi^m k' = K\pi^m k' \pi^{-m}\pi^m = K\pi^m$. □

Let $\mathbb{D}_m = \{\mu \in \mathbb{D} \mid \int_X \psi_{m,\chi} d\mu = 0\}$. Then by Lemma 6.5 we know that a cocycle z in $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$ takes values in $I_\chi \mathbb{D}$ if and only if $z(f) \in \mathbb{D}_m$ for all $m \geq 1$ and all $f \in F$.

Lemma 6.7

Fix $m \geq 1$. Let ζ be a cohomology class in $H^k(\Gamma, \mathbb{D})$. Suppose $\zeta = T_{\pi^m}\beta$, for some cohomology class β that can be represented by a cocycle b in $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$ with the property that $\int_{X'} \chi(x) db(f)(x) = 0$ for every $f \in F_k$. Then ζ can be represented by a cocycle z_m in $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$ which takes values in $\mathbb{D}_m$.

Proof. Apply Lemma 6.1 to β and b with $\mathbb{D}^* = \{\mu \in \mathbb{D} \mid \int_{X'} \chi(x) d\mu(x) = 0\}$. We get that $\zeta = T_{\pi^m}\beta$ is represented by $z_m = \sum e_t$, for some cochains e_t taking values in $\mathbb{D}^* E_t$. We index the E's so that $E_1 = A_1 = \pi^m$.

Now by Lemma 6.6, for any $\mu \in \mathbb{D}$,

$$\int_X \psi_{m,\chi}(x) d(\mu E_t)(x) = \int_X \psi_{m,\chi}(xE_t) d\mu(x) = \int_{X'} \psi_{m,\chi}(xE_t) d\mu(x) = 0$$

unless $t = 1$, in which case $X'E_1 = X_m$ and the integral equals

$$\int_{X'} \psi_{m,\chi}(xE_1) d\mu(x) = \int_{X'} \chi(x) d\mu(x).$$

It follows that for any $f \in F_k$, $\int_X \psi_{m,\chi}(x) d(z_m(f))(x) = \int_{X'} \chi(x) d\mu(x)$ where $e_1 = \mu E_1$ and $\mu \in \mathbb{D}^*$. But this integral vanishes by definition of $\mathbb{D}^*$. □

Now we set $\chi = \lambda\varepsilon$.

Lemma 6.8

Fix $m \geq 1$. Let ζ be a cohomology class in $H^k(\Gamma, \mathbb{D})^0$ which is in the kernel of $\phi_{\lambda,\varepsilon}$. Suppose $\zeta = T_{\pi^m}\beta$, for some cohomology class β. Then ζ can be represented by a cocycle z_m in $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$ which takes values in $\mathbb{D}_m$.

Proof. We need to show that β can be represented by a cocycle b in $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$ that takes values in $\mathbb{D}^*$.

Since T_π commutes with $\phi_{\lambda,\varepsilon}$ (Lemma 4.2) and acts injectively on the ordinary part of cohomology, we have $\phi_{\lambda,\varepsilon}(\beta) = 0$. Recall that $\phi_{\lambda,\varepsilon}$ is defined on the chain level by $\phi_{\lambda,\varepsilon}(b)(f) = \int_{X'} \varepsilon(x) vx \, d(b(f))(x)$. (We use the same resolution F as before for both Γ and Γ_ν.) From Lemma 7.6 below, we see that we may choose b so that $\phi_{\lambda,\varepsilon}(b)$ is not only a coboundary, but is identically zero as a function of f.

In particular, we see that the coefficient c of v in $\int_{X'} \varepsilon(x)vx\,d(b(f))(x)$ with respect to a basis of weight vectors of V is 0. But $x \in X'$ can be represented by upper triangular matrices, and v is a highest weight vector for the lower triangular matrices. Thus,

$$c = \int_{X'} \varepsilon(x)\lambda(x)d\big(b(f)\big)(x) = \int_{X'} \chi(x)d\big(b(f)\big)(x).$$

Since $c = 0$, this says $b(f) \in \mathbb{D}^*$. □

End of the Proof of Theorem 5.1. Let ζ be a cohomology class in $H^k(\Gamma, \mathbb{D})^0$ which is in the kernel of $\phi_{\lambda,\varepsilon}$. From Lemma 6.8 for every $m \geq 1$, ζ can be represented by a cocycle z_m in $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$ which takes values in $\mathbb{D}_m$. Since $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$ is compact, we may assume the z_m have a limit z_∞. Since the coboundary map is continuous, z_∞ is again a cocycle. Since the coboundaries are a compact subspace of the cocycles, z_∞ still represents ζ. And since the topology on the measures is defined by pointwise convergence on continuous functions, we have z_∞ taking values in $\mathbb{D}_m$ for every m. So z_∞ takes values in $I_\chi\mathbb{D}$ by Lemma 6.5. □

Section 7: Auxiliary Lemmas

Let Γ be a group, Γ_ν a subgroup of finite index in Γ. Let M be a right $\mathbb{O}\Gamma$-module and N be a right $\mathbb{O}\Gamma_\nu$-module. Suppose $\phi: M \to N$ is a map of $\mathbb{O}\Gamma_\nu$-modules. Let I be the induced module $\mathrm{Ind}(\Gamma_\nu, \Gamma, N) = \{f: \Gamma \to N \mid f(x\gamma) = f(x)\gamma$, for all $x \in \Gamma$, $\gamma \in \Gamma_\nu\}$. Make I a right module by $(fy)(x) = f(yx)$ for $x, y \in \Gamma$. We define $s: I \to N$ by $s(f) = f(1)$. It is a Γ_ν-map.

Define $\psi: M \to I$ by $\psi(m)(x) = \sigma(mx)$. One checks that ψ is a well-defined Γ-map and $s \circ \psi = \sigma$.

Lemma 7.1

Suppose ψ is surjective. Let P_* be a resolution of $\mathbb{O}$ by free $\mathbb{O}\Gamma$-modules, and a be a coboundary in $\mathrm{Hom}_{\Gamma_\nu}(P_k, N)$. Then we can choose a coboundary b in $\mathrm{Hom}_\Gamma(P_k, M)$ such that $\sigma b = a$ as cochains.

Proof. We have the isomorphism induced by $s: H_\Gamma(P_*, I) \to \mathrm{Hom}_{\Gamma_\nu}(P_*, N)$, and it commutes with d. Write $a = dsv$ for some v in $\mathrm{Hom}_\Gamma(P_{k-1}, I)$. Since P_{k-1} is a free $\mathbb{O}\Gamma$-module and ψ is surjective, there exists v' in $\mathrm{Hom}_\Gamma(P_{k-1}, M)$ such that $\psi v' = v$. Hence, $\sigma(dv') = s\psi(dv') = a$ as cochains. So we can take $b = dv'$. □

Now fix $\nu \geq 1$ and let Γ and Γ_ν be the groups used in the first six sections of this paper. Set $M = \mathbb{D}$, $N = L_\lambda(\varepsilon)$. If we take $\phi = \phi_{\lambda,\varepsilon}$ and define I and ψ as above, it is unfortunately not true that $\psi\colon \mathbb{D} \to I$ is surjective. We will have to take a slightly roundabout approach. We will see that we also need the ordinary condition in this step.

We let Y_ν denote the subset of X represented by matrices y such that $p^{(n+1-i-j)\nu}$ divides y_{ij} for all i, j such that $n > i + j - 1$. It is easy to see that $\{\gamma \in \Gamma \mid Y_\nu\gamma = Y_\nu\} = \Gamma_\nu$. Therefore, we can define a Γ_ν-equivariant map $\sigma\colon \mathbb{D} \to V_\lambda(\varepsilon)$ by the integral $\int_{Y_\nu} \varepsilon'(y)vyd\mu(y)$ where $\varepsilon'(y) = \varepsilon(\operatorname{diag}(y_{1n}, \ldots, y_{n1}))$.

To see where σ takes values, we choose a matrix $w \in \Gamma$ such that w is congruent modulo p^M for some $M > (n-1)\nu$ to the matrix

$$\begin{bmatrix} 0 & 0 & . & 1 \\ . & . & . & . \\ 0 & 1 & . & 0 \\ \pm 1 & 0 & . & 0 \end{bmatrix}$$

where the sign is chosen to make the determinant $+1$. Note that $\pi^\nu w^{-1}\pi^\nu = p^{(n-1)\nu}w'^{-1}$ where w' is in $G(\mathbb{Z})$ and we have $w' \equiv$

$$\begin{bmatrix} 0 & 0 & . & \pm 1 \\ . & . & . & . \\ 0 & 1 & . & 0 \\ 1 & 0 & . & 0 \end{bmatrix}$$

modulo $p^{M-(n-1)\nu}$. Taking inverses we also see that $\pi^\nu w'\pi^\nu = p^{(n-1)\nu}w$. Moreover, w acts on X and $Y_\nu w = Y_\nu w' = Y_\nu w^{-1} = Y_\nu w'^{-1} = X_\nu \subset X'$.

From now on set $L = L_\lambda(\varepsilon)$ and $V = V_\lambda(\varepsilon)$. Since $v \in L$ and L is stable under I^o, we see that σ takes values in $L * \pi^\nu w^{-1}$.

Lemma 7.2

If z in $\operatorname{Hom}_\Gamma(F_k, \mathbb{D})$ is a cocycle and $\sigma(z)$ is a coboundary, then there exists u in $\operatorname{Hom}_\Gamma(F_{k-1}, \mathbb{D})$ such that $\sigma(z + du) = 0$ as a cochain.

Proof. Let ψ correspond to σ as in Lemma 7.1. All we have to do is show that ψ is surjective. Since the image of ψ is a Γ-submodule of I, it will suffice to show that for b running through a $\mathbb{Z}_p$-basis of $L * \pi^\nu w^{-1}$ the function that sends 1 to b and $\Gamma - \Gamma_\nu$ to 0 lies in the image.

Now for any μ in $\mathbb{D}$ and x in Γ, we have

$$\psi(\mu)(x) = \sigma(\mu x) = \int_{Y\nu} \varepsilon'(y)vyd(\mu x)(y) = \int_X \mathrm{char}_\nu(yx)vyxd(\mu)(y),$$

where char_ν denotes the characteristic function of Y_ν.

Let μ be the dirac measure at y_0. We can write y_0 as the image in X of some matrix $\pi^{-\nu}gt_0\pi^\nu w^{-1}$ where g_0 is in $B^0(\mathbb{Z}_p)$. Note that $\varepsilon'(y_0) = \varepsilon(g_0)$. Then

$$\psi(\mu)(x) = \mathrm{char}_\nu(y_0x)\varepsilon(g_0x)v * \pi^{-\nu}g_0\pi^\nu w^{-1}x\,.$$

It is easy to see that y_0 and y_0x both in Y_ν implies that x is in Γ_ν. Therefore, $\psi(\mu)(x) = 0$ unless x is in Γ_ν. If $x = 1$, $\psi(\mu)(x) = \varepsilon(g_0)v\pi^{-\nu}g_0\pi^\nu w^{-1}$. Now $\varepsilon(g_0)$ is a unit and as g_0 varies, $v * \pi^{-\nu}g_0\pi^\nu w^{-1} = (v * g_0)\pi^\nu w^{-1}$ runs over a $\mathbb{Z}_p$-basis of $L * \pi^\nu w^{-1}$. (Remark: if you put back the x in Γ_ν, it is helpful to note that $\pi^\nu w$ normalizes Γ_ν.) □

We extend the $*$ action of G^π on L to an action of the group generated by $G(\mathbb{Z}_p)$ and π on V. Set $W_\nu = \pi^\nu w'$ and $c = \lambda(p^{(n-1)\nu}\pi^{-2\nu})$. Then $L * \pi^\nu w^{-1}W_\nu = L * \pi^\nu w^{-1}\pi^\nu w' = cL * w'^{-1}w' = cL$. Note also that W_ν stabilizes Γ_ν under conjugation. We will also denote by W_ν the induced action on cohomology that goes from $H^*(\Gamma_\nu, L * \pi^\nu w^{-1})$ to $H^*(\Gamma_\nu, cL)$.

Recall that the Hecke operator $T(\pi^\nu)$ on the cohomology of Γ_ν with coefficients in a G^π-module E equals $tr \circ \mathrm{res} \circ \pi^\nu$ where π^ν acts by conjugation $H^*(\Gamma_\nu, E) \to H^*(\pi^{-\nu}\Gamma_\nu\pi^\nu, E\pi^\nu)$, res is restriction onto $\Gamma_\nu \cap \pi^{-\nu}\Gamma_\nu\pi^\nu$, and tr is transfer back up to Γ_ν.

We need to lift the composite operator $T(\pi^\nu)W_\nu$ to the cochain level in a certain way. To do this, recall that we have a fixed set of upper triangular unipotent coset representatives γ_i of $\Gamma_\nu \cap \pi^{-\nu}T_\nu\pi^\nu$ in Γ_ν. Now extend the action of Γ on the resolution F_* to the p-power scalar matrices by letting them act trivially. Then write $T(\pi^\nu)W_\nu = \mathrm{tr} \circ \mathrm{res} \circ \pi^\nu \circ w' \circ \pi^\nu$ (the conjugation action is contravariant on cohomology). But this equals $\mathrm{tr} \circ \mathrm{res} \circ \pi^\nu w'\pi^\nu = \mathrm{tr} \circ \mathrm{res} \circ p^{(n-1)\nu}w$. We define TW_ν on a cochain z in $\mathrm{Hom}_{\Gamma_\nu}(F_k, V)$ by setting $TW_\nu(z)(f) = c\Sigma z(f\gamma_i^{-1}w^{-1})w\gamma_i$. This gives a lifting of $T(\pi^\nu)W_\nu$ to the cochain level. (The constant c comes from the $*$ action of the scalar $p^{(n-1)\nu}$ on V.)

Lemma 7.3

For z in $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$, we have $c\phi_*(z) = TW_\nu\sigma_*(z)$, where $c = \lambda(p^{(n-1)\nu}\pi^{-2\nu})$. Hence on cohomology $c\phi_* = T(\pi^\nu)W_\nu\sigma_*$.

Proof. Compute:

$$\begin{aligned} TW_\nu\sigma_*(z)(f) &= c\Sigma\sigma_*(z)(f\gamma_i^{-1}w^{-1})w\gamma_i \\ &= c\Sigma\left\{\int_X \operatorname{char}_\nu(y)\varepsilon'(y)vyd[z(f\gamma_i^{-1}w^{-1})](y)\right\}w\gamma_i \\ &= c\Sigma\int_X \operatorname{char}_\nu(y)\varepsilon'(y)vyw\gamma_i d\mu(y) \end{aligned}$$

where we have written μ for $z(f\gamma_i^{-1}w^{-1})$. Now z is Γ-equivariant, so $\mu = z(f)\gamma_i^{-1}w^{-1}$ and the right hand side

$$= c\Sigma\int_X \operatorname{char}_\nu(y\gamma_i^{-1}w^{-1})\varepsilon'(y\gamma_i^{-1}w^{-1})vydz(f)(y)\,.$$

Now $y\gamma_i^{-1}w^{-1}\in Y_\nu$ if and only if $y\in Y_\nu w\gamma_i$, in which case $\varepsilon'(y\gamma_i^{-1}w^{-1})=\varepsilon(y)$. It is easily checked that as i varies, $Y_\nu w\gamma_i$ runs over a partition of X' into open and closed subsets. Letting char_i denote the characteristic function of $Y_\nu w\gamma_i$, we see the right hand side equals

$$c\Sigma\int_X \operatorname{char}_i(y)\varepsilon(y)vydz(f)(y) = c\phi_*(z)(f)\,.\ \square$$

Lemma 7.4

(i) For z in $\operatorname{Hom}_\Gamma(F_k,\mathbb{D})$, we have $\sigma_*(T(w\pi^\nu w^{-1})z)=\phi_*(z')*\pi^\nu w^{-1}$ where $z'(f)=z(\tau(f))w$ and $\tau\colon F_*\to F_*$ is a homotopy equivalence that satisfies $\tau(f\gamma)=\tau(f)w\pi^\nu w^{-1}\gamma w\pi^{-\nu}w^{-1}$ for $\gamma\in w\pi^{-\nu}w^{-1}\Gamma w\pi^\nu w^{-1}\cap\Gamma=\Gamma_\nu$.

(ii) If $\phi_*(\zeta)=0$ in $H^k(\Gamma_\nu,L)$, then $\sigma_*(T(\pi^\nu)\zeta)=0$ in $H^k(\Gamma_\nu,L*\pi^\nu w^{-1})$.

Proof. Note that $z'(f)=z(\tau(f)w)$ is Γ_ν-equivariant up to an automorphism of Γ_ν. Indeed, for any γ in Γ_ν, $z'(f\gamma)=z(\tau(f\gamma))w=z(\tau(f))w\pi^\nu w^{-1}\gamma w\pi^{-\nu}w^{-1}w=[z'(f)]\pi^\nu w^{-1}\gamma w\pi^{-\nu}$. Therefore, since ϕ_* is Γ_ν-equivariant, $\phi_*(z')*\pi^\nu w^{-1}$ will be Γ_ν-equivariant on the nose, as it should be.

To prove (i), let $\{E_t\}$ be as in the proofs of Lemmas 6.1 and 6.7. Thus $\Gamma\pi^\nu\Gamma$ is the disjoint union of ΓE_t, and we write $E_t=\pi^\nu\gamma_t$. Then $\Gamma w\pi^\nu w^{-1}\Gamma$ is the disjoint union of ΓwE_tw^{-1}, and $wE_tw^{-1}=w\pi^\nu ww^{-1}\gamma_tw^{-1}$.

We compute $T(w\pi^\nu w^{-1})z$ in a manner similar to Formulae 4.3, where we replace π in 4.3 by $w\pi^\nu w^{-1}$. Thus for any $f\in F_k$,

$$\{T(w\pi^\nu w^{-1})z\}(f)=\Sigma_t z(\tau(fw\gamma_t^{-1}w^{-1})wE_tw^{-1}).$$

Therefore,

$$\begin{aligned}
&\sigma_*\big(T(w\pi^\nu w^{-1})z\big)(f)\\
&= \Sigma \int_X \operatorname{char}_\nu(y)\varepsilon'(y)vyd\big[z(\tau(fw\gamma_t^{-1}w^{-1})wE_tw^{-1})\big](y)\\
&= \Sigma \int_X \operatorname{char}_\nu(ywE_tw^{-1})\varepsilon'(ywE_tw^{-1})vywE_tw^{-1}d\big[z(\tau(fw\gamma_t^{-1}w^{-1}))\big](y)\,.
\end{aligned}$$

Since $Y_\nu w = X_\nu$, it follows from Lemma 6.6, as in the proof of Lemma 6.7, that $ywE_tw^{-1} \in Y_\nu$ if and only if $t = 1$, i.e., $E_t = \pi^\nu$ and $\gamma_t = 1$, in which case yw must be in X'. So the sum has only one nonzero term in it and

$$\sigma_*\big(T(w\pi^\nu w^{-1})z\big)(f) = \int_{X'w^{-1}} \varepsilon'(yw\pi^\nu w^{-1})vyw\pi^\nu w^{-1}d[z(\tau f)](y)\,.$$

Now for yw in X', $\varepsilon'(yw\pi^\nu w^{-1}) = \varepsilon(yw)$, so we get

$$\begin{aligned}
\sigma_*\big(T(w\pi^\nu w^{-1})z\big)(f) &= \left\{\int_{X'w^{-1}} \varepsilon(yw)vyw\, d[z(\tau f)](y)\right\} * \pi^\nu w^{-1}\\
&= \left\{\int_{X'} \varepsilon(x)vx\, d[z(\tau f)w](x)\right\} * \pi^\nu w^{-1}
\end{aligned}$$

which gives (i).

As for (ii), first note that $\Gamma w\pi^\nu w^{-1}\Gamma = \Gamma\pi^\nu\Gamma$, so that $T(w\pi^\nu w^{-1}) = T(\pi^\nu)$. Next, represent ζ by a cocycle z in $\operatorname{Hom}_\Gamma(F_k, L)$, so that $z(f)h = z(fh)$ for any h in Γ. Now by hypothesis, we can find a cochain b in $\operatorname{Hom}_{\Gamma_\nu}(F_{k-1}, L)$ so that $\int_{X'} \varepsilon(x)vsd[z(f)](x) = b(\partial f)$ for every f in F_k. Then by (i), $\sigma_*(T(\pi^\nu)\zeta)$ is represented by $\{\int_{X'} \varepsilon(x)vxd[z(\tau f)w](x)\}*\pi^\nu w^{-1} = b(\partial(\tau f)w)*\pi^\nu w^{-1}$. Thus $\sigma_*(T(\pi^\nu)\zeta)$ is represented by the coboundary db', where $b'(f) = b((\tau f)w) * \pi^\nu w^{-1}$. We only need to check that b' is Γ_ν-equivariant, which is the same calculation as in the first paragraph of this proof. □

Lemma 7.5

If ζ is an ordinary class in $H^k(\Gamma, \mathbb{D})^0$, and $\phi_*(\zeta) = 0$, then $\sigma_*(\zeta) = 0$.

Proof. Write $\zeta = T(\pi^\nu)\eta$. Then $\phi_*(\zeta) = 0$ implies $\phi_*(\eta) = 0$ since $T(\pi^\nu)$ commutes with ϕ_* and η, and hence $\phi_*(\eta)$, is ordinary. Then by Lemma 7.4 $\sigma_*(T(\pi^\nu)\eta) = 0$. □

Lemma 7.6

If ζ is an ordinary class in $H^k(\Gamma, \mathbb{D})^0$, and $\phi_*(\zeta) = 0$, then ζ can be represented by a cocycle such that $\phi_*(z) = 0$ as cochains.

Proof. We have $c\phi_*(\zeta) = T(\pi^\nu)W_\nu\sigma_*(\zeta)$. From Lemma 7.5, we get that $\sigma_*(\zeta) = 0$. By Lemma 7.2, ζ can be represented by a cocycle z such that $\sigma_*(z) = 0$ as cochains. Then by Lemma 7.3, $\phi_*(z) = c^{-1}TW_\nu\sigma_*(z) = 0$ as cochains. □

Section 8: Lifting eigenvalues

In this last chapter we address the purely algebraic aspects of the problem of lifting systems of eigenvalues. Throughout this section R will denote a complete noetherian local domain (hence commutative), $\mathcal{H}$ will be a commutative R-algebra, and A will be an R-module on which $\mathcal{H}$ acts. We are interested in the "$\mathcal{H}$-eigenvectors" in A. There are two possible complications. First, A may have torsion elements; and second, we may need to pass to a finite extension of the base before we obtain eigenvectors. To account for both of these possibilities we make the following definition.

DEFINITION 8.1.

a. A system of eigenvalues for $\mathcal{H}$ is a triple $(\varphi, I_\varphi, R_\varphi)$ where I_φ is an ideal in R, R_φ is a finite extension of R/I_φ, and $\varphi : \mathcal{H} \to R_\varphi$ is a homomorphism of R-algebras.
b. If φ is a system of eigenvalues for $\mathcal{H}$, then a φ-eigenvector on A is any nonzero $x \in A[I_\varphi] \otimes_{R/I_\varphi} R_\varphi$ for which $h(x) = \varphi(h)x$ for all $h \in \mathcal{H}$. If there exists a φ-eigenvector on A, we say that φ occurs in A.

We have the following Lemma.

Lemma 8.2

If A is finitely generated, then $\mathcal{H}$ has a system of eigenvalues occurring on A.

Proof. We will prove the lemma by induction on the Krull dimension of R. The statement is obviously true if R is a field. So we assume that R has Krull dimension $d \geq 1$ and that the assertion is true for all complete noetherian local domains with Krull dimension less than d. Let A be a non-zero finitely generated R-module. We consider two cases.

(1) *A is torsion-free.* Tensoring A with the fraction field K of R, we obtain an inclusion $A \subseteq A_K := A \otimes K$. By a standard argument, we can find an $\mathcal{H}$-eigenvector in $A_{\tilde{K}}$ for some finite extension $\tilde{K}$ of K. Since A is finite over R, the eigenvalues must lie in the integral closure $\tilde{R}$ of R in $\tilde{K}$. Let $\varphi : \mathcal{H} \longrightarrow \tilde{R}$ be the associated system of eigenvalues. Then $(\varphi, (0), \tilde{R})$ occurs on A.

(2) *A has non-trivial torsion elements.* In this case, there is a non-zero $\alpha \in R$ such that $A[\alpha] \neq 0$. If P_i, $i = 1, \ldots, r$ are the height one primes containing α, then $P_1 P_2 \cdots P_r$ is nilpotent modulo (α), say $(P_1 P_2 \cdots P_r)^n \subseteq (\alpha)$. In particular, it follows that $A[P_i] \neq 0$ for some i. So we conclude that there is a non-zero prime ideal P in R such that $A[P] \neq 0$. But $A[P]$ is an $\mathcal{H}$-submodule of A. Moreover $A[P]$ is an R/P-module. But R/P is a complete Noetherian local domain with Krull dimension $< d$. Hence, letting $\mathcal{H}_P$ denote the R/P-algebra obtained as the image of $\mathcal{H}$ in $\mathrm{End}_{R/P}(A[P])$, our induction assumption implies that there is a system $(\overline{\varphi}, I_{\overline{\varphi}}, R_{\overline{\varphi}})$ of eigenvalues for $\mathcal{H}_P$ occurring on $A[P]$. Letting φ be the composition of $\overline{\varphi}$ with the map $\mathcal{H} \to \mathcal{H}_P$, and I_φ be the inverse image of $I_{\overline{\varphi}}$ in R, we obtain a system of eigenvalues $(\varphi, I_\varphi, R_{\overline{\varphi}})$ for $\mathcal{H}$ that occurs on A.

Thus, in either case $\mathcal{H}$ has a system of eigenvalues occurring on A. This completes the proof of Lemma 8.2. □

DEFINITION 8.3. If $(\varphi, I_\varphi, R_\varphi)$ and (ψ, I_ψ, R_ψ) are two systems of eigenvalues for $\mathcal{H}$, we say that ψ is a lifting of φ if (a) $I_\psi \subseteq I_\varphi$; and (b) there is a homomorphism $R_\psi \longrightarrow R_\varphi$ such that the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\psi} & R_\psi \\ \| & & \downarrow \\ \mathcal{H} & \xrightarrow{\varphi} & R_\varphi \end{array}$$

is commutative.

Theorem 8.4

Let R be a complete noetherian local domain with maximal ideal $\mathcal{M}$ and residue field k. Let $\mathcal{H}$ be a commutative R-algebra and let A, B be finitely generated R-modules on which $\mathcal{H}$ acts. Suppose the maximal ideal $\mathcal{M} \subseteq R$ annihilates B. Let

$$\pi : A \longrightarrow B$$

be a surjective morphism of $\mathcal{H}$-modules. Then any system of $\mathcal{H}$-eigenvalues occurring on B can be lifted to a system of eigenvalues occurring on A.

Lemma 8.5

Let A be a finitely generated R-module with an endomorphism T. Then A admits a unique decomposition

$$A = A^{nil} \oplus A^{ord}$$

such that T preserves the decomposition and acts topologically nilpotently on A^{nil} and invertibly on A^{ord}.

Proof. Let $a_1,\ldots,a_n$ be a set of generators for A. Then there is an $n\times n$ matrix $M=(r_{ij})$ with R-coefficients such that

$$T(a_i)=\sum_{j=1}^{n} r_{ij}a_j.$$

Let $f(t)=\det(tI-M)\in R[t]$ be the characteristic polynomial of M. By the Weierstrass preparation theorem, $f(t)$ can be factored in $R[t]$ as

$$f(t)=f^{nil}(t)f^{ord}(t)$$

where f^{nil} is a distinguished polynomial and $f^{ord}(t)$ is invertible in $R[[t]]$, i.e. $f^{ord}(0)\in R^{\times}$. Define

$$A^{nil}:=\ker\big(f^{nil}(T)\big)$$
$$A^{ord}:=\ker\big(f^{ord}(T)\big).$$

Clearly, T acts topologically nilpotently on A^{nil} and invertibly on A^{ord}, so we must show $A=A^{nil}\oplus A^{ord}$. First note that since T is topologically nilpotent on A^{nil} and since $f^{ord}(t)$ is invertible in $R[[t]]$, we have $f^{ord}(T)$ acts invertibly on A^{nil}. Hence $A^{nil}\cap A^{ord}=0$. If $a\in A$, $f(T)$ annihilates a and therefore $f^{ord}(T)a\in A^{nil}$. Since $f^{ord}(T)$ is invertible on A^{nil}, there is an element $a^{nil}\in A^{nil}$ for which $f^{ord}(T)a^{nil}=f^{ord}(T)a$. Letting $a^{ord}=a-a^{nil}$ we have $a=a^{nil}+a^{ord}$ with $a^{nil}\in A^{nil}$ and $a^{ord}\in A^{ord}$. Thus

$$A=A^{nil}\oplus A^{ord}$$

as desired. Now suppose $A=A_1\oplus A_2$ is another decomposition for which T acts topologically nilpotently on A_1 and invertibly on A_0. Then $f^{nil}(T)$ is invertible on A_2 and $f^{ord}(T)$ is invertible on A_1. Hence $A_1\subseteq A^{nil}$ and $A_2\subseteq A^{ord}$. The opposite inclusions are clear. Hence $A_1=A^{nil}$ and $A_2=A^{ord}$. This completes the proof of Lemma 8.5. □

Lemma 8.6

Let R be a complete local noetherian ring and let A be a finitely generated R-module equipped with a mutually commutative denumerable set S of endomorphisms. Suppose there is an element $a_0\in A$ such that $\overline{a}_0\neq 0$ and $T\overline{a}_0=0$ for every $T\in S$. Then the set

$$A^{S-nil}:=\{a\in A\mid \forall T\in S, T \text{ acts topologically nilpotently on } a\ \}$$

is non-zero.

Proof. Suppose $S = \{T_1, T_2, \ldots, \}$ and for each $n \geq 1$ let $S_n := \{T_1, \ldots, T_n\}$. Define $A_1 := A^{T_1-nil}$ and $B_1 := A^{T_1-ord}$. For each $n \geq 1$ let

$$A_n := A_{n-1}^{T_n-nil}$$
$$B_n := A_{n-1}^{T_n-ord} \oplus B_{n-1}.$$

A simple induction argument shows that $\overline{a}_0 \in \overline{A}_n$ for each $n \geq 1$. Hence $A_n \neq 0$ for each n. Since R is noetherian and A is finitely generated, there is an integer $n_0 > 0$ such that $B_n = B_{n_0}$ for all $n > n_0$. Hence $A^{S-nil} = A_{n_0} \neq 0$, as claimed. This proves Lemma 8.6. □

Proof of Theorem 8.4. It suffices to prove the theorem when $B = \overline{A}$ and $\pi : A \to \overline{A}$ is the reduction map. Let $(\varphi, \mathcal{M}, k_\varphi)$ be a system of eigenvalues occurring in $\overline{A}$. Then k_φ is a finite field extension of $k := R/\mathcal{M}$. Hence there is a complete noetherian local domain S finite over R whose residue class field is k_φ. We let $\mathcal{H}$ act on $A \otimes S$ through the first factor and set

$$A_0 = (A \otimes S)^{\ker\varphi - nil}.$$

By Lemma 8.6 A_0 is a nonzero S-module. Let (ψ, I_ψ, S_ψ) be a system of eigenvalues for $\mathcal{H}$ occurring in A_0. As in the proof of Lemma 8.2, we may suppose I_ψ is a prime ideal in S, that S_ψ is a complete noetherian local domain, and that S_ψ is generated as S-algebra by the image of ψ. We will show that ψ is a lifting of φ.

Let k_ψ be the residue class field of S_ψ. Then k_ψ is a finite field extension of k_φ. Let φ' denote the composition of $\varphi : \mathcal{H} \longrightarrow k_\varphi$ with the embedding $k_\varphi \hookrightarrow k_\psi$. Let $\pi : S_\psi \longrightarrow k_\psi$ be the reduction map. We will first show $\pi \circ \psi = \varphi'$.

Let $T \in \mathcal{H}$, let $\lambda_0 = \psi(T)$ and let $\lambda \in S$ be any element for which $\pi(\lambda) = \varphi(T)$. Since $T - \lambda \in \ker\varphi$ we have $T - \lambda$ acts topologically nilpotently on A_0. Let $a_0 \in A_0[I_\psi] \otimes S_\psi$ be a ψ-eigenvector. Then

$$\lim_{n\to\infty} (T - \lambda)^n a_0 = 0, \text{ and}$$
$$(T - \lambda_0) a_0 = 0.$$

From this it follows at once that

$$\lim_{n\to\infty} (\lambda_0 - \lambda)^n a_0 = 0.$$

Hence $\lambda_0 - \lambda$ lies in the maximal ideal of S_ψ. Consequently, $\pi(\psi(T)) = \pi(\lambda_0) = \pi(\lambda) = \varphi(T) = \varphi'(T)$. This proves $\pi \circ \psi = \varphi'$, as desired.

This proves that ψ is a lifting of φ'. But since S_ψ is generated by the image of ψ, we see that $k_\psi = k_\varphi$. In particular, $\varphi = \varphi'$. Hence ψ is a lifting of φ and the proof of Theorem 8.4 is complete. □

Corollary 8.7

Let φ be a system of eigenvalues for the Hecke algebra H occurring on the p-torsion of the image of $\phi_{\lambda,\epsilon}$ in $H^*(\Gamma_\nu, L_\lambda(\epsilon))^0$. Then φ has a lifting ψ that occurs on $H^*(\Gamma, \mathbb{D})^0$.

Proof. This is an immediate consequence of Corollary 5.2 and Theorem 8.4. □

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