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# $p$-adic $L$-functions and $p$-adic periods of modular forms*

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## Introduction

Let $E$ be an elliptic curve which is defined over $\mathbf{Q}$ and has stable reduction modulo a given prime $p$. Assuming that $E$ is modular, one can associate to $E$ a $p$-adic $L$-function $L_p(E, s)$. (See [Mz-SwD, A-V, Vi, Mz-T-T] for its construction in various cases.) This function is defined by a certain interpolation property and is analytic for $s \in \mathbf{Z}_p$. In this paper, we will assume that $E$ has split multiplicative reduction at $p$. Under this assumption the interpolation property implies that $L_p(E, 1) = 0$. We will prove a formula for $L'_p(E, 1)$ which was discovered experimentally by Mazur, Tate, and Teitelbaum [Mz-T-T].

By Tate's $p$-adic uniformization theory, there is a $p$-adic integer $q_E \in p\mathbf{Z}_p$ (which we refer to as the Tate period for $E$) and a $p$-adic analytic isomorphism

$$E(\bar{\mathbf{Q}}_p) \cong \bar{\mathbf{Q}}_p^* / q_E^{\mathbf{Z}} \tag{0.1}$$

which is defined over $\mathbf{Q}_p$. Let $\log_p\colon \mathbf{Q}_p^* \to \mathbf{Z}_p$ be the usual $p$-adic logarithm on $\mathbf{Z}_p^*$, extended to $\mathbf{Q}_p^*$ by the convention $\log_p(p) = 0$. Let $\mathrm{ord}_p\colon \mathbf{Q}_p^* \to \mathbf{Z}$ be the normalized valuation. We define the $\mathfrak{L}$-invariant of $E$ by

$$\mathfrak{L}_p(E) = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}. \tag{0.2}$$

Our main result (Theorem 7.1) specializes to the following.

(0.3) **Theorem.** *Let $p$ be a prime $\geqq 5$ and let $E$ be a modular elliptic curve with split multiplicative reduction at $p$. Then*

$$L'_p(E, 1) = \mathfrak{L}_p(E) \cdot \frac{L_\infty(E, 1)}{\Omega_E}.$$

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*Here* $L_\infty(E, z)$ *is the Hasse-Weil L-function for* $E/\mathbf{Q}$ *and* $\Omega_E$ *is the real period for* $E$.

Let $w_\infty = \pm 1$ denote the sign in the functional equation for $L_\infty(E, z)$. Then, under our assumption that $E$ has split multiplicative reduction at $p$, $L_p(E, s)$ satisfies the functional equation

$$L_p(E, 2-s) = w_p \langle N \rangle^{s-1} L_p(E, s)$$

where $w_p = -w_\infty$. Here $\langle \cdot \rangle$ is projection to the subgroup $1+p\mathbf{Z}_p$ of principal units in $\mathbf{Z}_p^*$. If $w_\infty = -1$, then $w_p = +1$ and the above theorem is trivially true. Both sides of the equation vanish. If $w_\infty = +1$, then $L_p(E, s)$ has a zero at $s=1$ of odd order. As a consequence of the above theorem, we see that this order is 1 if and only if both $\log_p(q_E)$ and $L_\infty(E, 1)$ are nonzero. Manin has conjectured that $\log_p(q_E) \neq 0$ whenever $E$ is a Tate curve with algebraic $j$-invariant (see [Man1], §4.12). The Birch and Swinnerton-Dyer conjecture predicts that $L_\infty(E, 1) \neq 0$ precisely when the Mordell-Weil group $E(\mathbf{Q})$ is finite. It is conjectured in [Mz-T-T] that

$$\mathrm{ord}_{s=1}(L_p(E, s)) = 1 + \mathrm{ord}_{z=1}(L_\infty(E, z)).$$

At the moment, all we can prove is that the order of vanishing of $L_p(E, s)$ at $s=1$ is at least 2 if $w_\infty = -1$ and at least 3 if $w_\infty = +1$ and $L_\infty(E, 1) = 0$.

To explain the idea behind the proof of Theorem 0.3, we will give an outline in the special case $E = X_0(11)$ and $p=11$. For more details see (2.11) and (5.18). In this case $E$ has split multiplicative reduction at $p=11$, $L_\infty(E, 1) \neq 0$, and $w_\infty = +1$. Let $Ta_p(E)$ be the $p$-adic Tate module of $E$ and let

$$\rho_E\colon\ G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}(Ta_p(E))$$

be the associated Galois representation. Let $f_E = q \prod (1-q^n)^2 (1-q^{11n})^2$ be the unique normalized weight two newform over $\Gamma_0(11)$. Then the Mellin transform of $f_E$ is $L_\infty(E, z)$. We have $f_E | T_p = f_E$ and, in particular, $f_E$ is ordinary at $p$.

The basic ingredient in our proof of Theorem 0.3 is Hida's universal ordinary deformation of $Ta_p(E)$. In our example, the universal ordinary Hecke algebra turns out to be the completed group ring, $\Lambda = \mathbf{Z}_p[[1+p\mathbf{Z}_p]]$. Thus Hida's universal ordinary deformation of $Ta_p(E)$ is a free rank two $\Lambda$-module $\mathbf{T}$ equipped with an action of the Galois group

$$\rho\colon\ \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}_\Lambda(\mathbf{T})$$

such that $\mathbf{T}/P_0\mathbf{T} \cong Ta_p(E)$ where $P_0$ is the augmentation ideal in $\Lambda$. The Galois module $\mathbf{T}$ has a number of remarkable properties, which we will describe below.

For each $k \in \mathbf{Z}_p$, let $\sigma_{k-2}\colon \Lambda \to \mathbf{Z}_p$ be the unique continuous $\mathbf{Z}_p$-algebra homomorphism extending the character $1+p\mathbf{Z}_p \to \mathbf{Z}_p^*$, $t \mapsto t^{k-2}$. For $\alpha \in \Lambda$ we will write $\alpha(k)$ instead of $\sigma_{k-2}(\alpha)$ and refer to $\sigma_{k-2}$ as specialization to weight $k$. Let $P_k \subseteq \Lambda$ be the kernel of $\sigma_{k-2}$. Let $\mathbf{T}_k = \mathbf{T}/P_k\mathbf{T} \cong \mathbf{T} \otimes_{\Lambda, \sigma_{k-2}} \mathbf{Z}_p$ and let

$$\rho_k\colon\ \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}(\mathbf{T}_k)$$

be the reduction of $\rho$ modulo $P_k$. Now fix, once and for all, an embedding

(0.4) $$\bar{\mathbf{Q}} \subseteq \bar{\mathbf{Q}}_p.$$

Then $\rho$ has the following properties.

(0.5)a. $\rho_2 = \rho_E$.

(0.5)b. For each integer $k \geqq 2$, there is a normalized newform $f_k$ of weight $k$ and conductor dividing $p$ such that $\rho_k$ is equivalent to Deligne's $p$-adic Galois representation [D] associated to $f_k$ and our fixed embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_p$. The conductor of $f_k$ is 1 precisely when $k > 2$ and $k \equiv 2$ modulo $p-1$. For example, $f_{12}$ = Ramanujan's $\Delta$-function. For other values of $k$ the conductor is $p$ and the Nebentype character is the Dirichlet character of conductor $p$ associated to $\omega^{2-k}$ where $\omega\colon \mathbf{Z}_p^* \to \mu_{p-1} \subseteq \mathbf{Z}_p^*$ is the Teichmüller character.

(0.5)c. Let $G_{\mathbf{Q}_p} = \mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ be a fixed decomposition group over $p$. Then the restriction $\rho|_{G_{\mathbf{Q}_p}}$ of $\rho$ to $G_{\mathbf{Q}_p}$ is equivalent to an upper triangular representation

$$\rho|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \chi\varphi^{-1} & * \\ 0 & \varphi \end{pmatrix}$$

where $\varphi\colon G_{\mathbf{Q}_p} \to \Lambda^*$ is an unramified character and $\chi\colon G_{\mathbf{Q}_p} \to \Lambda^*$ is the unique character for which $\sigma_{k-2} \circ \chi = \chi_0^{k-1}\, \omega^{2-k}$, where $\chi_0$ is the cyclotomic character and $\omega$ is the Galois character associated to the Teichmüller character by class field theory.

The $\Lambda$-adic representation $\rho$ can be used to construct $p$-adic analytic functions (in fact Iwasawa functions) which interpolate various data attached to the newforms $f_k$, $k \geqq 2$. For example, for each prime $l \neq p$ let $\mathrm{Frob}_l$ be a Frobenius element at $l$ and let $a_l = Tr(\rho(\mathrm{Frob}_l)) \in \Lambda$. It follows from (0.5) b and the Eichler-Shimura relations that $a_l(k)$ is the $l$-th Fourier coefficient of $f_k$ for each integer $k \geqq 2$.

The Euler factors at $p$ will play an important role in our proof of Theorem 0.3. These factors can be described in terms of the representation $\rho$. Let $a_p = \varphi(\mathrm{Frob}_p) \in \Lambda^*$. Then for each integer $k \geqq 2$, the $p$-th Euler factor of the complex $L$-function $L_\infty(f_k, z)$ of $f_k$ has the form $[(1-\alpha_k p^{-z})(1-\beta_k p^{-z})]^{-1}$, where $\alpha_k = a_p(k)$ and

$$\beta_k = \begin{cases} p^{k-1}/\alpha_k & \text{if } k>2 \text{ and } k \equiv 2 \text{ modulo } (p-1), \\ 0 & \text{otherwise.} \end{cases}$$

These numbers satisfy the congruences $\alpha_k \equiv \alpha_2 = 1$ and $\beta_k \equiv 0$ modulo $p$. We will refer to $\alpha_k$ as the unit root of Frobenius and to $\beta_k$ as the non-unit root of Frobenius. From the above description we see that the family of unit roots of Frobenius $\{\alpha_k\}_{k \geqq 2}$ is interpolated by the Iwasawa function $a_p(k)$, but that the family of non-unit roots of Frobenius $\{\beta_k\}$ cannot be interpolated by any $p$-adic analytic function of $k$.

The remarks of the last two paragraphs may be summarized by saying that there is a formal $q$-expansion

(0.6) $$\mathbf{f} = \sum_{n=1}^{\infty} a_n q^n \in \Lambda[[q]]$$

such that for each integer $k \geqq 2$, the specialization $\mathbf{f}_k \overset{\text{def}}{=} \sum_{n=1}^{\infty} a_n(k)\, q^n$ of $\mathbf{f}$ to weight $k$ is the $q$-expansion of the '$p$-stabilized newform' associated to $f_k$. This is the cusp form $f_k^*$ defined by

$$(0.7) \qquad f_k^*(z) = f_k(z) - \beta_k f_k(pz)$$

for $z$ in the upper half-plane [W]. In particular, the $L$-function of $f_k^*$ is equal to the $L$-function of $f_k$ with the Euler factor $(1-\beta_k p^{-z})^{-1}$ removed.

The final ingredient we require for the proof of Theorem 0.3 is a two variable $p$-adic $L$-function $L_p(k, s)$, $k, s \in \mathbf{Z}_p$. In case $E$ is an elliptic curve with complex multiplication, which has ordinary reduction at $p$, the construction of such a $p$-adic $L$-function having the four properties below is due to Katz [Kz]. When $E$ is a modular elliptic curve with ordinary reduction at $p$ (e.g. if $E$ has multiplicative reduction at $p$), this function was constructed in special cases by Mazur [Mz2]. Mazur's construction was generalized by Kitagawa [K] for arbitrary ordinary $\Lambda$-adic cusp forms. Property (0.8)d was not treated. In this paper we will give another construction of the two variable $p$-adic $L$-function. The properties of it which we need for the case $E = X_0(11)$ are as follows (see Theorem 5.15 for the general case, where we also prove an important functional equation for the 'improved' $p$-adic $L$-function $L_p^*$).

(0.8)a. (Analyticity) $L_p(k, s)$ is analytic for $k, s \in \mathbf{Z}_p$.
(0.8)b. (Specialization to weight two) $L_p(2, s) = L_p(E, s)$.
(0.8)c. (Functional equation) $L_p(k, k-s) = -L_p(k, s)$.
(0.8)d. (Specialization to the critical value $s=1$) There is a factorization

$$L_p(k, 1) = (1 - a_p(k)^{-1})\, L_p^*(k, 1)$$

where $L_p^*(k, 1)$ is a $p$-adic analytic function of $k$ for which

$$L_p^*(2, 1) = \frac{L_\infty(E, 1)}{\Omega_E}.$$

In fact, much more is true. The two variable $p$-adic $L$-function $L_p(k, s)$ interpolates the one variable $p$-adic $L$-functions $L_p(f_k, s)$ associated to the newforms $f_k$, $k \geqq 2$, as in [A-V, Vi]. More precisely, recall that the definition of $L_p(f_k, s)$ depends on the choice of a complex period $\Omega_{f_k} \in \mathbf{C}^*$. This period is determined only up to multiplication by a non-zero element of the field generated by the Fourier coefficients of $f_k$. Fix, once and for all, a choice of complex periods $\Omega_{f_k}$, $k \geqq 2$, with $\Omega_{f_2} = \Omega_E$. Then the two variable $p$-adic $L$-function $L_p(k, s)$ interpolates the functions $L_p(f_k, s)$ in the following sense. For each integer $k \geqq 2$ there is a 'period' $\Omega_k \in \mathbf{Q}_p$ such that

$$L_p(k, s) = \Omega_k \cdot L_p(f_k, s).$$

Note that (0.8)b says that $\Omega_2 = 1$, and in particular that not all $\Omega_k$ vanish. Note also that (0.8)c follows from the above interpolation property and the functional equation of $L_p(f_k, s)$.

The fourth property (0.8)d lies somewhat deeper. For each integer $k \geqq 2$, and each integer $s_0$ with $0 < s_0 < k$ and $s_0 \equiv 1 \pmod{p-1}$, the $p$-adic $L$-function $L_p(f_k, s)$ satisfies the following interpolation property

$$(0.9) \qquad L_p(f_k, s_0) = (1 - \beta_k p^{-s_0})(1 - \alpha_k^{-1} p^{s_0 - 1}) \cdot \frac{L_\infty(f_k, s_0)}{\Omega_{f_k}}.$$

For a discussion of the Euler factors which occur in this expression, see [Gr]. For $s_0 = 1$, the second Euler factor is interpolated by an Iwasawa function, namely $(1 - a_p(k)^{-1})$. This vanishes at $k = 2$ and so is a nonunit in the Iwasawa algebra $\Lambda$. The function $L_p(k, 1)$ is also an Iwasawa function in $k$. Moreover, $L_p(k, 1)$ can be shown to be divisible in $\Lambda$ by $(1 - a_p(k)^{-1})$. The quotient $L_p^*(k, 1)$ is an Iwasawa function in $k$, which we regard as an "improved" $p$-adic $L$-function. It satisfies the interpolation property

$$(0.10) \qquad L_p^*(k, 1) = (1 - \beta_k p^{-1}) \frac{L_\infty(f_k, 1)}{\Omega_{f_k}} \cdot \Omega_k$$

for all integers $k \geqq 2$. When $k = 2$, this reduces to (0.8)d since $\beta_2 = 0$, $\Omega_{f_2} = \Omega_E$, and $\Omega_2 = 1$.

Theorem 0.3 is proved by calculating the linear term in the Taylor expansion of $L_p(k, s)$ about $(k, s) = (2, 1)$. From the functional equation (0.8)c we see that $L_p(k, k/2) = 0$ for all $k \in \mathbf{Z}_p$. Hence the linear term has the form $c \cdot (-\frac{1}{2}(k-2) + (s-1))$ for some constant $c \in \mathbf{Z}_p$. We calculate $c$ in two ways. From (0.8)b we see that $c = L'_p(E, 1)$. But from (0.8)d and the fact that $a_p(2) = \alpha_2 = 1$, it follows that $c = -2a'_p(2) \cdot L_\infty(E, 1)/\Omega_E$. Comparing these expressions for $c$ we obtain

$$L'_p(E, 1) = -2a'_p(2) \cdot \frac{L_\infty(E, 1)}{\Omega_E}.$$

Theorem 0.3 is therefore a consequence of the following special case of Theorem 3.18.

(0.11) **Proposition.** $\mathfrak{L}_p(E) = -2a'_p(2)$.

This proposition can be interpreted as follows. The analytic isomorphism (0.1) leads to an exact sequence

$$(0.12) \qquad 0 \to \mathbf{Q}_p(1) \to V \to \mathbf{Q}_p \to 0$$

where $V = Ta_p(E) \otimes \mathbf{Q}_p$ is the Tate module of $E$ tensored with $\mathbf{Q}_p$. For an arbitrary continuous $G_{\mathbf{Q}_p}$-module $M$, we let $H^n(M)$ denote the continuous Galois cohomology $H^n(G_{\mathbf{Q}_p}, M)$. Then the isomorphism class of the exact sequence (0.12) is determined by a nontrivial extension class $\xi \in H^1(\mathbf{Q}_p(1))$ and the isomorphism class of the middle term $V$ is determined by the line spanned by $\xi$ in $H^1(\mathbf{Q}_p(1))$. Now, by Kummer theory, $\operatorname{ord}_p$ and $\log_p$ give rise to coordinates on $H^1(\mathbf{Q}_p(1))$ in terms of which we get an isomorphism

$$(0.13) \qquad H^1(\mathbf{Q}_p(1)) \cong \mathbf{Q}_p^2.$$

With respect to these coordinates, the $\mathfrak{L}$-invariant $\mathfrak{L}_p(E)$ is the slope of the line spanned by $\xi$. In this way the isomorphism class of $V$ is determined by $\mathfrak{L}_p(E)$ and vice-versa.

Now let $\phi, \chi: G_{\mathbf{Q}_p} \to \Lambda^*$ be the characters of (0.5)c. Let $\mathbf{T}(\varphi^{-1})$ denote the underlying free rank two $\Lambda$-module of the representation $\rho$ with $G_{\mathbf{Q}_p}$ acting on $\mathbf{T}$ via $\rho \otimes \varphi^{-1}$. Then (0.5)c gives us an exact sequence $0 \to \Lambda(\chi\psi) \to \mathbf{T}(\varphi^{-1}) \to \Lambda \to 0$ where $\psi = \varphi^{-2}$. For each $k \in \mathbf{Z}_p$ we "specialize this exact sequence to weight $k$" and tensor with $\mathbf{Q}_p$ to obtain an exact sequence

$$0 \to \mathbf{Q}_p(\chi_k \psi_k) \to \mathbf{V}_k \to \mathbf{Q}_p \to 0 \tag{0.14}$$

where $\chi_k = \sigma_{k-2} \circ \chi$ and $\psi_k = \sigma_{k-2} \circ \psi$ and $\mathbf{V}_k = \mathbf{T}(\varphi^{-1})_k \otimes \mathbf{Q}_p$. Now, as before, the isomorphism class of $\mathbf{V}_k$ is determined by an extension class $\xi_k \in H^1(\mathbf{Q}_p(\chi_k \psi_k))$ up to homothety. Clearly, $\xi_k \neq 0$ for $k$ sufficiently close to 2. But for $k \neq 2$ the cohomology group $H^1(\mathbf{Q}_p(\chi_k \psi_k))$ is one-dimensional. Hence the isomorphism class of $\mathbf{V}_k$ is completely determined by the character $\chi_k \psi_k$. As $k$ approaches 2 in $\mathbf{Z}_p$, the sequence (0.14) flows into the sequence (0.12). Thus we should expect that the sequence (0.12) is completely determined by the characters $\psi_k$ for $k$ in a neighborhood of 2. More precisely, what we prove is

$$\left.\frac{d\psi_k(\mathrm{Frob}_p)}{dk}\right|_{k=2} = \mathfrak{L}_p(E). \tag{0.15}$$

Since $\psi_k(\mathrm{Frob}_p) = a_p(k)^{-2}$, this is equivalent to proposition (0.11). This completes our outline of the proof of Theorem 0.3 in the special case where $E = X_0(11)$ and $p = 11$.

In general, we may start with an arbitrary newform $f$ of weight 2 over $\Gamma_1(Np)(p \nmid N)$ which is split multiplicative at $p$ (i.e. the $p$th Hecke eigenvalue of $f$ is $+1$). In this case $f$ corresponds to a simple quotient of the Jacobian variety of $X_1(Np)$ which has multiplicative reduction at $p$. Our Theorem 7.1 is a strengthened form of Theorem 0.3 in which the $\mathfrak{L}$-invariant of $f$ is defined as in [Mz-T-T]. For example, if $E$ is a modular elliptic curve with split multiplicative reduction at $p$ and if $\psi$ is a primitive Dirichlet character (not necessarily quadratic) for which $\psi(p) = 1$, then we have $L_p(E, \psi, 1) = 0$ and Theorem 7.1 implies $L_p'(E, \psi, 1) = \mathfrak{L}_p(E) \dfrac{L_\infty(E, \psi, 1)}{\Omega_{E,\psi}}$ where $\Omega_{E,\psi} = \Omega_E^{\pm}/\tau(\bar{\psi})$ with $\Omega_E^{\pm}$ being the real or imaginary period of $E$ depending on the sign of $\psi$. This is the "local" property of the $\mathfrak{L}$-invariant discovered numerically by Mazur, Tate, and Teitelbaum in [Mz-T-T].

Some interesting problems arise in the general case which are not apparent in the special case $E = X_0(11)$ described above. In the general case, as in the special case, Hida has constructed a finite $\Lambda$-algebra $\mathcal{R}$ called the universal ordinary Hecke algebra and the newform $f$ corresponds to a $\bar{\mathbf{Q}}_p$-valued homomorphism $\kappa$ of $\mathcal{R}$. But, in general, $\mathcal{R}$ is larger than $\Lambda$, and we can no longer say that $k \in \mathbf{Z}_p$ parametrizes an analytic family of $\bar{\mathbf{Q}}_p$-valued homomorphisms of $\mathcal{R}$. However, it turns out that by restricting $k$ to a neighborhood of 2, we do get such a parametrization locally (see (2.7) and the remarks thereafter for a precise formulation). We then obtain, as before, an analytic family of ordinary

Galois representations $\rho_k$ and a $p$-adic $L$-function $L_p(k, s)$ which is defined for all $s \in \mathbf{Z}_p$ and all $k$ in a neighborhood of 2. In general, however, the functional equation relates the $p$-adic $L$-function to the contragredient family of Galois representations, which will differ from the original family if the Nebentype is nontrivial. This means that the $p$-adic $L$-function need not vanish identically along the line $s=\frac{k}{2}$ as it did in our special case. However, in Theorem 5.15e we will also prove a 'functional equation' for the improved two-variable $p$-adic $L$-function. This allows us to show that the restriction of the standard two-variable $p$-adic $L$-function to the line $s=\frac{k}{2}$ vanishes to order $\geqq 2$ at the point $(k, s)=(2, 1)$. This is sufficient for our purposes.

We may also inquire about the 'denominators' in the various $p$-adic $L$-functions. In general, we expect the two-variable $p$-adic $L$-function to have no denominator at all. On the other hand, the improved $p$-adic $L$-function may have a denominator. This denominator is a 'divisor' of the characteristic power series of a certain torsion $\Lambda$-module – the $\Lambda$-adic cuspidal group – which arises in our calculations (see 6.12). It would be interesting to analyze the structure of this group and to perform a descent analogous to Mazur's Eisenstein descent [Mz1].

We close this introduction by mentioning the following question. Assume that $E$ is a modular elliptic curve with good, ordinary reduction or multiplicative reduction at $p$. Then there is an associated two-variable $p$-adic $L$-function $L_p(k, s)$. Let $n=\mathrm{ord}_{z=1} L_\infty(E, z)$. Then it seems reasonable to believe that the expansion of $L_p(k, s)$ at $k=2, s=1$ should begin with the homogeneous term of degree $n$ (or $n+1$ in the case of split multiplicative reduction). If this degree is odd, then $-\frac{1}{2}(k-2)+(s-1)$ will be a linear factor in this term. Can one determine the other linear factors?

## 1. Hecke operators and ordinary eigenforms

In this section we fix some of the terminology and conventions which will be used in the rest of the paper.

Following [Sh2] we define Hecke algebras as double coset algebras. Let $\Sigma=GL_2(\mathbf{Q}) \cap M_2(\mathbf{Z})$ be the semigroup of $2 \times 2$ integral matrices with nonzero determinant. For each arithmetic group $\Gamma$ in $SL_2(\mathbf{Z})$ we let $D(\Gamma, \Sigma)$ denote the double coset algebra associated to the pair $(\Gamma, \Sigma)$. The elements of this algebra are the $\mathbf{Z}$-valued functions on $\Sigma$ which are bi-invariant with respect to $\Gamma$ and which are supported on the union of finitely many double cosets of $\Gamma$. Clearly $D(\Gamma, \Sigma)$ is generated by the characteristic functions $T(g) \in D(\Gamma, \Sigma)$ of the double cosets $\Gamma g \Gamma$, for $g \in \Sigma$. If $\Sigma_1$ is a subsemigroup of $\Sigma$ containing $\Gamma$, then we will denote by $D(\Gamma, \Sigma_1)$ the subalgebra of $D(\Gamma, \Sigma)$ consisting of functions supported on $\Sigma_1$. For each such $\Sigma_1$ we will denote by $\Sigma_1^+$ the subsemigroup of $\Sigma_1$ consisting of matrices with positive determinant.

In this paper, $\Sigma$-modules will be *contravariant* (i.e. $\Sigma$ acts *on the right*) unless otherwise stated. The algebra $D(\Gamma, \Sigma)$ acts contravariantly and functorially on the cohomology of $\Sigma$-modules. If $\Gamma$ is preserved by the anti-involution $g \mapsto g^*=\det(g)\, g^{-1}$, then $*$ induces an anti-involution on $D(\Gamma, \Sigma)$ by

$T(g) \mapsto T(g)^* = T(g^*)$. In this case we can also define a covariant action of $D(\Gamma, \Sigma)$ on the $\Gamma$-cohomology of $\Sigma$-modules by defining

$$(1.1) \qquad T(g) \cdot \Phi \overset{\text{def}}{=} \Phi | T(g)^*$$

for a cohomology class $\Phi$ and $g \in \Sigma$.

Fix a positive integer $N$, a prime $p$ which does not divide $N$, and an integer $r \geqq 0$. Let $\Sigma_1(p^r) = \left\{ g \in \Sigma \,\middle|\, g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \text{ modulo } p^r \right\}$. If $N > 1$, the algebra $D(\Gamma_1(Np^r), \Sigma_1(p^r))$ is not commutative, but we can construct a central subalgebra as follows. Let $\mathbf{Z}'$ denote the multiplicative set of integers which are prime to $p$. For each $a \in \mathbf{Z}'$ choose $\gamma_a \in \Gamma_1(N) \cap \Gamma_0(p^r)$ whose lower right hand entry is congruent to $a$ modulo $p^r$ and let $[a]_p = T\left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \gamma_a \right)$ in $D(\Gamma_1(Np^r), \Sigma_1(p^r))$. The map $\mathbf{Z}' \to D(\Gamma_1(Np^r), \Sigma_1(p^r))$, $a \mapsto [a]_p$, is multiplicative, hence extends to a $\mathbf{Z}$-algebra morphism $\mathbf{Z}[\mathbf{Z}'] \to D(\Gamma_1(Np^r), \Sigma_1(p^r))$. The image of this map is a central subalgebra. On the other hand, $\mathbf{Z}[\mathbf{Z}']$ embeds naturally in the completed group ring $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$. Hence we may form the $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$-algebra

$$(1.2) \qquad D_p(Np^r) \overset{\text{def}}{=} D(\Gamma_1(Np^r), \Sigma_1(p^r)) \otimes_{\mathbf{Z}[\mathbf{Z}']} \mathbf{Z}_p[[\mathbf{Z}_p^*]].$$

If $A$ is a $\mathbf{Z}_p[\Sigma_1(p^r)]$-module which satisfies Hypothesis P below, then the (contravariant or covariant) action of $D(\Gamma_1(Np^r), \Sigma_1(p^r))$ on the $\Gamma_1(Np^r)$-cohomology of $A$ extends uniquely to a continuous action of $D_p(Np^r)$.

(1.3) **Hypothesis P**. *The action of the scalar matrices $aI$, $a \in \mathbf{Z}'$, extends to a continuous action of the scalar matrices $aI$ for $a \in \mathbf{Z}_p^*$.*

We are going to view $D_p(N)$ as a universal algebra which acts on the $\Gamma_1(Np^r)$-cohomology, for every $r \geqq 0$, of every $\mathbf{Z}_p[\Sigma]$-module $A$ satisfying Hypothesis P. To define this action we note that the Hecke pair $(\Gamma_1(Np^r), \Sigma_1(p^r))$ is weakly compatible to $(\Gamma_1(N), \Sigma)$ in the sense of [A-S]. Hence, as in [A-S], there is a natural surjective $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$-morphism

$$(1.4) \qquad D_p(N) \to D_p(Np^r)$$

induced by restriction of functions on $\Sigma$ to $\Sigma_1(p^r)$. Let $D_p(N)$ act on the $\Gamma_1(Np^r)$-cohomology of $A$ via the composition of this morphism with the natural action of $D_p(Np^r)$.

The elements of $D(\Gamma_1(Np^r), \Sigma_1(p^r))$ supported on $\Sigma_1^+(p^r)$ (elements of positive determinant) generate a $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$-subalgebra of $D_p(Np^r)$ which we denote $D_p^+(Np^r)$. As before, the algebra $D_p^+(N)$ acts naturally on the $\Gamma_1(Np^r)$-cohomology of any $\mathbf{Z}_p[\Sigma_1^+(p^r)]$-module $A$ satisfying Hypothesis P.

(1.5) **Definition.** We define the following standard elements of $D_p(N)$.

**a.** For each positive integer $n$, let $T_n = T\left( \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \right)$.

**b.** For each integer $a$ prime to $N$ choose an element $\beta_a \in \Gamma_0(N)$ whose lower right entry is congruent to $a$ modulo $N$ and define $[a]_N = T(\beta_a)$.

**c.** We extend $[\cdot]_p\colon \mathbf{Z}_p^* \to D_p(N)$ to a multiplicative function on all nonzero $p$-adic integers by defining $[p]_p = T\left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\right)$.

**d.** $\iota = T\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$.

**e.** $W_N = T\left(\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}\right)$.

All of these elements, except $\iota$, are in the subalgebra $D_p^+(N)$. Moreover, $D_p(N) = D_p^+(N)[\iota]$. The elements $[a]_N$, for $a \in \mathbf{Z}$ prime to $N$, generate a subgroup $\Delta_N$ in $D_p^+(N)$ isomorphic to $(\mathbf{Z}/N\mathbf{Z})^*$. Let $\mathscr{H}$ be the $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$-subalgebra of $D_p^+(N)$ given by

(1.6) $$\mathscr{H} = \mathbf{Z}_p[[\mathbf{Z}_p^*]]\,[\Delta_N, T_n\,(n \in \mathbf{Z}^+), [p]_p].$$

Then $\mathscr{H}$ is commutative and is centralized by $\iota$. If we define $[a] = [a]_N \cdot [a]_p \in \mathscr{H}$ for integers $a$ which are prime to $Np$, then the map $a \mapsto [a]$ extends uniquely to a continuous multiplicative map from $\mathbf{Z}_{p,N}^* = \varprojlim (\mathbf{Z}/Np^r\mathbf{Z})^*$ to $\mathscr{H}$. In this way we obtain a $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$-morphism

(1.7) $$\mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]] \to \mathscr{H}$$

in terms of which we may view $\mathscr{H}$ as a $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]]$-algebra. The element $W_N$ does not centralize $\mathscr{H}[\iota]$. Indeed, $W_N$ does not even normalize $\mathscr{H}$ (e.g. $W_N T_n W_N^{-1} \notin \mathscr{H}$ if $(n, N) > 1$). However, we do have the following relations in $\mathscr{H}[\iota, W_N]$:

(1.8) $$\begin{aligned} W_n \cdot [a] \cdot W_N^{-1} &= [a]_N^{-1} \cdot [a]_p && \text{for all } a \in \mathbf{Z}_{p,N}^*; \\ W_N \cdot T_n \cdot W_N^{-1} &= [n]_N^{-1} \cdot T_n && \text{for every } n \text{ which is relative prime to } N; \\ W_N \cdot \iota \cdot W_N^{-1} &= [-1]_N \cdot \iota; \\ W_N^2 &= [-N]_p. \end{aligned}$$

Let $k$ be an integer $\geqq 2$ and, for each congruence group $\Gamma$, let $\mathscr{S}_k(\Gamma, \bar{\mathbf{Q}})$ denote the space of holomorphic weight $k$ cusp forms over $\Gamma$ whose $q$-expansions have algebraic coefficients. Let

(1.9) $$\mathscr{S}_k(\bar{\mathbf{Q}}) = \bigcup \mathscr{S}_k(\Gamma, \bar{\mathbf{Q}})$$

be the union over all congruence groups $\Gamma$. We let the subsemigroup $\Sigma^+ \subset \Sigma$ of elements with positive determinant act on $\mathscr{S}_k(\bar{\mathbf{Q}})$ by the weight $k$ action: if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma^+$ and $f \in \mathscr{S}_k(\bar{\mathbf{Q}})$ then $(f|g)(z) = \det(g)^{k-1}(cz+d)^{-k} f(gz)$ for $z$ in the upper half plane. We extend this by linearity to an action of $\Sigma^+$ on $\mathscr{S}_k(\bar{\mathbf{Q}}_p) = \mathscr{S}_k(\bar{\mathbf{Q}}) \otimes_{\bar{\mathbf{Q}}} \bar{\mathbf{Q}}_p$ and define

(1.10) $$\mathscr{S}_k(\Gamma_1(Np^r), \bar{\mathbf{Q}}_p) = \mathscr{S}_k(\bar{\mathbf{Q}}_p)^{\Gamma_1(Np^r)}.$$

Since the nonzero scalar matrices over $\mathbf{Z}$ act on $\mathscr{S}_k(\bar{\mathbf{Q}}_p)$ via $aI \mapsto a^{k-2}$, Hypothesis P is satisfied and we obtain an action of $\mathscr{H}[W_N]$ on $\mathscr{S}_k(\Gamma_1(Np^r), \bar{\mathbf{Q}}_p)$.

(1.11) **Definition. a.** If $r$ is a nonnegative integer and $\psi$ is a Dirichlet character whose conductor is a power of $p$, then we let $\sigma_{r,\psi}\colon \mathbf{Z}_p^* \to \bar{\mathbf{Q}}_p^*$ be the character defined by $a \mapsto \psi(a)\, a^r$. Such a character will be called an *arithmetic character*. If $\psi$ is trivial we will suppress it from the notation and write simply $\sigma_r$ instead of $\sigma_{r,\psi}$.
**b.** If $R$ is a commutative $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]]$-algebra we let $\mathscr{X}(R) = \mathrm{Hom}_{\mathrm{cont}}(R, \bar{\mathbf{Q}}_p)$ denote the set of continuous $\bar{\mathbf{Q}}_p$-valued homomorphisms on $R$. We will refer to the elements of $\mathscr{X}(R)$ as the $\bar{\mathbf{Q}}_p$-valued *points* on $R$. A continuous homomorphism $\kappa\colon R \to \bar{\mathbf{Q}}_p$ will be called an *arithmetic point* if its restriction to $\mathbf{Z}_p^*$ is an arithmetic character. In that case we say that $\kappa$ has weight $r+2$ and character $\varepsilon$ if $\kappa([a]) = \varepsilon(a)\, a^r$ for every rational integer $a$ prime to $Np$. Let

$$\mathscr{X}^{\mathrm{arith}}(R) = \text{the arithmetic points on } R.$$

**c.** We will write $\mathscr{X}_0$ and $\mathscr{X}_0^{\mathrm{arith}}$ for the points and the arithmetic points, respectively, of our base algebra $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$.

It will often be convenient to view the elements of $R$ as functions on $\mathscr{X}(R)$. When we wish to emphasize this point of view, we will write $\alpha(\kappa)$ instead of $\kappa(\alpha)$ for $\alpha \in R$ and $\kappa \in \mathscr{X}(R)$.

The homomorphism $\kappa\colon \mathscr{H} \to \bar{\mathbf{Q}}_p$ associated to any eigenform $f$ is easily seen to be an arithmetic point on the $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]]$-algebra $\mathscr{H}$ (1.7) whose weight is the weight of $f$ and whose character is the nebentype character of $f$.

## 2. Deformations of ordinary Galois representations

Fix a positive integer $N$ and a prime $p \geqq 5$ which does not divide $N$. For each integer $n \geqq 0$, let $J_{n/\mathbf{Q}}$ be the Jacobian of the modular curve $X_1(Np^n)$ equipped with Shimura's canonical model [Sh2] associated to the adelic group

$$\left\{ g \in GL_2(\mathbf{A}_f) \,\middle|\, g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{Np^n} \right\}.$$

There is a natural covariant action of the Hecke algebra $\mathscr{H}[\iota, W_N]$ on $J_n(\mathbf{C})$ induced by the action of the double coset algebra $D(\Gamma_1(Np^n), \Sigma_1^+(p^n))$ via algebraic correspondences and letting $\iota$ act by complex conjugation. The elements $[a]$, for $a \in \mathbf{Z}_{p,N}^*$ operate through the nebentype.

Let $Ta_p(J_n)$ denote the $p$-adic Tate module of $J_n$. The action of $\mathscr{H}$ on $J_n$ is defined over $\mathbf{Q}$, and therefore induces an action of $\mathscr{H}$ on $Ta_p(J_n)$ which commutes with the action of the Galois group $G_{\mathbf{Q}}$. For each pair of nonnegative integers $m \geqq n$, the natural projection $X_1(Np^m) \to X_1(Np^n)$ is defined over $\mathbf{Q}$, hence induces a Galois equivariant map of Tate modules $Ta_p(J_m) \to Ta_p(J_n)$. If $m, n$ are positive then this projection also commutes with $D_p(N)$ and in particular with $\mathscr{H}$. (When $n=0$ and $m>0$, it respects all of the generators $T_q$, $[a]$ of $\mathscr{H}$ except $T_p$ and $[p]_p$.) We may therefore form the projective limit over $n>0$ and define an $\mathscr{H}[G_{\mathbf{Q}}]$-module

$$(2.1) \qquad Ta_p(J_\infty) = \varprojlim_n Ta_p(J_n).$$

The importance of $Ta_p(J_\infty)$ for the study of $p$-adic Galois representations attached to modular forms was first recognized by Shimura [Sh1] (see [O] for a published account). More recently, Hida defined a certain factor of $Ta_p(J_\infty)$ called the ordinary part and made a careful and beautiful analysis of its structure. The following discussion is based on his works [H1, H2].

(2.2) **Definition.** Let $\mathbf{A}$ be a profinite abelian group and $T_p\colon \mathbf{A} \to \mathbf{A}$ be a continuous homomorphism. The ordinary submodule of $\mathbf{A}$ is defined to be $\mathbf{A}^0 = \bigcap_{n=1}^{\infty} T_p^n(\mathbf{A})$.

(2.3) **Proposition.** *Let $\mathbf{A} = \varprojlim \mathbf{A}_n$ be a profinite abelian group and let $T_p$ be an operator on $\mathbf{A}$ which is equal to a limit of operators on the finite quotients $\mathbf{A}_n$. Then $T_p$ acts invertibly on $\mathbf{A}^0$ and there is a canonical decomposition $\mathbf{A} = \mathbf{A}^0 \oplus \mathbf{A}^{\mathrm{nil}}$ where $\mathbf{A}^{\mathrm{nil}} = \{a \in \mathbf{A} \mid \lim_{n\to\infty} T_p^n(a) = 0\}$ is the subgroup on which $T_p$ acts topologically nilpotently.*

*Proof.* In case $\mathbf{A}$ is finite, $\mathbf{A}^0$ is the subgroup on which $T_p$ acts periodically. Clearly, $T_p$ acts invertibly on $\mathbf{A}^0$ in this case. The asserted decomposition then follows in the finite case from the fact that every orbit of $T_p$ is eventually periodic. The general case follows from the finite case by a simple compactness argument.

Since the Tate modules $Ta_p(J_n)$ are profinite, so also is $Ta_p(J_\infty)$. Moreover, $Ta_p(J_\infty)$ satisfies the hypotheses of Proposition 2.3, so we may define the ordinary part $Ta_p(J_\infty)^0$. Since the operator $T_p$ commutes with $\mathscr{H}[G_{\mathbf{Q}}]$, we see that $Ta_p(J_\infty)^0$ is a direct factor of the $\mathscr{H}[G_{\mathbf{Q}}]$-module $Ta_p(J_\infty)$.

Let $\Lambda$ be the Iwasawa algebra $\mathbf{Z}_p[[1+p\mathbf{Z}_p]]$ embedded in the natural way in $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ and let $\mathscr{L}$ be the fraction field of $\Lambda$. Let $H^0$ be the image of $\mathscr{H}$ in the endomorphism ring of $Ta_p(J_\infty)^0$. Hida has proven ([H1], Thm. 3.1) that $Ta_p(J_\infty)^0$ is a free $\Lambda$-module of finite rank. Moreover, he has constructed an idempotent $e_{\mathrm{prim}} \in H^0_{\mathscr{L}} = H^0 \otimes_\Lambda \mathscr{L}$ ([H2], pp. 250, 252) analogous to projection to the space of $N$-primitive eigenforms in Atkin-Lehner theory. We define the *primitive part* of $Ta_p(J_\infty)^0$ to be the $\mathscr{H}[G_{\mathbf{Q}}]$-submodule $\mathbf{T} = Ta_p(J_\infty)^0_{\mathrm{prim}}$ obtained as the intersection of $Ta_p(J_\infty)^0$ and $e_{\mathrm{prim}} \cdot Ta_p(J_\infty)^0 \otimes_\Lambda \mathscr{L}$. Then $\mathbf{T}$ is a reflexive $\Lambda$-module and is therefore free of finite rank. Since $\iota$ and $W_N$ preserve the $N$-primitive part of $Ta_p(J_n)$, they induce operators on $\mathbf{T}$. Note, however, that $W_N$ does not in general commute with $\mathscr{H}$ and that neither $\iota$ nor $W_N$ commutes with the action of the Galois group.

We associate to $\mathbf{T}$ the following data which will be used throughout the paper.

(2.4) **Definition. a.** The *universal ordinary $p$-adic Hecke algebra* of tame conductor $N$ is defined to be the image $\mathscr{R}$ of $\mathscr{H}$ in $\mathrm{End}_\Lambda(\mathbf{T})$. The natural map $h\colon \mathscr{H} \to \mathscr{R}$ endows $\mathscr{R}$ with the structure of $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]]$-algebra inherited from $\mathscr{H}$. The induced character $\eta\colon \mathbf{Z}_{p,N}^* \to \mathscr{R}^*$ will be called the *canonical character*. Let $\mathscr{K} = \mathscr{R} \otimes_\Lambda \mathscr{L}$ where $\mathscr{L}$ is the fraction field of $\Lambda$ and let $\tilde{\mathscr{R}}$ be the normalization of $\mathscr{R}$ in $\mathscr{K}$. Let $\mathscr{X} = \mathscr{X}(\tilde{\mathscr{R}}) = \mathrm{Hom}_{\mathrm{cont}}(\tilde{\mathscr{R}}, \bar{\mathbf{Q}}_p)$ and set

$$\mathscr{X}^{\mathrm{arith}} = \mathscr{X}^{\mathrm{arith}}(\tilde{\mathscr{R}}) = \text{the arithmetic points on } \tilde{\mathscr{R}} \text{ (see (1.11))}.$$

**b.** Let $\mathbf{T}^\pm$ denote the $\pm$ eigenmodules of $\iota$.

**c.** The *universal ordinary p-stabilized newform* of tame conductor $N$ is defined to be the formal $q$-expansion $\mathbf{f}\in\mathscr{R}[[q]]$ given by $\mathbf{f}=\sum_{n=1}^{\infty} a_n q^n$ where the coefficients are given by $a_n=h(T_n)$ for each integer $n>0$.
**d.** The *universal ordinary p-adic Galois representation* of tame conductor $N$ is defined to be the representation $\rho\colon G_{\mathbf{Q}}\to \mathrm{Aut}_{\mathscr{R}}(\mathbf{T})$.

(2.5) **Definition. a.** We will say that an eigenform $f\in\mathscr{S}_k(\Gamma_1(Np^m), \bar{\mathbf{Q}}_p)$ $(k\geqq 2)$ is *ordinary* if the eigenvalue $a_p$ of $T_p$ on $f$ is a unit, that is, if $|a_p|_p=1$.
**b.** An ordinary eigenform $f$ will be called a *p-stabilized newform* of tame conductor $N$ if it is normalized (its leading Fourier coefficient is 1) and the following two conditions hold.
(1) The conductor of $f$ is divisible by $N$.
(2) The level of $f$ is divisible by $p$.

It is not hard to see that an ordinary $p$-stabilized newform $f$ is either already a newform, or is related to a newform $g$ of conductor $N$ as described in (0.7). In the latter case, $f$ has level $Np$.

(2.6) **Theorem.** *Let $p$ be a prime $\geqq 5$ and suppose $p\nmid N$. Let $r$ be the number of ordinary p-stabilized newforms of tame conductor $N$ in $\mathscr{S}_2(\Gamma_1(Np), \bar{\mathbf{Q}}_p)$.*

**a.** (Hida) *The $\mathscr{L}$-algebra $\mathscr{K}=\mathscr{R}\otimes_\Lambda \mathscr{L}$ is a finite product of finite field extensions of $\mathscr{L}$* ([H2], Thm. 3.5). *Moreover, $\dim_{\mathscr{L}}\mathscr{K}=r$. For each $\kappa\in\mathscr{X}^{\mathrm{arith}}$, the localization $\mathscr{R}_{(\kappa)}$ of $\mathscr{R}$ at $\kappa$ is a discrete valuation ring which is unramified over $\Lambda$* ([H1], Cor. 1.4).
**b.** (Hida) *The map $\kappa\mapsto\mathbf{f}_\kappa$ establishes a one-one correspondence*

$$\left\{\begin{matrix}\text{Arithmetic points}\\ \text{on } \mathscr{R}\end{matrix}\right\}\leftrightarrow\left\{\begin{matrix}\text{Ordinary } p\text{-stabilized newforms}\\ \text{of tame conductor } N\end{matrix}\right\}.$$

**c.** (Hida) *The $\Lambda$-modules $\mathbf{T}^\pm$ are free of rank $r$. As $\mathscr{K}$-modules, $\mathbf{T}^\pm_{\mathscr{L}}=\mathbf{T}^\pm\otimes_\Lambda\mathscr{L}$ are free of rank one. We may therefore regard $\rho$ as a two-dimensional Galois representation over $\mathscr{K}$. This representation is unramified outside $Np$ and for each prime $l$ outside $Np$, the characteristic polynomial of $\rho(\mathrm{Frob}_l)$ is $X^2-a_l X+l\eta(l)$ where $\eta\colon \mathbf{Z}^*_{p,N}\to\mathscr{R}^*$ is the canonical character* (2.4) a.
**d.** (Mazur, Wiles) ([Mz-W], and [W] Theorem 2.2.2) *Let $\varphi\colon G_{\mathbf{Q}_p}\to\mathscr{R}^*$ be the unramified character for which $\varphi(\mathrm{Frob}_p)=a_p$. Let $\chi_0\colon G_{\mathbf{Q}_p}\to\mathbf{Z}_p^*$ be the cyclotomic character and let $\eta\colon G_{\mathbf{Q}_p}\to\mathbf{Z}^*_{p,N}\subseteq\mathscr{R}^*$ be the Galois character associated to the canonical character $\eta$* (2.4)a *by class field theory (i.e. compose $\eta$ with the homomorphism $G_{\mathbf{Q}_p}\to\mathbf{Z}^*_{p,N}$ induced by the action on $Np^\infty$ roots of unity). Then, as a $G_{\mathbf{Q}_p}$-module, $\mathbf{T}_{\mathscr{L}}$ has a filtration*

$$0\to\mathscr{K}(\chi_0\eta\varphi^{-1})\to\mathbf{T}_{\mathscr{L}}\to\mathscr{K}(\varphi)\to 0.$$

These results can be interpreted analytically as follows. For each arithmetic point $\kappa\in\mathscr{X}^{\mathrm{arith}}$ of weight $k_0$ and character $\varepsilon$, let $\mathscr{A}(\kappa)$ be the subring of $\bar{\mathbf{Q}}_p[[x-k_0]]$ consisting of formal power series in $x-k_0$ with a positive radius of convergence and let $\mathscr{M}(\kappa)$ be the field of fractions of $\mathscr{A}(\kappa)$. We endow $\mathscr{A}(\kappa)$ with an $\mathscr{R}_{(\kappa)}$-structure $\tilde{\kappa}\colon\mathscr{R}\to\mathscr{A}(\kappa)$ as follows. On the image of $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ in $\mathscr{R}$ we define $\tilde{\kappa}$ by associating to each $t\in\mathbf{Z}_p^*$ the power series in $\mathscr{A}(\kappa)$ representing the analytic function $k\mapsto\varepsilon\omega^{k_0}(t)\langle t\rangle^k$, where $\omega$ is the Teichmüller character and

$\langle\ \rangle$ is projection to the principal units $1+p\mathbf{Z}_p$. Now it is well-known that the ring of convergent power series is Henselian ([N], Thm. 45.5) and, from (2.6)a we have $\mathscr{R}_{(\kappa)}$ is unramified over $\Lambda$. Hence there is a unique extension of this map to a morphism

$$(2.7) \qquad \tilde{\kappa}\colon \mathscr{R}_{(\kappa)} \to \mathscr{A}(\kappa)$$

such that $\tilde{\kappa}(a)(k_0)=\kappa(a)$ for every $a\in\mathscr{R}$. Moreover, we can extend $\tilde{\kappa}$ by linearity to an $\mathscr{L}$-homomorphism $\tilde{\kappa}\colon \mathscr{K}\to\mathscr{M}(\kappa)$. We define the *domain of convergence* about $\kappa$ to be the intersection of the disks of convergence of the power series $\tilde{\kappa}(a)\in\mathscr{A}(\kappa)$ where $a$ ranges over $\tilde{\mathscr{R}}$. Since $\tilde{\mathscr{R}}$ is finite over $\Lambda$, $U_\kappa$ is an open disk centered at $k_0$.

(2.8) **Notational Conventions.** Let $\kappa$ be an arithmetic point on $\mathscr{R}$ and let $a\in\mathscr{K}$.
a. Define $a(\kappa, k)$ to be the meromorphic function of $k$ about $k_0$ represented by $\tilde{\kappa}(a)\in\mathscr{M}(\kappa)$. For each $k\in U_\kappa$, let $\kappa^{(k)}\in\mathscr{X}$ be the point defined by $\kappa^{(k)}(a)=a(\kappa, k)$ for $a\in\tilde{\mathscr{R}}$.
b. We say that $a$ is regular at $\kappa$ if $a(\kappa, k)$ does not have a pole at $k=k_0$. In that case, we will write $a(\kappa)$, $a'(\kappa)$ for the value, respectively the derivative, of $a(\kappa, k)$ at $k=k_0$.

For each $\kappa\in\mathscr{X}^{\mathrm{arith}}$ and $k\in U_\kappa$, we let $\mathbf{f}_{\kappa,k}$ denote the specialization $\Sigma a_n(\kappa, k)\, q^n$ of $\mathbf{f}$ to $\kappa^{(k)}$. As a function of $k\in U_\kappa$ this is an analytic family of formal $q$-expansions which interpolates the $q$-expansions of ordinary $p$-stabilized newforms at integers $k\geqq 2$ in $U_\kappa$. Similarly, we can specialize $\rho$ to obtain an analytic family $\rho_{\kappa,k}$, $k\in U_\kappa$, of Galois representations interpolating the Galois representations associated to the forms $\mathbf{f}_{\kappa,k}$ at integers $k\geqq 2$ in $U_\kappa$.

We close this section by describing an involution on $\mathscr{K}$ which will play an important role later.

(2.9) **Proposition.** *Conjugation by $W_N$ in $\mathrm{End}_{\mathscr{L}}(\mathbf{T}_{\mathscr{L}})$ preserves the subalgebra $\mathscr{K}$ and induces an involution $*$ on $\mathscr{K}$ satisfying the following properties.*
a. $[t]_N^* = [t]_N^{-1}$ *for all $t\in\Delta_N$; and*
b. $a_l^* = [l]_N^{-1}\, a_l$ *for all primes $l\nmid N$.*

*Proof.* Since, by Atkin-Lehner theory, the actions of $\mathscr{H}$ and of $W_N\mathscr{H}W_N^{-1}$ commute with one another on the $N$-primitive part of $Ta_p(J_n)$ for each $n$, they also commute on $\mathbf{T}$. In particular, $W_N\mathscr{K}W_N^{-1}$ centralizes $\mathscr{K}$ in $\mathrm{End}_{\mathscr{L}}(\mathbf{T}_{\mathscr{L}})$. Since $\mathbf{T}_{\mathscr{L}}^{\pm}$ are free of rank one as $\mathscr{K}$-modules and are preserved by $W_N\mathscr{K}W_N^{-1}$, there are involutions $i_\pm\colon\mathscr{K}\to\mathscr{K}$ over $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ such that, for every $a\in\mathscr{K}$, $i_\pm(a)=W_N\, a\, W_N^{-1}$ on $\mathbf{T}_{\mathscr{L}}^{\pm}$. By (1.8), each of these involutions satisfies properties a and b of the proposition. We need to show $i_+=i_-$. Let $\kappa\in\mathscr{X}^{\mathrm{arith}}$ be an arbitrary arithmetic point on $\tilde{\mathscr{R}}$, and let $\kappa_\pm=\kappa\circ i_\pm$. Then $\mathbf{f}_{\kappa_+}$ and $\mathbf{f}_{\kappa_-}$ are ordinary $p$-stabilized newforms which, according to b, have identical eigenvalues for the Hecke operators $T_n$, $(n, N)=1$. Hence, by the strong multiplicity one theorem, we have $\mathbf{f}_{\kappa_+}=\mathbf{f}_{\kappa_-}$. By Theorem 2.6b we conclude that $\kappa_+=\kappa_-$ and, since $\kappa$ was arbitrary, that $i_+=i_-$. This completes the proof.

The involution $*$ on $\mathscr{K}$ induces an involution on $\mathscr{X}$ which we will denote $\kappa\mapsto\kappa^*$. If $\kappa$ is an arithmetic point of weight $k_0$ and Nebentype character $\varepsilon=\varepsilon_N\,\varepsilon_p$, then $\kappa^*$ is an arithmetic point of the same weight $k_0$, but with character $\varepsilon_N^{-1}\,\varepsilon_p$.

Moreover, for any $a\in\mathscr{K}$ we have the following identity of meromorphic functions in a neighborhood of $k=k_0$:

$$a^*(\kappa^*, k)=a(\kappa, k). \tag{2.10}$$

(2.11) **Example.** In the example of the introduction, where $E=X_0(11)$, $p=11$, and $N=1$, the space $\mathscr{S}_2(\Gamma_1(11))$ is one dimensional and is spanned by $f_E$. Since $f_E$ is ordinary at $p=11$ we have $r=1$ in Theorem 2.6. Hence the universal ordinary Hecke algebra (2.4)a is given by $\mathscr{R}=\Lambda$. Moreover, we have $\mathbf{T}=Ta_p(J_\infty)^0_{\mathrm{prim}}=Ta_p(J_\infty)^0$. If we set

$$\mathbf{f}=\sum a_n q^n\in\Lambda[[q]], \qquad \rho\colon G_{\mathbf{Q}}\to \mathrm{Aut}_\Lambda(\mathbf{T})$$

as in (2.4)c, d, then $\mathbf{f}$, and $\rho$ are the deformations of $f_E$ and $Ta_p(E)$, respectively, whose properties are described in (0.5). For each integer $k\geqq 2$ let $\mathbf{f}_k$ (respectively, $\rho_k$) be the specialization of $\mathbf{f}$ (respectively, $\rho$) to weight $k$ and trivial character and let $f_k$ be the associated normalized newform. The assertions (0.5) are then immediate consequences of Theorem 2.6. The precise prescription of the conductor of each $f_k$ given in (0.5)b follows from [A-L]. Indeed, in [A-L] it is proven that if $f$ is a newform of weight $k\geqq 2$ with prime level $p$ and trivial character, then $a_p(f)=\pm p^{\frac{k-2}{2}}$. Hence $f$ can be ordinary at $p$ only if $k=2$.

## 3. The $\mathfrak{L}$-invariant

*The $\mathfrak{L}$-invariant of an abelian variety with split multiplicative reduction*

Let $p$ be a rational prime and let $A_{/\mathbf{Q}_p}$ be an abelian variety over $\mathbf{Q}_p$ with split multiplicative reduction. Then the dual abelian variety $B_{/\mathbf{Q}_p}$ also has split multiplicative reduction. Let $X$, $Y$ be the character groups of $B^0_{\mathbf{F}_p}$, $A^0_{\mathbf{F}_p}$, respectively. Then $X$ and $Y$ are free abelian groups of rank $\dim(A)$ on which the local Galois group $G_{\mathbf{Q}_p}=\mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts trivially. From the theory of $p$-adic uniformization [M<sup></sup>cC, Mo] we obtain a bi-multiplicative pairing

$$j\colon X\times Y\to \mathbf{Q}_p^* \tag{3.1}$$

and exact sequences of $G_{\mathbf{Q}_p}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{j} & \mathrm{Hom}(Y,\bar{\mathbf{Q}}_p^*) & \longrightarrow & A(\bar{\mathbf{Q}}_p) & \longrightarrow & 0 \\ 0 & \longrightarrow & Y & \xrightarrow{j} & \mathrm{Hom}(X,\bar{\mathbf{Q}}_p^*) & \longrightarrow & B(\bar{\mathbf{Q}}_p) & \longrightarrow & 0 \end{array} \tag{3.2}$$

where the maps labeled $j$ are induced by (3.1). Moreover, the pairing $\alpha=\mathrm{ord}_p\circ j\colon X\times Y\to\mathbf{Z}$ is nondegenerate. Let $X_p=X\otimes\mathbf{Q}_p$, $Y_p=Y\otimes\mathbf{Q}_p$ and let $\alpha_p$ be the non-degenerate pairing of $\mathbf{Q}_p$-vector spaces induced by $\alpha$:

$$\alpha_p\colon X_p\times Y_p\to\mathbf{Q}_p. \tag{3.3}$$

Recall that $\log_p\colon \mathbf{Q}_p^* \to \mathbf{Z}_p$ is the unique group homomorphism for which $\log_p p = 0$ and $\log_p(1+x) = x - x^2/2 + x^3/3 - \ldots + (-1)^{n+1} x^n/n + \ldots$ whenever $x \in p\mathbf{Z}_p$. The composition of $j$ with $\log_p$ induces another pairing

$$(3.4) \qquad \beta_p\colon X_p \times Y_p \to \mathbf{Q}_p.$$

(3.5) **Definition.** The $\mathfrak{L}$-invariant of $A$ is the $\mathbf{Q}_p$-endomorphism $\mathfrak{L}_p(A)\colon X_p \to X_p$ for which $\beta_p(x, y) = \alpha_p(\mathfrak{L}_p(A)\, x, y)$ for all $x \in X_p$, $y \in Y_p$.

(3.6) **Example.** If $A$ is an elliptic curve then $A$ is canonically isomorphic to its dual abelian variety. Hence we may take $A = B$ and $X = Y$. Moreover, $X$ is free of rank one over $\mathbf{Z}$. Tate's multiplicative period $q_A$ is given by $q_A = j(x_0, x_0)$ where $j$ is the pairing (3.1) and $x_0$ is a generator of $X = Y$. Hence $\beta_p(x_0, x_0) = \log_p(q_A)$, $\alpha(x_0, x_0) = \mathrm{ord}_p(q_A)$ and it follows that the $\mathfrak{L}$-invariant defined in (3.5) agrees with the $\mathfrak{L}$-invariant defined in the introduction for elliptic curves over $\mathbf{Q}_p$ with split multiplicative reduction.

Returning now to the general case we will show how the $\mathfrak{L}$-invariant can also be described in terms of the $p$-adic Galois representation associated to $A$. Indeed, we will generalize the above definition by first introducing the notion of a *split multiplicative* Galois representation and then associating an $\mathfrak{L}$-invariant to an arbitrary split multiplicative Galois representation.

### *The $\mathfrak{L}$-invariant of a split multiplicative representation*

Let $\mathrm{art}\colon \mathbf{Q}_p^* \to G_{\mathbf{Q}_p}^{ab}$ be the Artin symbol, where we observe the conventions of [Ser]. Thus, if $\chi_0$ is the cyclotomic character then $\chi_0(\mathrm{art}(u)) = u$ for all $u \in \mathbf{Z}_p^*$. We will write $\mathrm{Frob}_p$ for $\mathrm{art}(p)^{-1}$. This is a lifting to $G_{\mathbf{Q}_p}^{ab}$ of the Frobenius element on the maximal unramified extension of $\mathbf{Q}_p$. If $W$ is a finite dimensional vector space over a finite extension $K$ of $\mathbf{Q}_p$ and if $W$ is equipped with the trivial action of $G_{\mathbf{Q}_p}$, then there is a canonical isomorphism $H^1(W) \cong \mathrm{Hom}(G_{\mathbf{Q}_p}^{ab}, W)$ of $K$-vector spaces. Moreover, for any nontrivial principal unit $u \in 1 + p\mathbf{Z}_p$ the map

$$H^1(W) \to W \times W$$
$$\xi \mapsto \left(\xi(\mathrm{Frob}_p), \frac{1}{\log_p u}\, \xi(\mathrm{art}(u))\right)$$

is an isomorphism whose definition is independent of the choice of $u$. The space $H^1(W)$ therefore decomposes into a corresponding direct sum

$$H^1(W) = H^1(W)_{\mathrm{unr}} \oplus H^1(W)_{\mathrm{cyc}}$$

where $H^1(W)_{\mathrm{unr}}$ is the space of unramified homomorphisms and $H^1(W)_{\mathrm{cyc}}$ is the space of homomorphisms which factor through the basic cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$. Let

$$(3.7) \qquad \lambda_{\mathrm{unr}}\colon W \to H^1(W)_{\mathrm{unr}} \quad \text{and} \quad \lambda_{\mathrm{cyc}}\colon W \to H^1(W)_{\mathrm{cyc}}$$

be the induced linear isomorphisms. Hence for each $w \in W$, $\lambda_{\mathrm{unr}}(w)$ is the unique unramified homomorphism for which $\lambda_{\mathrm{unr}}(w)(\mathrm{Frob}_p) = w$ and $\lambda_{\mathrm{cyc}}(w)$ is the unique cyclotomic homomorphism for which $\lambda_{\mathrm{cyc}}(w)(\mathrm{art}(u))=\log_p(u)\cdot w$ for every $u\in\mathbf{Z}_p^*$.

(3.8) **Definition.** A finite dimensional $G_{\mathbf{Q}_p}$-representation $V$ over $K$ will be called split multiplicative if the following conditions are satisfied.
**a.** There is an exact sequence of $G_{\mathbf{Q}_p}$-representations

$$0\to V^0(1)\to V\to V^{et}\to 0$$

where $G_{\mathbf{Q}_p}$ acts trivially on $V^{et}$ and $V^0$ (hence via the cyclotomic character on $V^0(1)$).
**b.** The degree one coboundary map $\delta\colon H^1(V^{et})\to H^2(V^0(1))$ associated to the long exact cohomology sequence of **a** induces an isomorphism $\delta\colon H^1(V^{et})_{\mathrm{unr}}\xrightarrow{\sim} H^2(V^0(1))$.

Since $H^2(V^0(1))$ is canonically isomorphic to $V^0$, the composition of $\delta$ with $\lambda_{\mathrm{unr}}$ and $\lambda_{\mathrm{cyc}}$ (3.7) gives rise to maps $\delta_{\mathrm{unr}}$, $\delta_{\mathrm{cyc}}$: $V^{et}\to V^0$. Condition **b** of (3.8) is equivalent to the assertion that $\delta_{\mathrm{unr}}$ is an isomorphism. In particular we see that a split multiplicative representation must be even dimensional.

(3.9) **Definition.** Let $V$ be a split multiplicative $G_{\mathbf{Q}_p}$-representation. Then the $\mathfrak{L}$-invariant of $V$ is defined to be the endomorphism $\mathfrak{L}_p(V)\in\mathrm{End}_K(V^{et})$ given by

$$\mathfrak{L}_p(V)=-\delta_{\mathrm{unr}}^{-1}\circ\delta_{\mathrm{cyc}}.$$

To compare the definitions (3.5) and (3.9) we will need some well known facts from Kummer theory and Tate duality theory. For each integer $n\geqq 0$, let $\gamma^{(n)}\colon \mathbf{Q}_p^*\to H^1(\mu_{p^n})$ be the Kummer homomorphism. This sends $q\in\mathbf{Q}_p^*$ to the cohomology class $\gamma_q^{(n)}\in H^1(\mu_{p^n})$ represented by the 1-cocycle which sends $\sigma\in G_{\mathbf{Q}_p}$ to $(q^{1/p^n})^{\sigma-1}$ where $q^{1/p^n}$ is a fixed choice of a $p^n$-th root of $q$ in $\bar{\mathbf{Q}}_p^*$. The family $\{\gamma_q^{(n)}\}_{n\geqq 0}$ corresponds to an element of $\varprojlim_n H^1(\mu_{p^n})=H^1(\mathbf{Z}_p(1))$. Let $\gamma_q$ denote the image of this element in $H^1(\mathbf{Z}_p(1))\otimes\mathbf{Q}_p=H^1(\mathbf{Q}_p(1))$. Then the map $q\mapsto\gamma_q$ defines a continuous group homomorphism, $\gamma\colon \mathbf{Q}_p^*\to H^1(\mathbf{Q}_p(1))$ whose image spans $H^1(\mathbf{Q}_p(1))$.

To a finite dimensional continuous $\mathbf{Q}_p[G_{\mathbf{Q}_p}]$-representation $V$ we associate the contragredient representation $V^*$. From Tate duality we know that cup product induces perfect pairings

$$H^i(V)\times H^{2-i}(V^*(1))\to\mathbf{Q}_p=H^2(\mathbf{Q}_p(1))$$

for $i=0,1,2$. In the important special case when $V=\mathbf{Q}_p$ and $i=1$, the pairing is explicitly given by $(\xi,\gamma_q)\mapsto\xi(\mathrm{art}(q))$ for $\xi\in H^1(\mathbf{Q}_p)$ and $\gamma_q\in H^1(\mathbf{Q}_p(1))$. It follows that the transposes of the maps $\lambda_{\mathrm{unr}}\colon\mathbf{Q}_p\to H^1(\mathbf{Q}_p)$ and $\lambda_{\mathrm{cyc}}\colon\mathbf{Q}_p\to H^1(\mathbf{Q}_p)$ are given by

$$(3.10)\quad \begin{array}{ccc} H^1(\mathbf{Q}_p(1)) & \xrightarrow{\lambda^*_{\mathrm{unr}}} & \mathbf{Q}_p \\ \gamma_q & \longmapsto & \mathrm{ord}_p(q) \end{array} \quad\text{and}\quad \begin{array}{ccc} H^1(\mathbf{Q}_p(1)) & \xrightarrow{\lambda^*_{\mathrm{cyc}}} & \mathbf{Q}_p \\ \gamma_q & \longmapsto & \log_p(q). \end{array}$$

(3.11) **Theorem.** *Let $Ta_p(A)$ be the Tate module of an abelian variety $A/\mathbf{Q}_p$ with split multiplicative reduction and let $V=Ta_p(A)\otimes\mathbf{Q}_p$. Then $V$ is a split multiplicative Galois representation and $\mathfrak{L}_p(A)=\mathfrak{L}_p(V)$.*

*Proof.* For each integer $n \geqq 0$ a simple application of the snake lemma to the $p^n$-power map acting on the exact sequences (3.2) gives us exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(Y, \mu_{p^n}) & \xrightarrow{t_n} & A[p^n] & \xrightarrow{\delta} & X/p^n X & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathrm{Hom}(X, \mu_{p^n}) & \xrightarrow{t_n} & B[p^n] & \xrightarrow{\delta} & Y/p^n Y & \longrightarrow & 0. \end{array}$$

Passing to the projective limit over $n$, and tensoring with $\mathbf{Q}_p$ we obtain the exact sequences

$$\begin{gathered} 0 \to Y_p^*(1) \to V \to X_p \to 0 \\ 0 \to X_p^*(1) \to V^* \to Y_p \to 0. \end{gathered}$$

These sequences are $\mathbf{Q}_p(1)$-dual to each other, hence Tate duality gives us a perfect pairing between the associated long exact cohomology sequences. In particular, the coboundary map $\delta\colon H^1(X_p) \to H^2(Y_p^*(1)) = Y_p^*$ of degree one associated to the first of these sequences is the transpose of the coboundary map $\delta^*\colon H^0(Y_p) \to H^1(X_p^*(1))$ of degree zero associated to the second sequence. If we identify $H^0(Y_p)$ with $Y_p$ and $H^1(X_p^*(1))$ with $X_p^* \otimes H^1(\mathbf{Q}_p(1))$ in the natural way, then a simple calculation shows that $\delta^*$ is given by

(3.12)
$$\begin{gathered} \delta^*\colon\ Y_p \to X_p^* \otimes H^1(\mathbf{Q}_p(1)) \\ y \mapsto (x \mapsto \gamma_{j(x,y)}). \end{gathered}$$

Let $\lambda_{\mathrm{unr}}, \lambda_{\mathrm{cyc}}\colon X_p \to H^1(X_p)$ be as in (3.7) and define $\delta_{\mathrm{unr}}, \delta_{\mathrm{cyc}}\colon X_p \to Y_p^*$ by setting $\delta_{\mathrm{unr}} = \delta \circ \lambda_{\mathrm{unr}}$ and $\delta_{\mathrm{cyc}} = \delta \circ \lambda_{\mathrm{cyc}}$. We will show that $\delta_{\mathrm{unr}}$ and $\delta_{\mathrm{cyc}}$ induce the pairings $-\alpha_p$ (3.3) and $\beta_p$ (3.4). Indeed, their duals are given by $\delta^*_{\mathrm{unr}} = \lambda^*_{\mathrm{unr}} \circ \delta^*$ and $\delta^*_{\mathrm{cyc}} = \lambda^*_{\mathrm{cyc}} \circ \delta^*$, where the maps $\lambda^*_{\mathrm{unr}}, \lambda^*_{\mathrm{cyc}}\colon H^1(X_p^*(1)) = X_p^* \otimes H^1(\mathbf{Q}_p(1)) \to X_p^*$ are induced by (3.10). Hence, using (3.12) we find that $\delta^*_{\mathrm{unr}}, \delta^*_{\mathrm{cyc}}\colon Y_p \to X_p^*$ are given by $\delta^*_{\mathrm{unr}}(y)(x) = -\mathrm{ord}_p(j(x,y)) = -\alpha_p(x,y)$ and $\delta^*_{\mathrm{cyc}}(y)(x) = \log_p(j(x,y)) = \beta_p(x,y)$. By duality we therefore have

$$\delta_{\mathrm{unr}}(x)(y) = -\alpha_p(x,y) \quad \text{and} \quad \delta_{\mathrm{cyc}}(x)(y) = \beta_p(x,y)$$

for all $x \in X_p$ and $y \in Y_p$. From the nondegeneracy of $\alpha_p$ we see that $\delta_{\mathrm{unr}}$ is an isomorphism, hence $V$ is split multiplicative. The above identities together with the definition of $\mathfrak{L}_p(A)$ (3.5) imply $\delta_{\mathrm{unr}} \circ \mathfrak{L}_p(A) = -\delta_{\mathrm{cyc}}$. Hence, by the definition of $\mathfrak{L}_p(V)$ (3.9) we conclude that $\mathfrak{L}_p(A) = -\delta_{\mathrm{unr}}^{-1} \circ \delta_{\mathrm{cyc}} = \mathfrak{L}_p(V)$ and the theorem is proved.

### *The $\mathfrak{L}$-invariant and infinitesimal deformations*

The $\mathfrak{L}$-invariant exerts a strong influence on the $p$-adic deformations of a split multiplicative representation. To explain this, we begin with a few remarks about infinitesimal deformations. Let $\tilde{\mathbf{Q}}_p = \mathbf{Q}_p[T]/T^2$.

(3.13) **Definition.** An infinitesimal deformation of a $\mathbf{Q}_p[G_{\mathbf{Q}_p}]$-module $W$ is a $\tilde{\mathbf{Q}}_p[G_{\mathbf{Q}_p}]$-module $\mathbf{W}$ such that $\mathbf{W}/T\mathbf{W} \cong W$.

If $W$ is a trivial $\mathbf{Q}_p[G_{\mathbf{Q}_p}]$-module, we let $\tilde{W}=W\otimes\tilde{\mathbf{Q}}_p$ denote the trivial infinitesimal deformation of $W$. If $\psi\colon G_{\mathbf{Q}_p}\to \mathrm{Aut}_{\tilde{\mathbf{Q}}_p}(\tilde{W})$ is a continuous Galois representation then $\tilde{W}(\psi)$ will denote $\tilde{W}$ with $G_{\mathbf{Q}_p}$ acting via $\psi$. Clearly, $\tilde{W}(\psi)$ is an infinitesimal deformation of $W$ if and only if $\psi\equiv I$ modulo $T$. Assume now that $\psi$ is such a character. Then $\psi$ factors through $G^{ab}_{\mathbf{Q}_p}$ and after differentiation with respect to $T$ gives rise to an additive homomorphism

$$\frac{d\psi}{dT}\colon\ G^{ab}_{\mathbf{Q}_p}\to \mathrm{End}(W)$$

for which $\psi=I+\dfrac{d\psi}{dT}\cdot T$.

(3.14) **Theorem.** *Let $V$ be a split multiplicative Galois representation and fix a homomorphism $\psi\colon G_{\mathbf{Q}_p}\to \mathrm{Aut}_{\tilde{\mathbf{Q}}_p}(\tilde{V}^{et})$ and an exact sequence $0\to\tilde{V}^0(1)\to\mathbf{V}\to\tilde{V}^{et}(\psi)\to 0$ whose terms are infinitesimal deformations of the terms in the exact sequence (3.8)a. Then for any nontrivial principal unit $u\in 1+p\mathbf{Z}_p$*

$$\frac{d\psi}{dT}(\mathrm{Frob}_p)=\mathfrak{L}_p(V)\circ\frac{1}{\log_p(u)}\,\frac{d\psi}{dT}(\mathrm{art}(u)).$$

*Proof.* Multiplication by $T$ gives rise to a Galois equivariant map $V^{et}\to\tilde{V}^{et}(\psi)$ whose cokernel is $V^{et}$. Thus we obtain an exact sequence

$$0\to V^{et}\to\tilde{V}^{et}(\psi)\to V^{et}\to 0. \tag{3.15}$$

Let $\delta_\psi\colon V^{et}=H^0(V^{et})\to H^1(V^{et})$ be the degree zero coboundary map of the associated long exact cohomology sequence. A simple calculation shows that for each $x\in V^{et}$, $\delta_\psi(x)$ is the homomorphism whose value on an element $\sigma\in G_{\mathbf{Q}_p}$ is given by $\delta_\psi(x)(\sigma)=\dfrac{d\psi}{dT}(\sigma)(x)$. Hence the homomorphism $\delta_\psi\colon V^{et}\to H^1(V^{et})$ is given by

$$\delta_\psi=\lambda_{\mathrm{unr}}\circ\frac{d\psi}{dT}(\mathrm{Frob}_p)+\lambda_{\mathrm{cyc}}\circ\frac{1}{\log_p(u)}\,\frac{d\psi}{dT}(\mathrm{art}(u)). \tag{3.16}$$

Now consider the following diagram.

$$\begin{array}{ccccccccc}
\longrightarrow & H^1(\mathbf{V}) & \longrightarrow & H^1(\tilde{V}^{et}) & \longrightarrow & H^2(\tilde{V}^0(1)) & \longrightarrow & 0 \\
 & \uparrow & & \uparrow{\scriptstyle i_2} & & \uparrow{\scriptstyle i_1} & & \\
\longrightarrow & H^1(V) & \longrightarrow & H^1(V^{et}) & \xrightarrow{\ \delta\ } & H^2(V^0(1)) & \longrightarrow & 0 \\
 & & & \uparrow{\scriptstyle \delta_\psi} & & & & \\
 & & & H^0(V^{et})\cong V^{et} & & & &
\end{array}$$

Since $i_2\circ\delta_\psi=0$ and the diagram is commutative, we have $i_1\circ\delta\circ\delta_\psi=0$. By a simple application of Tate-duality it follows that $i_1$ is injective. Hence $\delta\circ\delta_\psi=0$. The theorem now follows by composing $\delta$ on the left with (3.16), and using the identities $\delta_{\mathrm{unr}}=\delta\circ\lambda_{\mathrm{unr}}$, $\delta_{\mathrm{cyc}}=\delta\circ\lambda_{\mathrm{cyc}}$, and $\mathfrak{L}_p(V)=-\delta_{\mathrm{unr}}^{-1}\circ\delta_{\mathrm{cyc}}$.

*The $\mathfrak{L}$-invariant of a split multiplicative newform*

Let $f$ be a classical weight two newform. Let $K_f$ be the completion in $\bar{\mathbf{Q}}_p$ of the field of Hecke eigenvalues and let $V_f$ be the two dimensional $G_{\mathbf{Q}}$-representation over $K_f$ constructed by Deligne [D]. We will say that $f$ is split multiplicative at $p$ if $V_f$ is split multiplicative at $p$. From the work of Deligne and Rapoport [D-R] we know that $f$ is split multiplicative at $p$ if and only if (1) the conductor of $f$ is $Np$ where $p \nmid N$ and (2) $f|T_p = f$. In particular, these two conditions show that if $f$ is split multiplicative at $p$ then $f$ is an ordinary $p$-stabilized newform of tame conductor $N$.

(3.17) **Definition.** The $\mathfrak{L}$-invariant of a weight two split multiplicative newform $f$ is defined to be the $\mathfrak{L}$-invariant of its Galois representation $V_f$. Hence $\mathfrak{L}_p(f) = \mathfrak{L}_p(V_f) \in K_f$.

(3.18) **Theorem.** *Let $f$ be a weight two newform of conductor $Np$ with $p \geqq 5$, and suppose $f$ is split multiplicative at p. Let $\mathscr{R}$ be the universal ordinary Hecke algebra of tame conductor $N$ and let $a_p = h(T_p) \in \mathscr{R}$. If $\kappa$ is the arithmetic point on $\mathscr{R}$ which corresponds to $f$ by Theorem* 2.6 *then*

$$a_p'(\kappa) = -\tfrac{1}{2}\mathfrak{L}_p(f).$$

*Proof.* Let $\varphi\colon G_{\mathbf{Q}_p} \to \mathscr{R}^*$ be the unramified character with $\varphi(\mathrm{Frob}_p) = a_p$. Let $\mathscr{R}_{(\kappa)}$ be the localization of $\mathscr{R}$ at $\kappa$. From Theorem 2.6d we obtain an exact sequence

$$(3.19) \qquad 0 \to \mathscr{R}_{(\kappa)}(\chi_0\,\eta\,\varphi^{-1}) \to \mathbf{T}^0_{\mathrm{prim}} \otimes_{\mathscr{R}} \mathscr{R}_{(\kappa)} \to \mathscr{R}_{(\kappa)}(\varphi) \to 0.$$

Since the specialization of this to $\kappa$ is the sequence $0 \to V_f^0\,(1) \to V_f \to V_f^{et} \to 0$, we see that $\varphi$ and $\eta$ are congruent to 1 modulo the maximal ideal $P_\kappa$ in $\mathscr{R}_{(\kappa)}$. In particular we see that $\kappa(a_p) = 1$, that $\eta(\mathrm{art}(u)) = [u]_p$ for any principle unit $u \in 1 + p\mathbf{Z}_p$, and that $\eta(\mathrm{Frob}_p) = 1$. Now tensor (3.19) with $\varphi\eta^{-1}$ and reduce modulo $P_\kappa^2$. This gives us an exact sequence

$$0 \to \tilde{K}_f(1) \to \tilde{V} \to \tilde{K}_f(\varphi^2\eta^{-1}) \to 0$$

where $\tilde{V}$ is an infinitesimal deformation of $V_f$. Since $\varphi\eta^{-1} \equiv 1$ modulo $P_\kappa$, we may apply Theorem 3.14 with $\psi = \varphi^2\eta^{-1}$ to obtain the identity

$$(3.20) \qquad \psi(\mathrm{Frob}_p)'(\kappa) = \mathfrak{L}_p(f) \cdot \frac{\psi(\mathrm{art}(u))'(\kappa)}{\log_p(u)}$$

for any nontrivial principal unit $u \in 1 + p\mathbf{Z}_p$. But $\varphi(\mathrm{Frob}_p) = a_p$, $\eta(\mathrm{Frob}_p) = \varphi(\mathrm{art}(u)) = 1$, and $\eta(\mathrm{art}(u)) = [u]_p \in \Lambda$. Hence $\psi(\mathrm{Frob}_p) = a_p^2$ and $\psi(\mathrm{art}(u)) = [u]_p^{-1}$. The theorem now follows from (3.20) and the simple fact that $[u]_p'(\kappa) = \log_p(u)$.

## 4. Modular symbols and special values of $L$-functions

In this section we review the basic definitions and properties of modular symbols. Modular symbols will be useful to us in two ways. First of all they give us a concrete realization of the one-dimensional compactly supported cohomology groups of a congruence subgroup of $SL_2(\mathbf{Z})$ as well as a concrete description of the action of the Hecke operators on these groups. On the other hand they provide a powerful tool for studying the arithmetic properties of critical values of $L$-functions associated to modular forms. We will recall how modular symbols are used to attach one-variable $p$-adic $L$-functions to $p$-stabilized ordinary newforms as in [Mz-T-T].

### *Definitions and first properties*

Fix a commutative ring $R$ and let $A$ be a contravariant $R[\Sigma]$-module where $\Sigma = M_2(\mathbf{Z}) \cap GL_2(\mathbf{Q})$ as in "section 1".

(4.1) **Definition.** Let $\mathscr{D} \overset{\text{def}}{=} \operatorname{Div}(\mathbf{P}^1(\mathbf{Q}))$ denote the group of divisors supported on the rational cusps $\mathbf{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$ of the upper half plane $\mathbf{H}$. Let $\mathscr{D}_0 \subseteq \mathscr{D}$ be the subgroup of divisors of degree zero. Note that $\Sigma$ acts by fractional linear transformations on $\mathscr{D}$ and on $\mathscr{D}_0$. Let $\Gamma$ be a congruence group.
**a.** An additive homomorphism $\Phi: \mathscr{D}_0 \to A$ will be called *modular symbol* over $\Gamma$ if $\Phi$ is a $\Gamma$-homomorphism, i.e. if $\Phi(\gamma D)|\gamma = \Phi(D)$ for all $D \in \mathscr{D}_0$ and $\gamma \in \Gamma$. We will denote the $R$-module of all $A$-valued modular symbols over $\Gamma$ by $\operatorname{Symb}_\Gamma(A)$.
**b.** An $A$-valued *boundary symbol* over $\Gamma$ is a $\Gamma$-homomorphism $\Phi: \mathscr{D} \to A$. We will denote the group of all $A$-valued boundary symbols over $\Gamma$ by $\operatorname{Bound}_\Gamma(A)$.
**c.** More generally, we define

$$\operatorname{Symb}(A) = \bigcup_\Gamma \operatorname{Symb}_\Gamma(A), \qquad \operatorname{Bound}(A) = \bigcup_\Gamma \operatorname{Bound}_\Gamma(A)$$

where $\Gamma$ runs over all congruence groups. We let $\Sigma$ act on these groups according to the formula $\Phi|g: D \mapsto \Phi(gD)|\, g$, for $g \in \Sigma$ and $\Phi \in \operatorname{Symb}(A)$, $D \in \mathscr{D}_0$ (respectively $\Phi \in \operatorname{Bound}(A)$, $D \in \mathscr{D}$).

Our interest in $\operatorname{Symb}_\Gamma(A)$ and $\operatorname{Bound}_\Gamma(A)$ is motivated by the following theorem which allows us to relate these groups to the cohomology of $\Gamma$ (see 4.3). Let $\tilde{\mathbf{H}}$ be the Borel-Serre completion of $\mathbf{H}$. Also, let $t(\Gamma)$ be the least common multiple of the orders of the torsion elements of $\Gamma$.

(4.2) **Theorem.** *Let $\Gamma$ be a congruence subgroup of $SL_2(\mathbf{Z})$ and suppose $t(\Gamma)$ is invertible in $R$. Then the long exact cohomology sequence of the pair $(\Gamma\backslash\tilde{\mathbf{H}}, \partial(\Gamma\backslash\tilde{\mathbf{H}}))$ with coefficients in $A$ is isomorphic to the right-shift of the long exact $\Gamma$-cohomology sequence of the exact sequence $0 \to A \to \operatorname{Hom}_{\mathbf{Z}}(\mathscr{D}, A) \to \operatorname{Hom}_{\mathbf{Z}}(\mathscr{D}_0, A) \to 0$. More precisely, for each integer $i \geqq 0$ we have the following commutative diagram:*

$$\begin{array}{ccccccc}
\longrightarrow & H^{i-1}(\Gamma, \operatorname{Hom}(\mathscr{D}_0, A)) & \longrightarrow & H^i(\Gamma, A) & \longrightarrow & H^i(\Gamma, \operatorname{Hom}(\mathscr{D}, A)) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^i_c(\Gamma\backslash\mathbf{H}, A)) & \longrightarrow & H^i(\Gamma\backslash\mathbf{H}, A) & \longrightarrow & H^i(\partial(\Gamma\backslash\tilde{\mathbf{H}}), A)) & \longrightarrow
\end{array}$$

*where the vertical arrows are isomorphisms.*

For a proof of this, see [A-S]. Recall that the parabolic cohomology group $H^1_{\mathrm{par}}(\Gamma, A)$ is defined to be the image of the map $H^1_c(\Gamma\backslash \mathbf{H}, A) \to H^1(\Gamma\backslash \mathbf{H}, A)$. The following theorem is an immediate consequence of Theorem 4.2.

(4.3) **Theorem.** *Suppose $t(\Gamma)$ is invertible in $R$. Then there is a canonical isomorphism* $\mathrm{Symb}_\Gamma(A) \cong H^1_c(\Gamma\backslash \mathbf{H}, A)$. *Moreover, there is a canonical exact sequence*

$$0 \to H^0(\Gamma, A) \to \mathrm{Bound}_\Gamma(A) \to \mathrm{Symb}_\Gamma(A) \to H^1_{\mathrm{par}}(\Gamma, A) \to 0.$$

Now let $\Gamma$ be a congruence group and let $D(\Gamma, \Sigma)$ be the double coset algebra associated to the pair $(\Gamma, \Sigma)$. The action of the algebra $D(\Gamma, \Sigma)$ on $A$-valued modular symbols over $\Gamma$ can be made explicit as follows. If $T(g) \in D(\Gamma, \Sigma)$ is the element associated to the double coset $\Gamma g \Gamma$, $g \in \Sigma$, then we can write $\Gamma g \Gamma$ as a finite disjoint union of right cosets, $\bigcup_i \Gamma g_i$. For a modular symbol $\Phi \in \mathrm{Symb}_\Gamma(A)$ we then have

$$\Phi \,|\, T(g) = \sum_i \Phi \,|\, g_i \in \mathrm{Symb}_\Gamma(A). \tag{4.4}$$

The matrix $\iota = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \Sigma$ induces an involution $\Phi \mapsto \Phi|\iota$ on modular symbols. If 2 is invertible in $R$, then we can decompose any modular symbol $\Phi$ in a unique way as a sum

$$\Phi = \Phi^+ + \Phi^- \tag{4.5}$$

where $\Phi^\pm|\iota = \pm \Phi^\pm$. Let $\mathrm{Symb}_\Gamma(A) = \mathrm{Symb}_\Gamma(A)^+ \oplus \mathrm{Symb}_\Gamma(A)^-$ be the corresponding decomposition of the space of modular symbols.

## *Modular symbols of integral weight $k \geqq 2$*

For each non-negative integer $r \geqq 0$, let $\mathrm{Sym}^r(R^2)$ denote the $R$-module of homogeneous polynomials of degree $r$ in two variables $X$, $Y$ with coefficients in $R$. We let $\Sigma$ act on $\mathrm{Sym}^r(R^2)$ by the formula $(F|g)(X, Y) = F((X, Y)\, g^*)$ for $g \in \Sigma$ and $F \in \mathrm{Sym}^r(R^2)$, where $*$ is the adjoint involution defined in "section 1".

(4.6) **Definition.** Fix an integer $k \geqq 2$. Then the $R[\Sigma]$-module $\mathrm{Symb}(\mathrm{Sym}^{k-2}(R^2))$ is called the module of modular symbols of weight $k$ over $R$.

This terminology is motivated by the following well known example of Eichler and Shimura. Recall from "section 1" that $\mathscr{S}_k(\bar{\mathbf{Q}})$ is the space of weight $k$ cusp forms of all levels having algebraic $q$-expansions and that $\Sigma^+$ acts on $\mathscr{S}_k(\bar{\mathbf{Q}})$ via the weight $k$ action.

(4.7) **Definition.** The *standard weight $k$ modular symbol* associated to a cusp form $f \in \mathscr{S}_k(\bar{\mathbf{Q}})$ is the modular symbol $\Phi_f \in \mathrm{Symb}(\mathrm{Sym}^{k-2}(\mathbf{C}^2))$ defined on divisors $\{c_2\} - \{c_1\} \in \mathscr{D}_0$, $c_1, c_2 \in \mathbf{P}^1(\mathbf{Q})$ by

$$\Phi_f(\{c_2\} - \{c_1\}) = 2\pi i \int_{c_1}^{c_2} f(z)\,(zX + Y)^{k-2}\, dz$$

where the integral is over the geodesic in the upper half-plane joining $c_1$ to $c_2$.

A straightforward calculation shows that the map $f \mapsto \Phi_f$ commutes with the action of $\Sigma^+$. We have the following theorem of Shimura [Sh].

(4.8) **Theorem.** *Let $f \in \mathcal{S}_k(\Gamma_1(Np^r))$ be a Hecke eigenform of weight $k \geqq 2$ and let $K(f)$ be the field generated by the Hecke eigenvalues of $f$. Then for either choice of sign $\pm$, the Hecke eigenspace associated to $f$ in $\mathrm{Symb}_{\Gamma_1(Np^r)}(\mathrm{Sym}^{k-2}(K(f)^2))^{\pm}$ is one dimensional over $K(f)$. Moreover, there are 'periods' $\Omega_f^{\pm} \in \mathbf{C}^*$ such that the modular symbols $\Psi_f^{\pm} = (\Omega_f^{\pm})^{-1} \Phi_f^{\pm}$ are defined over $K(f)$ and span the associated eigenspaces, that is*

$$0 \neq \Psi_f^{\pm} \in \mathrm{Symb}_{\Gamma_1(Np^r)}(\mathrm{Sym}^{k-2}(K(f)^2))^{\pm}.$$

Recall from Theorem 2.6b that $\mathcal{X}^{\mathrm{arith}} = \mathcal{X}^{\mathrm{arith}}(\tilde{\mathcal{R}})$ parametrizes the ordinary $p$-stabilized newforms of level $Np^r$, $r > 0$.

(4.9) **Definition.** For each $\kappa \in \mathcal{X}^{\mathrm{arith}}$ we fix the following data and notations.

a. $K_\kappa$ is the $p$-acid completion of $K(\mathbf{f}_\kappa)$ with respect to our fixed embedding (0.4).
b. $\mathbf{W}_\kappa$ is the Hecke eigenspace in $\mathrm{Symb}_{\Gamma_1(Np^r)}(\mathrm{Sym}^{k-2}(K_\kappa^2))$ associated to $\mathbf{f}_\kappa$. Here $k$ is the weight of $\mathbf{f}_\kappa$.
c. We fix, once and for all, two periods $\Omega_{\mathbf{f}_\kappa}^{\pm} \in \mathbf{C}^*$ as in Theorem 4.8 and let $\Psi_{\mathbf{f}_\kappa}^{\pm} \in \mathbf{W}_\kappa^{\pm}$ be the associated generators.

Consistent with the notational conventions of "section 2" (see (2.8)) we will write $K_{\kappa,k}$ and $\mathbf{W}_{\kappa,k}$ to denote $K_{\kappa(\kappa)}$ and $\mathbf{W}_{\kappa(\kappa)}$, respectively.

*Special values of L-functions*

Modular symbols provide us with a convenient tool for studying values of $L$-functions. We will attach "special values of $L$-functions" to modular symbols and show how the critical values of the $L$-function of a cusp form of weight $k \geqq 2$ can be described in terms of the "special values of the $L$-function" of the associated modular symbol.

(4.10) **Definition.** Let $\Phi \in \mathrm{Symb}(A)$. Then the *special value of the L-function of* $\Phi$ is defined to be the element $L(\Phi)$ of $A$ given by $L(\Phi) = \Phi(\{0\} - \{i\infty\})$.

Let $\psi$: $\mathbf{Z} \to R$ be a primitive Dirichlet character of conductor $m \geqq 1$. Then we may also define "special values of $L$-functions twisted by $\psi$". For simplicity, suppose $\Gamma = \Gamma_1(M)$ for some positive integer $M$. Define the *twist operator* $R_\psi$: $\mathrm{Symb}_\Gamma(A) \to \mathrm{Symb}(A)$ by the formula

$$\Phi | R_\psi \stackrel{\mathrm{def}}{=} \sum_{a=0}^{m-1} \psi(a)\, \Phi \left| \begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix} \right. .$$

(4.11) **Definition.** The *special value of the L-function of $\Phi$ twisted by $\psi$* is $L(\Phi, \psi) = L(\Phi | R_\psi)$. In case $\Phi$ is a modular symbol of weight $k \geqq 2$ over $R$ then the special value $L(\Phi, \psi)$ is a homogeneous polynomial of degree $k-2$ in $X$

and $Y$. If the binomial coefficients $\binom{k-2}{r}$, $0 \leqq r \leqq k-2$, are not zero divisors in $R$, then we define the 'special values' $L(\Phi, \psi, s_0) \in R$, for integers $s_0$ with $0 < s_0 < k$, to be the unique elements of $R$ for which

$$L(\Phi, \psi) = \sum_{s_0=1}^{k-1} \binom{k-2}{s_0-1} (-1)^{s_0-1} \cdot L(\Phi, \psi, s_0) \cdot X^{s_0-1}\, Y^{k-s_0-1}. \tag{4.12}$$

These definitions are motivated by the following well known example. If $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ is the Fourier expansion of a weight $k$ cusp form $f \in \mathcal{S}_k(\bar{\mathbf{Q}})$ and if $\psi: \mathbf{Z} \to \mathbf{C}$ is a primitive Dirichlet character with conductor $m > 0$, then the complex $L$-function of $f$ twisted by $\psi$ is defined by the Dirichlet series

$$L_\infty(f, \psi, s) = \sum_{n=1}^{\infty} \psi(n)\, a_n\, n^{-s} \quad \text{for } \operatorname{Re}(s) > \frac{k+1}{2}.$$

This function extends to an entire function in $s$. We are interested in its values at integers $s = s_0$ in the 'critical strip' $0 < s_0 < k$. For such values of $s_0$ we define

$$\Lambda(f, \psi, s_0) \stackrel{\text{def}}{=} m^{s_0-1} (s_0-1)!\, \frac{\tau(\psi)\, L_\infty(f, \bar{\psi}, s_0)}{(2\pi i)^{s_0-1}} \tag{4.13}$$

where $\tau(\psi)$ is the Gauss sum $\sum_{a=0}^{m-1} \psi(a)\, e^{2\pi i a/m}$. If $\psi$ is the trivial Dirichlet character, then we will suppress it from the notation and write simply $L_\infty(f, s)$ and $\Lambda(f, s_0)$ instead of $L_\infty(f, \psi, s)$ and $\Lambda(f, \psi, s_0)$ respectively.

(4.14) **Theorem.** *For every primitive Dirichlet character $\psi$ and each integer $s_0$ with $0 < s_0 < k$ we have*

$$L(\Phi_f, \psi, s_0) = \Lambda(f, \psi, s_0).$$

The proof is a straightforward calculation.

*p-adic L-functions attached to p-stabilized ordinary newforms*

For an arbitrary topological $\mathbf{Z}_p$-algebra $R$, we let $\operatorname{Meas}(\mathbf{Z}_p^*, R)$ denote the $R$-module of all bounded $R$-valued distributions on $\mathbf{Z}_p^*$ (see [Mz-SwD]). For fixed $\mu \in \operatorname{Meas}(\mathbf{Z}_p^*, R)$ we associate to each continuous character $\sigma: \mathbf{Z}_p^* \to R^*$ the element

$$L_p(\mu, \sigma) = \int_{\mathbf{Z}_p^*} \sigma(t)\, d\mu(t) \in R \tag{4.15}$$

in the usual way (see, for example, [Mz-SwD]). The resulting function of $\sigma$ will be called the $R$-valued *Iwasawa function* associated to $\mu$.

For each $\kappa\in\mathscr{X}^{\text{arith}}$ we follow [Mz-T-T] and define measures $\mu_\kappa^\pm\in\text{Meas}(\mathbf{Z}_p^*, K_\kappa)$ by setting

$$(4.16)\qquad \mu_\kappa^\pm(a+p^m\mathbf{Z}_p)=a_p(\kappa)^{-m}\,\Psi_{\mathbf{f}_\kappa}^\pm\left(\left\{\frac{a}{p^m}\right\}-\{i\infty\}\right)\Big|_{X=0,\,Y=1}$$

for each $a\in\mathbf{Z}$ prime to $p$, and each $m>0$. We then define the $p$-adic $L$-function

$$(4.17)\qquad L_p(\mathbf{f}_\kappa,\psi,s)=L_p(\mu_\kappa^{\text{sgn}(\psi)},\psi\langle\cdot\rangle^{s-1})$$

for each $\kappa\in\mathscr{X}^{\text{arith}}$. If $\psi$ is the trivial character, then we will suppress it from the notation and write simply $L_p(\mathbf{f}_\kappa, s)$. We have the following theorem.

(4.18) **Theorem.** *Let $\kappa\in\mathscr{X}_{\text{arith}}$ be an arithmetic point of weight $k$. Let $\psi$ be a finite character of $\mathbf{Z}_p^*$ of conductor $p^m$, $m\geqq 0$ and let $s_0$ be an integer with $0<s_0<k$. Then*

$$L_p(\mathbf{f}_\kappa,\psi,s_0)=a_p(\kappa)^{-m}\cdot(1-a_p(\kappa)^{-1}\,\psi\omega^{1-s_0}(p)\,p^{s_0-1})\cdot\frac{\Lambda(\mathbf{f}_\kappa,\psi\omega^{1-s_0},s_0)}{\Omega_{\mathbf{f}_\kappa}^{\text{sgn}(\psi)}}.$$

For more details of the construction of the $p$-adic $L$-function and a proof of Theorem 4.18, see [Mz-T-T].

## 5. $\Lambda$-adic modular symbols and two-variable $p$-adic $L$-functions

Fix a prime number $p>0$ and a positive integer $N$ which is not divisible by $p$. Let $\Gamma=\Gamma_1(N)$. In this section we examine the structure of the group of modular symbols over $\Gamma$ which take values in the module $\mathbf{D}$ of $\mathbf{Z}_p$-valued measures on $(\mathbf{Z}_p^2)'$ (=the set of primitive elements of $\mathbf{Z}_p^2$). Such modular symbols will be referred to as $\Lambda$-adic modular symbols. Our interest in $\text{Symb}_\Gamma(\mathbf{D})$ stems from two facts. First of all, the module $\mathbf{D}$ is rich enough to admit non-trivial morphisms to each of the modules $\text{Sym}^r(\mathbf{Z}_p^2)$, $r\geqq 0$. Thus a measure gives rise to a family of elements in $\text{Sym}^r$ as $r$ varies, and correspondingly, a $\Lambda$-adic modular symbol gives rise to a family of modular symbols of varying weights. The second reason for our interest in $\mathbf{D}$ rests on the fact that the elements of $\mathbf{D}$ give rise in a natural way to two variable $p$-adic $L$-functions. The main results of this section are Theorems 5.13 and 5.15. Theorem 5.13, which is proved in the next section, asserts the existence of ordinary $\Lambda$-adic modular eigensymbols which are $p$-adic deformations of the modular symbols associated to any given ordinary $p$-stabilized newform. In Theorem 5.15 we describe the analytic properties of the two-variable $p$-adic $L$-function associated to such an eigensymbol.

Let $\text{Cont}(\mathbf{Z}_p^2)$ denote the $\mathbf{Z}_p$-module of continuous $\mathbf{Z}_p$-valued functions on $\mathbf{Z}_p^2$ and let $\text{Step}(\mathbf{Z}_p^2)$ be the submodule of locally constant functions. The group $\tilde{\mathbf{D}}$ of $\mathbf{Z}_p$-valued measures on $\mathbf{Z}_p^2$ is defined to be $\tilde{\mathbf{D}}=\text{Hom}_{\mathbf{Z}_p}(\text{Step}(\mathbf{Z}_p^2),\mathbf{Z}_p)$. As is well known, every $\mu\in\tilde{D}$ has a unique extension to a $\mathbf{Z}_p$-homomorphism $\text{Cont}(\mathbf{Z}_p^2)\to\mathbf{Z}_p$ which is continuous with respect to the supremum norm on

$\mathrm{Cont}(\mathbf{Z}_p^2)$. If $\mu\in\tilde{\mathbf{D}}$, $\varphi\in\mathrm{Cont}(\mathbf{Z}_p^2)$ and $K\subseteq\mathbf{Z}_p^2$ is a compact open set, we will use the integral notation and write

$$\int_K \varphi\, d\mu$$

for the value of $\mu$ on the product of $\varphi$ with the characteristic function of $K$. In case $\varphi$ is identically 1, we will also write $\mu(K)$ for this integral.

We will be particularly interested in a certain direct summand $\mathbf{D}$ of $\tilde{\mathbf{D}}$ which may defined as follows. Let $(\mathbf{Z}_p^2)'$ denote the primitive elements of $\mathbf{Z}_p^2$, i.e. those elements which are not divisible by $p$. Then $\mathbf{D}$ is the submodule of $\tilde{\mathbf{D}}$ consisting of measures which are supported on $(\mathbf{Z}_p^2)'$. Restriction of measures from $\mathbf{Z}_p^2$ to $(\mathbf{Z}_p^2)'$ gives a projection from $\tilde{\mathbf{D}}$ to $\mathbf{D}$, hence $\mathbf{D}$ is a direct summand of $\tilde{\mathbf{D}}$.

There is a natural continuous action of $M_2(\mathbf{Z}_p)$ on $\tilde{\mathbf{D}}$. To describe this action we regard the elements of $\mathbf{Z}_p^2$ as row vectors and let $M_2(\mathbf{Z}_p)$ act by matrix multiplication on the right. Then $M_2(\mathbf{Z}_p)$ acts covariantly on $\mathrm{Step}(\mathbf{Z}_p^2)$ by the formula $\varphi\mapsto(g\varphi\colon \mathbf{x}\in\mathbf{Z}_p^2\mapsto\varphi(\mathbf{x}g))$. The contravariant action on $\tilde{\mathbf{D}}$ is given by $\mu\mapsto\mu|g$ where $\mu|g$ is given by the integration formula

$$\int_{\mathbf{Z}_p^2} \varphi\, d(\mu|g)=\int_{\mathbf{Z}_p^2} g\varphi\, d\mu.$$

Since the kernel of the natural projection $\tilde{\mathbf{D}}\to\mathbf{D}$ is preserved by this action of $M_2(\mathbf{Z}_p)$ we also obtain an induced action of $M_2(\mathbf{Z}_p)$ on $\mathbf{D}$. We will take this induced action as the natural action of $M_2(\mathbf{Z}_p)$ on $\mathbf{D}$. Note, that while the action has been defined to commute with the natural surjection $\tilde{\mathbf{D}}\to\mathbf{D}$, it does not respect the natural inclusion $\mathbf{D}\hookrightarrow\tilde{\mathbf{D}}$.

The group $\mathbf{Z}_p^*$ acts continuously on $\mathbf{D}$ and $\tilde{\mathbf{D}}$ via the scalar matrices in $M_2(\mathbf{Z}_p)$. We extend this to a continuous action of the algebra $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$. Note that the action of $M_2(\mathbf{Z}_p)$ commutes with these $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$-structures on $\mathbf{D}$ and $\tilde{\mathbf{D}}$. Now restrict the action of $M_2(\mathbf{Z}_p)$ to an action of $\Sigma$. Since Hypothesis P (1.3) is satisfied, we may form the $\mathscr{H}$-modules

(5.1) $$\mathbf{W}=\mathrm{Symb}_\Gamma(\mathbf{D})\quad\text{and}\quad\tilde{\mathbf{W}}=\mathrm{Symb}_\Gamma(\tilde{\mathbf{D}}).$$

Restriction of measures induces a natural surjective $\mathscr{H}$-morphism $\tilde{\mathbf{W}}\to\mathbf{W}$.

We now describe a simple procedure for attaching $p$-adic $L$-functions to the elements of $\mathbf{W}$. Recall (1.11)c that $\mathscr{X}_0$ is the space of $\bar{\mathbf{Q}}_p$-valued points on $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$. To each $\Phi\in\mathbf{W}$ we attach its 'special value of the $L$-function' $\mu_\Phi=L(\Phi)\in\mathbf{D}$ as in the last section (4.10) and define the *standard 2-variable p-adic L-function* associated to $\Phi$ to be the $\bar{\mathbf{Q}}_p$-valued function $L_p(\Phi)$ on $\mathscr{X}_0\times\mathscr{X}_0$ given by

(5.2)a. $$L_p(\Phi,\kappa,\sigma)=\int_{\mathbf{Z}_p^*\times\mathbf{Z}_p^*}\kappa(x)\,\sigma(y/x)\,d\mu_\Phi(x,y)$$

for $(\kappa,\sigma)\in\mathscr{X}_0\times\mathscr{X}_0$. If $\sigma$ is an arithmetic point, then its restriction to $\mathbf{Z}_p^*$ is an arithmetic character of the form $\sigma_{r,\psi}$. We can then extend $\sigma$ to a continuous multiplicative function $\mathbf{Z}_p\to\bar{\mathbf{Q}}_p$ by the convention $\sigma(p)=p^r$ if $\psi$ is the trivial character and $\sigma(p)=0$ otherwise. With this convention, we define the *improved 2-variable p-adic L-function* associated to $\Phi$ to be the function $L_p^*(\Phi)$ on $\mathscr{X}_0 \times \mathscr{X}_0^{\mathrm{arith}}$ given by

$$(5.2)\text{b.} \qquad L_p^*(\Phi, \kappa, \sigma) = \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \kappa(x)\, \sigma(y/x)\, d\mu_\Phi(x, y)$$

for $(\kappa, \sigma) \in \mathscr{X}_0 \times \mathscr{X}_0^{\mathrm{arith}}$.

It is clear from the definitions that $L_p(\Phi, \kappa, \sigma)$ is analytic in $(\kappa, \sigma)$ and that $L_p^*(\Phi, \kappa, \sigma)$ is analytic in $\kappa$ for each $\sigma \in \mathscr{X}_0^{\mathrm{arith}}$. Moreover, for fixed $\sigma$ there are unique $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$-morphisms

$$(5.3) \qquad \begin{aligned} &L_p(\cdot, \sigma)\colon \mathbf{W} \to \mathbf{Z}_p[[\mathbf{Z}_p^*]] \quad \text{for fixed } \sigma \in \mathscr{X}_0, \quad \text{and} \\ &L_p^*(\cdot, \sigma)\colon \mathbf{W} \to \mathbf{Z}_p[[\mathbf{Z}_p^*]] \quad \text{for fixed } \sigma \in \mathscr{X}_0^{\mathrm{arith}}, \end{aligned}$$

such that $\kappa(L_p(\Phi, \sigma)) = L_p(\Phi, \kappa, \sigma)$ and $\kappa(L_p^*(\Phi, \sigma)) = L_p^*(\Phi, \kappa, \sigma)$ for all $\kappa \in \mathscr{X}_0$.

Now fix a continuous $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$-algebra $R$. Let $\mathbf{D}_R = \mathbf{D} \otimes_{\mathbf{Z}_p[[\mathbf{Z}_p^*]]} R$ and $\mathbf{W}_R = \mathbf{W} \otimes_{\mathbf{Z}_p[[\mathbf{Z}_p^*]]} R$. There is a natural isomorphism $\mathbf{W}_R \cong \mathrm{Symb}_\Gamma(\mathbf{D}_R)$ of $\mathscr{H}$-modules. Extending the maps (5.3) by $R$-linearity, we obtain $R$-homomorphisms $L_p(\cdot, \sigma)$, $L_p^*(\cdot, \sigma)\colon \mathbf{W}_R \to R$. We may therefore extend our definitions (5.2) and associate to each $\Phi \in \mathbf{W}_R$ a two-variable $p$-adic $L$-function $L_p(\Phi)$ on $\mathscr{X}(R) \times \mathscr{X}_0$ and an improved $p$-adic $L$-function $L_p^*(\Phi)$ on $\mathscr{X}(R) \times \mathscr{X}_0^{\mathrm{arith}}$ by the formulas

$$(5.4) \qquad \begin{aligned} &L_p(\Phi, \kappa, \sigma) = \kappa(L_p(\Phi, \sigma)) \quad \text{for } (\kappa, \sigma) \in \mathscr{X}(R) \times \mathscr{X}_0, \quad \text{and} \\ &L_p^*(\Phi, \kappa, \sigma) = \kappa(L_p^*(\Phi, \sigma)) \quad \text{for } (\kappa, \sigma) \in \mathscr{X}(R) \times \mathscr{X}_0^{\mathrm{arith}}. \end{aligned}$$

In Proposition 5.8 we will prove a fundamental interpolation property of these $p$-adic $L$-functions. In particular we will show that they interpolate special values of $L$-functions attached to a certain family $\Phi_\kappa$, $\kappa \in \mathscr{X}^{\mathrm{arith}}(R)$, of modular symbols of integral weights attached to $\Phi$. This family of modular symbols is defined as follows. For each $\kappa \in \mathscr{X}^{\mathrm{arith}}(R)$ with weight $k \geqq 2$ and character $\varepsilon$ define the specialization map $\phi_\kappa\colon \mathbf{D} \to \mathrm{Sym}^{k-2}(\bar{\mathbf{Q}}_p^2)$ by the integration formula

$$(5.5)\text{a.} \qquad \phi_\kappa(\mu) = \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \varepsilon(x) \cdot (xY - yX)^{k-2}\, d\mu(x, y)$$

and extend this to the unique map $\phi_\kappa\colon \mathbf{D}_R \to \mathrm{Sym}^{k-2}(\bar{\mathbf{Q}}_p^2)$ which intertwines $\kappa$. If, moreover, the character $\varepsilon$ is trivial, then we also define a map $\tilde{\phi}_\kappa\colon \tilde{\mathbf{D}} \to \mathrm{Sym}^{k-2}(\bar{\mathbf{Q}}_p^2)$ by

$$(5.5)\text{b.} \qquad \tilde{\phi}_\kappa(\mu) = \int_{\mathbf{Z}_p^2} (xY - yX)^{k-2}\, d\mu(x, y)$$

and extend this to a map $\tilde{\phi}_\kappa\colon \tilde{\mathbf{D}}_R \to \mathrm{Sym}^{k-2}(\bar{\mathbf{Q}}_p^2)$ intertwining $\kappa$ as well. A simple calculation shows that if the conductor of $\varepsilon$ divides $p^r$ and $r > 0$ then $\phi_\kappa$ commutes with the action of $\Sigma_1(p^r)$. Hence, $\phi_\kappa$ induces an $\mathscr{H}[\iota, W_N]$-morphism $\phi_{\kappa,*}\colon \mathbf{W}_R \to \mathrm{Symb}_{\Gamma_1(Np^r)}(\mathrm{Sym}^{k-2}(\bar{\mathbf{Q}}_p^2))$. If $\varepsilon$ is trivial, then $\tilde{\phi}_\kappa$ commutes with all of $\Sigma$, hence $\tilde{\phi}_\kappa$ induces an $\mathscr{H}[\iota, W_N]$-morphism

$$\tilde{\phi}_{\kappa,*}\colon \tilde{\mathbf{W}}_R \to \mathrm{Symb}_{\Gamma_1(N)}(\mathrm{Sym}^{k-2}(\bar{\mathbf{Q}}_p^2)).$$

We may therefore make the following definitions.

(5.6) **Definition.** Let $\Phi \in \mathbf{W}_R$.

**a.** For each arithmetic point $\kappa\in\mathscr{X}^{\mathrm{arith}}(R)$ we define $\Phi_\kappa$ to be the image of $\Phi$ under $\phi_{\kappa,*}$.
**b.** For each $\kappa$ with trivial character, we also define $\tilde{\Phi}_\kappa$ to be the image of $\Phi$ under $\tilde{\phi}_{\kappa,*}$.

In Proposition 5.8 we will describe the behavior of the $p$-adic $L$-functions when $\Phi$ is transformed by the operator $T_p$. For this purpose, it will be useful to first record a few facts about the Hecke operators at $p$ acting on $\mathbf{W}$. We will say that a modular symbol $\Phi\in\tilde{\mathbf{W}}$ is supported on a given compact open subset $U$ of $\mathbf{Z}_p^2$, if, for every $D\in\mathscr{D}_0$, the measure $\Phi(D)\in\tilde{\mathbf{D}}$ is supported on $U$. Hence, $\mathbf{W}$ may be described as the set of modular symbols in $\tilde{\mathbf{W}}$ which are supported on $(\mathbf{Z}_p^2)'$. Since the scalar matrix $pI$ transforms any modular symbol $\Phi\in\tilde{\mathbf{W}}$ to one supported on $p\mathbf{Z}_p^2$, we see that $[p]_p$ annihilates $\mathbf{W}$.

The action of the operator $T_p$ and its powers $T_p^m$, $m>0$, on $\mathbf{W}$ can be described as follows. Consider the reduction map $(\mathbf{Z}_p^2)'\to\mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})$. For each $\mathbf{x}\in\mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})$ the preimage of $\mathbf{x}$ in $(\mathbf{Z}_p^2)'$ is a compact open set which we denote by $U(\mathbf{x},p^m)$. Choose an element $g_{\mathbf{x},p^m}\in\Sigma_1(N)$ with determinant $p^m$ for which $U(\mathbf{x},p^m)\subseteq((\mathbf{Z}_p)^2)'g_{\mathbf{x},p^m}$. The coset $\Gamma g_{\mathbf{x},p^m}$ is independent of the choice of $g_{\mathbf{x},p^m}$ with this property. The $m$th power of $T_p$ acting on $\Phi\in\mathbf{W}$ is then given by

$$(5.7)\qquad \Phi\,|\,T_p^m=\sum_{\mathbf{x}\in\mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})}\Phi\,|\,g_{\mathbf{x},p^m}.$$

This decomposes $\Phi\,|\,T_p^m$ into a sum of modular symbols which are supported on the disjoint compact open sets $U(\mathbf{x},p^m)$, $\mathbf{x}\in\mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})$.

We will say that a pair of arithmetic points $(\kappa,\sigma)\in\mathscr{X}^{\mathrm{arith}}(R)\times\mathscr{X}_0^{\mathrm{arith}}$ is *critical* if the weight of $\kappa$ is greater than or equal to the weight of $\sigma$. This is equivalent to saying that $\kappa\sigma^{-1}$ defines an arithmetic character on $\mathbf{Z}_p^*$.

(5.8) **Proposition.** *Let* $\Phi\in\mathbf{W}_R$ *and* $(\kappa,\sigma)\in\mathscr{X}(R)\times\mathscr{X}_0$.
**1.** (Relation Between the Standard and Improved $p$-adic $L$-functions.) *If* $\sigma\in\mathscr{X}_0^{\mathrm{arith}}$ *then*

$$L_p(\Phi\,|\,T_p,\kappa,\sigma)=L_p^*(\Phi\,|\,T_p,\kappa,\sigma)-\sigma(p)\cdot L_p^*(\Phi,\kappa,\sigma).$$

*Here* $\sigma(p)$ *is defined as in the paragraph preceding* (5.2)b.
**2.** (Interpolation.) *Suppose the pair* $(\kappa,\sigma)$ *is critical and suppose* $\sigma=\sigma_{r,\psi}$ *where* $r$ *is an integer* $\geqq 0$ *and* $\psi$ *is a finite character of* $\mathbf{Z}_p^*$ *of conductor* $p^m$, *with* $m\geqq 0$. *Then*

$$L_p^*(\Phi\,|\,T_p^m,\kappa,\sigma)=L(\Phi_\kappa,\psi,r+1)$$

*where* $\Phi_\kappa$ *is the specialization defined by* (5.6)a *and* $L(\Phi_\kappa,\psi,r+1)$ *is defined by* (4.12).
**3.** (Functional Equation for the standard $p$-adic $L$-function.) *Let* $W_N=\begin{pmatrix}0&1\\-N&0\end{pmatrix}$. *Then*

$$L_p(\Phi,\kappa,\sigma)=-\sigma^{-1}(-N)\cdot L_p(\Phi|W_N,\kappa,\kappa\sigma^{-1}).$$

**4.** (Functional Equation for the Improved $p$-adic $L$-functions.) *Suppose the pair* $(\kappa,\sigma)$ *is critical and that both* $\kappa$ *and* $\sigma$ *have trivial characters and fix* $r\geqq 0$ *so that* $\sigma=\sigma_r$. *Then*

$$L_p^*(\Phi,\kappa,\sigma)-\sigma(-N)^{-1}L_p^*(\Phi\,|\,W_N,\kappa,\kappa\sigma^{-1})=L_p(\Phi,\kappa,\sigma)+L(\tilde{\Phi}_\kappa,r+1).$$

*Proof.* Each of these statements is verified by a straightforward calculation. We will only sketch the details. We restrict ourselves to the special case $R=\mathbf{Z}_p[[\mathbf{Z}_p^*]]$, since the general case then follows by linearity.

Let $\mu=L(\Phi)\in\mathbf{D}$. Then for each $m>0$ we can write $L(\Phi|T_p^m)$ in the form

$$L(\Phi|T_p^m)=\sum_{\mathbf{x}\in\mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})}\mu_{\mathbf{x}}|g_{\mathbf{x},p^m} \tag{5.9}$$

where $\mu_{\mathbf{x}}=\Phi(g_{\mathbf{x},p^m}\cdot(\{0\}-\{i\infty\}))$. As remarked above, the decomposition (5.9) exhibits $L(\Phi|T_p^m)$ as a sum of measures supported on the disjoint compact open sets $U(\mathbf{x},p^m)$, for $\mathbf{x}\in\mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})$. Since the standard and improved $p$-adic $L$-functions are defined as integrals over $\mathbf{Z}_p^*\times\mathbf{Z}_p^*$ and $\mathbf{Z}_p^*\times\mathbf{Z}_p$, respectively, only those terms in (5.9) associated to $\mathbf{x}$ of the form $\mathbf{x}=[1,a]$ with $a\in\mathbf{Z}_p/p^m\mathbf{Z}_p$ will enter. Moreover, when $\mathbf{x}=[1,a]$ we may choose $g_{\mathbf{x},p^m}=\begin{pmatrix}1 & a\\ 0 & p^m\end{pmatrix}$. Now a simple calculation shows that when $\begin{pmatrix}1 & a\\ 0 & p^m\end{pmatrix}$ operates on the characteristic function of $U([1,a],p^m)$ the result is the characteristic function of $\mathbf{Z}_p^*\times\mathbf{Z}_p$. Hence we easily obtain

$$\begin{aligned} L_p(\Phi|T_p^m,\kappa,\sigma)&=\sum_{a\in(\mathbf{Z}/p^m\mathbf{Z})^*}\int_{\mathbf{Z}_p^*\times\mathbf{Z}_p}\kappa(x)\,\sigma(a+p^m y/x)\,d\mu_a(x,y)\\ L_p^*(\Phi|T_p^m,\kappa,\sigma)&=\sum_{a\in\mathbf{Z}/p^m\mathbf{Z}}\int_{\mathbf{Z}_p^*\times\mathbf{Z}_p}\kappa(x)\,\sigma(a+p^m y/x)\,d\mu_a(x,y)\end{aligned} \tag{5.10}$$

where we have written $\mu_a$ for the measure $\Phi(\{a/p^m\}-\{i\infty\})$.

Now consider the first assertion of the proposition. Setting $m=1$ in (5.10) and calculating the difference of the two expressions occurring there we see that only the term corresponding to $a=0$ survives. Thus

$$L_p^*(\Phi|T_p,\kappa,\sigma)-L_p(\Phi|T_p,\kappa,\sigma)=\sigma(p)\int_{\mathbf{Z}_p^*\times\mathbf{Z}_p}\kappa(x)\,\sigma(y/x)\,d\mu_0(x,y).$$

But $\mu_0=L(\Phi)$ so **1** follows.

We now turn to the proof of **2**. Fix an arithmetic point $\kappa\in\mathscr{X}^{\mathrm{arith}}$ of weight $k\geqq 2$. We must prove the identity $L_p^*(\Phi|T_p^m,\kappa,\sigma_{r,\psi})=L(\Phi_\kappa,\sigma_{r,\psi})$ for every finite character $\psi$ on $\mathbf{Z}_p^*$ of conductor $p^m$ and every integer $r$ with $0\leqq r\leqq k-2$. Fix the character $\psi$ of conductor $p^m$. Using (5.10) one easily calculates

$$\begin{aligned}&\sum_{r=0}^{k-2}\binom{k-2}{r}(-1)^r\cdot L_p^*(\Phi|T_p^m,\kappa,\sigma_{r,\psi})\cdot X^r\,Y^{k-2-r}\\ &\quad=\sum_{a\in\mathbf{Z}/p^m\mathbf{Z}}\psi(a)\int_{\mathbf{Z}_p^*\times\mathbf{Z}_p}\varepsilon(x)\,(x(y-aX)-y(p^m X))^{k-2}\,d\mu_a(x,y).\end{aligned}$$

On the other hand, from the definition of $\Phi_\kappa$ (5.6)a and the definition of $L(\Phi_\kappa,\psi,r+1)$ (4.12) we have

$$\begin{aligned}&\sum_{r=0}^{k-2}\binom{k-2}{r}(-1)^r\cdot L(\Phi_\kappa,\psi,r+1)\cdot X^r\,Y^{k-2-r}\\ &\quad=\sum_{a\in\mathbf{Z}/p^m\mathbf{Z}}\psi(a)\int_{\mathbf{Z}_p^*\times\mathbf{Z}_p}\varepsilon(x)\,(xY-yX)^{k-2}\,d\mu_a(x,y)\Bigg|\begin{pmatrix}1 & a\\ 0 & p^m\end{pmatrix}\end{aligned}$$

where the last |-operator denotes the action on the given homogeneous polynomial of degree $k-2$ in $\mathrm{Sym}^{k-2}(\bar{\mathbf{Q}}_p^2)$. Now **2** follows easily by comparing the last two displayed equalities.

To prove the functional equation **3**, we just notice that since $W_N$ interchanges the cusps $\infty$ and 0, we have $\mu_{\Phi|W_N} = -\mu_\Phi | W_N$. Now a simple calculation proves **3**.

The last property **4** follows from an application of the inclusion-exclusion principle. We have

$$L(\tilde{\Phi}_\kappa, \sigma) = \int_{(Z_p^2)'} \kappa\sigma^{-1}(x)\,\sigma(y)\,d\mu(x,y)$$
$$= \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} + \int_{\mathbf{Z}_p \times \mathbf{Z}_p^*} - \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p^*}$$

The first of these three integrals is $L_p^*(\Phi, \kappa, \sigma)$ and the last is $L_p(\Phi, \kappa, \sigma)$. The middle integral can easily be calculated as in the proof of the functional equation **3** and is equal to $-\sigma([-N]_p)^{-1} L_p^*(\Phi | W_N, \kappa, \kappa\sigma^{-1})$. This completes the proof of Proposition 5.8.

Recall the notations of (2.4) where $\mathscr{R}$ is the universal ordinary Hecke algebra, $\mathscr{K} = \mathscr{R} \otimes_\Lambda \mathscr{L}$, and $\mathscr{X} = \mathscr{X}(\tilde{\mathscr{R}})$. For each $\kappa \in \mathscr{X}$ let $\mathscr{R}_{(\kappa)}$ be the localization of $\tilde{\mathscr{R}}$ at $\kappa$. The fraction field $\mathscr{K}_{(\kappa)}$ of $\mathscr{R}_{(\kappa)}$ is a direct factor of $\mathscr{K}$ and correspondingly, the $\mathscr{K}_{(\kappa)}$-space $\mathbf{W}_{\mathscr{K}_{(\kappa)}}$ is a direct factor of $\mathbf{W}_{\mathscr{K}}$. We will say that an element $\Phi \in \mathbf{W}_\kappa$ is *regular at* $\kappa$ if the projection of $\Phi$ to $\mathbf{W}_{\mathscr{K}_{(\kappa)}}$ lies in the $\mathscr{R}_{(\kappa)}$-submodule $\mathbf{W}_{\mathscr{R}_{(\kappa)}}$. Every $\Phi \in \mathbf{W}_{\mathscr{K}}$ is regular at all but finitely many $\kappa$ in $\mathscr{X}$. To each $\Phi \in \mathbf{W}_{\mathscr{K}}$ we associate elements $L_p(\Phi, \sigma) \in \mathscr{K}$, for $\sigma \in \mathscr{X}_0$, and $L_p^*(\Phi, \sigma_0) \in \mathscr{K}$, for arithmetic $\sigma_0 \in \mathscr{X}_0^{\mathrm{arith}}$, as in (5.3). If $\Phi$ is regular at $\kappa$, then $L_p(\Phi, \kappa, \sigma)$ and $L_p^*(\Phi, \kappa, \sigma_0)$ are defined as in (5.4).

We are going to recast Proposition 5.8 in terms of the local charts (2.8). Fix an arithmetic point $\kappa \in \mathscr{X}^{\mathrm{arith}}$ of weight $k_0$ and character $\varepsilon$. Let $\Phi \in \mathbf{W}_{\mathscr{K}}$ be a modular symbol which is regular at $\kappa$. Let $U_\kappa$ be the domain of convergence about $\kappa$ (see the remarks before (2.8)) and define the *domain of analyticity* of $\Phi$ about $\kappa$ to be the open set

$$U_{\kappa,\Phi} \stackrel{\mathrm{def}}{=} \{k \in U_\kappa \mid \Phi \text{ is regular at } \kappa^{(k)}\}.$$

This is just $U_\kappa$ minus a finite set. For each rational integer $k \geqq 2$ in $U_{\kappa,\Phi}$, let $\Phi_{\kappa,k}$ denote the specialization of $\Phi$ to $\kappa^{(k)}$. Then $\Phi_{\kappa,k}$ is a modular symbol of weight $k$ and character $\varepsilon\omega^{k_0-k}$. For arbitrary $k \in U_{\kappa,\Phi}$, and for $\psi$ a finite character of $\mathbf{Z}_p^*$, $s \in \mathbf{Z}_p$, and $s_0 \in \mathbf{Z}^+$ we define

(5.11)
$$L_p(\Phi, \kappa, k, \psi, s) \stackrel{\mathrm{def}}{=} L_p(\Phi, \kappa^{(k)}, \psi\langle\cdot\rangle^{s-1});$$
$$L_p^*(\Phi, \kappa, k, \psi, s_0) \stackrel{\mathrm{def}}{=} L_p^*(\Phi, \kappa^{(k)}, \psi\langle\cdot\rangle^{s_0-1}).$$

We will be especially interested in these functions when $\Phi$ is an eigensymbol for the Hecke operators.

(5.12) **Definition.** For each arithmetic point $\kappa \in \mathscr{X}$, let $h_{(\kappa)}\colon \mathscr{H} \to \mathscr{R}_{(\kappa)}$ be the composition of the natural map $h\colon \mathscr{H} \to \mathscr{R}$ defined in (2.4)a with the localization morphism. Define $\mathbf{W}_{(\kappa)}$ to be the $h_{(\kappa)}$-eigenmodule in $\mathbf{W}_{\mathscr{R}_{(\kappa)}}$.

The involution $\iota$ preserves these modules. Let $\mathbf{W}^{\pm}_{(\kappa)}$ denote the $\pm$ eigenmodules for $\iota$. The following version of Hida's Control Theorem will be proved in the next section.

(5.13) **Theorem.** *Suppose $p \geqq 5$. Then for every $\kappa \in \mathscr{X}^{\text{arith}}$, and for either choice of sign $\pm$, the following are true.*
**a.** *$\mathbf{W}^{\pm}_{(\kappa)}$ is a free rank one $\mathscr{R}_{(\kappa)}$-module.*
**b.** *The specialization map $\phi_{\kappa,*}: \Phi \mapsto \Phi_\kappa$ (5.6) induces an isomorphism*

$$\mathbf{W}^{\pm}_{(\kappa)}/P_\kappa \mathbf{W}^{\pm}_{(\kappa)} \cong \mathbf{W}^{\pm}_{\kappa}.$$

If $\Phi$ is in one of the spaces $\mathbf{W}^{\pm}_{(\kappa)}$, then for each integer $k \geqq 2$ in the domain of analyticity of $\Phi$ about $\kappa$, the specialization $\Phi_{\kappa,k}$ lies in the Hecke eigenspace $\mathbf{W}^{\pm}_{\kappa,k}$ associated to $\kappa^{(k)}$ by (4.9). This is a one dimensional $K_{\kappa,k}$-vector space generated by the element $\Psi^{\pm}_{\mathbf{f}_{\kappa,k}}$ defined in (4.9). Thus there is a unique 'period' $\Omega_{\kappa,k}(\Phi) \in K_{\kappa,k}$ such that

$$\Phi_{\kappa,k} = \Omega_{\kappa,k}(\Phi) \cdot \Psi^{\pm}_{\mathbf{f}_{\kappa,k}}. \tag{5.14}$$

If $k$ is equal to the weight of $\kappa$, then we will suppress it from the notation and write simply $\Omega_\kappa(\Phi)$. Of course, our definition of the periods $\Omega_{\kappa,k}$ depends on the choice of complex periods used to define $\Psi^{\pm}_{\mathbf{f}_\kappa}$. It is interesting to ask whether there is a natural choice of complex periods and a choice of $\Phi$ so that $\Omega_\kappa(\Phi)$ extends to an analytic function of $\kappa \in \mathscr{X}$? It follows from Theorem 5.13 that there is an element $\Phi \in \mathbf{W}^{\pm}_{(\kappa)}$ for which $\Omega_\kappa(\Phi) = 1$. This is enough for our purposes.

Consider the sesquilinear map $*: \mathbf{W}_{\mathscr{K}} \to \mathbf{W}_{\mathscr{K}}$ defined on $\mathbf{W}$ by $\Phi \mapsto \Phi^* \overset{\text{def}}{=} \Phi | W_N$ and extended to $\mathbf{W}_{\mathscr{K}}$ by sesquilinearity: $(a\Phi)^* = a^* \Phi^*$ where $*$ is the involution on $\mathscr{K}$ defined in (2.9). A simple calculation shows that, for $\kappa \in \mathscr{X}^{\text{arith}}$, if $\Phi \in \mathbf{W}_{(\kappa)}$ then $\Phi^* \in \mathbf{W}_{(\kappa^*)}$. Note that $*$ is *not an involution* though, by (1.8), we do have the simple relation $(\Phi^*)^* = \Phi | [-N]_p$ for any $\Phi \in \mathbf{W}_{\mathscr{K}}$. In particular, since $[-N]_p$ acts invertibly on $\mathbf{W}$, the map $*: \mathbf{W}_{(\kappa)} \to \mathbf{W}_{(\kappa^*)}$ is an isomorphism.

(5.15) **Theorem.** *Let $\kappa \in \mathscr{X}^{\text{arith}}$ be an arithmetic point of weight $k_0$ and character $\varepsilon$ and let $\psi$ be a finite character of $\mathbf{Z}_p^*$. Fix an eigensymbol $\Phi \in \mathbf{W}^{\text{sgn}(\psi)}_{(\kappa)}$ and let $U = U_{\kappa,\Phi}$ be the domain of analyticity of $\Phi$ about $\kappa$. Then the following assertions are true.*
**a.** (Analyticity.) *$L_p(\Phi, \kappa, k, \psi, s)$ is analytic for $(k, s) \in U \times \mathbf{Z}_p$ and is an Iwasawa function in the variable $s$ (up to multiplication by a constant). For each positive integer $s_0$, $L_p^*(\Phi, \kappa, k, \psi, s_0)$ is analytic for $k \in U$.*
**b.** (Specialization of the weight variable.) *For each rational integer $k \geqq 2$ in $U$ we have the following identity of Iwasawa functions in $s$.*

$$L_p(\Phi, \kappa, k, \psi, s) = \Omega_{\kappa,k}(\Phi) \cdot L_p(\mathbf{f}_{\kappa,k}, \psi, s)$$

**c.** (Functional Equation.) *Let $\varepsilon_p$ be the $p$-component of $\varepsilon$. Then for $(k, s) \in U \times \mathbf{Z}_p$ we have*

$$L_p(\Phi, \kappa, k, \psi, s) = -\psi^{-1}(-N)\langle -N \rangle^{1-s} \cdot L_p(\Phi^*, \kappa^*, k, \varepsilon_p\, \omega^{k_0-2} \psi^{-1}, k-s).$$

**d.** (Specialization to critical values.) *For every $k\in U$ and every positive integer $s_0$ we have*

(i) $$L_p(\Phi,\kappa,k,\psi,s_0)=(1-a_p(\kappa,k)^{-1}\psi\omega^{1-s_0}(p)\,p^{s_0-1})\cdot L_p^*(\Phi,\kappa,k,\psi,s_0).$$

*Moreover, if k is an integer $\geqq 2$ in U and if $0<s_0<k$, then*

(ii) $$L_p^*(\Phi,\kappa,k,\psi,s_0)=\Omega_\kappa(\Phi)\cdot a_p(\kappa,k)^{-m}\cdot\frac{\Lambda(\mathbf{f}_{\kappa,k},\psi\omega^{1-s_0},s_0)}{\Omega_{\mathbf{f}_{\kappa,k}}^{\operatorname{sgn}(\psi)}}.$$

**e.** (Functional Equation for the improved p-acid L-function.) *Suppose the p-component of $\varepsilon$ is a power of the Teichmüller character, say $\varepsilon_p=\omega^n$. Then for every integer $k\geqq 2$ in U satisfying the congruence $k\equiv n+k_0-2 \pmod{p-1}$ and for every integer $s_0$ with $0<s_0<k$ we have*

$$\begin{aligned}&L_p^*(\Phi,\kappa,k,\omega^{s_0-1},s_0)-(-N)^{1-s_0}L_p^*(\Phi^*,\kappa^*,k,\omega^{k-s_0-1},k-s_0)\\&\quad=L_p(\Phi,\kappa,k,\omega^{s_0-1},s_0)+L(\tilde{\Phi}_{\kappa,k,s_0}).\end{aligned}$$

*Proof.* The first assertion a follows at once from the definitions. The assertions **c**, **d**(*i*), and **e** follow at once from **1**, **3**, and **4** of Proposition 5.8. Assertion **d**(*ii*) follows from **2** of (5.8), together with (5.14) and Theorem 4.14. To prove **b** we note that from **d** and Theorem 4.18 the desired equality holds if $s_0=1$ and $\psi$ is any nontrivial character. Since both sides of the equation are Iwasawa functions they are determined by these values. This completes the proof of Theorem 5.15.

Note that since the map $\Phi\mapsto\Phi^*$ is not an involution, the functional equations **c** and **e** are not symmetric in $\Phi$ and $\Phi^*$. Using the identity $(\Phi^*)^*=\Phi|[-N]_p$ and applying **c** with $(\Phi^*,\kappa^*)$ replacing $(\Phi,\kappa)$ we obtain

$$\begin{aligned}&L_p(\Phi^*,\kappa^*,k,\psi,s)\\&\quad=-\varepsilon_p\,\omega^{k_0-2}\psi^{-1}(-N)\langle -N\rangle^{k-s-1}\cdot L_p(\Phi,\kappa,k,\varepsilon_p\,\omega^{k_0-2}\psi^{-1},k-s).\end{aligned}$$

A similar identity is easily derived for the functional equation of the improved p-adic L-function. In case the N-component of $\varepsilon$ is trivial the next lemma shows that $\Phi$ is actually an eigensymbol for the operator $*$. In that case the functional equations (5.15)**c** and **e** take on a simpler, more symmetric form which we will exhibit in Corollary 5.17 below.

(5.16) **Lemma.** *Let $\kappa\in\mathscr{X}$ be an arithmetic point of weight $k_0$ and character $\varepsilon$ and suppose the conductor of $\varepsilon$ is prime to N. Then $*$ acts on $\mathbf{W}_{(\kappa)}$ as multiplication by an element $\mathbf{w}\in\mathscr{R}_{(\kappa)}$ where $\mathbf{w}^2=h_{(\kappa)}([-N]_p)$. Hence for k in the domain of convergence about $\kappa$, we have $\mathbf{w}(\kappa,k)=w\cdot\langle -N\rangle^{\frac{k-2}{2}}$ where $w\in\mathbf{Z}_p^*$ is a square root of $\varepsilon\omega^{k_0-2}(-N)$.*

*Proof.* The condition that $\varepsilon_N$ is trivial guarantees that the involution $*$ fixes $\mathscr{R}_{(\kappa)}$ elementwise. Hence $*$ induces an automorphism of $\mathbf{W}_{(\kappa)}$ and this automorphism is identical with $W_N$. Moreover, using (1.8) and the fact that $\varepsilon_N(-1)=1$, we see that $*$ preserves the $\pm$ submodules $\mathbf{W}_{(\kappa)}^{\pm}$. Since these are free of rank one over $\mathscr{R}_{(\kappa)}$, $*$ acts on each of them as multiplication by a unit in $\mathscr{R}_{(\kappa)}$. Now specialize to $\mathbf{W}_\kappa$ and use the fact that $W_N$ acts by a scalar on $\mathbf{W}_\kappa$ to deduce that $\mathbf{w}$ acts on all of $\mathbf{W}_{(\kappa)}$ by the same unit in $\mathcal{R}_{(\kappa)}$. The identity $\mathbf{w}^2=h_{(\kappa)}([-N]_p)$ now follows from the fact that $(\Phi^*)^*=\Phi|[-N]_p$ for all $\Phi\in\mathbf{W}$. The last assertion follows from the identity $\tilde{\kappa}([-N]_p, k)=\varepsilon\omega^{k_0-2}(-N)\langle -N\rangle^{k-2}$. This completes the proof of the lemma.

(5.17) **Corollary.** *Let $\Phi\in\mathbf{W}_{(\kappa)}$ where $\kappa\in\mathcal{X}^{\text{arith}}$ is an arithmetic point of weight $k_0$ and character $\varepsilon$ with conductor prime to $N$. Let $w$ be the square root of $\varepsilon\omega^{k_0-2}(-N)$ determined by $\kappa$ as in Lemma 5.16. Then*
**a.** $L_p(\Phi,\kappa,k,\psi,s)=-w\cdot\psi^{-1}(-N)\cdot\langle -N\rangle^{\frac{k}{2}-s}\cdot L_p(\Phi,\kappa,k,\varepsilon\omega^{k_0-2}\psi^{-1},k-s)$; *and*
**b.** *Suppose $\varepsilon=\omega^n$. Then for each integer $k\geqq 2$ in $U$ satisfying $k\equiv n+k_0-2 \pmod{p-1}$ and for each integer $s_0$ satisfying $0<s_0<k$ we have*

$$L_p^*(\Phi,\kappa,k,\omega^{s_0-1},s_0)-w\cdot\langle -N\rangle^{\frac{k}{2}-s_0}\cdot L_p^*(\Phi,\kappa,k,\omega^{k-s_0-1},k-s_0)$$
$$=L_p(\Phi,\kappa,k,\omega^{s_0-1},s_0)+L(\tilde{\Phi}_{\kappa,k},s_0).$$

(5.18) **Example.** As an illustration we will construct the two-variable $p$-adic $L$-function described in the introduction associated to $E=X_0(11)$, $p=11$, and verify the properties (0.8). Let $\Omega_E$ be the real period of $E$ and let

$$\Psi_E^+=\frac{1}{\Omega_E}\cdot\Phi_{f_E}^+\in\mathrm{Symb}_{\Gamma_1(11)}(\mathbf{Q})$$

where $\Phi_{f_E}^+$ is the plus part of the modular symbol associated to $f_E$ by (4.7). Then the $p$-adic $L$-function $L_p(E,s)$, $s\in\mathbf{Z}_p$, is given by (4.16) and (4.17). In (2.11) we showed that $\mathcal{R}=\Lambda$ in this case. Let $\kappa=\sigma_2$ be the unique arithmetic point on $\Lambda$ of weight two and trivial character and use Theorem 5.13 to choose a modular symbol $\Phi\in\mathbf{W}_{(\kappa)}^+$ with nonzero specialization to $\kappa$. Since $\mathcal{R}=\Lambda$ we can also assume that $\Phi^+$ is integral, i.e. $\Phi\in\mathbf{W}^+$, by 'clearing denominators'. Now define $L_p(k,s)=\Omega_\kappa(\Phi)^{-1}L_p(\Phi,\kappa,k,s)$ and $L_p^*(\ ,k,1)=\Omega_\kappa(\Phi)^{-1}L_p^*(\Phi,\kappa,k,1)$ for $k,s\in\mathbf{Z}_p$. It is clear from the definitions (5.2) that these are Iwasawa functions in both variables (up to multiplication by a scalar). Since the tame level is equal to 1, we have $\Phi^*=\Phi|W_1$. But $\Phi$ is invariant with respect to $SL(2,\mathbf{Z})$ and is therefore fixed by $W_1$. Thus $\Phi^*=\Phi$. The properties (0.8) now follow from Theorem 5.15 and its corollary 5.17.

## 6. Existence of $\Lambda$-adic eigensymbols

In this section we will prove Theorem 5.13. The proof is based on the following two propositions.

(6.1) **Proposition.** *The group $\mathbf{W}^0$ of ordinary $\Lambda$-adic modular symbols is a free $\Lambda$-module of finite rank. For each arithmetic point $\kappa\in\mathcal{X}_0^{\text{arith}}$ let $P_\kappa\subseteq\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ be the prime ideal associated to $\kappa$. Then for $\Phi\in\mathbf{W}^0$ we have $\Phi_\kappa=0\Leftrightarrow\Phi\in P_\kappa\mathbf{W}^0$.*

(6.2) **Proposition.** *There is a natural injective $\mathcal{H}$-homomorphism $Ta_p(J_\infty)^0\to\mathbf{W}_{\mathcal{L}}^0$.*

*Proof of Theorem 5.13.* Let $\kappa\in\mathcal{X}^{\text{arith}}$. Since $h(T_p)=a_p$ is a unit in $\mathcal{R}$, the module $\mathbf{W}_{(\kappa)}$ is contained in the ordinary part $\mathbf{W}^0\otimes_\Lambda\mathcal{R}_{(\kappa)}$. Since, by Proposition 6.1, $\mathbf{W}^0$ is a free $\Lambda$-module of finite rank, $\mathbf{W}_{(\kappa)}$ is a free $\mathcal{R}_{(\kappa)}$-module of finite rank.

From the last assertion of Proposition 6.1 and the fact that $\kappa$ is unramified over $\Lambda$ (Hida's Theorem (2.6)a), we conclude that the kernel of $\phi_{\kappa,*}$ in $\mathbf{W}^0_{\mathscr{R}(\kappa)}$ is $P_\kappa \mathbf{W}^0_{\mathscr{R}(\kappa)}$. In particular we see that the linear map $\mathbf{W}_{(\kappa)}/P_\kappa \mathbf{W}_{(\kappa)} \to \mathbf{W}_\kappa$ induced by $\phi_{\kappa,*}$ is injective.

Since $\mathbf{W}_\kappa$ has dimension two (Theorem 4.8), surjectivity of the map $\mathbf{W}_{(\kappa)}/P_\kappa \mathbf{W}_{(\kappa)} \to \mathbf{W}_\kappa$ will follow if we show that $\mathbf{W}_{(\kappa)}$ has $\mathscr{R}_{(\kappa)}$-rank at least two. Recall the submodule $\mathbf{T} = Ta_p(J_\infty)^0_{\text{prim}}$ of $Ta_p(J_\infty)^0$ from "section 2". By Hida's Theorem 2.6c we know that $\mathbf{T}_{\mathscr{L}}$ is a free $\mathscr{K}$-module of rank two. Using Proposition 6.2 we lift this to a free rank two $\mathscr{K}$-submodule of $\mathbf{W}^0_{\mathscr{L}}$. Since $\mathscr{K}$ is a semisimple $\mathscr{L}$-algebra, this space projects injectively to the $h$-eigenspace in $\mathbf{W}^0 \otimes_\Lambda \mathscr{K}$. The intersection of this eigenspace with $\mathbf{W}_{\mathscr{R}(\kappa)}$ is a rank two $\mathscr{R}_{(\kappa)}$-submodule of $\mathbf{W}_{(\kappa)}$. Hence $\mathbf{W}_{(\kappa)}$ has rank exactly 2 and the specialization morphism induces an isomorphism $\mathbf{W}_{(\kappa)}/P_\kappa \mathbf{W}_{(\kappa)} \cong \mathbf{W}_\kappa$. Since specialization commutes with the complex conjugation involution, (5.13)b follows. Assertion (5.13)a is a consequence of (5.13)b. This completes the proof.

## *Proof of Proposition 6.1*

Our proof of Proposition 6.1 will be based on two simple lemmas. Fix $\kappa \in \mathscr{X}_0^{\text{arith}}$. We will say that a function $\varphi: (\mathbf{Z}_p^2)' \to \bar{\mathbf{Q}}_p$ is homogeneous of degree $\kappa$ if $\varphi(t\mathbf{x}) = \kappa(t)\,\varphi(\mathbf{x})$ for every $t \in \mathbf{Z}_p^*$ and every $\mathbf{x} \in (\mathbf{Z}_p^2)'$. The following lemma follows easily from the definitions.

(6.3) **Lemma.** *A measure $\mu \in \mathbf{D}$ lies in $P_\kappa \mathbf{D}$ if and only if $\int \varphi\, d\mu = 0$ for every continuous function $\varphi$ on $(\mathbf{Z}_p^2)'$ which is homogeneous of degree $\kappa$.*

For each integer $m > 0$ let $\varphi_\kappa^{(m)}$ be the continuous function on $(\mathbf{Z}_p^2)'$ given by

$$\varphi_\kappa^{(m)}(a,b) = \begin{cases} \kappa(a) & \text{if } b \equiv 0 \bmod p^m; \\ 0 & \text{otherwise.} \end{cases}$$

(6.4) **Lemma.** *Let $\Phi \in \mathbf{W}$ be a $\Lambda$-adic modular symbol. Then the following are equivalent.*

**a.** $\Phi \in P_\kappa \mathbf{W}$.
**b.** $\int \varphi\, d\Phi(D) = 0$ *for all* $D \in \mathscr{D}_0$ *and all continuous functions* $\varphi$ *homogeneous of degree* $\kappa$.
**c.** $\int \varphi_\kappa^{(m)}\, d\Phi(D) = 0$ *for all* $D \in \mathscr{D}_0$ *and all* $m > 0$.

*Proof.* Since $P_\kappa$ is a principal ideal, we have $P_\kappa \mathbf{W} = \mathrm{Symb}_\Gamma(P_\kappa \mathbf{D})$. From this it follows that $\Phi \in P_\kappa \mathbf{W} \Leftrightarrow \Phi(D) \in P_\kappa \mathbf{D}$ for all $D \in \mathscr{D}_0$. So the equivalence $\mathbf{a} \Leftrightarrow \mathbf{b}$ follows from Lemma 6.3. The implication $\mathbf{b} \Rightarrow \mathbf{c}$ follows *a priori*. Now assume $\mathbf{c}$ is true. Then for every $\gamma \in \Gamma$, $\int \gamma \varphi_\kappa^{(m)}\, d\Phi(D) = \int \varphi_\kappa^{(m)}\, d\Phi(\gamma D) = 0$. So b follows from the fact that every continuous function $\varphi$ which is homogeneous of degree $\kappa$ is the uniform limit of a sequence of linear combinations of the functions $\gamma \varphi_\kappa^{(m)}$. This completes the proof of Lemma 6.4.

*Proof of Proposition 6.1.* We first prove the equality $\ker(\phi^0_{\kappa,*}) = P_\kappa \mathbf{W}^0$. Recall from (5.6)a that for $\Phi \in \mathbf{W}$ the specialization $\Phi_\kappa$ is the element of $\mathrm{Symb}_\Gamma(\mathrm{Sym}^{k-2}(\bar{\mathbf{Q}}_p^2))$ whose value on a divisor $D \in \mathscr{D}_0$ is given by

$$\Phi_\kappa(D) = \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \varepsilon(x)\,(xY - yX)^{k-2}\, d\Phi(D). \tag{6.5}$$

Since the integrand is homogeneous of degree $\kappa$, the inclusion $\ker(\phi_{\kappa,*}) \supseteq P_\kappa \mathbf{W}$ follows from the implication $\mathbf{a} \Rightarrow \mathbf{b}$ of (6.4). Conversely, suppose $\Phi \in \mathbf{W}^0$ and that $\Phi_\kappa = 0$. We will show that $\Phi \in P_\kappa \mathbf{W}^0$ by using $\mathbf{c} \Rightarrow \mathbf{a}$ from the last lemma. Fix $m > 0$ and $D \in \mathscr{D}_0$. Since $\Phi$ is ordinary, there is a $\Psi \in \mathbf{W}^0$ such that $\Psi | T_p^m = \Phi$. Using the notation of (5.7) we then have the identity

$$\int \varphi_\kappa^{(m)} \, d\Phi(D) = \sum_{\mathbf{x}} \int g_{\mathbf{x}, p^m} \varphi_\kappa^{(m)} \, d\Psi(g_{\mathbf{x}, p^m} \cdot D)$$

where the above sum runs over $\mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})$. But $g_{\mathbf{x}, p^m} \varphi_\kappa^{(m)} = 0$ unless $\mathbf{x} = [1, 0]$. Hence the above integral is equal to

$$\int g_{[1,0], p^m} \varphi_\kappa^{(m)} \, d\Psi(g_{[1,0], p^m} \cdot D) = \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \varepsilon(x) \, x^{k-2} \, d\Psi(g_{[1,0], p^m} \cdot D).$$

But this vanishes since it is the coefficient of $Y^r$ in $\phi_\kappa^*(\Phi)(g_{[1,0], p^m} \cdot D)$ (see (6.5)). We have therefore proven the equality $\ker(\phi_{\kappa,*}^0) = P_\kappa \mathbf{W}^0$, or equivalently, that $\Phi_\kappa = 0 \Leftrightarrow \Phi \in P_\kappa \mathbf{W}^0$. It follows from this and the compact version of Nakayama's lemma that $\mathbf{W}^0$ is a free $\Lambda$-module of finite rank. The proof of Proposition 6.1 is complete.

*Proof of Proposition 6.2*

The proof of Proposition 6.2 is based on a study of the cohomology exact sequence attached to $\mathbf{D}$ by Theorem 4.3. It will be convenient to simplify the notation and write

(6.6) $$\mathbf{B} = \mathrm{Bound}_\Gamma(\mathbf{D}), \quad \mathbf{W} = \mathrm{Symb}_\Gamma(\mathbf{D}), \quad \mathbf{V} = H^1_{\mathrm{par}}(\Gamma, \mathbf{D}).$$

It is easy to see that are no nonzero $\Gamma$-invariant measures in $\mathbf{D}$, hence Theorem 4.3 gives us an exact sequence

(6.7) $$0 \to \mathbf{B} \to \mathbf{W} \to \mathbf{V} \to 0.$$

(6.8) **Lemma.** *There is a natural $\mathscr{H}$-isomorphism $Ta_p(J_\infty) \cong \mathbf{V}$.*

*Proof.* Since the operator $W_N$ intertwines the covariant and the contravariant actions of $\mathscr{H}$ on $\mathbf{V}$, it will suffice to give an isomorphism $\pi\colon \mathbf{V} \cong Ta_p(J_\infty)$ for the covariant $\mathscr{H}$-structure on $\mathbf{V}$. For each integer $n \geqq 0$ let $H_1(X_n(\mathbf{C}), \mathbf{Z}_p)$, $H^1(X_n(\mathbf{C}), \mathbf{Z}_p)$ be the singular homology, respectively cohomology of the compact Riemann surface $X_n(\mathbf{C})$. Then the Albanese map gives us a canonical isomorphism Alb: $H_1(X_n(\mathbf{C}), \mathbf{Z}_p) \cong Ta_p(J_n)$ of $\mathscr{H}$-modules. Moreover, by Poincaré duality the intersection pairing gives us an isomorphism $H^1(X_n(\mathbf{C}), \mathbf{Z}_p) \cong H_1(X_n(\mathbf{C}), \mathbf{Z}_p)$ which intertwines the covariant action of $\mathscr{H}$ on cohomology with the natural (covariant) action on homology. Now write $\mathbf{V}_n$ for $H^1_{\mathrm{par}}(\Gamma_1(Np^n), \mathbf{Z}_p)$ equipped with the covariant action of $\mathscr{H}$. Then the Eichler-Shimura theorem gives us a canonical isomorphism $ES$: $\mathbf{V}_n \cong H^1(X_n(\mathbf{C}), \mathbf{Z}_p)$ of $\mathscr{H}$-modules. We define $\xi_n$ to be the composition

$$\xi_n\colon \mathbf{V}_n \xrightarrow{ES} H^1(X_n(\mathbf{C}), \mathbf{Z}_p) \xrightarrow{PD} H_1(X_n(\mathbf{C}), \mathbf{Z}_p) \xrightarrow{\mathrm{Alb}} \mathrm{Ta}_p(J_n).$$

For positive integers $m, n$ with $m>n$, we have a commutative diagram of $\mathscr{H}$-modules

$$\begin{array}{ccc} \mathbf{V}_m & \xrightarrow{\xi_m} & Ta_p(J_m) \\ \downarrow & & \downarrow \\ \mathbf{V}_n & \xrightarrow{\xi_n} & Ta_p(J_n) \end{array}$$

where the vertical arrow on the left is the corestriction morphism $\text{cores}_{m,n}$. We may therefore construct the limit $\mathbf{V}_\infty = \varprojlim \mathbf{V}_n$ and patch together an isomorphism

$$\xi_\infty\colon \mathbf{V}_\infty \to Ta_p(J_\infty).$$

For each $n>0$, consider the map $\pi_n\colon \mathbf{D} \to \mathbf{Z}_p$, given by $\mu \mapsto \mu((0,1)+p^n\mathbf{Z}_p^2)$. This map commutes with the action of $\Gamma_n$, hence induces a map $\pi_{n*}\colon \mathbf{V} \to \mathbf{V}_n$. A simple verification shows $\pi_{n*} = \text{cores}_{m,n} \circ \pi_{m*}$. Thus there is a natural homomorphism

$$\pi_*\colon \mathbf{V} \to \mathbf{V}_\infty.$$

We will prove that $\pi_*$ is an isomorphism by using Shapiro's Lemma and the simple observation that $\mathbf{D}$ is naturally isomorphic to a projective limit of induced modules. For each integer $n \geqq 0$, let $\mathbf{M}_n = ((\mathbf{Z}/p^n\mathbf{Z})^2)'$ denote the primitive vectors in $(\mathbf{Z}/p^n\mathbf{Z})^2$. Let $\mathbf{D}_n = \{\mu_n\colon \mathbf{M}_n \to \mathbf{Z}_p\}$ be the $\mathbf{Z}_p$-valued functions on $\mathbf{M}_n$ and let $\Gamma$ act on $\mathbf{D}_n$ by the rule $(\mu_n|\gamma)(v_n) = \mu_n(v_n\gamma^{-1})$. Since $\Gamma$ acts transitively on $\mathbf{M}_n$ and $\Gamma_n$ is the stabilizer of $(0,1)$ we see that $\mathbf{D}_n$ is an induced $\Gamma$-module. Hence, by Shapiro's lemma, the map $\mathbf{D}_n \to \mathbf{Z}_p$, $\mu_n \mapsto \mu_n((0,1))$ induces an isomorphism $H^1_{\text{par}}(\Gamma_1(N), \mathbf{D}_n) \cong H^1_{\text{par}}(\Gamma_n, \mathbf{Z}_p) = \mathbf{V}_n$.

For each $m \geqq n$, let $\mathbf{M}_m \to \mathbf{M}_n$ be the natural projection and define the connecting homomorphism $\delta_{m,n}$; $\mathbf{D}_m \to \mathbf{D}_n$ by $\delta_{m,n}(\mu_m) = \mu_n$ where for each $v_n \in \mathbf{M}_n$, $\mu_n(v_n) = \sum \mu_m(v_m)$ where the sum is over all $v_m \in \mathbf{M}_m$ lying over $v_n$. The maps $\mathbf{D} \to \mathbf{D}_n$ given by $\mu \mapsto \mu_n$ where $\mu_n(v_n) = \mu(v_n + p^n\mathbf{M})$ for $v_n \in \mathbf{M}_n$ commute with the connecting homomorphisms $\delta_{m,n}$ and induce an isomorphism

$$\mathbf{D} \cong \varprojlim \mathbf{D}_n.$$

Since $\Gamma$-cohomology commutes with projective limits in the category of compact $\Gamma$-modules we conclude

$$\mathbf{V} \cong \varprojlim H^1(\Gamma(N), \mathbf{D}_n) \to \varprojlim \mathbf{V}_n.$$

This completes the proof of Lemma 6.8.

To complete the proof of Proposition 6.2 we need to construct a Hecke equivariant splitting of the exact sequence (6.7). In general such a splitting does not exist over $\Lambda$. The obstruction is a group analogous to the classical cuspidal divisor class group. We need to first extend scalars to the quotient field $\mathscr{L}$ of $\Lambda$. Then, as in the classical setting, we will apply the Manin-Drinfeld principle to produce the desired splitting over $\mathscr{L}$. In order to use the Manin-Drinfeld principle we first need to analyze the structure of the $\mathscr{H}$-module $\mathbf{B} = \text{Bound}_\Gamma(\mathbf{D})$. The next lemma shows that $\mathbf{B}$ is a free $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$-module of finite rank and gives an explicit description of the action of the Hecke operators $T_l$ for $l \equiv 1 \pmod N$).

For each $\mathbf{x} \in \mathbf{P}^1(\mathbf{Q})$ let $\Gamma_{\mathbf{x}} \subseteq \Gamma$ be the unipotent stabilizer of $\mathbf{x}$ in $\Gamma$. Also, let $\mathbf{M}_{\mathbf{x}}$ denote the 'line' in $(\mathbf{Z}_p^2)'$ which is stabilized by $\Gamma_{\mathbf{x}}$.

(6.9) **Lemma.**

**a.** *Let* cusps$(\Gamma) \subseteq \mathbf{P}^1(\mathbf{Q})$ *be a complete set of representatives of the* $\Gamma$*-orbits in* $\mathbf{P}^1(\mathbf{Q})$. *Then there is a natural isomorphism of* $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$*-modules*

$$\mathbf{B} \cong \bigoplus_{\mathbf{x} \in \mathrm{cusps}(\Gamma)} \mathrm{Dist}(\mathbf{M}_{\mathbf{x}}).$$

*Hence* $\mathbf{B}$ *is a free* $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$*-module of rank* $\#$ cusps$(\Gamma)$.

**b.** *If* $l$ *is a prime different from* $p$ *which is congruent to* $1$ *modulo* $N$ *then* $T_l - l[l] - 1$ *annihilates* $\mathbf{B}$.

*Proof.* A function $\Phi\colon \mathbf{P}^1(\mathbf{Q}) \to \mathbf{D}$ represents a boundary symbol $\Phi \in \mathbf{B}$ if and only if $\Phi(\gamma \mathbf{x}) = \Phi(\mathbf{x})|\gamma$ for all $\mathbf{x} \in \mathbf{P}^1(\mathbf{Q})$ and $\gamma \in \Gamma$. Hence $\Phi(\mathbf{x}) \in \mathbf{D}^{\Gamma_{\mathbf{x}}}$ and the map

$$\mathbf{B} \to \bigoplus_{\mathbf{x} \in \mathrm{cusps}(\Gamma)} \mathbf{D}^{\Gamma_{\mathbf{x}}}$$

$$\Phi \mapsto \sum_{\mathbf{x} \in \mathrm{cusps}(\Gamma)} \Phi(\mathbf{x})$$

is an isomorphism. Thus our problem is reduced to a determination of all measures on $(\mathbf{Z}_p^2)'$ which are invariant under $\Gamma_{\mathbf{x}}$. We can obtain examples of such measures by 'extending by zero' measures on $\mathbf{M}_{\mathbf{x}}$. More precisely, define $i_{\mathbf{x}}\colon \mathrm{Dist}(\mathbf{M}_{\mathbf{x}}) \to \mathbf{D}^{\Gamma_{\mathbf{x}}}$ by $i_{\mathbf{x}}(\nu) = \mu$ where $\mu \in \mathbf{D}$ is given by the integration formulas

$$\int \varphi(x)\, d\mu(x) = \int_{\mathbf{M}_{\mathbf{x}}} \varphi(v)\, d\nu(v)$$

for all locally constant functions $\varphi \in \mathrm{Step}((\mathbf{Z}_p^2)')$. Lemma 6.9a follows immediately from the following lemma.

(6.10) **Lemma.** *The map* $i_{\mathbf{x}}\colon \mathrm{Dist}(\mathbf{M}_{\mathbf{x}}) \to \mathbf{D}^{\Gamma_{\mathbf{x}}}$ *is an isomorphism of* $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$*-modules.*

*Proof.* We will prove this assertion for any congruence group $\Gamma$ whose level is prime to $p$. The collection of all such groups is closed under conjugation by elements of $SL_2(\mathbf{Z})$. Since $\mathbf{P}^1(\mathbf{Q})$ is acted on transitively by $SL_2(\mathbf{Z})$, it suffices to prove the lemma in the special case $\mathbf{x} = \infty$. The map $i_\infty$ is clearly injective. We must show that it is surjective. So let $\mu \in \mathbf{D}^{\Gamma_\infty}$. We need to show that $\mu$ is supported on the line $\mathbf{M}_\infty = \{(0, r) | r \in \mathbf{Z}_p^*\}$. Let $U$ be an arbitrary compact open subset of $(\mathbf{Z}_p^2)'$ which is disjoint from $\mathbf{M}_\infty$. We will show $\mu(U) = 0$.

Choose a positive integer $m_0$ such that $v + p^{m_0}\mathbf{Z}_p^2 \subseteq U$ for every $v \in U$. Each $v$ in $U$ can be expressed as $v = (p^r a, b)$ for some $a \in \mathbf{Z}_p^*$, $b \in \mathbf{Z}_p$, and $r \geqq 0$. Since the set $v + p^{m_0}\mathbf{Z}_p^2$ is contained in $U$ it does not intersect $\mathbf{M}_\infty$. Hence $0 \notin p^r a + p^{m_0}\mathbf{Z}_p$ and consequently $r < m_0$. Now fix an arbitrary integer $n \geqq 0$. Then $U$ is the disjoint union of sets of the form $V = v + p^{m_0+n}\mathbf{Z}_p^2$. The open sets $V_k = (p^r a + p^{m_0+n}\mathbf{Z}_p) \times (b + p^{n+m_0}k + p^{2m_0+2n-r}\mathbf{Z}_p)$, $k = 0, 1, \ldots, p^{m_0+n-r} - 1$, are all

$\Gamma_\infty$-equivalent to one another. Indeed $V_k = V_0 \cdot \begin{pmatrix} 1 & p^{m_0+n-r} k a^{-1} \\ 0 & 1 \end{pmatrix}$. Hence all of these sets have the same measure under $\mu$. Since $V$ is the disjoint union of the $V_k$ we conclude that $\mu(V) = p^{m_0+n-r} \mu(U_0) \equiv 0 (\bmod p^n)$. Since $U$ is a disjoint union of such sets it follows that $\mu(U) \equiv 0 (\bmod p^n)$. But $n$ is arbitrary. Thus $\mu(U) = 0$. This completes the proof of Lemma 6.10 and hence also the proof of assertion **a** of Lemma 6.9.

To prove **b** of Lemma 6.9, we let $l$ be a prime which is congruent to one modulo $Np$ and compute $\Psi | T_l$ for an arbitrary element $\Psi \in \mathbf{B}$. Let $\mathbf{x} \in \mathbf{P}^1(\mathbf{Q})$ and choose $g \in SL_2(\mathbf{Z})$ so that $g\mathbf{x} = \infty$. For $k = 0, \ldots l-1$ let $\sigma_k = g^{-1} \begin{pmatrix} 1 & k \\ 0 & l \end{pmatrix} g$ and let $\sigma_l = g^{-1} \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} g$. Using the fact that $l \equiv 1 (\bmod N)$ it is easy to see that the Hecke operator $T_l$ is represented by the double coset $\Gamma \sigma_l \Gamma$. Moreover this double coset can be expressed as the disjoint union of the right cosets $\Gamma \sigma_k$, $k = 0, \ldots, l$. Thus

$$(\Phi | T_l)(\mathbf{x}) = \sum_{k=0}^{l} \Phi(\sigma_k \mathbf{x}) | \sigma_k.$$

Since each $\sigma_k$ fixes $\mathbf{x}$ we have $(\Phi | T_l)(\mathbf{x}) = \sum_{k=0}^{l} \Phi(\mathbf{x}) | \sigma_k$. A simple calculation shows that $\sigma_l$ acts trivially on $\mathbf{M}_\mathbf{x}$ and that for $k = 0, \ldots, l-1$, $\sigma_k$ acts on $\mathbf{M}_\mathbf{x}$ by multiplication by $l$. Hence $\Phi(\mathbf{x}) | \sigma_k = [l]\, \Phi(\mathbf{x})$ for $k = 0, \ldots, l-1$ and $\Phi(\mathbf{x}) | \sigma_l = \Phi(\mathbf{x})$. Now **b** follows easily, and Lemma 6.9 is proved.

(6.11) **Lemma.** *Let $l \neq p$ be a positive prime with $l \equiv 1$ modulo $N$ and let $\eta_l = T_l - l[l] - 1 \in \mathscr{H}$. Then $\eta_l$ acts invertibly on $\mathbf{V}^0_{\mathscr{L}}$ and annihilates $\mathbf{B}$. Moreover, the exact sequence*

$$0 \to \mathbf{B}^0_{\mathscr{L}} \to \mathbf{W}^0_{\mathscr{L}} \to \mathbf{V}^0_{\mathscr{L}} \to 0$$

*admits a unique splitting which commutes with $\mathscr{H}$.*

*Proof.* The fact that $\eta_l$ annihilates $\mathbf{B}$ was proved in Lemma 6.9b. By the Weil bounds we see that the kernel of $\eta_l$ acting on $Ta_p(J_\infty)$ is trivial. So, by Lemma 6.8, the same is true of $\mathbf{V}$. Hence $\eta_l$ acts invertibly on the finite dimensional $\mathscr{L}$ vector space $\mathbf{V}^0_{\mathscr{L}}$. This proves the first assertion of the proposition. To construct a section $s$: $\mathbf{V}^0_{\mathscr{L}} \to \mathbf{W}^0_{\mathscr{L}}$ we proceed as follows. For each $v \in \mathbf{V}^0_{\mathscr{L}}$ choose an element $\tilde{w} \in \mathbf{W}^0_{\mathscr{L}}$ lying over $\eta_l^{-1} v$ and set $s(v) = \eta_l \tilde{w}$. Clearly $s(v)$ lies over $v$. It does not depend on the choice of $\tilde{w}$ since $\eta_l$ annihilates $\mathbf{B}$. This splitting is Hecke equivariant since $\mathscr{H}$ is commutative. If $s'$ is another such section, then $s - s'$ is a morphism from $\mathbf{V}^0_{\mathscr{L}}$ to $\mathbf{B}^0_{\mathscr{L}}$ which intertwines $\eta_l$. Hence $s = s'$ and Lemma 6.11 is proved.

*Proof of Proposition 6.2.* By composing the isomorphism (6.8) $Ta_p(J_\infty)^0 \to \mathbf{V}^0$ with the section $\mathbf{V}^0_{\mathscr{L}} \to \mathbf{W}^0_{\mathscr{L}}$ constructed in Lemma 6.11, we obtain a natural injective $\mathscr{H}$-homomorphism $Ta_p(J_\infty)^0 \to \mathbf{W}^0_{\mathscr{L}}$. This completes the proof of Proposition 6.2.

(6.12) **Remark**. A more careful analysis of the proof of Lemma 6.9b reveals explicit formulas for the action of the Hecke operators $T_l$ for any prime $l$ not dividing $N$, including $l=p$. In particular, we have $\mathbf{B}=\mathbf{B}^0$. As a consequence we find that the splitting of Lemma 6.11 extends to a splitting of the sequence

$$0 \to \mathbf{B}_{\mathscr{L}} \to \mathbf{W}_{\mathscr{L}} \to \mathbf{V}_{\mathscr{L}} \to 0.$$

Indeed, we have $\mathbf{W}^{\text{nil}}=\mathbf{V}^{\text{nil}}$. By analogy with the classical situation, it is natural to regard the $p$-adic $L$-functions defined in the last section as being associated to elements of $Ta_p(J_\infty)$. More precisely, for each $x \in Ta_p(J_\infty)$, we view $x$ as an element of $\mathbf{V}$ by Lemma 6.8 and then use the splitting $s$ of the above sequence to lift $x$ to a $\Lambda$-adic modular symbol $\Phi_x \in \mathbf{W}_{\mathscr{L}}$. Since $s$ is not in general defined over $\Lambda$, we cannot say that $\Phi_x \in \mathbf{W}$. There will, in general, be denominators and we can ask what kind of poles these denominators will pass on to the $p$-adic $L$-functions associated to $\Phi_x$. We can say the following.

First, it is not difficult to see that the standard 2-variable $p$-adic $L$-function $L_p(\Phi_x)$ has no poles. Indeed, we can find another modular symbol $\Phi'_x \in \mathbf{W}$ which is congruent to $\Phi_x$ modulo $\mathbf{B}_{\mathscr{L}}$, i.e. $\Phi_x = \Phi'_x + \Psi$ for some $\Psi \in \mathbf{B}_{\mathscr{L}}$. But $L(\Psi) \in \mathbf{D}$ is a measure which, by (6.10) and the definition of $L(\Psi)$, is supported on the set $\{(a, b) \in \mathbf{Z}_p^2 \mid ab=0\}$. Since $L_p(\Psi)$ is defined by an integral over $\mathbf{Z}_p^* \times \mathbf{Z}_p^*$ (see 5.2a) we have $L_p(\Psi)=0$. So $L_p(\Phi_x)=L_p(\Phi'_x)$ and this is everywhere regular since $\Phi'_x$ is everywhere regular.

Second, we can bound the denominators which arise in the $\Lambda$-adic modular symbols $\Phi_x$ for $x \in Ta_p(J_\infty)$ as follows. The submodule $s^{-1}(\mathbf{W}) \subseteq \mathbf{V}$ is clearly preserved by the Hecke operators. In fact, it can be shown that $s^{-1}(\mathbf{W})$ corresponds to a Galois invariant submodule $S$ of $Ta_p(J_\infty)$. The quotient $C = Ta_p(J_\infty)/S$ is a natural analog of the classical cuspidal divisor class group. It would be interesting to analyze the structure of $C$ and to attempt an Eisenstein descent along the lines of [Mz1]. From the above discussion we see that $C$ is annihilated by the operators $T_l - 1 - l[l]$ for primes $l \equiv 1$ modulo $N$ with $l \neq p$. Hence $C$ is a torsion $\Lambda$-module which is annihilated by $a_l - 1 - l[l]$. From this it follows that the denominator in $\Phi_x$ is a divisor of $a_l - 1 - l[l]$ for every positive prime $l \equiv 1$ modulo $N$. While the denominator in $\Phi_x$ does not contribute poles to the standard 2-variable $p$-adic $L$-function (see last paragraph), we expect that it will contribute poles to the improved $p$-adic $L$-function $L_p^*(\Phi_x)$. Note that, by the Weil bounds, $a_l - 1 - l[l]$ does not vanish at any arithmetic point $\kappa \in \mathscr{X}^{\text{arith}}$. Hence each $\Phi_x$ is regular at these points, and correspondingly $L_p^*(\Phi_x, \kappa, \sigma)$ is regular at arithmetic points.

## 7. The main theorem

We are now ready to prove our main theorem.

(7.1) **Theorem.** *Suppose $f$ is a weight 2 newform which is split multiplicative at a prime $p \geqq 5$. Then $L_p(f, 1)=0$ and*

$$L'_p(f, 1) = \mathfrak{L}_p(f) \cdot \frac{L_\infty(f, 1)}{\Omega_f^+}.$$

*Proof.* The assertion $L_p(f, 1)=0$ follows from the interpolation properties of $L_p(f, s)$ described in Theorem 4.18. Indeed, the eigenvalue of $T_p$ acting on $f$

is 1, hence the 'Euler factor' appearing in 4.18 vanishes when $s_0=1$ and $\psi$ is trivial.

Let $N$ be the tame level of $f$ and let $\kappa\in\mathscr{X}^{\text{arith}}$ be the arithmetic point of weight two for which $f=\mathbf{f}_\kappa$. Using Theorem 5.13 we can choose an eigensymbol $\Phi\in\mathbf{W}^+_{(\kappa)}$ so that $\Phi_\kappa=\Psi^+_{\mathbf{f}_\kappa}$. Let $L_p(\Phi,\kappa,k,s)$, $L_p(\Phi^*,\kappa^*,k,s)$, $L^*_p(\Phi,\kappa,k,s)$ and $L^*_p(\Phi^*,\kappa^*,k,s)$ be the functions defined in (5.11). They are defined and analytic for all $(k,s)\in U\times\mathbf{Z}_p$ for some neighborhood $U$ of 2 in $\mathbf{Z}_p$. Since $f$ is split multiplicative at $p$, its nebentype character $\varepsilon$ satisfies $\varepsilon(p)=1$. Hence, by (2.9)b, we have $a_p(\kappa,k)=a_p(\kappa^*,k)$. We call this function $a_p(k)$ and note that $a_p(2)=1$. From 5.15d($i$) we deduce the identities

$$(7.2)\qquad \begin{aligned} L_p(\Phi,\kappa,k,1)&=(1-a_p(k)^{-1})\cdot L^*_p(\Phi,\kappa,k,1),\\ L_p(\Phi^*,\kappa^*,k,1)&=(1-a_p(k)^{-1})\cdot L^*_p(\Phi^*,\kappa^*,k,1).\end{aligned}$$

In particular, each of the functions $L_p(\Phi,\kappa,k,s)$ and $L_p(\Phi^*,\kappa^*,k,s)$ vanishes at $(k,s)=(2,1)$. In fact, we will show that the Taylor expansions of these two functions have the same linear terms around the point (2, 1). To see this we define constants $c,d\in\bar{\mathbf{Q}}_p$ (in fact, we have $c,d\in K_\kappa$) for which

$$L_p(\Phi,\kappa,k,s)\sim c(-\tfrac{1}{2}(k-2)+(s-1))+d(k-2)$$

where $f(k,s)\sim g(k,s)$ means that $f$ and its first partials agree with $g$ and its first partials at the point (2, 1). Replacing $s$ by $k-s$ in the functional equation 5.15c, we obtain

$$L_p(\Phi,\kappa,k,k-s)=-\langle -N\rangle^{1+s-k}L_p(\Phi^*,\kappa^*,k,s).$$

Hence $L_p(\Phi^*,\kappa^*,k,s)\sim -L_p(\Phi,\kappa,k,k-s)$ and we easily calculate its linear terms

$$L_p(\Phi^*,\kappa^*,k,s)\sim c(-\tfrac{1}{2}(k-2)+(s-1))-d(k-2).$$

Hence $L_p(\Phi,\kappa,k,s)-L_p(\Phi^*,\kappa^*,k,s)\sim 2d(k-2)$. To see that $d=0$ we set $s=1$ and calculate this difference using (7.2).

$$(7.3)\qquad \begin{aligned} &L_p(\Phi,\kappa,k,1)-L_p(\Phi^*,\kappa^*,k,1)\\ &\quad=(1-a_p(k)^{-1})\cdot(L^*_p(\Phi,\kappa,k,1)-L^*_p(\Phi^*,\kappa^*,k,1)).\end{aligned}$$

From 5.15e we have $L^*_p(\Phi,\kappa,2,1)-L^*_p(\Phi^*,\kappa^*,2,1)=L_p(\Phi,\kappa,2,1)+L(\tilde{\Phi}_\kappa,1)$ where $\tilde{\Phi}_\kappa$ is the tame specialization of $\Phi$ defined in (5.6)b. We have already seen $L_p(\Phi,\kappa,2,1)=0$. As for $L(\tilde{\Phi}_\kappa,1)$, we note that $\tilde{\Phi}_\kappa$ is a weight 2 modular symbol over $\Gamma_1(N)$ with the same eigenvalues as $f$ for the Hecke operators $T_l$, $l\nmid Np$. But $f$ is a newform whose level is divisible by $p$. Hence we must have $\tilde{\Phi}_\kappa=0$. In particular, $L(\tilde{\Phi}_\kappa,1)=0$, and it follows that $L^*_p(\Phi,\kappa,2,1)-L^*_p(\Phi^*,\kappa^*,2,1)=0$. Thus the expression in (7.3) has a double zero at $k=2$, and consequently $d=0$. We therefore have

$$(7.4)\qquad L_p(\Phi,\kappa,k,s)\sim c(-\tfrac{1}{2}(k-2)+(s-1)).$$

We complete the proof, as in the introduction, by calculating the constant $c$ in two ways. From 5.15b we have $L_p(\Phi, \kappa, 2, s) = L_p(f, s)$. Hence, setting $k=2$ in (7.4) we obtain

$$c = L'_p(f, 1).$$

On the other hand, setting $s=1$ in (7.4) and differentiating the first identity of (7.2) with respect to $k$ we find

$$-\tfrac{1}{2} \cdot c = a'_p(2) \cdot L^*_p(\Phi, \kappa, 2, 1).$$

But from Theorem 3.18 we have $a'_p(2) = -\tfrac{1}{2}\mathfrak{L}_p(f)$ and since $\Omega^+_\kappa(\Phi) = 1$, Theorem 5.15d(*ii*) gives us the identity $L^*_p(\Phi, \kappa, 2, 1) = L_\infty(f, 1)/\Omega^+_f$. Hence

$$c = \mathfrak{L}_p(f) \cdot \frac{L_\infty(f, 1)}{\Omega^+_f}$$

and the proof is complete.

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