Vol. 53, No. 3 DUKE MATHEMATICAL JOURNAL © September 1986

MODULAR FORMS IN CHARACTERISTIC ℓ AND SPECIAL VALUES OF THEIR L-FUNCTIONS

AVNER ASH AND GLENN STEVENS

In this paper we specialize our results in [A-S] to $G = GL_2$ to obtain information about congruences between classical holomorphic modular forms. In the final section we indicate how the methods can be adapted to prove congruences between special values of L-functions of modular forms of possibly different weights.

We begin with a study of the q-expansions in characteristic $\ell > 0$ of Hecke eigenforms of weight $k \geqslant 2$ and level N prime to ℓ. By a theorem of Eichler and Shimura this is equivalent to a study of the systems of eigenvalues of Hecke operators acting on the group cohomology of $\Gamma_1(N)$ (see §2). Using the functorial properties of cohomology, we show (Theorems 3.4 and 3.5) that the systems of Hecke eigenvalues (mod ℓ) occurring in the space, $\mathcal{M}_{>2}(\Gamma_1(N))$, of modular forms of level N and all weights > 2 coincide, up to twist, with those occurring in the space, $\mathcal{M}_2(\Gamma_1(N\ell))$, of weight two forms of level $N\ell$. In particular, we see that there are only finitely many systems of eigenvalues (mod ℓ) occurring in the infinite dimensional space $\mathcal{M}_{>2}(\Gamma_1(N))$, a fact proved by Jochnowitz [J] for prime $N \leqslant 17$.

Group cohomology has been used before in this theory. For example, a proof of Theorem 3.4(a) was given by Hida [H1] who refers to much earlier but unpublished work of Shimura [S1]. An account of Shimura's work can also be found in Ohta's article [O]. Kuga, Parry, and Sah [K-P-S] have proved similar statements and extended them to quaternionic groups. Haberland [Ha] has used cohomological methods in his study of congruences of Cartan type. Apparently new in our treatment is the use of the operator θ, a cohomological analog of "twisting" of modular forms, and of the operator Ψ (see the proof of 3.3(b)).

By taking the point of view of group cohomology we lose some structure. For example, we do not see the algebra structure of the space of modular forms. Nor do we see the Hodge decomposition of the cohomology groups. Notice also that we obtain no information about weight one forms. Nevertheless, the functoriality of group cohomology provides a powerful tool for studying congruences between eigenforms. Moreover, as many authors have noted [M, Mz, St], group cohomology is well suited for the study of p-adic properties of special values of L-functions.

In §4 we develop the theory of higher weight ($k > 2$) modular symbols, and examine the special values of L-functions attached to them. In Theorem 4.5 and

Received December 10, 1985. Research partially supported by grants from the National Science Foundation.

its corollary we show how the theory can be used to prove congruences modulo ℓ between special values of L-functions of higher weight cusp forms over $SL(2, \mathbf{Z})$ and those of weight two cusp forms over $\Gamma_1(\ell)$.

§1. Modular forms in characteristic ℓ. For purposes of comparison and to introduce notation which will be used later, we review the literature on modular forms in characteristic ℓ.

We assume throughout that ℓ is a prime > 3.

Let $N > 0$ and $\Gamma = \Gamma_1(N)$. The classical Hecke algebra $\mathscr{H} = \mathbf{Z}[T_n, \langle a \rangle]$, where n runs through the positive integers and a through $(\mathbf{Z}/N\mathbf{Z})^*$, acts on the space $\mathscr{M}_k(\Gamma)$ of weight k holomorphic modular forms over Γ. This action respects the decomposition of $\mathscr{M}_k(\Gamma)$ into cusp forms, $\mathscr{S}_k(\Gamma)$, and Eisenstein series $\mathscr{E}_k(\Gamma)$.

We will always assume $k \geqslant 2$. The ℓ-adic properties of the systems of eigenvalues of $\mathscr{H}$ occurring in $\mathscr{S}_k(\Gamma)$ reflect the structure of the ℓ-adic Galois representation attached to $\mathscr{S}_k(\Gamma)$ by Deligne [D]. The case of weight 2 is of special interest because in this case the ℓ-adic representation is the Tate module of the ℓ-divisible group of the Jacobian of the modular curve $X_1(N)/\mathbf{Q}$. Thus congruences modulo ℓ for systems of eigenvalues occurring in $\mathscr{S}_2(\Gamma)$ are related to the structure of the Galois module of ℓ-division points of the modular Jacobian. This point of view is used by Doi and Ohta [D-O] to prove congruences between weight two cusp forms.

An important principle, first observed by Shimura [S1] (see [O]), is that systems of eigenvalues of $\mathscr{H}$ occurring in $\mathscr{S}_k(\Gamma)$ are congruent modulo a prime $\underline{\lambda}$ above ℓ to systems occurring in $\mathscr{S}_2(\Gamma_1)$ for another congruence group Γ_1 of level $N\ell$. This principle offers the possibility of proving congruences between cusp forms of higher weight by first reducing to weight 2. This idea has been developed by Hida [H1, H2, H3] in his work on congruence primes, and also by Ribet [R1]. We view our Theorem 3.5 as a generalization of Shimura's principle.

Since each form $f \in \mathscr{M}_k(\Gamma)$ has a Fourier expansion we may view $\mathscr{M}_k(\Gamma)$ as a subspace of $\mathbf{C}[[q]]$, $q = e^{2\pi i z}$. For a Dirichlet character ε: $(\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*$ let $\mathscr{M}_k(\Gamma, \varepsilon)$ be the space of forms with character ε. Let $\mathscr{M}_k(\Gamma, \varepsilon; \mathbf{Z}[\varepsilon]) = \mathscr{M}_k(\Gamma, \varepsilon) \cap \mathbf{Z}[\varepsilon][[q]]$ where $\mathbf{Z}[\varepsilon]$ is the ring generated over $\mathbf{Z}$ by the values of ε. For a $\mathbf{Z}[\varepsilon]$-algebra R let $\mathscr{M}_k(\Gamma, \varepsilon; R) = \mathscr{M}_k(\Gamma, \varepsilon; \mathbf{Z}[\varepsilon]) \otimes_{\mathbf{Z}[\varepsilon]} R \subseteq R[[q]]$. Similarly define $\mathscr{S}_k(\Gamma, \varepsilon; R)$ and $\mathscr{E}_k(\Gamma, \varepsilon; R)$. Using the description of the action of $\mathscr{H}$ on $\mathscr{M}_k(\Gamma, \varepsilon)$ in terms of q-expansions [S2] we see that $\mathscr{H}$ acts on the spaces $\mathscr{M}_k(\Gamma, \varepsilon; R)$, $\mathscr{S}_k(\Gamma, \varepsilon; R)$ and $\mathscr{E}_k(\Gamma, \varepsilon; R)$ for every $\mathbf{Z}[\varepsilon]$-algebra R.

The following theorem is due to Shimura (cf. [S2], Thm. 3.52).

THEOREM 1.1.
(i) $\mathscr{M}_k(\Gamma, \varepsilon; \mathbf{C}) \cong \mathscr{M}_k(\Gamma, \varepsilon)$.
(ii) $\mathscr{S}_k(\Gamma, \varepsilon; \mathbf{C}) \cong \mathscr{S}_k(\Gamma, \varepsilon)$.
(iii) $\mathscr{E}_k(\Gamma, \varepsilon; \mathbf{C}) \cong \mathscr{E}_k(\Gamma, \varepsilon)$. ∎

Now let $\mathscr{O}$ be the ring of all algebraic integers and $\underline{\lambda} \subseteq \mathscr{O}$ be a prime ideal dividing ℓ. Fix an identification $\mathscr{O}/\underline{\lambda} = \overline{\mathbf{F}}_\ell$. For a system of eigenvalues

$\Phi: \mathscr{H} \to \mathcal{O}$ let $\bar{\Phi}: \mathscr{H} \to \bar{\mathbf{F}}_\ell$ be the reduction of Φ modulo $\underline{\lambda}$. The following corollary is an immediate consequence of the last theorem and Propositions 1.2.2 and 1.2.3 of [A-S].

COROLLARY 1.2. *Let $\Psi: \mathscr{H} \to \bar{\mathbf{F}}_\ell$ be a system of eigenvalues. Then Ψ occurs in $\mathscr{M}_k(\Gamma, \bar{\varepsilon}; \bar{\mathbf{F}}_\ell)$ if and only if there is a $\Phi: \mathscr{H} \to \mathcal{O}$ occurring in $\mathscr{M}_k(\Gamma, \varepsilon)$ such that $\Psi = \bar{\Phi}$.* ∎

The algebra $\mathscr{M}(SL_2(\mathbf{Z}); \mathbf{F}_\ell) = \mathbf{F}_\ell + \Sigma_{k \geq 2} \mathscr{M}_k(SL_2(\mathbf{Z}); \mathbf{F}_\ell) \subseteq \mathbf{F}_\ell[[q]]$ has been studied by Swinnerton-Dyer and Serre [Se1, Se2, Sw-D]. A useful tool in their theory is the derivation (or "twist") $\theta: \mathscr{M}(SL_2(\mathbf{Z}); \mathbf{F}_\ell) \to \mathscr{M}(SL_2(\mathbf{Z}); \mathbf{F}_\ell)$ defined by $\theta(\Sigma a_n q^n) = \Sigma n a_n q^n$. The operator θ maps $\mathscr{M}_k(SL_2(\mathbf{Z}); \mathbf{F}_\ell)$ to $\mathscr{M}_{k+\ell+1}(SL_2(\mathbf{Z}); \mathbf{F}_\ell)$ and intertwines the action of $\mathscr{H}$ on these two spaces. For a system of eigenvalues $\Phi: \mathscr{H} \to \bar{\mathbf{F}}_\ell$ and a nonnegative integer ν let $\Phi^{(\nu)}: \mathscr{H} \to \bar{\mathbf{F}}_\ell$ be the "ν-fold twist" of Φ defined by $\Phi^{(\nu)}(T_n) = n^\nu \Phi(T_n)$ and $\Phi^{(\nu)}(\langle a \rangle) = \Phi(\langle a \rangle)$. If Φ is the system of eigenvalues associated to a nonzero eigenform $f \in \mathscr{M}_k(SL_2(\mathbf{Z}); \bar{\mathbf{F}}_\ell)$ then $\Phi^{(1)}$ is the system associated to θf.

The following theorem is due to Serre and Tate. It has been strengthened to congruence groups of prime level ≤ 17 by N. Jochnowitz [J].

THEOREM 1.3.

(a) *There are only finitely many systems of eigenvalues $\Phi: \mathscr{H} \to \bar{\mathbf{F}}_\ell$ occurring in the space $\mathscr{M}(SL_2(\mathbf{Z}); \mathbf{F}_\ell) \otimes \bar{\mathbf{F}}_\ell$.*

(b) *If Φ occurs in this space then there is a system Ψ occurring in*

$$\bigoplus_{2 \leq k \leq \ell+1} \mathscr{M}_k(SL_2(\mathbf{Z}); \bar{\mathbf{F}}_\ell)$$

and an integer $\nu \geq 0$ such that $\Phi = \Psi^{(\nu)}$. ∎

The first part of this theorem follows easily from Corollary 2.5. We will give a cohomological proof of (b) for the groups $\Gamma_1(N)$ for arbitrary $N \geq 1$ in §3 (see Corollary 3.6).

The next theorem was proved by Serre.

THEOREM 1.4.

(a) ([Se1], Thm. 11). $\mathscr{S}_{\ell+1}(SL_2(\mathbf{Z}); \mathbf{F}_\ell) = \mathscr{S}_2(\Gamma_0(\ell); \mathbf{F}_\ell)$.

(b) [Se2]. *For $2 \leq k \leq \ell - 1$*

$$\mathscr{S}_k(SL_2(\mathbf{Z}); \mathbf{F}_\ell) \subseteq \mathscr{S}_2(\Gamma_1(\ell), \omega^{k-2}; \mathbf{F}_\ell)$$

where $\omega: (\mathbf{Z}/\ell\mathbf{Z})^* \to \mathcal{O}^*$ is the unique character satisfying $\omega(n) \equiv n \pmod{\underline{\lambda}}$ for all $n \in (\mathbf{Z}/\ell\mathbf{Z})^*$. ∎

Serre gives two proofs of (b). One uses the geometry of the modular curve $X_1(\ell)$ in characteristic ℓ. The other proof involves multiplication by an Eisenstein series and is similar to the proof of (a). Our Theorem 3.4(a), (b) provides a cohomological analog.

§2. Passing to cohomology. The goal of this section is Proposition 2.3 which relates systems of Hecke eigenvalues occurring in spaces of modular forms modulo $\underline{\lambda}$ to systems occurring in certain cohomology groups.

We will use the terminology of Hecke pairs and associated Hecke algebras as in [A-S]. Thus if Γ is a congruence subgroup of $SL_2(\mathbf{Z})$ and S is a subsemigroup of $GL_2(\mathbf{Q})$ containing Γ, then (Γ, S) is a Hecke pair. The associated Hecke algebra $\mathscr{H}(\Gamma, S)$ can be defined as a double coset algebra as in [A-S] or, equivalently, as the convolution algebra of $\mathbf{Z}$-valued Γ bi-invariant functions on S which vanish outside finitely many double cosets. This algebra acts on the right on the cohomology groups $H^*(\Gamma, E)$ of any right S-module E.

We want to compare systems of Hecke eigenvalues occurring in $\mathscr{M}_k(\Gamma_1(N))$ to systems occurring in $\mathscr{M}_2(\Gamma_1(N\ell))$. Thus we need to relate the Hecke algebra of $\Gamma_1(N)$ to the Hecke algebra of $\Gamma_1(N\ell)$. For this purpose the notion of compatibility of Hecke pairs was introduced in [An, A-S]. Unfortunately, if we wish to use this notion in the present situation we must exclude the Hecke operators T_n for $\ell \mid n$. Preferring not to do this we make instead the following definition.

DEFINITION 2.1. *A Hecke pair* (Γ_0, S_0) *is said to be weakly compatible to a Hecke pair* (Γ, S) *if*

(a) $(\Gamma_0, S_0) \subseteq (\Gamma, S)$;

(b) *the set* $S' = S \setminus \Gamma S_0$ *satisfies* $SS' \subseteq S'$ *and* $S'S_0 \subseteq S'$;

(c) $\Gamma \cap S_0 S_0^{-1} = \Gamma_0$.

Remark. If we replace (b) by the stronger condition $S' = \varnothing$ then we obtain the notion of compatibility ([A-S] Definition 1.1.2). ∎

If $(\Gamma_0, S_0) \subseteq (\Gamma, S)$ are weakly compatible then there is a canonical algebra homomorphism ι: $\mathscr{H}(\Gamma, S) \to \mathscr{H}(\Gamma_0, S_0)$. Viewing the Hecke algebras as convolution algebras this map is given by restriction of functions on S to functions on S_0.

The following lemma is the basic tool which allows us to relate systems of Hecke eigenvalues occurring in the cohomology of Γ to those occurring in Γ_0.

LEMMA 2.2. *Suppose* $(\Gamma_0, S_0) \subseteq (\Gamma, S)$ *are weakly compatible Hecke pairs.*

(a) *Let* E *be a right* S*-module,* F *be a right* S_0*-module and* ϕ: $E \to F$ *be an* S_0*-morphism. If* $E|\sigma \subseteq \ker(\phi)$ *for every* $\sigma \in S \setminus \Gamma S_0$ *then the composition*

$$H^r(\Gamma, E) \overset{\text{res}}{\to} H^r(\Gamma_0, E) \overset{\phi_*}{\to} H^r(\Gamma_0, F)$$

is Hecke equivariant; i.e. if $\xi \in H^r(\Gamma, E)$ *and* $h \in \mathscr{H}(\Gamma, S)$ *then*

$$(\phi_* \circ \mathrm{res})(\xi | h) = (\phi_* \circ \mathrm{res}(\xi))|\iota(h).$$

(b) *If* F *is a right* S_0*-module then the induced module* $\mathrm{Ind}(\Gamma_0, \Gamma; F)$ *inherits a natural right* S*-action. The Shapiro isomorphism*

$$\mathscr{S}\colon H^r(\Gamma, \mathrm{Ind}(\Gamma_0, \Gamma; F)) \to H^r(\Gamma_0, F)$$

is Hecke equivariant.

Proof. The proof of (a) is a simple calculation with cocycles.

Now consider (b). The right semigroup action of S on $\mathrm{Ind}(\Gamma_0, \Gamma; F)$ is defined as follows: for $\sigma \in S$, $f \in \mathrm{Ind}(\Gamma_0, \Gamma; F)$, and $x \in \Gamma$ set

$$(f|\sigma)(x) = \begin{cases} 0 & \text{if } x\sigma^{-1} \notin S_0^{-1}\Gamma, \\ f(y)|\tau & \text{if } x\sigma^{-1} = \tau^{-1}y \text{ with } \tau \in S_0,\ y \in \Gamma. \end{cases}$$

The Shapiro isomorphism is the composition of restriction to Γ_0 with the map on cohomology induced by the S_0-morphism

$$\mathrm{Ind}(\Gamma_0, \Gamma; F) \to F$$
$$f \mapsto f(1).$$

Clearly, if $\sigma \in S \setminus \Gamma S_0$ then $(f|\sigma)(1) = 0$. Thus the Hecke equivariance of $\mathscr{S}$ is a consequence of (a). ∎

For a nonnegative integer g let V_g be the right representation of GL_2 on $\mathrm{Sym}^g(\mathbf{A}^2)$. Thus for a commutative ring R, $V_g(R)$ is the space of homogeneous degree g polynomials in two variables over R. The action of an element $\sigma \in GL_2(R)$ on a polynomial $P \in V_g(R)$ is given by

$$(P|\sigma)(X, Y) = P((X, Y)\sigma^{-1}).$$

THEOREM 2.3 (Shimura, [S2] Chapter 8). *Let* Γ *be a congruence subgroup of* $SL_2(\mathbf{Z})$ *which is preserved by the involution* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, *and let* $g = k - 2$ *be nonnegative. Then there is an isomorphism of* $\mathscr{H}(\Gamma, GL_2(\mathbf{Q}))$*-modules*

$$H^1(\Gamma; V_g(\mathbf{C})) \cong \mathscr{S}_k(\Gamma) \oplus \mathscr{S}_k^{\mathrm{anti}}(\Gamma) \oplus \mathscr{E}_k(\Gamma)$$

where $\mathscr{S}_k^{\mathrm{anti}}(\Gamma)$ *is the space of antiholomorphic weight* k *cusp forms over* Γ. ∎

Now fix an integer $N > 0$ and a prime ℓ not dividing N. Let

$$\Gamma = \Gamma_1(N),$$

$$S = \left\{\sigma \in GL_2(\mathbf{Q}) | \sigma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}),\ c \equiv 0 (\operatorname{mod} N)\right\},$$

$$\Gamma_0 = \Gamma_0(\ell) \cap \Gamma_1(N),$$

$$S_0 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in S | c \equiv 0,\ d \not\equiv 0 \ (\operatorname{mod} \ell)\right\},$$

$$\Gamma_1 = \Gamma_1(N\ell),$$

$$S_1 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in S | c \equiv 0,\ d \equiv 1 \ (\operatorname{mod} \ell)\right\},$$

It is easy to see that the Hecke pairs $(\Gamma_1, S_1) \subseteq (\Gamma_0, S_0) \subseteq (\Gamma, S)$ are pairwise weakly compatible, and that the natural map ι: $\mathscr{H}(\Gamma, S) \to \mathscr{H}(\Gamma_1, S_1)$ is an isomorphism. These algebras are seen to be commutative as in [S2] Chapter 3. The group Γ_0 normalizes these Hecke pairs and the actions of Γ_0 induced by conjugation on the Hecke algebras are trivial.

For each integer n let $\sigma_n = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in S$ and for each $r \in (\mathbf{Z}/N\mathbf{Z})^*$ fix $\gamma_r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \cap \Gamma_1(\ell)$ with $d \equiv r \pmod N$. Then with the standard action of $\mathscr{H}(\Gamma, S)$ on $\mathscr{M}_k(\Gamma)$ ([S2], §8.3) we have for $f \in \mathscr{M}_k(\Gamma)$

$$f|[\Gamma\sigma_n\Gamma] = f|T_n$$

$$f|[\Gamma\gamma_r\Gamma] = f|\langle r\rangle.$$

The corresponding statement with (Γ, S) replaced by (Γ_1, S_1) is also true. Hence the classical Hecke algebra $\mathscr{H} = \mathbf{Z}[T_n, \langle r\rangle]$ acts on $\mathscr{M}_k(\Gamma_1)$ via the maps

$$\mathscr{H} \to \mathscr{H}(\Gamma, S) \xrightarrow{\iota} \mathscr{H}(\Gamma_1, S_1)$$

$$T_n \mapsto [\Gamma\sigma_n\Gamma] \mapsto [\Gamma_1\sigma_n\Gamma_1]$$

$$\langle r\rangle \mapsto [\Gamma\gamma_r\Gamma] \mapsto [\Gamma_1\gamma_r\Gamma_1].$$

If E is an arbitrary right S-module (respectively S_1-module) we let $\mathscr{H}$ act on $H^*(\Gamma; E)$ (respectively $H^*(\Gamma_1; E)$) via these maps. Moreover the group $\Gamma_0/\Gamma_1 \cong (\mathbf{Z}/\ell\mathbf{Z})^*$ acts on $H^*(\Gamma_1; E)$ as a group of Nebentype operators which commute with the action of $\mathscr{H}$.

For an $\mathcal{O}\Gamma_0$-module H and an integer g let $H^{(\omega^g)} = \{h \in H | h|\gamma = \omega^g(d)h$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0\}$.

DEFINITION 2.4.
(a) *For $M > 0$ let*

$$\tilde{\mathcal{M}}_k(\Gamma_1(M)) = \bigoplus_{\varepsilon} \mathcal{M}_k(\Gamma_1(M), \varepsilon; \overline{\mathbf{F}}_\ell)$$

where ε runs over all characters ε: $(\mathbf{Z}/M\mathbf{Z})^* \to \mathcal{O}^*$.

(b) *For $k \geqslant 2$, let $\Omega_k(\Gamma)$ (respectively $\tilde{\Omega}_k(\Gamma)$) be the set of systems of eigenvalues Φ: $\mathcal{H} \to \mathcal{O}$ (respectively Φ: $\mathcal{H} \to \overline{\mathbf{F}}_\ell$) occurring in $\mathcal{M}_k(\Gamma)$ (respectively $\tilde{\mathcal{M}}_k(\Gamma)$).*

(c) *For $g \geqslant 0$ let $\Omega_2(\Gamma_1, \omega^g)$ (respectively $\tilde{\Omega}_2(\Gamma_1, \omega^g)$) be the set of systems of eigenvalues Φ: $\mathcal{H} \to \mathcal{O}$ (respectively Φ: $\mathcal{H} \to \overline{\mathbf{F}}_\ell$) occurring in $\mathcal{M}_2(\Gamma_1)^{(\omega^g)}$ (respectively $\tilde{\mathcal{M}}_2(\Gamma_1)^{(\omega^g)}$).*

By [A-S] Propositions 1.2.2 and 1.2.3 we have surjective reduction maps

$$\Omega_k(\Gamma) \to \tilde{\Omega}_k(\Gamma)$$

$$\Omega_2(\Gamma_1, \omega^g) \to \tilde{\Omega}_2(\Gamma_1, \omega^g)$$

$$\Phi \mapsto \overline{\Phi}.$$

The map $\Gamma \to \mathbf{F}_\ell^2 \setminus \{0\}$ given by $\gamma \mapsto (0,1)\gamma$ gives a bijection $\Gamma_1 \setminus \Gamma \to \mathbf{F}_\ell^2 \setminus \{0\}$ which commutes with the right action of Γ. Thus we may identify the right S-module $\mathrm{Ind}(\Gamma_1, \Gamma; \mathbf{F}_\ell)$ with the module I of $\mathbf{F}_\ell$-valued functions on $\mathbf{F}_\ell^2$ which vanish at the origin. The action of S on I is given by

$$(f|\sigma)(a, b) = f((a, b)\sigma^{-1})$$

for $\sigma \in S$, $f \in I$, $(a, b) \in \mathbf{F}_\ell^2$.

For each integer g let I_g be the S-submodule of I consisting of homogeneous functions of degree g. Then I_g depends only on g modulo $\ell - 1$ and we have a decomposition

$$I \cong \bigoplus_{g=0}^{\ell-2} I_g.$$

PROPOSITION 2.5. *Let $g \geqslant 0$ and $k = g + 2$. Let Φ: $\mathcal{H} \to \overline{\mathbf{F}}_\ell$ be a system of eigenvalues.*

(a) *Φ occurs in $H^1(\Gamma; V_g(\overline{\mathbf{F}}_\ell))$ iff either* (i) *$\Phi \in \tilde{\Omega}_k(\Gamma)$ or* (ii) *$g > 0$ and Φ occurs in $H^0(\Gamma; V_g(\overline{\mathbf{F}}_\ell))$.*

(b) *Φ occurs in $H^1(\Gamma; I_g \otimes \overline{\mathbf{F}}_\ell)$ iff $\Phi \in \tilde{\Omega}_2(\Gamma_1, \omega^g)$.*

Proof of 2.5(*a*). Let K be a number field large enough to contain all eigenvalues of $\mathcal{H}$ acting on $\mathcal{M}_k(\Gamma)$. Let $R \subseteq K$ be the discrete valuation ring associated to the place below $\underline{\underline{\lambda}}$ and let $\lambda \in R$ be a generator of the maximal ideal. We may assume $R/\lambda \subseteq \overline{\mathbf{F}}_\ell$ contains the image of Φ.

In the diagram

$$\begin{array}{ccccc} & & & H^1(\Gamma; V_g(R/\lambda)) & \\ & & & \uparrow & \\ 0 \to H^1(\Gamma; V_g(R))_{\text{tor}} & \to & & H^1(\Gamma; V_g(R)) & \xrightarrow{i} H^1(\Gamma; V_g(\mathbf{C})) \end{array}$$

the vertical arrow is surjective (because $\ell > 3$, cf. proof of Theorem 1.3.5 of [A-S]) and the horizontal sequence is exact. Since the image of i spans $H^1(\Gamma; V_g(\mathbf{C}))$ we see that Φ occurs in $H^1(\Gamma; V_g(R/\lambda))$ iff there is a $\Psi\colon \mathscr{H} \to R$ occurring in $H^1(\Gamma; V_g(R))_{\text{tor}} \oplus H^1(\Gamma; V_g(\mathbf{C}))$ such that $\overline{\Psi} = \Phi$. By 2.3 and 1.2 we see that there is a $\Psi\colon \mathscr{H} \to R$ occurring in $H^1(\Gamma; V_g(\mathbf{C}))$ with $\overline{\Psi} = \Phi$ iff $\Phi \in \tilde{\Omega}_k(\Gamma)$. On the other hand such a Ψ occurs in $H^1(\Gamma; V_g(R))_{\text{tor}}$ iff Φ occurs in the kernel $H^1(\Gamma; V_g(R))_\lambda$ of multiplication by λ. To complete the proof of (a) we will show

$$H^1(\Gamma; V_g(R))_\lambda \cong \begin{cases} 0 & \text{if } g = 0 \\ H^0(\Gamma; V_g(R/\lambda)) & \text{if } g > 0. \end{cases}$$

The short exact sequence $0 \to V_g(R) \xrightarrow{\lambda} V_g(R) \to V_g(R/\lambda) \to 0$ gives rise to an exact sequence in cohomology

$$H^0(\Gamma; V_g(R)) \to H^0(\Gamma; V_g(R/\lambda)) \to H^1(\Gamma; V_g(R))_\lambda \to 0.$$

If $g > 0$, $V_g(\mathbf{C})$ is a nontrivial irreducible Γ-module and hence has no nonzero Γ-invariants. Thus $H^0(\Gamma; V_g(R)) = 0$ and $H^0(\Gamma; V_g(R/\lambda)) \cong H^1(\Gamma; V_g(R))_\lambda$ as desired. The case $g = 0$ is trivial. ■

LEMMA 2.6. *There is an isomorphism of $\mathscr{H}$-modules*

$$H^1(\Gamma; I_g) \cong H^1(\Gamma_1; \mathbf{F}_\ell)^{(\omega^g)}.$$

Proof. Let $(\mathbf{F}_\ell)_{\omega^g}$ be the rank one $\mathbf{F}_\ell$-module on which Γ_0 acts via ω^g. Then the induced module $\text{Ind}(\Gamma_0, \Gamma; (\mathbf{F}_\ell)_{\omega^g})$ is isomorphic as an S-module to I_g. By Lemma 2.2 the Shapiro isomorphism $H^1(\Gamma; I_g) \cong H^1(\Gamma_0; (\mathbf{F}_\ell)_{\omega^g})$ commutes with $\mathscr{H}$. This latter group is isomorphic to $H^1(\Gamma_1, \mathbf{F}_\ell)^{(\omega^g)}$ by [A-S] Lemma 1.1.5. ■

Proof of 2.5(*b*). As in 2.5(a) we know $\Phi \in \tilde{\Omega}_2(\Gamma_1, \omega^g)$ iff Φ occurs in $H^1(\Gamma_1; \bar{\mathbf{F}}_\ell)^{(\omega^g)}$. By the lemma this is equivalent to the occurrence of Φ in $H^1(\Gamma; I_g \otimes \bar{\mathbf{F}}_\ell)$. ■

COROLLARY 2.7. *The set* $\bigcup_{k \geqslant 2} \tilde{\Omega}_k(\Gamma)$ *is finite.*

Proof. This is an immediate consequence of 2.5(a) and [A-S] Theorem 2.2 ■

This result generalizes the first part of the theorem of Serre and Tate (1.3(a)).

§3. Systems of Hecke eigenvalues. To simplify the notation we will write $\bar{V}_g$ for $V_g(\bar{\mathbf{F}}_\ell)$.

LEMMA 3.1. *For* $0 \leqslant g < \ell$ *there are* Γ*-invariant perfect pairings.*
(1) $\bar{V}_g \times \bar{V}_g \to \bar{\mathbf{F}}_\ell$
(2) $I_g \times I_{-g} \to \bar{\mathbf{F}}_\ell$.

Proof. We leave it to the reader to verify that the following pairings are nondegenerate and Γ-invariant.
(1) For $P(X, Y) = \sum_{\nu=0}^{g} a_\nu X^{g-\nu} Y^\nu$, and $Q(X, Y) = \sum_{\nu=0}^{g} b_\nu X^{g-\nu} Y^\nu$

$$\langle P, Q \rangle_V = \sum_{\nu=0}^{g} \binom{g}{\nu}^{-1} (-1)^\nu a_\nu b_{g-\nu}.$$

This pairing is determined by the formula

$$\left\langle (aX + cY)^g, (bX + dY)^g \right\rangle_V = \left(\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^g.$$

(2) For $f_1 \in I_g$, $f_2 \in I_{-g}$

$$\langle f_1, f_2 \rangle_I = \sum_{x \in \mathbf{F}_\ell^2} f_1(x) f_2(x).$$ ■

Let α_g: $\bar{V}_g \to I_g$ be the Γ-morphism which sends a polynomial to its associated function on $\mathbf{F}_\ell^2$. For $g < \ell$ let β_g: $I_g \to \bar{V}_{\ell-1-g}$ be the dual morphism to $\alpha_{\ell-l-g}$. Then β_g is given explicitly by

$$\beta_g(f) = \sum_{(r,s) \in \mathbf{F}_\ell^2} f(r, s)(sX - rY)^{\ell-1-g}.$$

Remark. The maps α_g and β_g are special cases of maps occurring in the main diagrams of [A-S]. For example, if we let ϕ: $\bar{V}_g \to \bar{\mathbf{F}}_\ell$ be defined by $\phi(P) = P(0, 1)$, then α_g is the map $\alpha(\phi)$ of [A-S] (1.3.2). If ψ: $(\bar{\mathbf{F}}_\ell)_{\omega^g} \to \bar{V}_{\ell-1-g}$, is given by $\psi(r) = rX^{\ell-1-g}$ then β_g is the map $\beta(\psi)$ of [A-S] (1.3.3). ■

For an integer $\nu \geqslant 0$ we define the "ν-fold twist" of an $\mathbf{F}_\ell S$-module (M, π_M) to be the module $(M(\nu), \pi_{M(\nu)})$ whose underlying space is $M(\nu) = M$ and whose S-action is given by

$$\pi_{M(\nu)}(\sigma)m = \det(\sigma^{-1})^\nu \pi_M(\sigma)m$$

for $\sigma \in S$ and $m \in M$.

Let $\theta \in \overline{V}_{\ell+1}$ be the polynomial

$$\theta(X, Y) = X^\ell Y - XY^\ell = XY \prod_{a=1}^{\ell-1} (X - aY).$$

For $\sigma \in S$ a simple calculation shows $\theta\sigma = \det(\sigma^{-1})\theta$. Hence multiplication by θ induces an S-morphism

$$\overline{V}_g(1) \xrightarrow{\theta} \overline{V}_{g+\ell+1}$$

for every $g \geqslant 0$.

LEMMA 3.2.
(a) *If* $0 < g < \ell$ *the sequence*

$$0 \to \overline{V}_g \xrightarrow{\alpha_g} I_g \xrightarrow{\beta_g} \overline{V}_{\ell-1-g}(g) \to 0$$

is an exact sequence of S-modules.
(b) *The map*

$$\overline{V}_\ell \xrightarrow{\alpha_\ell} I_\ell$$

is an isomorphism of S-modules.
(c) *For* $g > \ell$ *the sequence*

$$0 \to \overline{V}_{g-\ell-1}(1) \xrightarrow{\theta} \overline{V}_g \xrightarrow{\alpha_g} I_g \to 0$$

is an exact sequence of S-modules.

Proof. For arbitrary $g > 0$ it is easy to verify that α_g, β_g are S-morphisms. Moreover, a polynomial $P \in \overline{V}_g$ $(g \geqslant 0)$ is in the kernel of α_g iff P vanishes at every point of $\mathbf{P}^1(\mathbf{F}_\ell)$ iff P is divisible by θ. Thus in each of the three cases (a), (b), (c) $\ker(\alpha_g)$ is as claimed.

Suppose $0 < g < \ell$. The last paragraph shows $\alpha_{\ell-1-g}$ is injective, and therefore its dual β_g is surjective.

Now let $Q \in \bar{V}_g$. We will show $\beta_g \circ \alpha_g(Q) = 0$. For $P \in \bar{V}_{\ell-1-g}$ we have

$$\left\langle P, \beta_g \circ \alpha_g(Q) \right\rangle_V = \left\langle \alpha_{\ell-1-g}(P), \alpha_g(Q) \right\rangle_I = \sum_{x \in \mathbf{F}_\ell^2} P(x)Q(x) = 0$$

since $\Sigma R(x) = 0$ for any homogeneous polynomial R of degree $\ell - 1$. Thus Image(α_g) $\subseteq$ ker(β_g). Counting dimensions shows that this inclusion is in fact an equality, proving (a).

Next suppose $g \geqslant \ell$. To show α_g is surjective it suffices to show that for each $[a, b]$ in $\mathbf{P}^1(\mathbf{F}_\ell)$ there is a $P \in \bar{V}_g$ such that

(i) $P([a, b]) \neq 0$

(ii) $P([r, s]) = 0$ for $[a, b] \neq [r, s]$.

Since Γ acts transitively on $\mathbf{P}^1(\mathbf{F}_\ell)$ we may take $[a, b] = [1, 0]$. In this case we let $P(X, Y) = X^{g-\ell+1}\prod_{a=1}^{\ell-1}(X - aY)$. This completes the proof of the lemma. ∎

For each integer $\nu \geqslant 0$ and $\mathbf{F}_\ell S$-module E we have an isomorphism of abelian groups $H^*(\Gamma; E) \cong H^*(\Gamma; E(\nu))$. The action of $\mathscr{H}$ on these groups is related by the formula

$$t_n^{(\nu)} = n^\nu t_n$$

where t_n (respectively $t_n^{(\nu)}$) is the endomorphism of $H^*(\Gamma; E)$ (respectively $H^*(\Gamma; E(\nu))$) induced by the Hecke operator T_n.

We define an action of $\mathscr{H}$ on the ν-fold twist $\mathbf{F}_\ell(\nu)$ of the trivial S-module $\mathbf{F}_\ell$ by

$$\mathscr{H} \times \mathbf{F}_\ell(\nu) \to \mathbf{F}_\ell(\nu)$$

$$(T_n, r) \mapsto n^\nu \deg(T_n) r$$

where

$$\deg(T_n) = \sum_{\substack{d \mid n \\ (N, d) = 1}} \frac{n}{d}$$

is the number of right Γ-cosets in the double coset $\Gamma\sigma_n\Gamma$. Then we have an isomorphism of $\mathscr{H}$-modules $H^0(\Gamma; \mathbf{F}_\ell(\nu)) \cong \mathbf{F}_\ell(\nu)$.

LEMMA 3.3. *We have the following isomorphisms of $\mathscr{H}$-modules.*

(a) $H^0(\Gamma; I_g) \cong \begin{cases} \mathbf{F}_\ell(\ell - 1) & \text{if } g \equiv 0 \pmod{\ell - 1} \\ 0 & \text{otherwise.} \end{cases}$

(b) $H^0(\Gamma; \bar{V}_g) \cong \oplus_\nu \mathbf{F}_\ell(\nu)$ *where ν ranges over all nonnegative integers such that*

(i) $(\ell + 1)\nu \leqslant g$, *and*

(ii) $(\ell + 1)\nu \equiv g \pmod{\ell^2 - \ell}$.

Proof. (a) Since Γ acts transitively on $\mathbf{F}_\ell^2 \setminus \{0\}$ the Γ-invariants in I are the functions which are constant on $\mathbf{F}_\ell^2 \setminus \{0\}$. Thus $H^0(\Gamma; I_0) \cong \mathbf{F}_\ell$ as abelian groups. The action of $\mathscr{H}$ is easily verified to be as claimed.

(b) L. E. Dickson [Di] has shown that the ring of Γ-invariants in Sym*(V) is generated by

$$\theta = X^\ell Y - XY^\ell$$

and

$$\Psi = (X^{\ell^2}Y - XY^{\ell^2})/\theta = \sum_{i=0}^{\ell} (X^{\ell-i}Y^i)^{\ell-1}$$

We have already observed $\sigma\theta = \det(\sigma)\theta$ for $\sigma \in S^{-1}$. One also easily verifies $\sigma\Psi = \Psi$. Thus

$$H^0(\Gamma; \overline{V}_g) = \bigoplus_{\nu, \mu} \mathbf{F}_\ell \theta^\nu \Psi^\mu \cong \bigoplus_{\nu} \mathbf{F}_\ell(\nu)$$

where the first sum is over ν, μ satisfying $\nu(\ell + 1) + \mu(\ell^2 - \ell) = g$ and the last sum is over ν satisfying (i), (ii) of the lemma. ■

The long exact cohomology sequences obtained from Lemma 3.2 together with the last lemma and Lemma 2.6 yield the following theorem.

THEOREM 3.4.
(a) *If* $0 < g < \ell$ *there is an exact sequence of* $\mathscr{H}$*-modules*

$$0 \to H^1(\Gamma; \overline{V}_g) \to H^1(\Gamma_1; \mathbf{F}_\ell)^{(\omega^g)} \to H^1(\Gamma; \overline{V}_{\ell-1-g}(g)) \to 0.$$

(b) *There is an isomorphism of* $\mathscr{H}$*-modules*

$$H^1(\Gamma; \overline{V}_\ell) \cong H^1(\Gamma_1; \mathbf{F}_\ell)^{(\omega)}.$$

(c) *For* $g > \ell$ *there is an exact sequence of* $\mathscr{H}$*-modules*

$$H^0(\Gamma; I_g) \to H^1(\Gamma; \overline{V}_{g-\ell-1}(1)) \xrightarrow{\theta} H^1(\Gamma; \overline{V}_g) \to H^1(\Gamma_1; \mathbf{F}_\ell)^{(\omega^g)} \to 0. \quad ■$$

It is now an easy matter using 2.6 to derive conclusions about systems of Hecke eigenvalues occurring in the spaces $\tilde{\mathscr{M}}_k(\Gamma)$ and $\tilde{\mathscr{M}}_2(\Gamma_1)^{(\omega^g)}$. If $\tilde{\Omega}$ is a set of systems of eigenvalues Φ: $\mathscr{H} \to \overline{\mathbf{F}}_\ell$ and $\nu \geqslant 0$ let

$$\tilde{\Omega}^{(\nu)} = \{\Phi^{(\nu)}: \Phi \in \tilde{\Omega}\}.$$

THEOREM 3.5. *Let* $g > 0$ *and* $k = g + 2$. *Recall that* $\tilde{\Omega}_k(\Gamma, \omega)^{(a)}$ *stands for the set of systems of Hecke eigenvalues occurring in the space of modular forms in* *characteristic ℓ for the group Γ with weight k, character ω and twisted by the ath power of the determinant. If ω (resp. a) is omitted from the notation, the trivial character (resp. zeroth power) is implied. (Cf. Definition 2.4).*

(a) *For* $2 < k < \ell + 2$, $\tilde{\Omega}_2(\Gamma_1, \omega^g) = \tilde{\Omega}_k(\Gamma) \cup \tilde{\Omega}_{\ell+3-k}(\Gamma)^{(g)}$.

(b) $\tilde{\Omega}_2(\Gamma_1, \omega) = \tilde{\Omega}_{\ell+2}(\Gamma)$.

(c) *If* $k > \ell + 2$, *and* $\Phi \in \tilde{\Omega}_2(\Gamma_1, \omega^g)$, *then either* $\Phi \in \tilde{\Omega}_k(\Gamma)$ *or* Φ *occurs in* $H^0(\Gamma; V_g(\bar{\mathbf{F}}_\ell))$ (*see* 3.3).

(d) *For* $k > 2$, $\tilde{\Omega}_k(\Gamma) \subseteq \bigcup_{0 \leqslant \nu \leqslant g/(\ell+1)} \tilde{\Omega}_2(\Gamma_1, \omega^{g-2\nu})^{(\nu)}$.

Proof. Statements (a), (b), and (c) follow from Theorem 3.4(a), (b), and (c) using Proposition 2.5. Now consider (d). By Proposition 2.5(a) it suffices to show that if Φ occurs in $H^1(\Gamma; V_g(\bar{\mathbf{F}}_\ell))$ then Φ is a member of the right hand side of (d). We will prove this by induction on g. If $g \leqslant \ell$ it follows from (a) and (b). We therefore suppose $g > \ell$ and that the statement holds for all $g' < g$. By Theorem 3.4(c) Φ occurs either in $H^1(\Gamma; V_{g-\ell-1}(\bar{\mathbf{F}}_\ell))$ or in $H^1(\Gamma_1; \bar{\mathbf{F}}_\ell)^{(\omega^g)}$. In the first case the induction hypothesis gives the desired result. In the second case Lemma 2.6 together with Proposition 2.5 completes the proof. ■

In particular this shows that, up to twisting, the set of weight 2 systems for Γ_1 is the same as the set of weight > 2 systems for Γ. Moreover from (a) and (d) we obtain the following strengthening of Jochnowitz's theorem (Theorem 1.3) to arbitrary level.

COROLLARY 3.6. *For every* $k > \ell + 2$

$$\tilde{\Omega}_k(\Gamma) \subseteq \bigcup_{\nu=0}^{\ell-1} \bigcup_{2 \leqslant r \leqslant \ell+1} \tilde{\Omega}_r(\Gamma)^{(\nu)}.$$ ■

§4. Special values of L-functions. In this section we examine the basic properties of special values of L-functions, $\Lambda(\xi, \chi)$, attached to compactly supported cohomology classes $\xi \in H_c^1(\Gamma; E)$ and primitive Dirichlet characters χ. We show how this theory can be used to prove congruences modulo $\underline{\lambda}$ between the algebraic parts of special values of L-functions of higher weight cusp forms over $SL(2, \mathbf{Z})$ and those of weight two cusp forms over $\Gamma_1(\ell)$.

Let $\mathscr{D} = \operatorname{div}(\mathbf{P}^1(\mathbf{Q}))$ be the group of divisors supported on the rational cusps, $\mathbf{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{i\infty\}$, of the upper half plane, and let $\mathscr{D}_0 \subset \mathscr{D}$ be the subgroup of divisors of degree zero. The natural action of $GL(2, \mathbf{Q})$ on $\mathbf{P}^1(\mathbf{Q})$ induces an action on $\mathscr{D}$ which preserves $\mathscr{D}_0$.

Let E be a $\mathbf{Z}\Gamma$-module where $\Gamma = \Gamma_1(N)$.

DEFINITION 4.1. *We will refer to* $\operatorname{Hom}_\Gamma(\mathscr{D}_0; E)$ *as the group of E-valued modular symbols over* Γ.

For the remainder of this section R will denote a commutative ring with identity in which the order of every torsion element of Γ is invertible. If E is an

$R\Gamma$-module then the next proposition shows

$$\mathrm{Hom}_\Gamma(\mathscr{D}_0; E) \cong H_c^1(\Gamma; E).$$

More precisely the next proposition compares two long exact cohomology sequences. One of these is the cohomology sequence of the short exact sequence of $R\Gamma$-modules

$$0 \to E \to \mathrm{Hom}_Z(\mathscr{D}; E) \to \mathrm{Hom}_Z(\mathscr{D}_0; E) \to 0.$$

The other is the long exact cohomology sequence of the pair $(\Gamma \backslash \tilde{\mathbf{H}}; \partial(\Gamma \backslash \tilde{\mathbf{H}}))$ where $\tilde{\mathbf{H}}$ is the Borel-Serre completion of the upper half plane, $\mathbf{H}$.

PROPOSITION 4.2. *Let E be an $R\Gamma$-module and $\tilde{E}$ be the associated local coefficients system on $\Gamma \backslash \tilde{\mathbf{H}}$. For each integer $i \geqslant 0$ we have the following commutative diagram*:

$$\begin{array}{ccccccc}
\to H^{i-1}(\Gamma; \mathrm{Hom}(\mathscr{D}_0; E)) & \to & H^i(\Gamma; E) & \to & H^i(\Gamma; \mathrm{Hom}(\mathscr{D}; E)) & \to \\
\downarrow & & \downarrow & & \downarrow & \\
\to H_c^i(\Gamma \backslash \mathbf{H}; \tilde{E}) & \to & H^i(\Gamma \backslash \mathbf{H}; \tilde{E}) & \to & H^i(\partial(\Gamma \backslash \tilde{\mathbf{H}}); \tilde{E}) & \to
\end{array}$$

where the vertical arrows are isomorphisms. If, moreover, E is an S-module then all maps in this diagram are $\mathscr{H}$-morphisms (S and $\mathscr{H}$ as in section 2).

Proof. Since the boundary components of $\tilde{\mathbf{H}}$ are in one-one correspondence with the points of $\mathbf{P}^1(\mathbf{Q})$ we have $H_0(\partial(\tilde{\mathbf{H}}); \mathbf{Z}) \cong \mathscr{D}$. This isomorphism respects the $GL(2, \mathbf{Q})$-action. The boundary map $H_1(\tilde{\mathbf{H}}, \partial(\tilde{\mathbf{H}}); \mathbf{Z}) \to H_0(\partial(\tilde{\mathbf{H}}); \mathbf{Z}) \cong \mathscr{D}$ gives rise to a $GL(2, \mathbf{Q})$-isomorphism $H_1(\tilde{\mathbf{H}}, \partial(\tilde{\mathbf{H}}); \mathbf{Z}) \cong \mathscr{D}_0$. We therefore have the following commutative $R\Gamma$-diagram:

$$\begin{array}{ccccccccc}
0 \to & H^0(\tilde{\mathbf{H}}; \tilde{E}) & \to & H^0(\partial(\tilde{\mathbf{H}}); \tilde{E}) & \to & H^1(\tilde{\mathbf{H}}, \partial(\tilde{\mathbf{H}}); \tilde{E}) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & E & \to & \mathrm{Hom}(\mathscr{D}; E) & \to & \mathrm{Hom}(\mathscr{D}_0; E) & \to 0
\end{array}$$

The diagram of the proposition follows by passing to the long exact cohomology sequences of the above short exact sequences and using the isomorphisms

$$H^i(\Gamma; H^j(\tilde{\mathbf{H}}; E)) \cong H^{i+j}(\Gamma \backslash \mathbf{H}; \tilde{E})$$

$$H^i(\Gamma; H^j(\partial(\tilde{\mathbf{H}}); E)) \cong H^{i+j}(\partial(\Gamma \backslash \tilde{\mathbf{H}}); \tilde{E})$$

$$H^i(\Gamma; H^j(\tilde{\mathbf{H}}, \partial(\tilde{\mathbf{H}}); E)) \cong H_c^{i+j}(\Gamma \backslash \mathbf{H}; \tilde{E}).$$

If E is also an S-module then all maps of the above diagram are S-morphisms, and thus the diagram of the proposition is $\mathscr{H}$-equivariant. ■

Let $U \subseteq GL(2)$ be the standard unipotent subgroup, $\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$. Let χ be a primitive Dirichlet character of conductor m, and let R be a $\mathbf{Z}[\chi]$-algebra.

DEFINITION 4.3. *Let E be an $R\Gamma U(\mathbf{Z}[1/m])$-module and $\xi \in \mathrm{Hom}_\Gamma(\mathscr{D}_0; E)$ be an E-valued modular symbol over Γ. The special value of the L-function of ξ twisted by χ is*

$$\Lambda(\xi, \chi) = \sum_{\substack{r=0 \\ (r,m)=1}}^{m-1} \bar{\chi}(r) \begin{pmatrix} 1 & -r/m \\ 0 & 1 \end{pmatrix} \xi\left(\left\{\frac{r}{m}\right\} - \{i\infty\}\right) \in E.$$

If β: $E \to E'$ is a morphism of $R\Gamma U(\mathbf{Z}[1/m])$-modules then we clearly have

$$\beta(\Lambda(\xi, \chi)) = \Lambda(\beta_* \xi, \chi).$$

Now let $k = g + 2 \geqslant 2$ and $f \in \mathscr{S}_k(\Gamma)$ be a weight k cusp form over Γ. Let

$$\omega(f) = f(z)(zX + Y)^g \, dz.$$

This is a $V_g(\mathbf{C})$-valued holomorphic differential 1-form on the upper half plane. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Q})$ with $\det(\gamma) > 0$ we let $(f|\gamma)(z) = \det(\gamma)(cz + d)^{-k} f(\gamma z)$ and write γ^* for pullback of differential forms under the map γ: $\mathbf{H} \to \mathbf{H}$. Then a straightforward calculation shows

$$\gamma^* \omega(f) = \gamma \circ \omega(f|\gamma).$$

Integration of $\omega(f)$ gives rise to a compactly supported cohomology class on $\Gamma \setminus \mathbf{H}$ whose associated modular symbol $\xi_f \in \mathrm{Hom}_\Gamma(\mathscr{D}_0; V_g(\mathbf{C}))$, is given by

$$\xi_f(\{x\} - \{y\}) = \int_y^x \omega(f) \in V_g(\mathbf{C}),$$

where the integral is over the geodesic in $\mathbf{H}$ joining y to x.

Suppose the Fourier expansion of f is given by

$$f(z) = \sum_{n \geqslant 1} a_n e^{2\pi i n z}.$$

The L-function of f twisted by a primitive Dirichlet character χ is defined by the Dirichlet series

$$L(f, \chi, s) = \sum_{n \geqslant 1} a_n \chi(n) n^{-s}$$

for $\mathrm{Re}(s) > (k+1)/2$ and extends to an entire function on the complex plane. For an integer $\nu (0 \leqslant \nu \leqslant g)$ let

$$\Lambda(f, \chi, \nu+1) = (-1)^{\nu} \nu! \frac{\tau(\bar{\chi}) L(f, \chi, \nu+1)}{(2\pi i)^{\nu+1}},$$

where

$$\tau(\bar{\chi}) = \sum_{\substack{r=0 \\ (r,m)=1}}^{m-1} \bar{\chi}(r) e^{2\pi i r/m}$$

is the Gauss sum of $\bar{\chi}$. The following proposition is well known (cf. [Ra]).

PROPOSITION 4.4. *If χ is a primitive Dirichlet character then*

$$\Lambda(\xi_f, \chi) = \sum_{\nu=0}^{g} \binom{g}{\nu} \Lambda(f, \chi, \nu+1) X^{\nu} Y^{g-\nu}. \qquad \blacksquare$$

We now specialize to the case $N = 1$, so that $\Gamma = SL(2, \mathbf{Z})$. Let $\ell > 3$ be a prime and $\Gamma_1 = \Gamma_1(\ell)$. Let $f \in \mathscr{S}_k(\Gamma)$ be a nonzero weight k (necessarily $\geqslant 12$) cusp form over Γ and suppose f is a Hecke eigenform. Let K_f be the field generated by the Hecke eigenvalues. Let $R \subseteq K_f$ be the discrete valuation ring determined by a place above ℓ and let $\lambda \in R$ be a generator of the maximal ideal. As in section 1 let $\underline{\lambda} \subseteq \mathscr{O}$ be a prime ideal above λ in the ring $\mathscr{O}$ of all algebraic integers and fix an identification $\mathscr{O}/\underline{\lambda} \cong \overline{\mathbf{F}}_{\ell}$. In particular this fixes an inclusion $R/\lambda \to \overline{\mathbf{F}}_{\ell}$.

THEOREM 4.5. *Let $k < \ell + 2$ and let f be a weight k eigencuspform for $SL(2, \mathbf{Z})$ whose eigenvalues are not congruent modulo $\underline{\lambda}$ to those of the weight k Eisenstein series. Then there are complex numbers $\Omega_f^+, \Omega_f^- \in \mathbf{C}^*$ and an $\mathscr{H}$-eigenvector $\xi \in H_c^1(\Gamma_1; \overline{\mathbf{F}}_{\ell})$ such that*

(a) *The $\mathscr{H}$-eigenvalues of ξ are congruent modulo $\underline{\lambda}$ to those of f;*

(b) $\Lambda(f, \chi, 1)/\Omega_f^{\mathrm{sgn}(\chi)} \in R[\chi]$, *and* $\Lambda(f, \chi, 1)/\Omega_f^{\mathrm{sgn}(\chi)} \equiv \Lambda(\xi, \chi) \pmod{\underline{\lambda}}$ *for every primitive Dirichlet character χ whose conductor is prime to ℓ; and*

(c) *for each choice of* $\mathrm{sgn} = \pm 1$, $\Lambda(\xi, \chi)$ *is not identically zero as a function of χ with* $\chi(-1) = \mathrm{sgn}$.

Proof. Let $\xi_f \in \mathrm{Hom}_{\Gamma}(\mathscr{D}_0; V_g(\mathbf{C}))$ be the modular symbol associated to f. By a theorem of Manin [M1], there are complex numbers $\Omega_f^{\pm} \in \mathbf{C}^*$ and modular symbols $\xi_f^{\pm} \in \mathrm{Hom}_{\Gamma}(\mathscr{D}_0; V_g(R))$ such that

$$\xi_f = \Omega_f^+ \xi_f^+ + \Omega_f^- \xi_f^-.$$

Moreover $\xi_f^{\pm}$ satisfy the symmetry relations:

$$\xi_f^{\pm}(\{-x\} - \{-y\}) = \pm\xi_f(\{x\} - \{y\})$$

for $x, y \in \mathbf{P}^1(\mathbf{Q})$, where the $\pm$ signs agree on both sides of this equation. Since $\mathrm{Hom}_\Gamma(\mathscr{D}_0; V_g(R))$ is torsion free we may assume that after reducing $\xi_f^{\pm}$ modulo λ we obtain nonzero symbols $\bar{\xi}_f^{\pm} \in \mathrm{Hom}_\Gamma(\mathscr{D}_0; V_g(\overline{\mathbf{F}}_\ell))$.

Let ϕ: $V_g(\overline{\mathbf{F}}_\ell) \to \overline{\mathbf{F}}_\ell$ be defined by $\phi(P) = \phi(0,1)$. Then ϕ is an S_1-morphism and therefore induces an $\mathscr{H}$-morphism

$$\phi_*: \mathrm{Hom}_\Gamma\big(\mathscr{D}_0; V_g(\overline{\mathbf{F}}_\ell)\big) \to \mathrm{Hom}_{\Gamma_1}(\mathscr{D}_0; \overline{\mathbf{F}}_\ell).$$

Let $\xi^{\pm} = \phi_*(\bar{\xi}_f^{\pm}) \in \mathrm{Hom}_{\Gamma_1}(\mathscr{D}_0; \overline{\mathbf{F}}_\ell)$ and set $\xi = \xi^+ + \xi^-$. Then

$$\Lambda(f, \chi, 1)/\Omega_f^{\mathrm{sgn}(\chi)} = \big[\Lambda(\xi_f, \chi)(0,1)\big]/\Omega_f^{\mathrm{sgn}(\chi)}$$

by 4.4. But this last expression is $\Lambda(\xi_f^{\mathrm{sgn}(\chi)}, \chi)(0,1) \equiv \Lambda(\xi^{\mathrm{sgn}(\chi)}, \chi) \equiv \Lambda(\xi, \chi)$ (modulo $\underline{\lambda}$) proving (a) and (b).

To prove (c) consider the diagram

$$\begin{array}{ccccc} 0 \to \mathrm{Hom}_\Gamma\big(\mathscr{D}; V_g(\overline{\mathbf{F}}_\ell)\big) & \to & \mathrm{Hom}_\Gamma\big(\mathscr{D}_0; V_g(\overline{\mathbf{F}}_\ell)\big) & \to & H^1\big(\Gamma; V_g(\overline{\mathbf{F}}_\ell)\big) \\ & & \downarrow \phi_* & & \downarrow \\ & & \mathrm{Hom}_{\Gamma_1}(\mathscr{D}_0; \overline{\mathbf{F}}_\ell) & \to & H^1(\Gamma_1; \overline{\mathbf{F}}_\ell) \end{array}$$

where the rightmost vertical map is induced by the inclusion of Theorem 3.4(a).

Fix a choice of sign, sgn $= \pm 1$. It suffices to show that there is a character χ such that $\Lambda(\xi^{\mathrm{sgn}}, \chi) \neq 0$. By Theorem 2.1 of [St1] this will follow if we can show that the image of ξ^{sgn} in $H^1(\Gamma_1; \overline{\mathbf{F}}_\ell)$ is nonzero. Thus the proof of (c) will be complete if we show that $\bar{\xi}_f^{\mathrm{sgn}}$ is not in the image of $\mathrm{Hom}_\Gamma(\mathscr{D}; V_g(\overline{\mathbf{F}}_\ell))$.

Let $\Gamma_\infty = U(\mathbf{Z}) \subseteq \Gamma$ be the unipotent subgroup of Γ which fixes the ∞-cusp. Since $\ell > g$ the Γ_∞-invariants in $V_g(\overline{\mathbf{F}}_\ell)$ are given by

$$V_g(\overline{\mathbf{F}}_\ell)^{\Gamma_\infty} = \overline{\mathbf{F}}_\ell X^g.$$

Moreover, because Γ acts transitively on $\mathbf{P}^1(\mathbf{Q})$, the map

$$\mathrm{Hom}_\Gamma\big(\mathscr{D}; V_g(\overline{\mathbf{F}}_\ell)\big) \to V_g(\overline{\mathbf{F}}_\ell)^{\Gamma_\infty}$$

$$\eta \mapsto \eta(\{i\infty\})$$

is an isomorphism. Thus $\mathrm{Hom}_\Gamma(\mathscr{D}; V_g(\overline{\mathbf{F}}_\ell))$ is one-dimensional and is spanned by the unique element η satisfying $\eta(\{i\infty\}) = X^g$. A straightforward calculation shows that for each prime p,

$$\eta T_p = \left(p^{g+1} + 1\right)\eta.$$

But $p^{g+1} + 1$ is the eigenvalue of T_p acting on the weight k Eisenstein series. By hypothesis this is different (modulo $\underline{\lambda}$) from the eigenvalue of T_p acting on f for at least one prime p. Hence $\bar{\xi}^{\text{sgn}}$ is not a multiple of the image of η in $\text{Hom}_\Gamma(\mathscr{D}_0; V_g(\overline{\mathbf{F}}_\ell))$. This completes the proof of (c). ∎

Remark. The condition that f not satisfy an Eisenstein congruence modulo $\underline{\lambda}$ is known to be fulfilled for fixed f and ℓ sufficiently large. In fact Ribet, [R2] Lemma 4.6, shows that f is not congruent even to a twist of an Eisenstein series if $\ell > k + 1$ and ℓ does not divide the numerator of the kth Bernoulli number. ∎

COROLLARY 4.6. *Let $k < \ell + 2$. Let $f \in \mathscr{S}_k(SL_2(\mathbf{Z}))$ be an eigenform with system of eigenvalues Φ: $\mathscr{H} \to \mathscr{O}$, and assume $\Phi(T_p) \not\equiv (p^{k-1} + 1) \pmod{\underline{\lambda}}$ for at least one prime p. Suppose there is only one normalized eigenform $f_1 \in \mathscr{S}_2(\Gamma_1(\ell))$ whose system of eigenvalues Φ_1: $\mathscr{H} \to \mathscr{O}$ satisfies the congruence $\Phi \equiv \Phi_1$ (modulo $\underline{\lambda}$). (The existence of at least one such f_1 is guaranteed by Theorem* 3.5(a).) *Then there exist periods $\Omega_{f_1}^{\pm} \in \mathbf{C}^*$ such that*

$$\frac{\Lambda(f, \chi, 1)}{\Omega_f^{\text{sgn}(\chi)}} \equiv \frac{\Lambda(f_1, \chi, 1)}{\Omega_{f_1}^{\text{sgn}(\chi)}} \pmod{\underline{\lambda}}$$

for every primitive Dirichlet character χ of conductor prime to ℓ. As in (c) *of the theorem we may assume this congruence is nontrivial as a function of χ of either sign.*

Proof. Let $\Omega_1^{\pm} \in \mathbf{C}^*$ and $\xi_1^{\pm} \in \text{Hom}_{\Gamma_1}(\mathscr{D}_0; \mathscr{O})$ be chosen so that

$$\xi_{f_1} = \Omega_1^+ \xi_1^+ + \Omega_1^- \xi_1^-.$$

We will find $\alpha^{\pm} \in \mathscr{O}_{\underline{\lambda}}$ such that $\alpha^{\pm}\xi_1^{\pm} \equiv \xi^{\pm} \pmod{\underline{\lambda}}$ where $\xi \in \text{Hom}_{\Gamma_1}(\mathscr{D}_0; \overline{\mathbf{F}}_\ell)$ is the modular symbol provided by the theorem. We then obtain the corollary by setting $\Omega_{f_1}^{\pm} = (\alpha^{\pm})^{-1} \cdot \Omega_1^{\pm}$.

Let $\tilde{\xi} \in \text{Hom}_{\Gamma_1}(\mathscr{D}_0; \mathscr{O}_{\underline{\lambda}})$ be any lifting of ξ. Then for either choice of sign $\pm$ we can write $\tilde{\xi}^{\pm}$ in the form

$$\tilde{\xi}^{\pm} = \alpha^{\pm}\xi_1^{\pm} + \eta^{\pm}$$

where $\alpha^{\pm} \in \overline{\mathbf{Q}}$ and $\eta^{\pm} \in \text{Hom}_{\Gamma_1}(\mathscr{D}_0; \overline{\mathbf{Q}})$ is a sum of $\mathscr{H}$-eigensymbols other than $\xi_1^{\pm}$.

The uniqueness of f_1 assures the existence of an $h \in \mathscr{H} \otimes \mathscr{O}_{\underline{\lambda}}$ such that $\Phi_1(h) = 1$ and $\Psi(h) \equiv 0 \pmod{\underline{\lambda}}$ for any system of eigenvalues $\Psi \in$

$\Omega_2(\Gamma_1, \omega^{k-2})$ different from Φ_1. For n sufficiently large we then have $h^n \cdot \eta^{\pm} \in \mathrm{Hom}_{\Gamma_1}(\mathscr{D}_0; \mathscr{O}_{\underline{\lambda}})$ and $h^n\eta^{\pm} \equiv 0 \pmod{\underline{\lambda}}$. Thus $\alpha^{\pm} \cdot h^n\xi_1^{\pm} = h^n\tilde{\xi}^{\pm} - h^n\eta^{\pm} \in \mathrm{Hom}_{\Gamma_1}(\mathscr{D}_0; \mathscr{O}_{\underline{\lambda}})$ and $\alpha^{\pm}\xi_1^{\pm} \equiv h^n\tilde{\xi} \equiv \xi^{\pm} \pmod{\underline{\lambda}}$, which was to be proved. ■

For example let $f = \Delta$ be the unique normalized weight 12 cusp form over $SL(2, \mathbf{Z})$. Let $f_1 = q\prod_{n \geq 1}(1 - q^n)^2(1 - q^{11n})^2$ be the unique normalized weight two newform over $\Gamma_0(11)$. Then $f \equiv f_1 \pmod{11}$ and there is a congruence between the special values of L-functions of these forms modulo any prime $\underline{\lambda}$ dividing 11. If instead we take f_1 to be the unique normalized weight two newform over $\Gamma_1(13)$ with Nebentypus character ω^{10} then $f \equiv f_1$ (modulo $\underline{\lambda}$) for a prime $\underline{\lambda}$ above 13 and again we have a congruence between the special values of their L-functions.

Doi and Ohta [D-O] have calculated "congruence primes" for the space of weight two cusp forms over $\Gamma_0(\ell)$, $\ell \leq 223$. Their tables reveal that for $\ell \leq 233$ the "congruence primes" are all less than ℓ; in fact the product of the "congruence primes" is less than ℓ. This suggests that the uniqueness of the form f_1 in the corollary may be a common phenomenon, if not a general one.

References

[An] A. N. Andrianov, *The multiplicative arithmetic of Siegel modular forms*, Russian Math. Surveys **34** No. **1** (1979), 75–148.

[A-S] A. Ash and G. Stevens, *Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues*, Journal reine angew. Math. **365** (1986), 192–220.

[D] P. Deligne, *Formes modulaires et representations ℓ-adiques*, Sem. Bourbaki, Lect. Notes in Math. **179** (1971), 139–186.

[Di] L. E. Dickson, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12** (1911), 75–98.

[D-O] K. Doi and M. Ohta, "On some congruences between cusp forms on $\Gamma_0(N)$," In *Modular Functions of One Variable V*, Lect. Notes in Math. **601** (1977), 91–105.

[E] M. Eichler, *Eine Verallgemeinerung der abelschen Integrale*, Math. Zeit. **67** (1957), 267–298.

[Ha] K. Haberland, *Perioden von Modulformen einer Variabler und Gruppencohomologie, I, II, III*, Math. Nachr. **112** (1983), 245–315.

[H1] H. Hida, *On congruence divisors of cusp forms as factors of the special values of their zeta-functions*, Invent. Math. **64** (1981), 221–262.

[H2] ______, *Congruences of cusp forms and special values of their zeta functions*, Invent. Math. **63** (1981), 225–261.

[H3] ______, *Kummer's criterion for the special values of Hecke L-functions of imaginary quadratic fields and congruences among cusp forms*, Invent. Math. **66** (1982), 415–459.

[J] N. Jochnowitz, *Congruences between systems of eigenvalues of modular forms*, Trans. Amer. Math. Soc. **270** (1982), 269–285.

[K-P-S] M. Kuga, W. Parry and C.-H. Sah, "Group cohomology and Hecke operators." In *Manifolds and Lie Groups*. Progress in Mathematics **14** Boston; Birkhäuser, 1980.

[M1] J. Manin, *Periods of parabolic forms and p-adic Hecke series*, Math. USSR Sbornik **21** (1973), 371–393.

[M2] ______, *The values of p-adic Hecke series at integer points of the critical strip*, Math. USSR Sbornik **22** (1974), 631–637.

[Mz] B. Mazur, *On the arithmetic of special values of L-functions*, Invent. Math. **66** (1977), 207–240.

[O] M. OHTA, *On p-adic representations attached to automorphic forms*, Japanese Journal of Math. (new series) **8** (1982), 1–47.

[Ra] M. RAZAR, *Dirichlet series and Eichler cohomology*, preprint.

[R1] K. RIBET, *Mod p Hecke operators and congruences between modular forms*, Invent. Math. **71** (1983), 193–205.

[R2] ______, *On ℓ-adic representations attached to modular forms*, Invent. Math. **28** (1975), 245–275.

[Se1] J.-P. SERRE, "Formes modulaires et fonctions zeta p-adique," in *Modular Functions of One Variable* III, Lect. Notes in Math. **350** (1973), 191–269.

[Se2] ______, Letter to J.-M. Fontaine (1979).

[S1] G. SHIMURA, *An ℓ-adic method in the theory of automorphic forms*, Unpublished (1968).

[S2] ______, *Introduction to the arithmetic theory of automorphic forms*, Publications of the Mathematical Society to Japan **11**. Princeton; Princeton University Press, 1971.

[St1] G. STEVENS, *The cuspidal group and special values of L-functions*, Trans. Amer. Math. Soc. **291** (1985), 519–550.

[St2] ______, *Arithmetic on Modular Curves*, Progress in Mathematics **20**, Boston, Birkhäuser 1982.

[Sw-D] H. P. F. SWINNERTON-DYER, "On ℓ-adic representations and congruences for coefficients of modular forms I, II," *Modular Functions of One Variable* III, IV, Lect. Notes in Math. **350** (1973), 1–55; **601** (1977), 63–90.

ASH: DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210

STEVENS: DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS 02138 AND DEPARTMENT OF MATHEMATICS, BOSTON UNIVERSITY, BOSTON, MASSACHUSETTS 02215