Math. Ann. 277, 25 51 (1987)

Mathematische Annalen

Poincaré Series on $GL(r)$ and Kloostermann Sums

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This paper is devoted to the study of certain Poincaré series on $GL(r)$, $r \geqq 2$, obtained by averaging Whittaker functions over discrete groups.

When $r=2$, these are the Poincaré series introduced by Petersson [10] and later used by Selberg [12] and others in their work on the Ramanujan conjecture. The case $r=3$ was initiated by Bump et al. [4]. They have extended most of Selberg's theory [12] to this case. In particular, they have calculated the Fourier coefficients of the Poincaré series in terms of certain trigonometric sums which they appropriately call Kloostermann sums and have indicated how good estimates for these sums would give information toward the generalized Ramanujan conjecture for $GL(3)$.

In the present work we take an adelic look at the general case $r \geqq 2$. We calculate the Fourier coefficients of the Poincaré series and begin an investigation into the resulting Kloostermann sums with the eventual aim of giving good estimates for them. The reader should compare our results with those of an upcoming paper of Friedberg [6] who has also considered the general case but from the classical point of view.

In the first section we define adelic Poincaré series with arbitrary K-type and show (Theorem 1.13) that they are dual in an appropriate sense to certain zeta functions associated to pairs of Whittaker functions. This extends the classical view of Poincaré series as being dual to Fourier coefficients (see Example 1.14).

In Sect. 2 we give an adelic description of the Fourier coefficients of Poincaré series (Theorem 2.7). The Kloostermann sums arise in this context from the calculation of certain p-adic integrals over unipotent groups (Definition 2.10, Theorem 2.12). In particular, the multiplicative property of the sums, proven in [4] for $r=3$, is inherent to the adelic approach.

In the remainder of the paper we investigate the properties of the local Kloostermann sums. Before outlining our results it may be useful to first describe the long range goals of this theory.

The Kloostermann sums $Kl(n, \psi, \psi')$ (Definition 2.10) are indexed by elements n of the normalizer $N_{\mathbf{Q}}$ of the standard torus T, and pairs of characters ψ, ψ' of the

* Research supported by a National Science Foundation Postdoctoral Fellowship

standard unipotent subgroup $U_{\mathbf{Q}}$ which are trivial on $U_{\mathbf{Z}}$. For fixed ψ, ψ' we define the Kloostermann zeta function

$$Z(\Lambda, \psi, \psi') = \sum_{n \in N_{\mathbf{Q}}} Kl(n, \psi, \psi') \cdot \|n\|_{\infty}^{\Lambda} \tag{0.1}$$

where the variable Λ ranges over the complexified root space $\mathscr{A}_{\mathbf{C}}$ of the standard torus and $\| \;\|_{\infty}^{\Lambda}$ is defined by (1.8). It is natural to decompose (0.1) into a sum of Dirichlet series indexed by the Weyl group $\mathscr{W}$:

$$\begin{aligned} Z(\Lambda, \psi, \psi') &= \sum_{w \in \mathscr{W}} Z_w(\Lambda, \psi, \psi'), \\ Z_w(\Lambda, \psi, \psi') &= \sum_{t \in T_{\mathbf{Q}}} Kl(tw, \psi, \psi') \cdot \|t\|_{\infty}^{\Lambda}. \end{aligned} \tag{0.2}$$

When ψ and ψ' are the trivial characters, the Z_w are the Dirichlet series which appear in the constant term of the Eisenstein series [7] induced from the trivial representation of the Borel subgroup and having trivial K-type. These Dirichlet series are known to converge for $\mathrm{Re}(\Lambda) \in 2\varrho + \mathscr{C}$ where ϱ is half the sum of the positive roots and $\mathscr{C}$ is the fundamental Weyl chamber. Since the Kloostermann sums associated to the trivial characters bound the general sums we obtain for arbitrary ψ, ψ':

(0.3) $\quad Z(\Lambda, \psi, \psi')$ converges absolutely for $\mathrm{Re}(\Lambda) \in 2\varrho + \mathscr{C}$.

The principal aim is to prove the following conjecture.

Conjecture 1. *Suppose ψ and ψ' are regular (Definition 1.2). Then $Z(\Lambda, \psi, \psi')$ extends to a holomorphic function in the region* $\mathrm{Re}(\Lambda) \in \varrho + \mathscr{C}$.

When $r = 2$ it is known that Conjecture 1 implies the generalized Ramanujan conjecture [12]. We expect this to be true for general r (compare [4]).

Of course when ψ, ψ' are regular we expect that the trivial estimates for the Kloostermann sums can be improved and that the region of absolute convergence in (0.3) can be extended. In fact we conjecture the following.

Conjecture 2. *Suppose ψ and ψ' are regular. Then $Z(\Lambda, \psi, \psi')$ converges absolutely in the region* $\mathrm{Re}(\Lambda) \in \frac{3}{2}\varrho + \mathscr{C}$.

When $r = 2$ this follows from Weil's bound [15] for the classical Kloostermann sums.

Note that Conjecture 2 is an assertion about the asymptotic distribution of absolute values of Kloostermann sums while the strengthening of Conjecture 2 to Conjecture 1 may be viewed as a statement about the distribution of arguments of Kloostermann sums.

In Sect. 5 we will give a proof of Conjecture 2 for $GL(3)$. This is accomplished by applying the general techniques of Sects. 3 and 4 to estimate the sums attached to the long element $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ of the Weyl group (Theorem 5.1). The sums associated to $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ have already been estimated by Larsen [9] and the sums associated to the remaining elements of the Weyl group are either trivial or are $GL(2)$ sums.

Sections 3 and 4 are devoted to a study of the local Kloostermann sums $Kl_p(n, \psi, \psi')$, $n \in N(\mathbf{Q}_p)$. Our results can be roughly divided into three categories: (1) identities among Kloostermann sums (Theorem 3.2); (2) factorization of Kloostermann sums for $GL(r)$ into Kloostermann sums for $GL(r_1)$ and $GL(r_2)$ with $r = r_1 + r_2$ (Theorem 3.7, Corollary 3.11); and (3) decompositions of Kloostermann sums into sums of simpler trigonometric sums (Sect. 4).

It is interesting to note that the orbit decomposition of the Kloostermann sums described in Sect. 4 leads to trigonometric sums $S_w(\theta; \ell)$ [Definition 4.9 (c)] which are easily described without mentioning the group $GL(r)$, but which do not seem to have appeared in the literature before. Examples 4.12 and 4.13 show that these sums generalize sums considered elsewhere (e.g. [3, 5, 8]). It seems likely that the techniques of [3, 5, 8] can be applied to obtain good estimates for $S_w(\theta; \ell)$ at least when $\ell = 1$.

It is perhaps significant that the local Kloostermann sums $Kl_p(n, \psi, \psi')$ can be given an algebraic geometric interpretation. Though we do not explicitly use this fact in this paper, it has motivated the decompositions described in Sects. 4 and 5. We therefore outline, at the beginning of Sect. 4, the description of $Kl_p(n, \psi, \psi')$ as a sum of character values over the rational points of an algebraic variety defined over $\mathbf{F}_p$. Examples show that the associated variety is in general not smooth. We wonder if one can construct intrinsically a smooth stratification of this variety. We refer the reader to the remarks following the proof of Theorem 5.1 for an example.

1. Poincaré Series

In this section we describe the formal construction of Poincaré series for the group $G = GL(r)$, $r \geqq 1$ [see Definition 1.5 and (1.11)]. We will write $\mathbf{Q}_v$ for the completion of $\mathbf{Q}$ at a place v and write $\mathbf{A}$ for the adeles of $\mathbf{Q}$.

Let

$$U = \left\{ \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \right\}$$

be the standard unipotent subgroup of G and let

$$\psi : U(\mathbf{A})/U(\mathbf{Q}) \to \mathbf{C}^*$$

be a character of $U(\mathbf{A})$ which is trivial on $U(\mathbf{Q})$. Every such character has the form $\psi=\psi_{\underline{v}}$ for some $\underline{v}\in\mathbf{Q}^{r-1}$ where $\psi_{\underline{v}}$ is given by

$$(1.1)\qquad \psi_{\underline{v}}\begin{pmatrix}1 & x_1 & * & \dots & * & * \\ 0 & 1 & x_2 & \dots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & x_{r-1} \\ 0 & 0 & 0 & \dots & 0 & 1\end{pmatrix}=\xi(v_1x_1+v_2x_2+\dots+v_{r-1}x_{r-1})$$

and $\xi:\mathbf{A}\to\mathbf{C}^*$ is the standard additive character.

(1.2) *Definition.* We will say that the character $\psi=\psi_{\underline{v}}$ is regular if $v_1v_2\dots v_{r-1}\neq 0$.

Let $K_\infty=SO(r)$, $K_p=G(\mathbf{Z}_p)$, and $K=K_\infty\times\prod_p K_p$ be the standard maximal compact subgroups respectively of the real group $G(\mathbf{R})$, the p-adic group $G(\mathbf{Q}_p)$, and the adelic group $G(\mathbf{A})$. For each place v of $\mathbf{Q}$ we fix a finite dimensional complex Hermitian space V_v with inner product $\langle\, ,\rangle_v$ and let $\sigma_v:K_v\to\mathrm{Aut}(V_v)$ be an irreducible unitary representation of K_v. We assume that (σ_v, V_v) is the trivial representation with a canonical unit vector [i.e. $\cong(\mathbf{1},\mathbf{C})$] for all but finitely many v. We can then form the tensor product representation

$$(1.3)\qquad (\sigma, V)=\bigotimes_v(\sigma_v, V_v)$$

of K, which is finite dimensional and unitary with respect to the inner product $\langle\, ,\rangle=\prod_v\langle\, ,\rangle_v$.

Let Z be the center of G, and fix a character

$$(1.4)\qquad \chi=\prod_v\chi_v:Z(\mathbf{A})/Z(\mathbf{Q})\to\mathbf{C}^*$$

extending the central character of (σ, V).

Let $\psi=\prod_v\psi_v$ be the factorization of ψ into local characters $\psi_v:U(\mathbf{Q}_v)\to\mathbf{C}^*$.

(1.5) *Definition.* (a) A ψ_v-Whittaker function is a function $W_v:G(\mathbf{Q}_v)\to V_v$ satisfying

$$W_v(u_vz_vg_vk_v)=\psi_v(u_v)\,\chi_v(z_v)\,\sigma_v(k_v^{-1})\,W_v(g_v)$$

for all $u_v\in U(\mathbf{Q}_v)$, $z_v\in Z(\mathbf{Q}_v)$, $g_v\in G(\mathbf{Q}_v)$, and $k_v\in K_v$.

(b) A ψ-Whittaker function is a function $W:G(\mathbf{A})\to V$ satisfying

$$W(uzgk)=\psi(u)\,\chi(z)\,\sigma(k^{-1})\,W(g)$$

for all $u\in U(\mathbf{A})$, $z\in Z(\mathbf{A})$, $g\in G(\mathbf{A})$, and $k\in K$.

(1.6) *Definition.* Let W be a ψ-Whittaker function. The Poincaré series associated to W is defined formally to be the function $P_W:G(\mathbf{A})\to V$ given by the series

$$P_W(g)=\sum_{\gamma\in Z_{\mathbf{Q}}U_{\mathbf{Q}}\backslash G_{\mathbf{Q}}} W(\gamma g).$$

Of course this last definition can only be understood formally unless wither the series is absolutely convergent or we have some other method of regularizing the sum.

In the applications we follow Selberg's example [12] and use the "Hecke trick" to regularize the sum. In particular, we introduce a complex analytic family of ψ-Whittaker functions containing the given one, W. The definition of P_W is then obtained by analytic continuation from a region where the series converges.

To describe the analytic family it is convenient to introduce some notation. Let $\mathscr{G}$ be the real Lie algebra of G, and let $\mathscr{A}$ be the standard Cartan subalgebra of $\mathscr{G}$ associated to the standard torus

$$T=\left\{\begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & * \end{pmatrix}\right\} \subseteqq G.$$

For each place v of $\mathbf{Q}$ we have the Iwasawa decomposition

$$G(\mathbf{Q}_v)=U(\mathbf{Q}_v)\,T(\mathbf{Q}_v)K_v.$$

The Harish-Chandra function $H_v: G(\mathbf{Q}_v)\to\mathscr{A}$ is given by

$$(1.7)\qquad H_v(utk)=H_v(t)=\begin{pmatrix} \log|t_1|_v & 0 & \dots & 0 \\ 0 & \log|t_2|_v & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \log|t_r|_v \end{pmatrix}$$

for $u\in U(\mathbf{Q}_v)$, $t=\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_r \end{pmatrix}\in T(\mathbf{Q}_v)$ and $k\in K_v$.

The Killing form, $(A,B)=\operatorname{Tr}(AB)$, restricts to a positive definite form on $\mathscr{A}$ which we extend by bilinearity to a positive definite Hermitian form on the complexification $\mathscr{A}_{\mathbf{C}}$ of $\mathscr{A}$. For $\Lambda\in\mathscr{A}_{\mathbf{C}}$ and $g_v\in G(\mathbf{Q}_v)$ we then set

$$(1.8)\qquad \|g_v\|_v^{\Lambda}\overset{\text{def}}{=}\exp((\Lambda, H_v(g_v)))\in\mathbf{C}^*.$$

Globally, we define for $g\in G(\mathbf{A})$

$$(1.9)\qquad \|g\|_{\mathbf{A}}^{\Lambda}=\prod_v \|g_v\|_v^{\Lambda}.$$

Now for a given ψ-Whittaker function W we obtain a complex analytic family of ψ-Whittaker functions W_Λ, $\Lambda\in\mathscr{A}_{\mathbf{C}}$ defined by

$$(1.10)\qquad W_\Lambda(g)=W(g)\cdot\|g\|_{\mathbf{A}}^{\Lambda}.$$

We will write $P_W(g,\Lambda)$ for $P_{W_\Lambda}(g)$.

Thus

$$(1.11)\qquad P_W(g,\Lambda)=\sum_{\gamma\in Z(\mathbf{Q})U(\mathbf{Q})\backslash G(\mathbf{Q})} W(\gamma g)\cdot\|\gamma g\|_{\mathbf{A}}^{\Lambda}.$$

We denote by $L^2(G(\mathbf{Q})\backslash G(\mathbf{A});\sigma,\chi)$ the space of square integrable V-valued automorphic forms ϕ which satisfy $\phi(gzk)=\chi(z)\,\sigma(k^{-1})\,\phi(g)$ for $g\in G(\mathbf{A})$, $z\in \mathbf{Z}(\mathbf{A})$, and $k\in K$. The inner product on this space is given by

$$(1.12)\qquad \langle\phi_1,\phi_2\rangle_2=\int_{Z_{\mathbf{A}}G_{\mathbf{Q}}\backslash G_{\mathbf{A}}}\langle\phi_1(g),\phi_2(g)\rangle\,dg.$$

(1.13) **Theorem.** *Let $W=\prod_v W_v$ be a factorizable ψ-Whittaker function and suppose $P_W(\cdot,\Lambda)$ converges to an element of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A});\sigma,\chi)$. Let ϕ be an automorphic form in $L^2(G(\mathbf{Q})\backslash G(\mathbf{A});\sigma,\chi)$ whose associated Whittaker function*

$$W'(g)=\int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}}\phi(ug)\,\overline{\psi(u)}\,du$$

is factorizable: $W'=\prod_v W'_v$. Then

$$\langle P_W(\cdot,\Lambda),\phi\rangle_2=\prod_v I_v(W_v,W'_v,\Lambda)$$

where

$$I_v(W_v,W'_v,\Lambda)=\int_{Z(\mathbf{Q}_v)\backslash T(\mathbf{Q}_v)}\langle W_v(t_v),W'_v(t_v)\rangle\cdot\|t_v\|_v^{\Lambda}\,dt_v.$$

Proof. We compute:

$$\begin{aligned}\langle P_W(\cdot,\Lambda),\phi\rangle_2&=\int_{Z_{\mathbf{A}}G_{\mathbf{Q}}\backslash G_{\mathbf{A}}}\langle P_W(g,\Lambda),\phi(g)\rangle\,dg\\&=\int_{Z_{\mathbf{A}}G_{\mathbf{Q}}\backslash G_{\mathbf{A}}}\ \sum_{\gamma\in Z_{\mathbf{Q}}U_{\mathbf{Q}}\backslash G_{\mathbf{Q}}}\langle W(\gamma g),\phi(g)\rangle\cdot\|\gamma g\|_{\mathbf{A}}^{\Lambda}\,dg.\end{aligned}$$

Since ϕ is automorphic we have $\phi(g)=\phi(\gamma g)$ and the last integral can be unfolded to obtain

$$\begin{aligned}&\int_{Z_{\mathbf{A}}U_{\mathbf{Q}}\backslash G_{\mathbf{A}}}\langle W(g),\phi(g)\rangle\cdot\|g\|_{\mathbf{A}}^{\Lambda}\,dg\\&=\int_{Z_{\mathbf{A}}U_{\mathbf{A}}\backslash G_{\mathbf{A}}}\int_{U_{\mathbf{Q}}\backslash U_{\mathbf{A}}}\langle W(ug),\phi(ug)\rangle\,du\cdot\|g\|_{\mathbf{A}}^{\Lambda}\,dg.\end{aligned}$$

Recalling that W is a ψ-Whittaker function we see that $W(ug)=\psi(u)\,W(g)$ and therefore that the last integrand is equal to $\langle W(g),\phi(ug)\,\overline{\psi(u)}\rangle$. Thus the inner integral above becomes $\langle W(g),W'(g)\rangle$. We then have

$$\begin{aligned}&\int_{Z_{\mathbf{A}}U_{\mathbf{A}}\backslash G_{\mathbf{A}}}\langle W(g),W'(g)\rangle\cdot\|g\|_{\mathbf{A}}^{\Lambda}\,dg\\&=\int_{B_{\mathbf{A}}\backslash G_{\mathbf{A}}}\int_{Z_{\mathbf{A}}U_{\mathbf{A}}\backslash B_{\mathbf{A}}}\langle W(\beta g),W'(\beta g)\rangle\cdot\|\beta g\|_{\mathbf{A}}^{\Lambda}\,d\beta\,dg\\&=\int_{(B_{\mathbf{A}}\cap K)\backslash K}\int_{Z_{\mathbf{A}}\backslash T_{\mathbf{A}}}\langle W(tk),W'(tk)\rangle\cdot\|t\|_{\mathbf{A}}^{\Lambda}\,dt\,dk.\end{aligned}$$

Applying the transformation law (1.5)(b) to W and W' and using the fact that σ is a *unitary* representation we see that $\langle W(tk),W'(tk)\rangle=\langle W(t),W'(t)\rangle$. We may assume that the Haar measure on $(B_{\mathbf{A}}\cap K)\backslash K$ has total measure 1. Then the last integral simplifies to

$$\int_{Z_{\mathbf{A}}\backslash T_{\mathbf{A}}}\langle W(t),W'(t)\rangle\cdot\|t\|_{\mathbf{A}}^{\Lambda}\,dt.$$

Since W and W' are factorizable this integral factors and the theorem is proved. □

Note that if $\pi = \bigotimes_v \pi_v$ is an irreducible constituent of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}); \sigma, \chi)$ and if $\phi \in L^2(\ )$ corresponds to a factorizable element of π then the factorizability of W' in the theorem is guaranteed by Shalika's multiplicity one theorem [13].

In words, we might say that Theorem 1.13 expresses a duality between Poincaré series and certain zeta functions. This should be compared to the classical view of Poincaré series as being dual to Fourier coefficients. The connection between these two points of view is illustrated by the next example.

(1.14) *Example.* It is not hard to relate our Poincaré series to the ones considered by Bump et al. [4]. We take χ to be the trivial character of $Z(\mathbf{A})$ and (σ, V) to be the trivial representation of K. Let

$$(1.15)\qquad \tau = \begin{pmatrix} 1 & x_1 & * & \dots & * & * \\ 0 & 1 & x_2 & \dots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & x_{r-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \cdots y_{r-1} & & & & 0 \\ & y_2 \cdots y_{r-1} & & & \\ & & \ddots & & \\ & & & y_{r-1} & \\ 0 & & & & 1 \end{pmatrix}$$

be a real upper triangular matrix with $y_1, \dots, y_{r-1}$ positive. Let E be the function on these matrices defined by

$$E(\tau) = e^{2\pi i \Sigma v_j(x_j + i y_j)}.$$

By the Iwasawa decomposition we can factor any element $g_\infty \in G(\mathbf{R})$ into a product $g_\infty = \tau z k$ with τ as in (1.15), $z \in Z(\mathbf{R})$ and $k \in O(r)$. We may therefore extend the function E to a function $W_\infty : G(\mathbf{R}) \to \mathbf{C}$ defined by

$$(1.16)\qquad W_\infty(\tau z k) = \begin{cases} E(\tau) & \text{if } \ zk \in Z(\mathbf{R}) \cdot SO(r), \\ 0 & \text{otherwise}. \end{cases}$$

Then W_∞ is a ψ_∞-Whittaker where ψ is defined by (1.1).

At the finite places we define

$$(1.17)\qquad W_p(u_p t_p k_p) = \begin{cases} \psi(u_p) & \text{if } \ t_p \in Z(\mathbf{Q}_p) \cdot T(\mathbf{Z}_p), \\ 0 & \text{otherwise}. \end{cases}$$

Then $W = \prod_v W_v$ is a global ψ-Whittaker function.

If we view τ (1.15) as an element of $G(\mathbf{A})$ by letting the finite components be 1, then we see at once that for $\gamma \in G(\mathbf{Q})$ we have

$$(1.18)\qquad W(\gamma\tau) = \begin{cases} E(\gamma\tau) & \text{if } \ \gamma \in Z(\mathbf{Q}) \cdot SL(r, \mathbf{Z}), \\ 0 & \text{otherwise}. \end{cases}$$

Substituting this into (1.10) we obtain

$$(1.19)\qquad P_W(\tau, \Lambda) = \sum_{\gamma \in U(\mathbf{Z})\backslash PSL(r, \mathbf{Z})} E(\gamma\tau) \cdot \|\gamma\tau\|_\infty^\Lambda.$$

The right hand side is the Poincaré series of [4, 6].

It is known that if ψ is regular then this series converges to a square integrable automorphic form when $\mathrm{Re}(\Lambda) \in 2\varrho + \mathscr{C}$ [1]. (Recall, ϱ is half the sum of the positive roots and $\mathscr{C}$ is the fundamental Weyl chamber.) So for Λ in this region we may apply Theorem 1.13. For $\phi \in L^2(\ )$ as in the theorem we find

$$I_p(W_p, W'_p, \Lambda) = W'_p(1).$$

If we write $W'_f = \prod_p W'_p$ then the theorem states

$$\langle P_W(\cdot, \Lambda), \phi \rangle_2 = W'_f(1) \cdot I_\infty(W_\infty, W'_\infty, \Lambda). \tag{1.20}$$

Recall that in the classical language $W'_f(1)$ is the ψ-Fourier coefficient of ϕ. Thus (1.20) expresses the classical duality between Poincaré series and Fourier coefficients.

2. Fourier Coefficients and Kloostermann Integrals

In this section we give the formal calculation of the Fourier coefficients of the Poincaré series P_W. Recall [11] that, for ϕ an automorphic form on G and $\psi' : U(\mathbf{A})/U(\mathbf{Q}) \to \mathbf{C}^*$ a character, the ψ'-Fourier coefficient of ϕ is the function $\phi_{\psi'}$ on $G(\mathbf{A})$ given by

$$\phi_{\psi'}(g) = \int_{U(\mathbf{Q}) \backslash U(\mathbf{A})} \phi(ug) \overline{\psi'(u)}\, du.$$

We normalize the Haar measure du on $U(\mathbf{A})$ as follows. The group U can be factored as a product $U = \prod_\lambda U_\lambda$ over the positive roots λ (the order is unimportant). Each $U_\lambda(\mathbf{A})$ is canonically isomorphic to $\mathbf{A}$ which is equipped with a canonical Haar measure. The Haar measure on $U(\mathbf{A})$ is the product of the measures on $U_\lambda(\mathbf{A})$.

Our calculation of the Fourier coefficients of P_W uses the Bruhat decomposition of G as described in [2]. Let N be the normalizer in G of the standard torus T. Thus N/T is the Weyl group $\mathscr{W}$ of T. For $n \in N(\mathbf{Q})$ we can decompose

$$U = U_n^+ U_n^- = U_n^- U_n^+,$$

where $U_n^\pm = U \cap n^{-1} U^\pm n$ with $U^+ = U$ and U^- = the opposite unipotent subgroup. The decomposition of an element of U is *unique* and the decomposition depends only on the image of n in the Weyl group. We will abbreviate the notation and set

$$U_n \overset{\text{def}}{=} U_n^-. \tag{2.1}$$

The Bruhat decomposition of G is given over $\mathbf{Q}$ by

$$G(\mathbf{Q}) = \coprod_{n \in N(\mathbf{Q})} U(\mathbf{Q}) n U_n(\mathbf{Q}). \tag{2.2}$$

The decomposition of an element of $G(\mathbf{Q})$ is *unique.*

The result of our calculation of the ψ'-Fourier coefficient of P_W is a sum of products of local integrals which we call *Kloostermann integrals* and which we now describe.

For our fixed character $\psi = \prod_v \psi_v$ and $n \in N(\mathbf{Q})$ we define $\psi_{v,n}: U_n^+(\mathbf{Q}_v) \to \mathbf{C}^*$ by

$$\psi_{v,n}(u) \overset{\text{def}}{=} \psi_v(nun^{-1}). \tag{2.3}$$

This is meaningful since $u \in U_n^+ \Rightarrow nun^{-1} \in U$. We then define the global character $\psi_n: U_n^+(\mathbf{A}) \to \mathbf{C}^*$ by

$$\psi_n = \prod_v \psi_{v,n}. \tag{2.4}$$

If $\psi' = \prod_v \psi'_v$ is another character of $U(\mathbf{A})$ set

$$\begin{aligned} \delta_n(\psi_v, \psi'_v) &= \begin{cases} 1 & \text{if} \quad \psi_{v,n} = \psi'_v|_{U_n^+(\mathbf{Q}_v)} \\ 0 & \text{otherwise} \end{cases} \\ \delta_n(\psi, \psi') &= \prod_v \delta_n(\psi_v, \psi'_v). \end{aligned} \tag{2.5}$$

(2.6) *Definition.* Let W_v be a ψ_v-Whittaker function, $\psi'_v: U(\mathbf{Q}_v) \to \mathbf{C}^*$ be a character, and $n \in N(\mathbf{Q})$. The associated *Kloostermann integral* $K_v = K_v(W_v, \psi'_v, n)$ is the function $K_v: G(\mathbf{Q}_v) \to V_v$ defined by

$$K_v(g) = \delta_n(\psi_v, \psi'_v) \cdot \int_{U_n(\mathbf{Q}_v)} W_v(nug)\, \overline{\psi'_v(u)}\, du.$$

Note that K_v is a ψ'_v-Whittaker function.

(2.7) **Theorem.** *Let $W = \prod_v W_v$ be a factorizable ψ-Whittaker function on $G(\mathbf{A})$. Then for $g = (g_v)_v \in G(\mathbf{A})$, the ψ'-Fourier coefficient of the Poincaré series $P(g) = P_W(g)$ is given by*

$$P_{\psi'}(g) = \sum_{n \in Z_{\mathbf{Q}} \backslash N_{\mathbf{Q}}} \prod_v K_v(W_v, \psi'_v, n)(g_v).$$

Proof. By the definition of Fourier coefficient

$$P_{\psi'}(g) = \int_{U(\mathbf{Q}) \backslash U(\mathbf{A})} P(ug)\, \overline{\psi'(u)}\, du.$$

Substituting the series for P and using the Bruhat decomposition for $G(\mathbf{Q})$ we obtain

$$\begin{aligned} P_{\psi'}(g) &= \int_{U(\mathbf{Q}) \backslash U(\mathbf{A})} \sum_{n \in Z_{\mathbf{Q}} \backslash N_{\mathbf{Q}}} \sum_{\gamma \in U(\mathbf{Q}) \backslash U(\mathbf{Q}) n U_n(\mathbf{Q})} W(\gamma u g)\, \overline{\psi'(u)}\, du \\ &= \sum_{n \in Z_{\mathbf{Q}} \backslash N_{\mathbf{Q}}} \int_{U(\mathbf{Q}) \backslash U(\mathbf{A})} \sum_{\mu \in U_n(\mathbf{Q})} W(n\mu u g)\, \overline{\psi'(u)}\, du. \end{aligned}$$

If $\mathscr{D}$ is a fundamental domain for $U(\mathbf{Q}) \backslash U(\mathbf{A})$ then $\coprod_{\mu \in U_n(\mathbf{Q})} \mu \mathscr{D}$ is a fundamental domain for $U_n^+(\mathbf{Q}) \backslash U(\mathbf{A})$ since $U(\mathbf{Q}) = U_n^+(\mathbf{Q})\, U_n(\mathbf{Q})$. Thus we can unfold the above integral to find

$$P_{\psi'}(g) = \sum_{n \in Z_{\mathbf{Q}} \backslash N_{\mathbf{Q}}} \int_{U_n^+(\mathbf{Q}) \backslash U(\mathbf{A})} W(nug)\, \overline{\psi'(u)}\, du.$$

Using the Bruhat decomposition again we see that if $\mathscr{D}_n^+$ is a fundamental domain for $U_n^+(\mathbf{Q}) \backslash U_n^+(\mathbf{A})$ then $\mathscr{D}_n^+ \times U_n(\mathbf{A})$ is a fundamental domain for $U_n^+(\mathbf{Q}) \backslash U(\mathbf{A})$. The last integral therefore becomes

$$\sum_{n\in Z_{\mathbf{Q}}\backslash N_{\mathbf{Q}}} \int_{U_n^+(\mathbf{Q})\backslash U_n^+(\mathbf{A})} \int_{U_n(\mathbf{A})} W(nu^+ug)\overline{\psi'(u^+u)}\,du\,du^+ .$$

Now if $u^+\in U_n^+(\mathbf{A})$ then $nu^+n^{-1}\in U(\mathbf{A})$ and therefore $W(nu^+ug)=\psi_n(u^+)\,W(nug)$. Thus

$$\begin{aligned} P_{\psi'}(g) &= \sum_{n\in Z_{\mathbf{Q}}\backslash N_{\mathbf{Q}}} \int_{U_n^+(\mathbf{Q})\backslash U_n^+(\mathbf{A})} \psi_n(u^+)\overline{\psi'(u^+)}\,du^+ \int_{U_n(\mathbf{A})} W(nug)\overline{\psi'(u)}\,du \\ &= \sum_{n\in Z_{\mathbf{Q}}\backslash N_{\mathbf{Q}}} \delta_n(\psi,\psi') \int_{U_n(\mathbf{A})} W(nug)\overline{\psi'(u)}\,du . \end{aligned}$$

Since W is factorizable the integral factors and the theorem is proved. □

We conclude this section with a calculation of the p-adic Kloostermann integrals in terms of generalized Kloostermann sums in the special case where W_p has level one, i.e. when the representation (σ_p, V_p) is trivial. Note that in the general case W_p has level one for almost all primes p.

For $n\in N(\mathbf{Q}_p)$ we will write

$$\text{(2.8)}\qquad \begin{aligned} C(n) &\stackrel{\text{def}}{=} U(\mathbf{Q}_p)\,nU(\mathbf{Q}_p)\cap G(\mathbf{Z}_p); \\ X(n) &\stackrel{\text{def}}{=} U(\mathbf{Z}_p)\backslash C(n)/U_n(\mathbf{Z}_p); \\ Y(n) &\stackrel{\text{def}}{=} U(\mathbf{Z}_p)\backslash C(n)/U(\mathbf{Z}_p). \end{aligned}$$

By the Bruhat decomposition we have natural maps

$$\text{(2.9)}\qquad \begin{aligned} u &: X(n)\to U(\mathbf{Z}_p)\backslash U(\mathbf{Q}_p), \\ u' &: X(n)\to U_n(\mathbf{Q}_p)/U_n(\mathbf{Z}_p) \end{aligned}$$

defined by the relation $x=u(x)\cdot n\cdot u'(x)$ for $x\in X(n)$.

(2.10) *Definition.* (a) Let $n\in N(\mathbf{Q}_p)$. Let ψ_p be a character of $U(\mathbf{Q}_p)$ which is trivial on $U(\mathbf{Z}_p)$ and let ψ'_p be a character of $U_n(\mathbf{Q}_p)$ trivial on $U_n(\mathbf{Z}_p)$. The local Kloostermann sum associated to this data is

$$Kl_p(n,\psi_p,\psi'_p)=\sum_{x\in X(n)} \psi_p(u(x))\cdot\psi'_p(u'(x)).$$

(b) Let $n\in N(\mathbf{Q})$. Let $\psi=\prod_p \psi_p$ $\left(\text{respectively } \psi'=\prod_p \psi'_p\right)$ be a character of $U(\mathbf{A})$ (respectively $U_n(\mathbf{A})$) which is trivial on $\prod_p U(\mathbf{Z}_p)$ $\left(\text{respectively } \prod_p U_n(\mathbf{Z}_p)\right)$. The global Kloostermann sum associated to this data is

$$Kl(n,\psi,\psi')=\prod_p Kl_p(n,\psi_p,\psi'_p).$$

The Kloostermann sums which appear in the Fourier coefficients of Poincaré series are those for which ψ'_p can be extended to a character (which we also denote ψ'_p) of $U(\mathbf{Q}_p)/U(\mathbf{Z}_p)$ which satisfies $\delta_n(\psi_p,\psi'_p)=1$. Under these assumptions, for $x\in X(n)$ the term $\psi_p(u(x))\cdot\psi'_p(u'(x))$ depends only on the image of x in $Y(n)$. Thus the sum in (a) can also be expressed as a weighted sum over $Y(n)$.

The general Kloostermann sums defined above will arise naturally in the next section. For example, in Corollary 3.11 we show how to factor certain Kloostermann sums in terms of lower dimensional sums of this more general type.

Let $W_p: G(\mathbf{Q}_p) \to \mathbf{C}$ be a ψ_p-Whittaker function of level one. The aim of Theorem 2.12 below is to describe the Kloostermann integrals $K_p(W_p, \psi'_p, n)(g)$ for $g \in G(\mathbf{Q}_p)$ in terms of the local Kloostermann sums. By the Iwasawa decomposition it suffices to calculate $K_p(t)$ for $t \in T(\mathbf{Q}_p)$. But a simple variable change shows

(2.11) $$K_p(W_p, \psi'_p, n)(t) = \|t\|_p^{2\delta} \cdot K_p(W_p, \psi'_{p,t}, nt)(1),$$

where $\psi'_{p,t}(u) = \psi'_p(tut^{-1})$. Thus we only need to calculate $K_p(1)$.

(2.12) **Theorem.** *Suppose W_p has level one. Then*

$$K_p(W_p, \psi'_p, n)(1) = \delta_n(\psi_p, \psi'_p) \sum_{t \in T(\mathbf{Z}_p) \backslash T(\mathbf{Q}_p)} W_p(t) \cdot Kl_p(t^{-1}n, \bar{\psi}_{p,t}, \overline{\psi'_p}).$$

Proof. By definition

$$K_p(1) = \delta_n(\psi_p, \psi'_p) \cdot \int_{U_n(\mathbf{Q}_p)} W_p(nu) \overline{\psi'_p(u)}\, du$$
$$= \delta \cdot \sum_{u' \in U_n(\mathbf{Q}_p)/U_n(\mathbf{Z}_p)} W_p(nu') \overline{\psi'_p(u')}\, du.$$

Now for each $u' \in U_n(\mathbf{Q}_p)$ there are $t \in T(\mathbf{Q}_p)$ and $u \in U(\mathbf{Q}_p)$ such that $ut^{-1}nu' \in G(\mathbf{Z}_p)$. Moreover, the Bruhat decomposition guarantees that the class of t in $T(\mathbf{Z}_p) \backslash T(\mathbf{Q}_p)$ is well defined and that once we have chosen t the class of u in $U(\mathbf{Z}_p) \backslash U(\mathbf{Q}_p)$ is determined. Thus we have

$$K_p(1) = \delta \cdot \sum_{t \in T(\mathbf{Z}_p) \backslash T(\mathbf{Q}_p)} \sum_{\substack{u \in U(\mathbf{Z}_p) \backslash U(\mathbf{Q}_p) \\ u' \in U_n(\mathbf{Q}_p)/U_n(\mathbf{Z}_p) \\ ut^{-1}nu' \in G(\mathbf{Z}_p)}} W_p(nu') \overline{\psi'_p(u')}$$
$$= \delta \cdot \sum_t \sum_{u,u'} \overline{\psi_p(tut^{-1})}\, W_p(tut^{-1}nu') \overline{\psi'_p(u')}.$$

Since $ut^{-1}nu' \in G(\mathbf{Z}_p)$ we have $W_p(tut^{-1}nu') = W_p(t)$ and the last sum simplifies to

$$K_p(1) = \delta \cdot \sum_t W_p(t) \sum_{u,u'} \overline{\psi_{p,t}(u)}\, \overline{\psi'_p(u')}$$

which proves the theorem. □

We will say that a global Whittaker function (and the associated Poincaré series) has level one if the representation (σ_p, V_p) is trivial for *every* finite prime p. Using Theorem 2.7 we can now easily calculate the ψ'-Fourier coefficient of a level one Poincaré series. The result is stated in the following corollary. As in the remarks preceding Theorem 2.12 it is enough to calculate $P_{\psi'}(g)$ when $g = g_\infty \in G(\mathbf{R})$ has all of its finite components equal to 1.

(2.13) **Corollary.** *Let $W = \prod_v W_v$ be a factorizable ψ-Whittaker function of level one and let $W_f = \prod_p W_p$. Then the ψ'-Fourier coefficient of the Poincaré series $P = P_W$ is given by the following formula for $g_\infty \in G(\mathbf{R})$:*

$$P_{\psi'}(g_\infty) = \sum_{n \in Z_{\mathbf{Q}} \backslash N_{\mathbf{Q}}} \delta_n(\psi, \psi') \cdot \sum_{t \in T(\hat{\mathbf{Z}}) \backslash T(\mathbf{Q})} W_f(t) \cdot Kl(t^{-1}n, \bar{\psi}_t, \overline{\psi'}) \cdot K_\infty(W_\infty, \psi'_\infty, n)(g_\infty).$$

(2.14) *Example.* We let $P=P_W(\cdot,\Lambda)$ where W is the ψ-Whittaker function of example 1.14 and calculate the ψ'-Fourier coefficient $P_{\psi'}(\tau)$ using Corollary 2.13. Here τ is given by (1.15).

By the definition of W we see that $W_f(t)$ vanishes unless $t\in T(\mathbf{Z})\cdot Z(\mathbf{Q})$. It is also evident from the definition of the Kloostermann sums that $Kl(t^{-1}n,\bar{\psi}_t,\overline{\psi'})=0$ unless $\det(t)=\pm\det(n)$. So we can rewrite the formula in Corollary 2.13 as

$$P_{\psi'}(\tau)=\sum_{n\in N(\mathbf{Q})/\pm 1}\delta_n(\psi,\psi')\cdot Kl(n,\bar{\psi},\overline{\psi'})\cdot K_\infty(W_\infty,\psi'_\infty,n)(\tau).$$

This should be compared to the results of [4, 6].

3. Kloostermann Sums

In this section the prime p and the ground field $\mathbf{Q}_p$ will be fixed unless otherwise specified. Thus we will write simply G for $G(\mathbf{Q}_p)$, N for $N(\mathbf{Q}_p)$, U for $U(\mathbf{Q}_p)$, etc.

To understand the Kloostermann sums $Kl_p(n,\psi,\psi')$, $n\in N$, it is clearly essential to study the structure of the coset spaces $X(n)$ and $Y(n)$ [see (2.8)]. In this section we examine two aspects of this structure: (1) isomorphisms among the $X(n)$ and among the $Y(n)$; and (2) factorizations of $X(n)$, $Y(n)$ into coset spaces coming from $GL(r_1)$ and $GL(r_2)$ with $r_1+r_2=r$. In the next section we will consider a third aspect, namely decomposition of $X(n)$ into a disjoint union of simpler sets. In Theorem 3.12 we give a necessary and sufficient condition for $X(n)$ to be nonempty.

We begin by observing that there are several symmetries of G which preserve $G(\mathbf{Z}_p)$ and respect the Bruhat decomposition. These symmetries lead to relations among the Kloostermann sums.

If we let $w_0\in\mathscr{W}$ be the long element of the Weyl group and ${}^tg$ denote the transpose of an element $g\in G$, then the involution

$$\begin{aligned}\iota:G&\to G\\ g&\mapsto g^\iota=w_0\,{}^tg^{-1}w_0\end{aligned}\tag{3.1}$$

is an example of such a symmetry. Note that ι preserves the unipotent group U and that it sends U_n to U_{n^ι} for $n\in N$. Thus if ψ is a character of U so also is $\psi\circ\iota$ and if ψ' is a character of U_n then $\psi'\circ\iota$ is a character of U_{n^ι}.

(3.2) **Theorem.** *Let $n\in N$ and let $\psi:U\to\mathbf{C}^*$, $\psi':U_n\to\mathbf{C}^*$ be characters which are trivial on $U(\mathbf{Z}_p)$, $U_n(\mathbf{Z}_p)$.*

(a) *If $t\in T(\mathbf{Z}_p)$ then*

$$Kl_p(n,\psi_t,\psi')=Kl_p(tn,\psi,\psi'),$$

$$Kl_p(n,\psi,\psi'_t)=Kl_p(nt^{-1},\psi,\psi').$$

(b) $Kl_p(n,\psi,\psi')=Kl_p(n^\iota,\psi\circ\iota,\psi'\circ\iota)$.

(c) *If the image of n in the Weyl group $\mathscr{W}$ is the long element w_0 then* $Kl_p(n,\psi,\psi')=Kl_p(n^{-1},\overline{\psi'},\bar{\psi})$.

Proof. The first statement (a) is an immediate consequence of the equalities $t\cdot C(n)=C(tn)$ and $C(n)\cdot t^{-1}=C(nt^{-1})$.

The involution ι maps $C(n)$ to $C(n^\iota)$ and U_n to U_{n^ι}. Thus ι induces a bijection $X(n)\to X(n^\iota)$ and (b) follows easily.

The map $\gamma\mapsto\gamma^{-1}$ sends $C(n)$ to $C(n^{-1})$, and since inversion preserves U this induces a bijection $Y(n)\to Y(n^{-1})$. For general n this map cannot be lifted to a map $X(n)\to X(n^{-1})$. But in case n lies over w_0 we have $U_n=U=U_{n^{-1}}$. Thus $Y(n)=X(n)$ and $Y(n^{-1})=X(n^{-1})$, and (c) follows easily. □

In our study of the double coset spaces $X(n)$, $Y(n)$, it is convenient to keep in mind the following invariants of a U double coset.

(3.3) *Definition.* Let $g\in G$ and let $I,J\subseteq\{1,\dots,r\}$ be subsets of order k. We say that the subdeterminant

$$g_{IJ}=\det(g_{ij})_{\substack{i\in I\\ j\in J}}$$

is *exposed* if

(1) $g_{IJ}\neq 0$, and

(2) $g_{I'J'}=0$ whenever $I'\geqq I$, $J'\leqq J$ but $(I,J)\neq(I',J')$.

Here, of course, I',J' denote subsets of $\{1,\dots,r\}$ of order k and subsets have been ordered lexicographically.

Visually, g_{IJ} is *exposed* if it is revealed by a glance at the matrix of $k\times k$ subdeterminants from the lower left hand corner, imagining zero determinants to be invisible.

The following lemma is evident.

(3.4) **Lemma.** *Let $g,g'\in G$ and suppose $g=u_1g'u_2$ with $u_1,u_2\in U$. Then g_{IJ} is exposed if and only if g'_{IJ} is exposed. Moreover, if they are exposed then they are equal, $g_{IJ}=g'_{IJ}$.* □

As a consequence of this we see that if $X(n)\neq\phi$, $n\in N$, then every exposed subdeterminant of n is integral. In fact we will see later that this is also a sufficient condition.

We identify the Weyl group $\mathcal{W}$ with the symmetric group on r letters as follows. First identify $\mathcal{W}$ with the subgroup of N consisting of permutation matrices whose entries are zeroes and ones in the usual way. Then define $w(j)$ for $w\in\mathcal{W}$ and $j\in\{1,\dots,r\}$ by the formula

$$w\cdot e_j=e_{w(j)}$$

where $e_1,\dots,e_r$ is the standard basis of column vectors. We let N act on $\{1,\dots,r\}$ via the canonical map $N\to\mathcal{W}$. These actions of N and $\mathcal{W}$ extend in a natural way to actions on the collection of subsets of $\{1,\dots,r\}$.

Now fix $r_1,r_2>0$ with $r=r_1+r_2$. We will write G_i for $GL(r_i)$, U_i for its standard unipotent subgroup, and N_i for the normalizer of the standard torus. We imbed $G_1\times G_2$ diagonally in G:

$$G_1\times G_2\hookrightarrow G$$
$$(g_1,g_2)\mapsto g_1\times g_2=\begin{pmatrix} g_1 & 0\\ 0 & g_2\end{pmatrix}.$$

Thus $U_1\times U_2$ will be viewed as a subgroup of U and $N_1\times N_2$ as a subgroup of N.

Now let $n \in N$, $J \subseteq \{1, \ldots, r\}$ be a subset with r_2 elements and $I = n(J)$. Then there is a unique choice of $v, w \in \mathscr{W}$, $n_1 \in N_1$, and $n_2 \in N_2$ such that the following conditions hold:

(3.5) (a) The restrictions of v and w to each of the two sets $I_1 = \{1, \ldots, r_1\}$ and $I_2 = \{r_1 + 1, \ldots, r\}$ is order preserving;
(b) $v(I_2) = I$, $w(I_2) = J$;
(c) $v^{-1} n w = n_1 \times n_2$.

Note that condition (a) is equivalent to

$$(3.6) \qquad \begin{aligned} v(U_1 \times U_2) v^{-1} &\subseteq U, \\ w(U_1 \times U_2) w^{-1} &\subseteq U. \end{aligned}$$

Let $U_{r_1, r_2} = \left\{ \begin{pmatrix} 1_{r_1} & * \\ 0 & 1_{r_2} \end{pmatrix} \right\} \subseteq U$ and $U^-_{r_1, r_2} = \left\{ \begin{pmatrix} 1_{r_1} & 0 \\ * & 1_{r_2} \end{pmatrix} \right\} \subseteq U^-$.

(3.7) **Theorem.** *Let $n \in N$, $J \subseteq \{1, \ldots, r\}$ be a subset of order r_2 and $I = n(J)$. Let $v, w \in \mathscr{W}$, $n_1 \in N_1$, and $n_2 \in N_2$ be as in* (3.5). *Thus, in particular, $v^{-1} n w = n_1 \times n_2$. Suppose n_{IJ} is exposed and $n_{IJ} \in \mathbf{Z}_p^*$. Then the following hold.*

(a) *If $\gamma_1 \in C(n_1)$, $\gamma_2 \in C(n_2)$, $\nu \in U(\mathbf{Z}_p) \cap v U_{r_1, r_2} v^{-1}$, and $\mu \in U(\mathbf{Z}_p) \cap w U^-_{r_1, r_2} w^{-1}$, then*

$$(*) \qquad \nu v(\gamma_1 \times \gamma_2) w^{-1} \mu \in C(n).$$

(b) *Every element of $C(n)$ has a unique decomposition in the form* $(*)$.

(c) *The map $C(n_1) \times C(n_2) \to C(n)$, $\gamma_1 \times \gamma_2 \mapsto v(\gamma_1 \times \gamma_2) w^{-1}$ induces an injection*

$$X(n_1) \times X(n_2) \hookrightarrow X(n),$$

and a bijection

$$Y(n_1) \times Y(n_2) \leftrightarrow Y(n).$$

If $v = 1$ then the first map is also bijective.

Proof. Let $\gamma_1, \gamma_2, \nu, \mu$ be as in the theorem. Then $\gamma_i = u_i n_i u_i'$ with $u_i, u_i' \in U_i$. By (3.6) we know that $u = v(u_1 \times u_2) v^{-1}$ and $u' = w(u_1' \times u_2') w^{-1}$ are in U. Thus $\nu v(\gamma_1 \times \gamma_2) w^{-1} \mu = \nu u v(n_1 \times n_2) w^{-1} u' \mu$ lies in UnU. This element is clearly also in $G(\mathbf{Z}_p)$ and is therefore in $C(n)$, proving (a).

Now let $\beta \in C(n)$. Then β_{IJ} is an exposed subdeterminant and $\beta_{IJ} = n_{IJ} \in \mathbf{Z}_p^*$. Thus a simple row and column reduction applied to β produces elements $\nu, \mu \in U(\mathbf{Z}_p)$ such that $\gamma = \nu \beta \mu$ satisfies

$$(3.8) \qquad \gamma_{ij} = 0 \text{ unless } \begin{cases} (i, j) \in I \times J; & \text{or} \\ i < \min(I) \text{ and } j \notin J; & \text{or} \\ i \notin I \text{ and } j > \max(J). \end{cases}$$

In fact we can achieve this even with the following assumptions for $i < j$

$$(3.9) \qquad \nu_{ij} = 0 \text{ unless } \begin{cases} j \in I, & \text{and} \\ i \notin I \end{cases}$$

and

(3.10) $$\mu_{ij}=0 \text{ unless } \begin{cases} i\in J, & \text{and} \\ j\notin J. \end{cases}$$

One verifies easily that under conditions (3.8), (3.9), (3.10) the elements v and μ are uniquely determined.

Now condition (a) is equivalent to $v^{-1}\gamma w\in G_1(\mathbf{Z}_p)\times G_2(\mathbf{Z}_p)$. So we may write

$$\beta=v^{-1}v(\gamma_1\times\gamma_2)w^{-1}\mu^{-1}$$

in a unique way with $\gamma_i\in G_i(\mathbf{Z}_p)$ subject to the conditions (3.9) and (3.10) on v and μ.

We have $(v^{-1}vv)_{ij}=v_{v(i),v(j)}$. Thus (3.9) is equivalent to the statement

$$(v^{-1}vv)_{ij}=0 \text{ unless } \begin{cases} j\in v^{-1}(I), & \text{and} \\ i\notin v^{-1}(I). \end{cases}$$

Since $v^{-1}(I)=\{r_1+1,\dots,r\}$ this is equivalent to $v^{-1}vv\in U_{r_1,r_2}$. Similarly, (3.10) is equivalent to $w^{-1}\mu w\in U^-_{r_1,r_2}$. This proves (b).

Let $\phi: C(n_1)\times C(n_2)\to X(n)$ and $\psi: C(n_1)\times C(n_2)\to Y(n)$ be the maps induced by $(\gamma_1,\gamma_2)\mapsto v(\gamma_1\times\gamma_2)w^{-1}$. If $\gamma_i,\gamma'_i\in C(n_i)$, $i=1,2$, then there are $u_i,u'_i\in U_i$ such that $\gamma'_i=u_i\gamma_i u'_i$. We then have

$$v(\gamma'_1\times\gamma'_2)w^{-1}=(v(u_1\times u_2)v^{-1})[v(\gamma_1\times\gamma_2)w^{-1}](w(u'_1\times u'_2)w^{-1}).$$

Since v, w are in $\mathscr{W}$ which is contained in $G(\mathbf{Z}_p)$, (3.6) shows $u_1\times u_2$ is integral if and only if $v(u'_1\times u'_2)v^{-1}\in U(\mathbf{Z}_p)$ and similarly $u'_1\times u'_2$ is integral if and only if $w(u'_1\times u'_2)w^{-1}\in U(\mathbf{Z}_p)$. Thus $\psi(\gamma_1,\gamma_2)=\psi(\gamma'_1,\gamma'_2)$ if and only if γ_i,γ'_i represent the same element of $Y(n_i)$ for $i=1,2$. This proves that ψ induces a well defined and injective map $Y(n_1)\times Y(n_2)\to Y(n)$. The surjectivity of this map is an immediate consequence of (b).

If in addition we have $u'_1\in U_{n_1}$, $u'_2\in U_{n_2}$ then $w(u'_1\times u'_2)w^{-1}\in U_n$. Indeed, we have $nw(u'_1\times u'_2)w^{-1}n^{-1}=v[(n_1u'_1n_1^{-1})\times(n_2u'_2n_2^{-1})]v^{-1}\in v(U_1^-\times U_2^-v^{-1})$. This is contained in U^- as can be seen by transposing the first inclusion of (3.6). Moreover, $w(u'_1\times u'_2)w^{-1}\in U_n(\mathbf{Z}_p)\Leftrightarrow u'_1\times u'_2\in U_{n_1}(\mathbf{Z}_p)\times U_{n_2}(\mathbf{Z}_p)$. It follows that ϕ induces an injective map $X(n_1)\times X(n_2)\to X(n)$.

Finally, suppose $v=1$, and let $x\in X(n)$. By (b) x is represented by an element $\gamma\in C(n)$ of the form $(\gamma_1\times\gamma_2)w^{-1}\mu$ with $\gamma_1\in C(n_1)$, $\gamma_2\in C(n_2)$, and $\mu\in U(\mathbf{Z}_p)\cap wU^-_{r_1,r_2}w^{-1}$. To show that x is in the image of ϕ it suffices to show $\mu\in U_n(\mathbf{Z}_p)$. If we write $\mu=wuw^{-1}$ with $u\in U^-_{r_1,r_2}$, then $n\mu n^{-1}=(n_1\times n_2)u(n_1\times n_2)^{-1}$. Since $N_1\times N_2$ normalizes $U^-_{r_1,r_2}$ it follows that $n\mu n^{-1}\in U^-$. We therefore have $\mu\in U(\mathbf{Z}_p)\cap n^{-1}U^-n=U_n(\mathbf{Z}_p)$ and the proof is complete. □

As an example of how this theorem can be used to prove identities among Kloostermann sums we mention the following corollary.

(3.11) **Corollary.** *Let* $n=(n_1\times n_2)w^{-1}$ *where* $n_i\in N_i$ $(i=1,2)$ *and* $w\in W$ *satisfies* (3.5)(a). *Then* $U_n=w(U_{n_1}\times U_{n_2})w^{-1}$. *If* $\psi: U/U(\mathbf{Z}_p)\to\mathbf{C}^*$, $\psi': U_n/U_n(\mathbf{Z}_p)\to\mathbf{C}^*$ *are characters and* $\psi_i: U_i\to\mathbf{C}^*$, $\psi'_i: U_{n_i}\to\mathbf{C}^*$ *are the characters for which* $\psi|U_1\times U_2=\psi_1\times\psi_2$, $\psi'_w=\psi'_1\times\psi'_2$ *(recall* $\psi'_w(u)=\psi'(wuw^{-1})$*) then*

$$Kl_p(n,\psi,\psi')=Kl_p(n_1,\psi_1,\psi'_1)\cdot Kl_p(n_2,\psi_2,\psi'_2).$$

The proof of this is straightforward using part (c) of the theorem. We leave it to the reader.

We are now in position to characterize those $n \in N$ for which $X(n) \neq \phi$.

(3.12) **Theorem.** *Let $n \in N$. Then $C(n) \neq \phi$ (and hence $X(n)$, $Y(n) \neq \phi$) if and only if $\det(n) \in \mathbf{Z}_p^*$ and every exposed subdeterminant of n is integral.*

Proof. From Lemma 3.4 we see that if $C(n) \neq \phi$ then every exposed determinant in n is integral. Clearly, we must also have $\det(n) \in \mathbf{Z}_p^*$.

So suppose $\det(n) \in \mathbf{Z}_p^*$ and every exposed determinant is integral. We will prove $C(n) \neq \phi$ by induction on r. The theorem is trivial when $r=1$. We therefore assume the theorem to be known for $GL(r')$ with $r'<r$.

We first consider the case where some *proper* exposed determinant n_{IJ} is in $\mathbf{Z}_p^*$. Let $r_2 = \# I = \# J$ and $r_1 = r - r_2$. Then as in Theorem 3.7 we can choose $v, w \in \mathscr{W}$ and $n_i \in N_i$ $(i=1,2)$ satisfying (3.5). We will show that n_1, n_2 satisfy the hypotheses of the theorem.

We have $\det(n_2) = n_{IJ} \in \mathbf{Z}_p^*$ and since $\det(n) \in \mathbf{Z}_p^*$ we also have $\det(n_1) \in \mathbf{Z}_p^*$.

Let $I', J' \subsetneqq I_2 = \{r_1+1, \ldots, r\}$ be subsets of order k for which $(n_2)_{I'J'}$ is exposed in n_2. Then $(n_2)_{I'J'} = n_{v(I'), w(J')}$. Since $v(I_2) = I$, $w(I_2) = J$ and w, v are order preserving on I_2 we see that $n_{v(I'), w(J')}$ is an exposed subdeterminant of n_{IJ} and therefore also of n. Thus $(n_2)_{I'J'} = n_{v(I'), w(J')}$ is integral and we conclude that n_2 satisfies the hypotheses of the theorem.

Now let $I', J' \subsetneqq I_1 = \{1, \ldots, r_1\}$ such that $(n_1)_{I'J'}$ is exposed in n_1. Let $I'' = v(I') \cup I$ and $J'' = w(J') \cup J$. Then $n_{I''J''}$ is an exposed subdeterminant of n and we therefore have $n_{I''J''} \in \mathbf{Z}_p$. Since $n_{I''J''} = \pm (n_1)_{I'J'} \cdot n_{IJ}$ and $n_{IJ} \in \mathbf{Z}_p^*$ it follows that $(n_1)_{I'J'} \in \mathbf{Z}_p$. Thus n_1 also satisfies the hypotheses of the theorem.

By the inductive hypothesis $C(n_1)$ and $C(n_2)$ are nonempty. Using Theorem 3.7 we conclude $C(n) \neq \phi$.

Finally, we consider the general case. Let $\pi = \begin{pmatrix} p & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$. If $n \in N$ and n_{IJ} is exposed then

$$\operatorname{ord}_p(n_{IJ}) - 1 \leqq \operatorname{ord}_p((\pi n \pi^{-1})_{IJ}) \leqq \operatorname{ord}_p(n_{IJ}) + 1$$

and if n is not a diagonal matrix then $\operatorname{ord}_p((\pi n \pi^{-1})_{IJ}) = \operatorname{ord}_p(n_{IJ}) - 1$ for some exposed determinant n_{IJ}. If n is diagonal and satisfies the hypotheses of the theorem then all of its entries are units and $n \in C(n) \neq \phi$.

For n not diagonal the above remarks guarantee the existence of a maximal k for which $m = \pi^k n \pi^{-k}$ has integral exposed determinants. Then some exposed determinant m_{IJ} of m is a unit. Thus by what has been proved above we have $C(m) \neq \phi$. Let $\beta \in C(m)$. Since $G(\mathbf{Z}_p) = U(\mathbf{Z}_p) \cdot B^-(\mathbf{Z}_p) \cdot U(\mathbf{Z}_p)$ we may assume $\beta \in B^-(\mathbf{Z}_p)$ is a lower triangular matrix. Let $\gamma = \pi^{-k} \beta \pi^k$. If we write $\beta = umu'$ with $u, u' \in U$, then $\gamma = (\pi^{-k} u \pi^k) n (\pi^{-k} u' \pi^k) \in UnU$. Since β is integral and lower triangular, so also is γ. Thus $\gamma \in UnU \cap G(\mathbf{Z}_p) = C(n)$ and we have proved $C(n) \neq \phi$. □

4. Decompositions of Kloostermann Sums

Motivated by the well known case $r=2$, we expect that to give good estimates for the Kloostermann sums, we need first to give them a more algebraic geometric interpretation. In fact, it is quite easy to construct algebraic varieties $\mathbf{X}(n)/\mathbf{F}_p$, $n\in N(\mathbf{Q}_p)$, whose $\mathbf{F}_p$-rational points are in one to one correspondence with the elements of $X(n)$ [see (2.8)].

However, two problems arise which prevent us from simply plugging into the general theory of exponential sums [3, 5, 8]. First, the general theory is concerned with characters of $\mathbf{F}_p$, but in our situation we have a character of $U(\mathbf{Q}_p)$ which should be viewed as a product of characters of $\mathbf{Q}_p/\mathbf{Z}_p$. Second, to prove purity of weights it is usually assumed that the varieties are nonsingular. Interestingly, examples show that the varieties $\mathbf{X}(n)$ are in general *not* smooth.

Judging from the ease with which the first problem is handled in the case $r=2$ (see [14]), we expect that the second problem is the more serious one. One approach to circumventing it would be to construct a smooth stratification of $\mathbf{X}(n)$ (see the remarks at the end of Sect. 5 for an example). This corresponds to decomposing the Kloostermann sums into smaller sums, one for each smooth strata in $\mathbf{X}(n)$. Estimating these smaller sums would then lead to good estimates for the total sums. Since this program has only barely been begun, we will only sketch the basic constructions here, and look at an example in the next section.

Most of this section will be devoted to another decomposition of $X(n)$ provided by the orbits of an action of $T(\mathbf{Z}_p)$ on $X(n)$. This decomposition is finer than the one we get from a smooth stratification. We therefore do not expect the resulting estimates for Kloostermann sums to be best possible, though they will improve the trivial estimates. The sums which arise from this orbit decomposition are easy to describe without reference to $GL(r)$ (see Definition 4.9) but do not seem to have appeared in the literature before.

We begin with the observation that $U(\mathbf{Z}_p)\backslash U(\mathbf{Q}_p)$ can be identified with a product of a number of copies of $\mathbf{Q}_p/\mathbf{Z}_p$. By the Witt construction $\mathbf{Q}_p/\mathbf{Z}_p$ is naturally identified with an inductive limit of affine spaces over $\mathbf{F}_p$. In this way $U(\mathbf{Z}_p)\backslash U(\mathbf{Q}_p)$ becomes the set of rational points of an inductive limit $\mathbf{U}/\mathbf{F}_p$ of affine spaces. Similarly we identify $U_n(\mathbf{Q}_p)/U_n(\mathbf{Z}_p)$ with a limit $\mathbf{U}_n/\mathbf{F}_p$ of affine spaces.

The maps u, u' of (2.9) provide an inclusion

$$(u,u'):X(n)\hookrightarrow \mathbf{U}\times\mathbf{U}_n(\mathbf{F}_p).$$

The condition $u(x)\cdot n\cdot u'(x)\in G(\mathbf{Z}_p)$ translates into a system of algebraic equations in the coordinates of $\mathbf{U}\times\mathbf{U}_n$. These equations define an algebraic variety, $\mathbf{X}(n)/\mathbf{F}_p$, contained in some finite layer of the inductive limit $\mathbf{U}\times\mathbf{U}_n$, and for which we have

$$X(n)=\mathbf{X}(n)(\mathbf{F}_p).$$

In this way we realize the Kloostermann sum $Kl_p(n,\psi,\psi')$ as a sum of character values over the $\mathbf{F}_p$-rational points of the affine variety $\mathbf{X}(n)$.

It would be quite interesting to give a description in terms of the algebraic group $GL(r)$ of a smooth stratification of $\mathbf{X}(n)$. We will return to this problem in a future paper. In the next section we look at the special case $r=3$.

We turn now to the orbit decomposition of $X(n)$. Let $t\in T(\mathbf{Z}_p)$ and set $s=n^{-1}tn\in T(\mathbf{Z}_p)$. If $\gamma\in C(n)$ then $\gamma=unu'$ with $u,u'\in U$ and $t\gamma s^{-1}$

$=(tut^{-1})n(su's^{-1})\in UnU\cap G(\mathbf{Z}_p)=C(n)$. Since conjugation by t and s preserves $U(\mathbf{Z}_p)$ and $U_n(\mathbf{Z}_p)$ the map $T(\mathbf{Z}_p)\times C(n)\to C(n)$, $(t,\gamma)\mapsto t\gamma s^{-1}$ descends to an action of $T(\mathbf{Z}_p)$ on $X(n)$:

$$(4.1)\qquad \begin{array}{ccc} T(\mathbf{Z}_p) & \times\ X(n)\longrightarrow & X(n) \\ t & ,\quad x & \longmapsto t*x. \end{array}$$

For characters $\psi,\psi': U/U(\mathbf{Z}_p)\to\mathbf{C}^*$ the decomposition of $X(n)$ into $T(\mathbf{Z}_p)$-orbits leads to the following decomposition of the Kloostermann sums:

$$(4.2)\qquad Kl_p(n,\psi,\psi')=\sum_{x\in T(\mathbf{Z}_p)\backslash X(n)}\ \sum_{T(\mathbf{Z}_p)*x}\psi(u(x))\cdot\psi'(u'(x)).$$

Here $T(\mathbf{Z}_p)\backslash X(n)$ is a set of representations $x\in X(n)$ for the $T(\mathbf{Z}_p)$-orbits and $T(\mathbf{Z}_p)*x$ is the orbit through x.

To describe the inner sum in (4.2) it helps to have some more notation. The roots of the standard torus T in $GL(r)$ are the characters $\lambda_{ij}: T\to GL(1)$ given by

$$(4.3)\qquad \lambda_{ij}\begin{pmatrix} t_1 & & 0\\ & \ddots & \\ 0 & & t_r\end{pmatrix}=t_it_j^{-1},\qquad 1\leqq i,j\leqq r,\ i\neq j.$$

Let $\Delta=\{\lambda_{i,i+1}\mid 1\leqq i<r\}$ be the root basis associated to the standard unipotent subgroup U. This induces the ordering on the roots given by $\lambda_{ij}>0\Leftrightarrow i<j$. The action of the Weyl group $\mathscr{W}$ on the roots is given by $w(\lambda)(t)=\lambda(w^{-1}tw)$ for $w\in\mathscr{W}$, $t\in T$ and λ a root. We then have $w(\lambda_{ij})=\lambda_{w(i),w(j)}$. Finally, let

$$(4.4)\qquad \Delta_w=\{\lambda\in\Delta\mid w(\lambda)<0\}=\{\lambda_{i,i+1}\mid w(i+1)<w(i)\}.$$

We return now to the inner sum of (4.2). If w is the element of the Weyl group associated to $n\in N$, then the condition $u\in U_n$ is equivalent to the statements $u_{ii}=1 (1\leqq i\leqq r)$ and, when $i\neq j$,

$$(4.5)\qquad u_{ij}=0 \text{ unless } \begin{cases} i<j & \text{and}\\ w(i)>w(j). \end{cases}$$

Now write

$$(4.6)\qquad u(x)=\begin{pmatrix} 1 & x_1 & * & & *\\ 0 & 1 & x_2 & \ddots & \\ & \ddots & \ddots & \ddots & *\\ & & 0 & 1 & x_{r-1}\\ 0 & & & 0 & 1\end{pmatrix},\quad u'(x)=\begin{pmatrix} 1 & x'_1 & * & & *\\ 0 & 1 & x'_2 & \ddots & \\ & \ddots & \ddots & \ddots & *\\ & & 0 & 1 & x'_{r-1}\\ 0 & & & 0 & 1\end{pmatrix}.$$

Since $u'(x)\in U_n$ we have $x'_i=0$ unless $w(i+1)<w(i)$. For $1\leqq i<r$ define $\kappa_i: X(n)\to\mathbf{Q}_p/\mathbf{Z}_p$ and if also $w(i+1)<w(i)$ define $\kappa'_i: X(n)\to\mathbf{Q}_p/\mathbf{Z}_p$ by

$$(4.7)\qquad \begin{aligned}\kappa_i(x)&=x_i,\\ \kappa'_i(x)&=x'_i.\end{aligned}$$

For $t \in T(\mathbf{Z}_p)$ we then have

$$\kappa_i(t * x) = \underline{\lambda}_{i,i+1}(t) \cdot \kappa_i(x), \qquad \kappa_i'(t * x) = \underline{\lambda}_{w(i),w(i+1)}(t) \cdot \kappa_i'(x). \tag{4.8}$$

We are therefore led to make the following definitions.

(4.9) *Definition.* (a) For $\ell > 0$ and $w \in \mathscr{W}$ let

$$A_w(\ell) = (\mathbf{Z}/p^\ell\mathbf{Z})^\Delta \times (\mathbf{Z}/p^\ell\mathbf{Z})^{\Delta_w} = \prod_{i=1}^{r-1} (\mathbf{Z}/p^\ell\mathbf{Z}) \times \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{r-1} (\mathbf{Z}/p^\ell\mathbf{Z}).$$

A typical element of $A_w(\ell)$ will be denoted

$$\underline{\lambda} \times \underline{\lambda}' = (\lambda_i)_{i=1,\dots,r-1} \times (\lambda_i')_{\substack{i=1,\dots,r-1 \\ w(i+1)<w(i)}}.$$

(b) Let

$$V_w(\ell) = \left\{ \underline{\lambda} \times \underline{\lambda}' \in A_w(\ell) \;\middle|\; \begin{matrix} \lambda_i \in (\mathbf{Z}/p^\ell\mathbf{Z})^* \qquad \text{and} \\ \lambda_i' \cdot \prod_{w(i+1) \leq j < w(i)} \lambda_j = 1 \end{matrix} \right\}.$$

(c) For a character $\theta : A_w(\ell) \to \mathbf{C}^*$ define

$$S_w(\theta;\ell) = \sum_{v \in V_w(\ell)} \theta(v).$$

Using the notation (1.1), let $\psi = \psi_{\underline{v}}$, $\psi' = \psi_{\underline{v}'}$ be characters of U where $\underline{v} = (v_1, \dots, v_{r-1})$, $\underline{v}' = (v_1', \dots, v_{r-1}')$ are in $\mathbf{Z}_p^{r-1}$.

(4.10) **Theorem.** *Let $n \in N$ and ℓ be large enough so that the matrix entries of $u(x)$, $u'(x)$ lie in $\frac{1}{p^\ell}\mathbf{Z}_p/\mathbf{Z}_p$ for every $x \in X(n)$. Let $\kappa_i(x)$, $\kappa_i'(x)$ be as in* (4.7) *and define the character $\theta_x : A_w(\ell) \to \mathbf{C}^*$ by*

$$\theta_x(\underline{\lambda} \times \underline{\lambda}') = \prod_{i=1}^{r-1} \xi(\lambda_i \cdot v_i \cdot \kappa_i(x)) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{r-1} \xi(\lambda_i' \cdot v_i' \cdot \kappa_i'(x)).$$

If $N(x)$ denotes the number of elements in the orbit through an element x of $X(n)$ then

$$Kl_p(n, \psi, \psi') = [p^\ell(1-p^{-1})]^{1-r} \cdot \sum_{x \in T(\mathbf{Z}_p)\backslash X(n)} N(x) \cdot S_w(\theta_x;\ell).$$

Proof. This is simply a restatement of (4.2). Note that $[p^\ell(1-p^{-1})]^{r-1}$ is the number of elements in $V_w(\ell)$. □

Giving good estimates for the sums $S_w(\theta;\ell)$ will therefore lead to improvements on the trivial estimates for the sums $Kl(n, \psi, \psi')$ once we have understood the orbit structure of $X(n)$.

We content ourselves here with a few examples. In each of these examples, $\theta : A_w(\ell) \to \mathbf{C}^*$ will be the character defined by

$$\theta(\underline{\lambda} \times \underline{\lambda}') = \prod_{i=1}^{r-1} \xi\left(\frac{v_i\lambda_i}{p^\ell}\right) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{r-1} \xi\left(\frac{v_i'\lambda_i'}{p^\ell}\right), \tag{4.11}$$

with $v_i, v'_i \in \mathbf{Z}_p$. As always $\xi: \mathbf{Q}_p/\mathbf{Z}_p \to \mathbf{C}^*$ is the standard additive character.

(4.12) *Example.* Let $w_0 = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}$ be the long element of the Weyl group.

Then $\Delta_{w_0} = \Delta$, $A_{w_0}(\ell) = (\mathbf{Z}/p^\ell\mathbf{Z})^{2(r-1)}$, and

$$V_{w_0}(\ell) = \{\underline{\lambda} \times \underline{\lambda}' \in A_{w_0}(\ell) \mid \lambda_i \cdot \lambda'_{r-i} = 1, \text{ for } 1 \leqq i \leqq r-1\}.$$

Thus if $\theta: A_{w_0}(\ell) \to \mathbf{C}^*$ is given by (4.11) then

$$S_{w_0}(\theta; \ell) = \prod_{i=1}^{r-1} S_2(v_i, v'_{r-i}; p^\ell)$$

where

$$S_2(v, v'; p^\ell) = \sum_{\substack{\lambda, \lambda' \in (\mathbf{Z}/p^\ell\mathbf{Z}) \\ \lambda\lambda' = 1}} \xi\left(\frac{v\lambda + v'\lambda'}{p^\ell}\right)$$

is the classical $GL(2)$-Kloostermann sum.

(4.13) *Example.* Let $r = r_1 + r_2$ with $r_1, r_2 > 0$ and let

$$w = \begin{pmatrix} 0 & 1_{r_1} \\ 1_{r_2} & 0 \end{pmatrix}.$$

Then

$$w(i) = \begin{cases} i + r_1 & \text{if } \ 1 \leqq i \leqq r_2, \\ i - r_2 & \text{if } \ r_2 < i \leqq r, \end{cases}$$

and Δ_w consists of the single element $\underline{\lambda}_{r_2, r_2+1}$. We have

$$A_w(\ell) = \{(\lambda_1, \dots, \lambda_{r-1}) \times (\lambda'_{r_2}) \in (\mathbf{Z}/p^\ell\mathbf{Z})^r\},$$
$$V_w(\ell) = \{(\lambda_1, \dots, \lambda_{r-1}) \times (\lambda'_{r_2}) \in A_w(\ell) \mid \lambda'_{r_2} \cdot \lambda_1 \dots \lambda_{r-1} = 1\}.$$

If $\theta: A_w(\ell) \to \mathbf{C}^*$ is given by (4.11) then

$$S_w(\theta; \ell) = \sum_{\substack{\lambda_i, \lambda'_{r_2} \in \mathbf{Z}/p^\ell\mathbf{Z} \\ \lambda'_{r_2} \prod \lambda_i = 1}} \xi\left(\frac{v_1\lambda_1 + \dots + v_{r-1}\lambda_{r-1} + v'_{r_2}\lambda'_{r_2}}{p^\ell}\right).$$

For $\ell = 1$ this sum was estimated by Deligne [5]. For $r = 3$ and ℓ arbitrary Larsen [9] gave estimates and more recently Friedberg [6] has given estimates in the general case.

(4.14) *Example.* In the preceding example we now set $r_1 = 1$, $r_2 = r - 1$ and let $n \in N(\mathbf{Q}_p)$ lie over w. We look at the orbit structure of $X(n)$.

As in Theorem 3.2(a) we may assume without loss of generality that n has the form

$$n=\begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^{r-1}p^{-b} \\ p^{a_1} & 0 & & & 0 \\ & p^{a_2} & \ddots & & \vdots \\ & & \ddots & 0 & 0 \\ 0 & & & p^{a_{r-1}} & 0 \end{pmatrix}.$$

By Theorem 3.12 we may suppose $a_i \geqq 0$, $i=1,\dots,r-1$, and $b=\sum a_i$.

If $a_i=0$ for some i then using Theorem 3.7 the orbit structure of $X(n)$ can be deduced from that of similar coset spaces arising from $GL(r-1)$.

If $a_i>0$ for all i and if $\gamma \in C(n)$ then Lemma 3.4 assures us that γ is congruent modulo p to an upper triangular matrix. Thus the diagonal entries of γ are in $\mathbf{Z}_p^*$ and we can find an element $u \in U(\mathbf{Z}_p)$ for which $u\gamma$ is lower triangular. We conclude that any $x \in X(n)$ is represented by a matrix of the form

$$x=\begin{pmatrix} t_1t_2^{-1} & & & & 0 \\ p^{a_1} & t_2t_3^{-1} & & & \\ & p^{a_2} & \ddots & & \\ & & \ddots & t_{r-1}t_r^{-1} & \\ 0 & & & p^{a_{r-1}} & t_rt_1^{-1} \end{pmatrix}$$

where $t_1,\dots,t_r \in \mathbf{Z}_p^*$. If we set $t=\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_r \end{pmatrix} \in T(\mathbf{Z}_p)$ and $s=n^{-1}tn$ then $x=tx_0s^{-1}$ $=t*x_0$ where

$$x_0=\begin{pmatrix} 1 & & & & 0 \\ p^{a_1} & 1 & & & \\ & p^{a_2} & \ddots & & \\ & & \ddots & 1 & \\ 0 & & & p^{a_{r-1}} & 1 \end{pmatrix}.$$

It follows that the orbit represented by x_0 in $X(n)$ is the only orbit.

Factoring x_0 according to its Bruhat decomposition we find

$$(4.15)\quad x_0=\begin{pmatrix} 1 & p^{-a_1} & & & 0 \\ & 1 & p^{-a_2} & & \\ & & \ddots & \ddots & \\ & & & 1 & p^{-a_{r-1}} \\ 0 & & & & 1 \end{pmatrix} n \begin{pmatrix} 1 & 0 & \dots & 0 & (-1)^r p^{-b_1} \\ & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & -p^{-b_{r-2}} \\ & & & 1 & p^{-b_{r-1}} \\ 0 & & & & 1 \end{pmatrix},$$

where $b_k=\sum_{i=k}^{r-1} a_i$.

Now let $\ell = \max(a_i)$, and suppose $\psi = \psi_{\underline{v}}$, $\psi' = \psi_{\underline{v}'}$ are characters of $U(\mathbf{Q}_p)$ with $\underline{v}, \underline{v}' \in (\mathbf{Z}_p)^{r-1}$. Then

$$Kl(n, \psi, \psi') = \frac{\# X(n)}{\# V_w(\ell)} \cdot S_w(\theta; \ell),$$

where $\theta: A_w(\ell) \to \mathbf{C}^*$ is given by

$$\theta(\underline{\lambda} \times \underline{\lambda}') = \xi\left(\frac{v'_{r-1}\lambda'_{r-1}}{p^{a_{r-1}}} + \sum_{i=1}^{r-1} \frac{v_i\lambda_i}{p^{a_i}}\right).$$

Since the map $u': X(n) \to U_n(\mathbf{Q}_p)/U_n(\mathbf{Z}_p)$ is injective, we can use (4.15) to calculate

$$\# X(n) = \prod_{i=1}^{r-1} \phi(p^{b_i}),$$

where ϕ is Euler's totient function. Since clearly $\# V_w(\ell) = \phi(p^\ell)^{r-1}$ we conclude

$$Kl(n, \psi, \psi') = p^{(\Sigma b_i) - (r-1)\ell} S_w(\theta; \ell).$$

(4.16) *Example.* In Example 4.13 set $r_1 = r-1$, $r_2 = 1$ and let $n \in N(\mathbf{Q}_p)$ lie over w. The involution ι of (3.1) sends n to an element of the form considered in Example 4.15. Theorem 3.2(b) then reduces this case to the preceding one.

5. *GL* (3)-Kloostermann Sums

In this section we set $r = 3$ and use Theorem 4.10 to estimate the Kloostermann sums attached to the long element $w_0 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$ of the Weyl group. Note that the sums attached to the identity element are trivial, those associated to the transpositions (12) and (23) are $GL(2)$-sums (Theorem 3.7) and those associated to the 3-cycles (123) and (132) have been estimated by Larsen [9]. This leaves only the long element considered here.

Theorem 5.1. *Let $\psi, \psi': U/U(\mathbf{Z}_p) \to \mathbf{C}^*$ be characters,* $n = \begin{pmatrix} & & p^{-r} \\ & -p^{r-s} & \\ p^s & & \end{pmatrix} \in N$, *and $\sigma = \min(r, s)$. Then there is a constant C depending only on ψ, ψ' for which*

$$|Kl_p(n, \psi, \psi')| \leqq C(r+1)(s+1)p^{\frac{\sigma+r+s}{2}}.$$

We will prove this by tabulating the $T(\mathbf{Z}_p)$ orbits in $X(n)$ and using Example 4.12 to estimate the sum over each orbit.

We begin with the observation that the map u' of (2.9) furnishes an injection

$$u': X(n) \hookrightarrow U/U(\mathbf{Z}_p).$$

This follows from the uniqueness of the Bruhat decomposition and the equality $U = U_n$. Since also

$$u'(t * x) = s \cdot u'(x) \cdot s^{-1}$$

for $t \in T(\mathbf{Z}_p)$ and $s = n^{-1}tn$, we see that the orbits in $X(n)$ correspond to $T(\mathbf{Z}_p)$-conjugacy classes in $U/U(\mathbf{Z}_p)$.

To determine which u in $U/U(\mathbf{Z}_p)$ lie in $u'(X(n))$ we will use the following easily established lemma.

(5.2) **Lemma.** *Let $r \geqq 1$ and for each $k = 1, \ldots, r$ let $I_k = \{r-k+1, \ldots, r\}$ be the final k-element subset of $\{1, \ldots, r\}$. Let $g, g' \in GL(r, \mathbf{Q}_p)$. Then $g' \in U(\mathbf{Q}_p)g$ if and only if for each $k = 1, \ldots, r$ and every k-element subset $I \subseteqq \{1, \ldots, r\}$ we have $g_{I_k, I} = g'_{I_k, I}$. (In words, this last equality asserts that the bottom row of $k \times k$ subdeterminants of g agrees with that for g'.)* □

We return now to $G = GL(3)$ and fix $n = \begin{pmatrix} & & p^{-r} \\ & -p^{r-s} & \\ p^s & & \end{pmatrix}$ once and for all.

Suppose we are given $\lambda \in \mathbf{Z}_p^*$ and nonnegative integers a, b satisfying

$$(5.3)\quad \begin{array}{ll} \text{(i)} & a \leqq s, \quad b \leqq r; \\ \text{(ii)} & \mu = p^r(p^{-a-b} - \lambda p^{-s}) \in \mathbf{Z}_p^*. \end{array}$$

Then there is an element $x_{a,b}^\lambda \in X(n)$ for which

$$(5.4)\qquad u'(x_{a,b}^\lambda) = \begin{pmatrix} 1 & p^{-a} & \lambda p^{-s} \\ 0 & 1 & p^{-b} \\ 0 & 0 & 1 \end{pmatrix} \quad (\operatorname{mod} U(\mathbf{Z}_p)).$$

Indeed, we have the matrix identity

$$(5.5)\ \begin{pmatrix} \mu^{-1} & 0 & 0 \\ \lambda^{-1}p^{r-b} & \lambda^{-1}\mu & 0 \\ p^s & p^{s-a} & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \mu^{-1}p^{s-r-a} & \mu^{-1}p^{-s} \\ 0 & 1 & \lambda^{-1}p^{r-s-b} \\ 0 & 0 & 1 \end{pmatrix} n \begin{pmatrix} 1 & p^{-a} & \lambda p^{-s} \\ 0 & 1 & p^{-b} \\ 0 & 0 & 1 \end{pmatrix}.$$

Our conditions on a, b, λ, μ assure that this matrix lies in $C(n)$. We take $x_{a,b}^\lambda$ to be the associated element of $X(n)$.

(5.6) *Definition.* Let ψ, ψ' be characters of $U/U(\mathbf{Z}_p)$.

(a) For a, b, and λ satisfying (5.3) let

$$X_{a,b}^\lambda(n) = T(\mathbf{Z}_p) * x_{a,b}^\lambda$$

be the orbit through $x_{a,b}^\lambda$ and let

$$S_{a,b}^\lambda(n, \psi, \psi') = \sum_{x \in X_{a,b}^\lambda(n)} \psi(u(x)) \cdot \psi'(u'(x))$$

be the Kloostermann sum restricted to this orbit.

(b) For a, b satisfying (5.3)(i) let

$$X_{a,b}(n) = \bigcup_\lambda X_{a,b}^\lambda(n)$$

where λ runs over the elements of $\mathbf{Z}_p^*$ satisfying (5.3)(ii). Let

$$S_{a,b}(n, \psi, \psi') = \sum_{x \in X_{a,b}(n)} \psi(u(x)) \cdot \psi'(u'(x)).$$

(5.7) **Lemma.** $X(n)=\coprod_{a,b} X_{a,b}(n)$ *where* $a,b\geqq 0$ *run over integers satisfying* (5.3)(i).

Proof. The union is clearly disjoint and is contained in $X(n)$ by definition. So let $x\in X(n)$. We will show $x\in X^{\lambda}_{a,b}(n)$ for some triple a,b,λ satisfying (5.3).

Since $G(\mathbf{Z}_p)=U(\mathbf{Z}_p)\cdot B^-(\mathbf{Z}_p)\cdot U(\mathbf{Z}_p)$ where B^- is the group of lower triangular matrices, we see that x is represented by an element β in $C(n)\cap B^-(\mathbf{Z}_p)$. Replacing x by some other element of its $T(\mathbf{Z}_p)$-orbit if necessary, we may assume $\beta=u\cdot n\cdot u'$ with $u,u'\in U$ and $u'=\begin{pmatrix}1 & p^{-a} & \lambda p^{-c}\\ 0 & 1 & p^{-b}\\ 0 & 0 & 1\end{pmatrix}$ where $a,b,c\geqq 0$ and $\lambda\in\mathbf{Z}_p^*$. Since the matrix

$$nu'=\begin{pmatrix}0 & 0 & p^{-r}\\ 0 & -p^{r-s} & -p^{r-s-b}\\ p^s & p^{s-a} & \lambda p^{s-c}\end{pmatrix}$$

lies in $U(\mathbf{Q}_p)\beta$ we can apply Lemma 5.2 first to the bottom row and then to the bottom row of 2×2 subdeterminants of the pair nu',β to conclude $s\geqq a$, $s=c$, and $r\geqq b$, $p^r(p^{-a-b}-\lambda p^{-c})\in\mathbf{Z}_p^*$. So we see that a,b,λ satisfy (5.3) and that $x=x^{\lambda}_{a,b}$ which lies in $X^{\lambda}_{a,b}(n)$. This proves the lemma. □

Let $v_1,v_2,v'_1,v'_2\in\mathbf{Z}_p$ and define the characters ψ,ψ' of $U/U(\mathbf{Z}_p)$ by

(5.8)
$$\psi\begin{pmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{pmatrix}=\xi(v_1x_1+v_2x_2),$$
$$\psi'\begin{pmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{pmatrix}=\xi(v'_1x_1+v'_2x_2).$$

(5.9) **Theorem.** *Let* $\sigma=\min(r,s)$ *and* $a\leqq s$, $b\leqq r$ *be nonnegative integers. Then*

$$|S_{a,b}(n,\psi,\psi')|\leqq|v_1v_2v'_1v'_2|_p^{-1/2}\cdot p^{\sigma+\frac{a+b}{2}}.$$

Proof. The involution ι of (3.1) sends $X_{a,b}(n)$ to $X_{b,a}(n^{\iota})$. Composing ψ and ψ' with ι has the effect of replacing (v_1,v_2) by $(-v_2,-v_1)$ and (v'_1,v'_2) by $(-v'_2,-v'_1)$. Applying ι to n reverses the roles of r and s. Thus we may assume $r\geqq s$ without loss of generality.

The inequalities $r\geqq s\geqq a$ and $r\geqq b$ imply that the matrix entries of $u(x)$ and $u'(x)$ lie in $p^{-r}\mathbf{Z}_p/\mathbf{Z}_p$ for every $x\in X(n)$. Indeed, by Lemma 5.7 it is enough to verify this for $x=x^{\lambda}_{a,b}$. But $x=x^{\lambda}_{a,b}$ is represented by the matrix (5.5) where $u(x)$ and $u'(x)$ are visibly displayed. The claim is now easily verified.

Now let $\mathcal{S}$ be a finite subset of $\mathbf{Z}_p^*$ such that $X_{a,b}(n)$ is the disjoint union of the $X^{\lambda}_{a,b}(n)$ with $\lambda\in\mathcal{S}$. Then as in Theorem 4.10 we have

(5.10)
$$S_{a,b}(n,\psi,\psi')=p^{-2r}(1-p^{-1})^{-2}\cdot\sum_{\lambda\in\mathcal{S}}\#(X^{\lambda}_{a,b}(n))\cdot S_{w_0}(\theta^{\lambda}_{a,b},r),$$

where S_{w_0} is defined in 4.9 and $\theta^\lambda_{a,b}: A_{w_0}(r) \to \mathbf{C}^*$ is the character given by

$$\theta^\lambda_{a,b}(\underline{\lambda} \times \underline{\lambda}') = \xi\left(\frac{(v_1\mu^{-1}p^{s-a})\lambda_1 + (v_2\lambda^{-1}p^{2r-s-b})\lambda_2 + (v'_1p^{r-a})\lambda'_1 + (v'_2p^{r-b})\lambda'_2}{p^r}\right).$$

By Example 4.12 we have

$$(5.11)\quad S_{w_0}(\theta^\lambda_{a,b}; r) = S_2(v_1\mu^{-1}p^{s-a}, v'_2p^{r-b}; p^r) \cdot S_2(v_2\lambda^{-1}p^{2r-s-b}, v'_1p^{r-a}; p^r)$$

where S_2 is the classical $GL(2)$-Kloostermann sum [see (4.12)].

The inequality

$$(5.12)\qquad |S_2(\lambda, \mu; p^r)| \leqq 2(p^r \cdot gcd(|\lambda|_p^{-1}, |\mu|_p^{-1}, p^r))^{1/2}$$

for $\lambda, \mu \in \mathbf{Z}_p$ is well known [14]. In order to apply this bound to (5.11) we first note

$$\begin{aligned} gcd(|v_1p^{s-a}|_p^{-1}, |v'_2p^{r-b}|_p^{-1}, p^r) &\leqq |v_1v'_2|_p^{-1} \cdot gcd(p^{s-a}, p^{r-b}, p^r) \\ &= |v_1v'_2|_p^{-1} \cdot gcd(p^{s-a}, p^{r-b}) \\ &\leqq |v_1v'_2|_p^{-1} \cdot p^{\frac{(s-a)+(r-b)}{2}}, \end{aligned}$$

and similarly

$$gcd(|v_2p^{2r-s-b}|_p^{-1}, |v'_1p^{r-a}|_p^{-1}, p^r) \leqq |v_2v'_1|_p^{-1} \cdot p^{\frac{(2r-s-b)+(r-a)}{2}}.$$

Combining these inequalities with (5.11) and (5.12) we obtain the bound

$$(5.13)\qquad |S_{w_0}(\theta^\lambda_{a,b}; r)| \leqq 4 \cdot |v_1v_2v'_1v'_2|_p^{-1/2} \cdot p^{2r-\frac{a+b}{2}}.$$

This inequality together with (5.10) give

$$|S_{a,b}(n)| \leqq 4 \cdot |v_1v_2v'_1v'_2|_p^{-1/2} \cdot (1-p^{-1})^{-2} \cdot p^{-\frac{a+b}{2}} \cdot \sum_{\lambda \in \mathcal{S}} \#(X^\lambda_{a,b}(n)).$$

The sum appearing on the right hand side is equal to $\#(X_{a,b}(n))$ which is $\leqq p^{a+b+s}$. Since $p \geqq 2$ we have $(1-p^{-1})^{-2} \leqq \frac{1}{4}$ and the theorem follows. □

Proof of Theorem 5.1. Let $\ell = \max(r,s) = r+s-\sigma$. If $X_{a,b}(n)$ is nonempty then there is a $\lambda \in \mathbf{Z}_p^*$ such that a, b, λ satisfy (5.3). The condition $p^r(p^{-a-b} - \lambda p^{-s}) \in \mathbf{Z}_p^*$ implies $a+b \leqq \ell$.

By Theorem 5.9 we therefore have

$$|S_{a,b}(n, \psi, \psi')| \leqq C \cdot p^{\sigma + \frac{a+b}{2}} \leqq C \cdot p^{\sigma + \frac{\ell}{2}}$$

where $C = |v_1v_2v'_1v'_2|_p^{-1/2}$ depends only on ψ, ψ'. Theorem 5.1 now follows from the equality $Kl(n, \psi, \psi') = \sum_{\substack{a \leqq s \\ b \leqq r}} S_{a,b}(n, \psi, \psi')$. □

There is no reason to believe that the exponent $\frac{\sigma+r+s}{2}$ appearing in Theorem 5.1 is best possible. Because we have treated the sums $S^\lambda_{a,b}$ as the basic sums our proof of Theorem 5.1 does not account for the quite likely possibility of cancellations among the $S^\lambda_{a,b}$ in the sum $S_{a,b} = \sum_\lambda S^\lambda_{a,b}$.

In Sect. 4 we indicated how to construct an algebraic variety $\mathbf{X}(n)/\mathbf{F}_p$ for which $X(n)=\mathbf{X}(n)(\mathbf{F}_p)$. A similar construction leads to subvarieties $\mathbf{X}_{a,b}(n)/\mathbf{F}_p \subsetneq \mathbf{X}(n)/\mathbf{F}_p$ for which $X_{a,b}(n)=\mathbf{X}_{a,b}(n)(\mathbf{F}_p)$.

Moreover, it can be shown that $\mathbf{X}(n)=\coprod_{a,b} \mathbf{X}_{a,b}(n)$ is a smooth stratification of $\mathbf{X}(n)$. The proof of the smoothness of $\mathbf{X}_{a,b}(n)$ is accomplished by exhibiting a transitive group of birational equivalences acting on $\mathbf{X}_{a,b}(n)$. This suggests strongly that the sums $S_{a,b}$ should be viewed as the basic sums and correspondingly that Theorem 5.1 can be improved.

Proof of the Case $r=3$ *of Conjecture* 2. (See the introduction.) We need to prove that the Kloostermann zeta function $Z_{w_0}(\Lambda, \psi, \psi')$ converges absolutely for $\mathrm{Re}(\Lambda)\in\frac{3}{2}\varrho+\mathscr{C}$. This is already known for the zeta functions $Z_w(\ )$ for $w \neq w_0$ (see the remarks at the beginning of this section). Now write $\mathrm{Re}(\Lambda)=s_1\underline{\mu}_1+s_2\underline{\mu}_2$ where $\underline{\mu}_1=\frac{2}{3}\underline{\lambda}_{1,2}+\frac{1}{3}\underline{\lambda}_{2,3}$ and $\underline{\mu}_2=\frac{1}{3}\underline{\lambda}_{1,2}+\frac{2}{3}\underline{\lambda}_{2,3}$ [notation as in (4.3)]. Note that $\underline{\mu}_1, \underline{\mu}_2$ is the dual basis to $\underline{\lambda}_{1,2}, \underline{\lambda}_{2,3}$. We need to show that $Z_{w_0}(\ )$ converges whenever $s_1, s_2>\frac{3}{2}$. By Theorem 3.12 we can write

$$Z_{w_0}(\Lambda, \psi, \psi')=\sum_{s\in T(\mathbf{Z})}\sum_{D_1, D_2=1}^{\infty} Kl(stw_0, \psi, \psi')\cdot \|t\|_{\infty}^{\Lambda},$$

where $t=\begin{pmatrix} D_1^{-1} & & \\ & D_1D_2^{-1} & \\ & & D_2\end{pmatrix}$. By Theorem 5.1 this is bounded by

$$C\cdot \sum_{D_1, D_2=1}^{\infty} \{gcd(D_1, D_2)(D_1D_2)\}^{1/2}\cdot D_1^{-s_1+\varepsilon}D_2^{-s_2+\varepsilon}$$

$$< C\cdot\sum_{d=1}^{\infty}\sum_{d_1, d_2=1}^{\infty} d^{\frac{3}{2}-s_1-s_2+2\varepsilon}d_1^{\frac{1}{2}-s_1+\varepsilon}d_2^{\frac{1}{2}-s_2+\varepsilon},$$

where C is a constant which depends only on ψ and ψ' and ε is positive. This clearly converges for $s_1, s_2>\frac{3}{2}$ and ε sufficiently small. □

It is amusing to observe that even if we could improve the exponent in Theorem 5.1 to $\frac{r+s}{2}$, this would not result in a larger region of convergence for the Kloostermann zeta function.

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Received February 12, 1986