

## Rigidity of $p$ -adic cohomology classes of congruence subgroups of $\mathrm{GL}(n, \mathbb{Z})$

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### ABSTRACT

This paper provides foundations for studying  $p$ -adic deformations of arithmetic eigenpackets, that is, of systems of Hecke eigenvalues occurring in the cohomology of arithmetic groups with coefficients in finite-dimensional rational representations. The concept of ‘arithmetic rigidity’ of an arithmetic eigenpacket is introduced and investigated. An arithmetic eigenpacket is said to be ‘arithmetically rigid’ if (modulo twisting) it does not admit a  $p$ -adic deformation containing a Zariski dense set of arithmetic specializations.

The case of  $\mathrm{GL}(n)$  and ordinary eigenpackets is worked out, leading to the construction of a ‘universal  $p$ -ordinary arithmetic eigenpacket’. Tools for explicit investigation into the structure of the associated eigenvarieties for  $\mathrm{GL}(n)$  are developed. Of note is the purely algebraic Theorem 5.1, which keeps track of the specializations of the universal eigenpacket. We use these tools to prove that known examples of non-selfdual cohomological cuspforms for  $\mathrm{GL}(3)$  are arithmetically rigid. Moreover, we conjecture that, in general, arithmetic rigidity for  $\mathrm{GL}(3)$  is equivalent to non-selfduality.

### Introduction

The main result of [4] describes a framework for constructing  $p$ -adic analytic families of  $p$ -ordinary arithmetic Hecke eigenclasses in the cohomology of congruence subgroups of  $\mathrm{GL}(n)/\mathbb{Q}$ , where the Hecke eigenvalues vary  $p$ -adically as functions of the weight. Unanswered in that paper was the question of whether or not every  $p$ -ordinary arithmetic eigenclass can be ‘deformed’ in a ‘positive-dimensional family’ of arithmetic eigenclasses. In this paper we make precise the notions of ‘deformation’ and ‘rigidity’ and investigate their properties. Rigidity corresponds to the non-existence of positive-dimensional deformations other than those coming from twisting by the powers of the determinant. Formal definitions are given in Section 3. When  $n = 3$ , we will give a necessary and sufficient condition for a given  $p$ -ordinary arithmetic eigenclass to be  $p$ -adically rigid and we will use this criterion to give examples of  $p$ -adically rigid eigenclasses on  $\mathrm{GL}(3)$ .

On the other hand, an argument of Hida in [12, Section 5] can be modified to prove the existence of non-trivial, non- $p$ -torsion  $p$ -adic deformations (modulo twisting) for the same examples. Details will appear elsewhere. Here we merely note that the use of quasicuspidality (see Definition 6.2) seems to be essential to make Hida’s arguments work in the  $\mathrm{GL}(3)$  case.

Coupled with our rigidity result (Theorem 9.4) this proves the existence of non-arithmetic  $p$ -adic deformations of  $p$ -ordinary arithmetic (cohomological) cuspforms for  $\mathrm{GL}(3)$ . In fact, both of our examples are deformations of Hecke eigenclasses in the cohomology of congruence subgroups of  $\mathrm{SL}(3, \mathbb{Z})$  with trivial coefficients.

More precisely, we define a ‘Hecke eigenpacket’ to be a map from the Hecke algebra to a coefficient ring which gives the Hecke eigenvalues attached to a Hecke eigenclass in the cohomology. We then consider  $p$ -adic analytic ‘deformations’ of the Hecke eigenpacket. There are several advantages to this point of view, as opposed to deforming the cohomology classes

themselves. For one thing, a surjective map of coefficient modules need not induce a surjective map on the cohomology, but the long exact sequence in cohomology enables us to keep track of Hecke eigenpackets. For example, Lemma 4.3 is a surjectivity result on the level of Hecke eigenpackets proved in this way (see Theorem 5.1). Standard tools of algebra and geometry can be used to study the Hecke algebra and the deformations of the eigenpackets. Moreover, it is the Hecke eigenvalues that carry the interesting arithmetic information attached to a cohomological eigenclass.

Our main theoretical tool is the cohomology of congruence subgroups  $\Gamma \subseteq \mathrm{GL}(n, \mathbb{Z})$  with coefficients in the space of bounded measures on the big cell of  $\mathrm{GL}(n)$  (see Section 2). In Section 4 we define the ring  $R_{\mathrm{ord}}$  as a quotient of the Hecke algebra acting on the  $p$ -ordinary part of this cohomology. In Theorem 4.2 we prove that  $R_{\mathrm{ord}}$  is *universal* in the sense that every  $p$ -ordinary arithmetic Hecke eigenpacket factors through  $R_{\mathrm{ord}}$ . We show that  $R_{\mathrm{ord}}$  is a complete semilocal noetherian ring and study the geometry of its associated  $p$ -adic rigid analytic space  $\mathcal{X}_{\mathrm{ord}}$ . This leads us to a necessary and sufficient condition for  $p$ -adic rigidity (Theorem 4.9), valid for any  $n$ . In Sections 6 through 8 we recall a number of representation theoretic facts, which in the case  $n = 3$  allow us to prove a criterion for rigidity (Corollary 8.5) that is accessible to numerical computation. Our particular computations are described in Section 9.

The first and third authors have also developed a theory of  $p$ -adic deformations for non-ordinary (but finite slope) Hecke eigenpackets. Using a more representation-theoretic approach, Emerton [11] has defined an eigenvariety in this context, generalizing to  $\mathrm{GL}(n)$  the Coleman–Mazur eigencurve for  $\mathrm{GL}(2)$ . Emerton states his results for any  $\mathbb{Q}_p$ -split reductive algebraic group and it is, perhaps, worth noting that our methods also apply *mutatis mutandis* in this more general context.

The question of  $p$ -adic rigidity arises again in the finite slope context. However, it will probably be impossible to verify by computations such as those in this paper whether or not any particular non-ordinary eigenpacket is  $p$ -adically rigid. The reason is that in the non-ordinary case, we lose control over the ‘size’ of the subset of the parameter (‘weight’) space over which the family exists. In the ordinary case, we know that the families exist over the whole parameter space, and so we can test  $p$ -adic rigidity by looking at weights that are  $p$ -adically far from the given weight. Since the only weights for which computer calculations are currently feasible are  $p$ -adically rather far from each other, this limits our computational methods to the ordinary case.

One knows that the symmetric square lift to  $\mathrm{GL}(3)$  of any  $p$ -ordinary classical newform  $f$  does indeed deform into a  $p$ -adic analytic family (lifting Hida’s  $p$ -ordinary deformation of  $f$ ). From [3] we know that the cohomology classes associated to the symmetric square lifts are essentially selfdual eigenclasses (Definition 8.1) for  $\mathrm{GL}(3)$ . In looking for examples of  $p$ -adically rigid eigenclasses we therefore examined the non-essentially selfdual  $p$ -ordinary eigenclasses constructed numerically in [2]. The numerical calculations described in Section 9 show that these eigenclasses satisfy our criterion and are therefore  $p$ -adically rigid.

The calculations suggest that, outside of symmetric square lifts, non-torsion arithmetic cohomology is very sporadic. On the other hand, failure of rigidity of any non-symmetric square lift would give rise to an abundance of non-essentially selfdual eigenclasses, at least if the Hecke eigenpacket is not essentially selfdual modulo  $p$ . Therefore we are led to the following conjecture.

**CONJECTURE 0.1.** If a finite slope cuspidal Hecke eigenclass in the cohomology of a congruence subgroup of  $\mathrm{GL}(3, \mathbb{Z})$  is not essentially selfdual, then it is  $p$ -adically rigid.

Recent work of Eric Urban suggests that the converse may be true for all  $\mathrm{GL}(n)$ ,  $n \geq 2$ . When  $n = 2$ , every cuspidal eigenclass is essentially selfdual, and in this case Hida (in the

ordinary case) and Coleman (in general) proved that all finite slope cuspidal eigenclasses can be deformed into non-trivial  $p$ -adic analytic families; that is, they are not  $p$ -adically rigid in our sense.

In order to prove  $p$ -adic rigidity in the cases we study in the last section of the paper, we have to compute the cuspidal cohomology of  $\Gamma_0(q)$  with small-dimensional irreducible coefficient modules when  $n = 3$ . Here,  $\Gamma_0(q)$  denotes the arithmetic subgroup of  $\mathrm{SL}(n, \mathbb{Z})$  consisting of matrices with top row congruent to  $(*, 0, \dots, 0)$  modulo  $q$ , where  $q$  is a prime different from  $p$ .

Here are the computational results. Let  $V_h$  denote the irreducible finite-dimensional  $\mathrm{GL}(3, \mathbb{C})$ -module of highest weight  $(2h, h, 0)$ . Now,  $V_h$  is a submodule of  $\mathrm{Sym}^h(A) \otimes \mathrm{Sym}^h(A^\dagger) \otimes \det^h$ , where  $A$  is the standard 3-dimensional complex representation of  $\mathrm{GL}(3, \mathbb{C})$  and  $A^\dagger$  is its contragredient. Let  $E$  be a rational complex representation of  $\mathrm{GL}(3, \mathbb{Z})$  with coefficients in  $E$  is known to vanish if  $E$  contains no submodule isomorphic to a twist by a power of the determinant of any  $V_h$ . Therefore to look for cuspidal classes, it suffices to consider cohomology with  $V_h$  coefficients.

**COMPUTATIONAL RESULTS.** *There is no cuspidal cohomology for  $\Gamma_0(q)$  with coefficients in  $V_h$  for  $h = 1, 2$ , and  $q$  any prime up to and including 223, and for  $h = 3, 4$ , and  $q$  any prime up to and including 97.*

The computations use the programs and methods outlined in [1, Section 8]. Cuspidal cohomology can only occur in  $H^2$  and  $H^3$ , and Poincaré duality shows that  $H_{\mathrm{cusp}}^2 \simeq H_{\mathrm{cusp}}^3$ . Therefore it suffices to compute  $H^3$ .

It is somewhat surprising that we found no non-essentially selfdual cuspidal cohomology in these ranges, since for  $h = 0$  (that is, trivial coefficients) there exists non-essentially selfdual cuspidal cohomology for  $q = 53, 61, 79, 89$  (see [2]).

As a consequence of these computations, also in Section 9, we deduce the 3-adic rigidity of the cuspidal cohomology with trivial coefficients of level 89 and the 5-adic rigidity of the cuspidal cohomology of level 61. The difficulty of testing other examples is purely owing to our inability to compute for large  $h$ .

In this paper we avoid discussing an eigenvariety for  $\mathrm{GL}(n)$ , preferring instead to formulate our results in terms of eigenpackets. We look only at ‘arithmetic eigenpackets’, that is, those with associated rigid analytic spaces that are the Zariski closure of their arithmetic points (see Definitions 3.2 and 3.7). In particular, we say that an arithmetic point is ‘rigid’ if it is isolated on any arithmetic eigenpacket that contains it (see Definition 3.10).

It may help to describe our picture roughly in terms of a putative eigenvariety. We focus on the  $n = 3$  situation, and let  $Y$  be the eigenvariety for  $\mathrm{GL}(3)$  modulo twisting. Now,  $Y$  projects to a two-dimensional weight space  $\mathcal{W}$ . If a point  $x$  on  $Y$  is a symmetric square lift, then we expect that it will lie on a component of  $Y$  which projects to a subspace of  $\mathcal{W}$  containing the one-dimensional selfdual line. If  $x$  is not a symmetric square lift, then our conjecture above implies that any component of  $Y$  containing  $x$  will project down to a set in  $\mathcal{W}$  that is either one-dimensional and transverse to the selfdual line or zero-dimensional.

The contents of this paper are as follows. In Sections 1 and 2 we introduce the Hecke algebras and the cohomology groups of interest to us and describe the action of the Hecke algebra on the cohomology. Section 2 also recalls the main results from [4] that form the basis of the rest of this paper. In Section 3 we develop a formalism of Hecke eigenpackets and define the key notions of  $p$ -adic deformation and  $p$ -adic rigidity. In Section 4, we introduce the  $p$ -ordinary arithmetic eigenvariety and in Section 5 we prove its universality.

In Sections 6 through 8 we collect the representation theoretic facts we need in order to give a testable criterion for  $p$ -adic rigidity. We then give our computational results in Section 9.

In particular, we give two examples of  $p$ -adically rigid eigenclasses, one with  $p = 3$  and level 89, the other with  $p = 5$  and level 61.

1. *Hecke operators and  $(T, S)$ -modules*

Let  $G = \text{GL}(n)$  ( $n \geq 2$ ), and fix a positive prime  $p$ . Let  $T \subseteq G$  be the group of diagonal matrices, and let  $S \subseteq G(\mathbb{Q})$  be a subsemigroup containing a given congruence subgroup  $\Gamma \subseteq \text{SL}_n(\mathbb{Z})$ .

DEFINITION 1.1. A  $(T, S)$ -module is a topological  $\mathbb{Z}_p$ -module  $V$  endowed with a continuous left action of  $T(\mathbb{Q}_p)$  and a right action of  $S$  that commutes with the  $T(\mathbb{Q}_p)$ -action.

In the examples,  $V$  can always be viewed as a ‘highest weight module’ in which the action of  $T$  is defined via multiplication by the highest weight character of  $T(\mathbb{Q}_p)$ .

Let  $\Lambda$  and  $\Lambda_T$  be the completed group rings on the topological groups  $T(\mathbb{Z}_p)$  and  $T(\mathbb{Q}_p)$ , respectively. Then  $\Lambda$  is the Iwasawa algebra of  $T(\mathbb{Z}_p)$  and, letting  $T_0 \subseteq T(\mathbb{Q}_p)$  be the subgroup of matrices with diagonal entries that are integral powers of  $p$ , we have  $\Lambda_T = \Lambda[T_0]$ . The group  $T_0$  is free abelian with generators  $\pi_k := \text{diag}(1, \dots, 1, p, \dots, p)$  ( $k$  copies of  $p$ ), for  $k = 1, \dots, n$ , and consequently we have

$$\Lambda_T = \Lambda[\pi_1, \pi_1^{-1}, \dots, \pi_n, \pi_n^{-1}],$$

a Laurent polynomial ring in  $n$  variables over  $\Lambda$ . The element

$$\pi := \pi_1 \cdot \dots \cdot \pi_{n-1}$$

plays a special role in our theory.

Since  $T(\mathbb{Q}_p)$  acts continuously on any  $(T, S)$ -module  $V$ , we may regard  $V$  as a right  $\Lambda_T[S]$ -module. Indeed, this defines an equivalence of categories between the category of  $(T, S)$ -modules and the category of continuous  $\Lambda_T[S]$ -modules.

Now fix a positive integer  $N$  with  $p \nmid N$  and define the congruence subgroups

$$\Gamma := \{g \in \text{SL}(n, \mathbb{Z}) \mid \text{top row of } g \equiv (*, 0, \dots, 0) \pmod{N}\}$$

and

$$\Gamma_0 := \{g \in \Gamma \mid g \text{ is upper triangular modulo } p\}.$$

We define the subsemigroups  $S'$  and  $S'_0$  of  $M_n^+(\mathbb{Z})$  (the plus denotes positive determinant) by

$$S' := \{s \in M_n^+(\mathbb{Z}) \mid (\det(s), Np) = 1, \text{ top row of } s \equiv (*, 0, \dots, 0) \pmod{N}\}$$

$$S'_0 := \{s \in S' \mid s \text{ is upper triangular modulo } p\}.$$

Finally, we define the subsemigroups  $S$  and  $S_0$  of  $G(\mathbb{Q}_p)$ :

$$S := \text{semigroup generated by } S' \text{ and } \pi,$$

$$S_0 := \text{semigroup generated by } S'_0 \text{ and } \pi.$$

For  $s \in S$  we let  $T(s)$  be the double coset  $\Gamma s \Gamma$ . It is well known that the double coset algebra  $D_{\Lambda_T}(\Gamma, S) = \Lambda_T[\Gamma \backslash S / \Gamma]$  is the free polynomial algebra over  $\Lambda_T$  generated by  $T(\pi)$  and the elements

$$T(\ell, k) := T(s), \quad s = \text{diag}(1, \dots, 1, \ell, \dots, \ell), \quad \text{with } k \text{ copies of } \ell (1 \leq k \leq n),$$

for all primes  $l$  not dividing  $Np$ . (See, [16, Propositions 3.16, 3.30 and Theorem 3.20].)

The double coset algebra  $D_{\Lambda_T}(\Gamma_0, S_0)$  is likewise a free polynomial algebra generated by the  $\Gamma_0$ -double cosets of  $\pi$  and of the  $\text{diag}(1, \dots, 1, \ell, \dots, \ell)$  with  $l$  a prime not dividing  $Np$  and

$1 \leq k \leq n$ . Thus the map sending a polynomial expression in the generators of  $D_{\Lambda_T}(\Gamma, S)$  to the corresponding polynomial expression in the generators of  $D_{\Lambda_T}(\Gamma_0, S_0)$  gives an isomorphism  $D_{\Lambda_T}(\Gamma, S) \rightarrow D_{\Lambda_T}(\Gamma_0, S_0)$ .

We define the element  $U$  to be  $T(\pi)$ , and let  $\mathcal{H}$  denote the  $\Lambda$ -subalgebra of  $D_{\Lambda_T}(\Gamma, S)$  generated by the elements  $U$  and the  $T(\ell, k)$ :

$$\mathcal{H} = \Lambda \left[ U, T(\ell, k) \mid \ell \nmid Np \text{ is prime and } 1 \leq k \leq n \right].$$

We let  $\mathcal{H}_0$  be the image of  $\mathcal{H}$  in  $D_{\Lambda_T}(\Gamma_0, S_0)$  and note that

$$\mathcal{H} \xrightarrow{\sim} \mathcal{H}_0$$

is an isomorphism of  $\Lambda$ -modules.

We define a modified action of  $S$ , which we call the  $\star$ -action, as follows. There is a unique multiplicative map  $\tau : S \rightarrow T_0$  that is trivial on  $S'$  and sends  $\pi$  to itself. For any  $(T, S)$ -module  $V$  we define the  $\star$ -action of  $S$  on  $V$  by the formula

$$w \star \sigma := \tau(\sigma)^{-1} \cdot w\sigma.$$

for  $w \in V$  and  $\sigma \in S$ . We then define  $V^\star$  to be  $V$  endowed with the  $\star$ -action of  $S$  on the right and the given action of  $\Lambda$  on the left. For any subquotient  $L$  of  $V^\star$  the double coset algebra  $D_\Lambda(\Gamma, S)$  acts naturally on the  $\Gamma$ -cohomology of  $L$ . We then let  $\mathcal{H}$  act on the cohomology of  $L$ .

We apply this construction verbatim for any  $(T, S_0)$ -module  $V$  and similarly obtain a  $\Lambda[S_0]$ -module  $V^\star$ . Then for any  $\Lambda[S_0]$ -subquotient  $L$  of  $V^\star$  the  $\Gamma_0$ -cohomology of  $L$  receives a natural  $\mathcal{H}_0$ -module structure. Using our fixed isomorphism  $\mathcal{H} \cong \mathcal{H}_0$ , we may (and do) regard the  $\Gamma_0$ -cohomology of  $L$  as an  $\mathcal{H}$ -module. For example, if  $V$  is a  $(T, S)$ -module and  $L$  is a subquotient of  $V^\star$  then we may also regard  $V$  as a  $(T, S_0)$ -module and  $L$  as a  $\Lambda[S_0]$ -subquotient of  $V^\star$ . Therefore both  $H^*(\Gamma, L)$  and  $H^*(\Gamma_0, L)$  are endowed with natural  $\mathcal{H}$ -module structures. Note, however, that in general the restriction map

$$H^*(\Gamma, L) \longrightarrow H^*(\Gamma_0, L)$$

does not commute with  $U$ , though it does commute with the Hecke operators  $T(\ell, k)$  for  $\ell \nmid Np$ .

Note that we use throughout the notation  $H^*$  to stand for  $\bigoplus_k H^k$ .

### 2. $p$ -adic interpolation of arithmetic cohomology

Let  $B$  be the Borel subgroup of upper triangular matrices and denote the group of lower triangular matrices by  $B^{\text{opp}}$ . Let  $\lambda$  denote the highest weight, with respect to  $(B^{\text{opp}}, T)$ , of a finite-dimensional irreducible right rational  $\mathbb{Q}_p[G]$ -module  $V_\lambda$ . Then  $V_\lambda$  inherits a natural action of  $S$  on the right, and we let  $T(\mathbb{Q}_p)$  act (continuously) on the left via the character  $\lambda$ . In a moment, we will also choose a  $\mathbb{Z}_p$ -lattice  $L_\lambda \subseteq V_\lambda$  that is invariant under  $S'_0$  and satisfies  $\pi^{-1}L_\lambda\pi \subseteq L_\lambda$ . Thus by the discussion of the last section, we obtain an action of  $\mathcal{H}$  on  $H^*(\Gamma, V_\lambda)$ ,  $H^*(\Gamma_0, V_\lambda)$ , and  $H^*(\Gamma_0, L_\lambda)$ .

Now let  $N^{\text{opp}}$  be the unipotent radical of  $B^{\text{opp}}$  and let  $Y := N^{\text{opp}}(\mathbb{Q}_p) \backslash G(\mathbb{Q}_p)$ . We also let

$$X := \text{the image of } G(\mathbb{Z}_p) \text{ in } Y,$$

and

$$X_0 := \text{the image of } I \text{ in } Y,$$

where  $I$  is the Iwahori subgroup of  $G(\mathbb{Z}_p)$  consisting of matrices that are upper triangular modulo  $p$ . The semigroup  $S$  acts on  $Y$  by multiplication on the right and  $T(\mathbb{Q}_p)$  acts by multiplication on the left. Both  $X$  and  $X_0$  are stable under  $T(\mathbb{Z}_p)$ . Also  $X$  is stable under the

action of  $S'$  and  $X_0$  is stable under  $S'_0$ . We define a new action of  $S$  on  $Y$  by setting for  $\sigma \in S$

$$N^{\text{opp}}(\mathbb{Q}_p)g \star \sigma = N^{\text{opp}}(\mathbb{Q}_p)\tau(\sigma)^{-1}g\sigma,$$

where  $\tau : S \rightarrow T_0$  is the homomorphism sending  $\pi$  to  $\pi$  and acting trivially on  $S'$  introduced above. This new  $\star$ -action is well defined since  $\pi$  normalizes  $N^{\text{opp}}(\mathbb{Q}_p)$ . Moreover,  $X_0$  is stable under the  $\star$ -action of  $S_0$  since every element of  $X_0$  is represented by some element  $b \in B(\mathbb{Z}_p)$  and  $\pi^{-1}B(\mathbb{Z}_p)\pi \subset B(\mathbb{Z}_p)$ . We will henceforth use this  $\star$ -action when  $S$  acts on  $Y$  or  $S_0$  on  $X_0$ .

For any open subset  $Z \subseteq Y$ , we let  $\mathbb{D}(Z)$  be the space of  $\mathbb{Z}_p$ -valued measures on  $Z$ , and so  $\mathbb{D}(Z) = \text{Hom}_{\mathbb{Z}_p}(C(Z), \mathbb{Z}_p)$ , where  $C(Z)$  is the space of all compactly supported continuous  $\mathbb{Z}_p$ -valued functions on  $Z$ . By functoriality (using the  $\star$ -action),  $\mathbb{D}(Y)$  inherits a natural  $(T, S)$ -structure. By extending measures by zero, we obtain a  $\Lambda[S_0]$ -embedding  $\mathbb{D}(X_0) \hookrightarrow \mathbb{D}(Y)$ . In [4], the space  $\mathbb{D} = \mathbb{D}(X)$  is given a  $\Lambda[S]$  structure by setting for  $\mu \in \mathbb{D}$  and  $f \in C(X)$

$$(\mu \star s)(f) = \mu((s \star \tilde{f})|_X),$$

where  $\tilde{f}$  is the extension by zero of  $f$  to  $Y$  and  $(s \star \tilde{f})(y) = \tilde{f}(y \star s)$ . Hence the cohomology groups  $H^*(\Gamma_0, \mathbb{D}(X_0))$ ,  $H^*(\Gamma, \mathbb{D})$  are endowed with natural  $\mathcal{H}$ -module structures. Moreover, restriction of distributions from  $X$  to  $X_0$  induces a  $\Lambda[S_0]$ -morphism  $\mathbb{D} \rightarrow \mathbb{D}(X_0)$  and thus induces a map

$$H^*(\Gamma, \mathbb{D}) \longrightarrow H^*(\Gamma_0, \mathbb{D}(X_0))$$

that commutes with action of  $\mathcal{H}$ . (To see that this map commutes with  $T(\pi)$  requires an explicit calculation similar to the proof of [4, Proposition 4.2].)

For any dominant integral weight  $\lambda$ , we define the map

$$\phi_\lambda : \mathbb{D}(Y) \longrightarrow V_\lambda$$

by  $\phi_\lambda(\mu) = \int_{X_0} v_\lambda x d\mu(x)$ , where  $v_\lambda$  denotes a fixed highest weight vector in  $V_\lambda$ . One checks easily that  $\phi_\lambda$  is a morphism of  $(T, S)$ -modules. Thus  $\phi_\lambda$  induces a  $\Lambda[S_0]$ -morphism

$$\phi_\lambda : \mathbb{D}(X_0) \longrightarrow V_\lambda^*.$$

We let  $L_\lambda$  be the image of this map and note that, since  $\mathbb{D}(X_0)$  is compact,  $L_\lambda$  is a finitely generated  $I$ -invariant submodule of  $V_\lambda$ , and hence is a lattice in  $V_\lambda$  (since  $V_\lambda$  is irreducible). This is the  $S_0$ -invariant lattice we promised to choose in the first paragraph of this section. Thus  $\phi_\lambda$  induces a natural  $\mathcal{H}$ -morphism

$$\phi_\lambda : H^*(\Gamma, \mathbb{D}) \longrightarrow H^*(\Gamma_0, L_\lambda).$$

For any compact  $\mathbb{Z}_p$ -module  $H$  with a continuous action of the operator  $U$  on it, we define the ordinary part of  $H$  to be  $H^0 = \cap U^i H$ , where  $i$  runs over all positive integers. Then  $H^0$  is the largest submodule of  $H$  on which  $U$  acts invertibly, and  $H \mapsto H^0$  is an exact functor.

The following theorem is [4, Theorem 5.1].

**THEOREM 2.1.** *The group  $H^*(\Gamma, \mathbb{D})^0$  is finitely generated as a  $\Lambda$ -module. Moreover, for every dominant integral weight  $\lambda$ , the kernel of the map*

$$\phi_\lambda : H^*(\Gamma, \mathbb{D})^0 \longrightarrow H^*(\Gamma_0, L_\lambda)^0$$

is  $I_\lambda H^*(\Gamma, \mathbb{D})^0$ , where  $I_\lambda$  is the kernel of the homomorphism  $\lambda^\dagger : \Lambda \rightarrow \mathbb{Z}_p$  induced by  $\lambda$ .

For easy reference, we will also cite here the two main lemmas from [4] that we need to prove Theorem 2.1. We first need a few definitions. We let

$$W := \text{Spf}(\Lambda)$$

be the formal scheme associated to  $\Lambda$ . We note that for each local field  $K/\mathbb{Q}_p$  the  $K$ -points of  $W$  are in one-to-one correspondence with continuous homomorphisms  $k : T(\mathbb{Z}_p) \rightarrow K^\times$ . If  $t \in T(\mathbb{Z}_p)$  and  $k \in W(K)$ , then we will write  $t^k \in K^\times$  for the value of  $k$  on  $t$ . Dominant integral weights can (and will) be viewed as elements of  $W(\mathbb{Q}_p)$ . We let  $W^+$  denote the set of dominant integral weights.

Now let  $k \in W(\mathbb{Q}_p)$ . We let  $C_k(X)$  denote the  $\mathbb{Z}_p$ -module

$$C_k(X) := \{f : X \rightarrow \mathbb{Z}_p \mid f \text{ is continuous and } f(tx) = t^k f(x) \text{ for all } t \in T(\mathbb{Z}_p), x \in X\}.$$

Let  $\mathbb{D}_k$  be the space of continuous  $\mathbb{Z}_p$ -valued linear functionals on  $C_k(X)$ . Clearly  $C_k(X) \subseteq C(X)$  is a closed subspace, and so restriction of linear functionals induces a surjective map  $\mathbb{D} \rightarrow \mathbb{D}_k$ . We regard both  $\mathbb{D}$  and  $\mathbb{D}_k$  as  $\Lambda[S]$ -modules (with the  $\star$ -action of  $S$ ). Let  $I_k \subseteq \Lambda$  be the prime associated to  $k \in W(\mathbb{Q}_p)$ . The action of  $\Lambda$  on  $\mathbb{D}_k$  factors through  $\Lambda/I_k$ .

LEMMA 2.2. *The ideal  $I_k$  is generated by a  $\mathbb{D}$ -regular sequence. Moreover, the canonical sequence*

$$0 \rightarrow I_k \mathbb{D} \rightarrow \mathbb{D} \rightarrow \mathbb{D}_k \rightarrow 0$$

is exact.

*Proof.* Lemma 1.1 of [4] shows that  $I_k$  is generated by a finite  $\mathbb{D}$ -regular sequence. We have already noted above that the map  $\mathbb{D} \rightarrow \mathbb{D}_k$  is surjective. We clearly have  $I_k \mathbb{D}$  is contained in the kernel and it follows from [4, Lemma 6.3] that  $I_k \mathbb{D}$  contains the kernel. Thus the sequence is exact. This completes the proof.  $\square$

LEMMA 2.3. *If  $\lambda \in W^+$  is a dominant integral weight, then the canonical map*

$$H^*(\Gamma, \mathbb{D}_\lambda)^0 \xrightarrow{\sim} H^*(\Gamma_0, L_\lambda)^0$$

is an  $\mathcal{H}$ -isomorphism.

*Proof.* The surjectivity of the map follows from Lemma 6.7 and the injectivity follows from [4, Lemma 6.8]. (Note, however, that the statement of Lemma 6.8 contains a typo. The conclusion of the statement should read: ‘Then  $\zeta$  can be represented by a cocycle  $z_m$  in  $\text{Hom}_\Gamma(F_k, \mathbb{D})$  which takes values in  $\mathbb{D}^*$ .’)  $\square$

REMARK 2.4. In a forthcoming paper by the first and third authors, we generalize the ideas of this section to the non-ordinary ‘finite slope’ situation.

### 3. Eigenpackets: twists and deformations

DEFINITION 3.1. If  $A$  is a  $\Lambda$ -module, a  $\Lambda$ -homomorphism  $\varphi : \mathcal{H} \rightarrow A$  will be called an  $A$ -valued eigenpacket. If  $\varphi' : \mathcal{H} \rightarrow A'$  is another eigenpacket then a morphism  $f : \varphi \rightarrow \varphi'$  is defined to be a  $\Lambda$ -morphism  $f : A \rightarrow A'$  such that the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\varphi} & A \\ & \searrow \varphi' & \downarrow f \\ & & A' \end{array}$$

is commutative. Conversely, if  $f : A \rightarrow A'$  is a  $\Lambda$ -morphism, then we may push  $\varphi$  forward by  $f$  to obtain an eigenpacket  $\varphi' := f_*\varphi$  and a morphism  $f : \varphi \rightarrow \varphi'$ . If  $\varphi$  is an eigenpacket, then we will denote its target by  $A_\varphi$ .

If  $A$  is either a  $\mathbb{Q}_p$ -Banach algebra or a  $\mathbb{Z}_p$ -algebra that is separated and complete in the  $p$ -adic topology, then there is a one-to-one correspondence between continuous homomorphisms  $\Lambda \rightarrow A$  and continuous characters  $T(\mathbb{Z}_p) \rightarrow A^\times$ . For a character  $\chi : T(\mathbb{Z}_p) \rightarrow A^\times$  we let  $\chi^\dagger : \Lambda \rightarrow A$  denote the associated homomorphism. Conversely, if  $A$  is a  $\Lambda$ -algebra, we let  $\chi_A$  denote the associated character and call  $\chi_A$  the structure character of  $A$ . For any eigenpacket  $\varphi$  we define the *weight* of  $\varphi$  to be the structure character  $\chi_{A_\varphi}$ , which we will denote simply by  $\chi_\varphi$ .

Let  $K/\mathbb{Q}_p$  be a finite field extension. Whenever we say ‘ $K$ -valued eigenpacket’ we will implicitly assume that  $K$  has been endowed with a  $\Lambda$ -algebra structure.

We are interested in eigenpackets that occur in the  $\Gamma_0$ -cohomology of finite-dimensional irreducible rational representations of  $GL(n)$ . The weight of any such eigenpacket must be a dominant integral character of  $T(\mathbb{Z}_p)$ .

**DEFINITION 3.2.** Let  $K/\mathbb{Q}_p$  be a finite field extension and let  $\varphi$  be a  $K$ -valued eigenpacket. We say  $\varphi$  is *arithmetic of weight*  $\lambda$ , where  $\lambda$  is a dominant integral character, if  $\varphi$  occurs in  $H^*(\Gamma_0, V_\lambda(K))$  (that is, if  $\varphi$  is the homomorphism of eigenvalues of some Hecke eigenvector).

**DEFINITION 3.3.** Given an  $A$ -valued eigenpacket  $\varphi$  and a continuous character  $\psi : \mathbb{Z}_p^\times \rightarrow A^\times$ , we may ‘twist’  $\varphi$  by  $\psi$  as follows. Let  $A(\psi)$  be the unique continuous  $\Lambda$ -algebra with underlying  $\mathbb{Z}_p$ -algebra  $A$  and with structure character  $\chi_A \cdot (\psi \circ \det)$ . The  $\psi$ -twist of  $\varphi$  is defined to be the unique eigenpacket

$$\text{Tw}_\psi(\varphi) : \mathcal{H} \longrightarrow A(\psi)$$

given on Hecke operators by  $U \mapsto \varphi(U)$  and  $T(\sigma) \mapsto \psi(\det(\sigma))\varphi(T(\sigma))$  for  $\sigma \in S'$ . In particular, we define the *standard twist* of  $\varphi$  to be

$$\text{Tw}(\varphi) := \text{Tw}_1(\varphi),$$

where  $1 : \mathbb{Z}_p^\times \rightarrow A^\times$  is the identity character. Iteration of the standard twist gives us the  $n$ -fold twist  $\text{Tw}^n(\varphi)$  for any integer  $n$ . We also define the *universal twist* of  $\varphi$  by

$$\widetilde{\text{Tw}}(\varphi) := \text{Tw}_{\tilde{\psi}}(i_*\varphi),$$

where  $i : A \rightarrow A[[\mathbb{Z}_p^\times]] := A \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$  is the base extension morphism and  $\tilde{\psi} : \mathbb{Z}_p^\times \rightarrow A[[\mathbb{Z}_p^\times]]$  the universal character.

Note that by definition, the weight of  $\text{Tw}_\psi(\varphi)$  is the product of  $\chi_\varphi$  and  $\psi \circ \det$ . Moreover, for any  $\psi : \mathbb{Z}_p^\times \rightarrow A^\times$  there is a unique  $A$ -morphism  $\widetilde{\text{Tw}}(\varphi) \rightarrow \text{Tw}_\psi(\varphi)$ . The next proposition states that arithmeticity of eigenpackets is invariant under twist.

**PROPOSITION 3.4.** *If  $\varphi$  is arithmetic, then  $\text{Tw}^n(\varphi)$  is arithmetic for every  $n \in \mathbb{Z}$ . If, moreover,  $\varphi$  has weight  $\lambda$ , then  $\text{Tw}^n(\varphi)$  has weight  $\lambda \cdot \det^n$ .*

*Proof.* Let  $\lambda$  be a dominant integral weight and let  $\varphi$  be an eigenpacket occurring in  $H^*(\Gamma_0, V_\lambda(K))$ . Now let  $V_\lambda(n)$  be  $V_\lambda$  with the  $G$ -action twisted by the  $n$ -fold power of the determinant. Then the identity map  $V_\lambda \rightarrow V_\lambda(n)$  is a  $\Gamma_0$ -isomorphism and therefore



induces an isomorphism  $H^*(\Gamma_0, V_\lambda(K)) \rightarrow H^*(\Gamma_0, V_\lambda(n)(K))$ . A simple calculation shows that this map sends any  $\varphi$ -eigenvector to a  $\text{Tw}^n(\varphi)$ -eigenvector. Thus  $\text{Tw}^n(\varphi)$  occurs in  $H^*(\Gamma_0, V_\lambda(n)(K))$ . It follows that  $\text{Tw}^n(\varphi)$  is arithmetic of weight  $\lambda \cdot \det^n$ . This completes the proof of Proposition 3.4.  $\square$

Consider a topological  $\Lambda$ -algebra  $A$  which is of one of the following two types:

- (1) a  $\mathbb{Q}_p$ -affinoid algebra; or
- (2) a complete semilocal noetherian  $\Lambda$ -algebra with finite residual fields.

In either case, we assume that the structure morphism  $\Lambda \rightarrow A$  is continuous.

REMARK. Alternatively, we could work in the category of affine formal schemes over  $\text{Spf}(\Lambda)$ , and indeed for all of our applications it suffices to work in that category. In that case, the ring of rigid analytic functions on an admissible affinoid open set is of type (1), while on an affine formal scheme finite over  $\text{Spf}(\Lambda)$ , the ring of global rigid analytic functions of norm at most 1 is of type (2).

If  $A$  is arbitrary of type (1) or (2) then we will say that  $A$  is  $p$ -analytic and in this case an  $A$ -valued eigenpacket will be called  $p$ -analytic. In case (1) we will say  $\varphi$  is an *affinoid* eigenpacket. These conventions are justified by the fact that when  $A$  is  $p$ -analytic there is a naturally defined  $p$ -adic rigid analytic space  $\mathcal{X}_A$  over  $\mathbb{Q}_p$  such that for every finite extension  $K/\mathbb{Q}_p$  the  $K$ -valued points of  $\mathcal{X}_A$  are given by continuous homomorphisms from  $A$  to  $K$ :

$$\mathcal{X}_A(K) = \text{Hom}_{\text{cont}}(A, K).$$

(For the construction of  $\mathcal{X}_A$  in case (2) see [13, Section 7], or [10, Section 1.1].) We will also write  $\mathcal{X}_A(\overline{\mathbb{Q}}_p) := \text{Hom}_{\text{cont}}(A, \overline{\mathbb{Q}}_p)$  for the set of  $\overline{\mathbb{Q}}_p$ -valued points and note that every  $\overline{\mathbb{Q}}_p$ -valued point must actually take values in some finite extension of  $\mathbb{Q}_p$ .

We give  $\mathcal{X}_A(\overline{\mathbb{Q}}_p)$  the Zariski topology with closed sets that are the zero sets of the ideals of  $A$ . Note that if  $A$  is of type (1) then this is the usual topology, but if  $A$  is of type (2) then it is coarser than the usual topology defined in terms of the full ring of rigid analytic functions.

Henceforth, all eigenpackets will be assumed to be  $p$ -analytic. Thus for every eigenpacket  $\varphi$ , the  $\Lambda$ -algebra  $A_\varphi$  is  $p$ -analytic. We denote the associated rigid analytic space by  $\mathcal{X}_\varphi$ . We will often use geometric language to describe properties of  $\varphi$ . For example, we say  $\varphi$  is *affinoid* if  $A_\varphi$  is an affinoid algebra and we say that  $\varphi$  is *irreducible* if  $\mathcal{X}_\varphi$  is an irreducible topological space.

DEFINITION 3.5. A subset  $\mathcal{S} \subseteq \mathcal{X}_A(\overline{\mathbb{Q}}_p)$  is said to be *Zariski dense* if 0 is the only element of  $A$  that vanishes at every point in  $\mathcal{S}$ .

We define the *weight space*  $\mathcal{W}$  for  $\text{GL}(n)$  by  $\mathcal{W} := \mathcal{X}_\Lambda$ . For any  $p$ -analytic  $\Lambda$ -algebra  $A$  the homomorphism  $\Lambda \rightarrow A$  induces a rigid analytic morphism

$$\text{wt} : \mathcal{X}_A \longrightarrow \mathcal{W},$$

which we call the *weight map*. We let  $\mathcal{W}^+$  be the set of all dominant integral weights and if  $A$  is a  $p$ -analytic  $\Lambda$ -algebra then for any field  $K$  finite over  $\mathbb{Q}_p$  we define

$$\mathcal{X}_A^+(K) := \{ x \in \mathcal{X}_A(K) \mid \text{wt}(x) \in \mathcal{W}^+ \}.$$

We let  $\mathcal{X}_A^+ := \mathcal{X}_A^+(\overline{\mathbb{Q}}_p)$  and call the points of  $\mathcal{X}_A^+$  the *dominant integral points* on  $\mathcal{X}_A$ .

REMARK 3.6. The set  $\mathcal{W}^+$  of dominant integral points is a Zariski dense subset of  $\mathcal{W}$ . However the set  $\mathcal{X}_A^+$  is not, in general, Zariski dense in  $\mathcal{X}_A$ .

If  $\varphi$  is an eigenpacket then we say  $\varphi$  is an eigenpacket over  $\mathcal{X}_\varphi$ . For each rigid analytic subspace  $Z \subseteq \mathcal{X}_\varphi$ , we let  $\varphi_Z$  be the composition of  $\varphi : \mathcal{H} \rightarrow A_\varphi$  with the map  $A_\varphi \rightarrow A(Z)$  defined by restriction to  $Z$  of rigid analytic functions on  $\mathcal{X}_\varphi$ . In this case  $\varphi_Z$  will be called the restriction of  $\varphi$  to  $Z$ . In particular, if  $x \in \mathcal{X}_\varphi(\overline{\mathbb{Q}_p})$ , then the restriction of  $\varphi$  to  $\{x\}$  will be called the specialization of  $\varphi$  at  $x$  and denoted by  $\varphi_x$ . We define

$$\mathcal{A}(\varphi) := \{ x \in \mathcal{X}_\varphi(\overline{\mathbb{Q}_p}) \mid \varphi_x \text{ is arithmetic} \}$$

and note that  $\mathcal{A}(\varphi) \subseteq \mathcal{X}_\varphi^+$ .

DEFINITION 3.7. An eigenpacket  $\varphi$  is said to be

- (1) *arithmetic* if the set  $\mathcal{A}(\varphi)$  is Zariski dense in  $\mathcal{X}_\varphi$ , and
- (2)  $\overline{\mathbb{Q}_p}$ -*valued* if the underlying ring of  $A_\varphi$  is a finite field extension of  $\mathbb{Q}_p$ .

If  $\varphi$  is an arithmetic eigenpacket, then we regard the family of all specializations  $\varphi_x$ ,  $x \in \mathcal{X}_\varphi(\overline{\mathbb{Q}_p})$ , as an analytic family of  $\overline{\mathbb{Q}_p}$ -valued eigenpackets that interpolates the arithmetic eigenpackets  $\varphi_x$ ,  $x \in \mathcal{A}(\varphi)$ . We sometimes prefer to take a more *local* point of view, starting with a  $\overline{\mathbb{Q}_p}$ -eigenpacket and then ‘deforming’ it to an analytic family. With this picture in mind, we make the following definition.

DEFINITION 3.8. Given a  $K$ -valued eigenpacket  $\varphi_0$ , a *deformation* of  $\varphi_0$  is a pair  $(\varphi, x)$  consisting of an eigenpacket  $\varphi$  and a point  $x \in \mathcal{X}_\varphi(K)$  such that

- (1)  $A_\varphi$  is a  $p$ -analytic integral domain; and
- (2)  $\varphi_x = \varphi_0$ .

We say that  $(\varphi, x)$  is an arithmetic deformation of  $\varphi_0$  if, in addition to (1) and (2), we also have

- (3)  $\varphi_Z$  is arithmetic for every admissible affinoid neighborhood  $Z$  of  $x$  in  $\mathcal{X}_\varphi$ .

Although we will not have use for it later, we record here a notion of equivalence of deformations. We say that two deformations  $(\varphi, x)$ ,  $(\varphi', x')$  of  $\varphi_0$  are locally equivalent if there exists an irreducible affinoid neighborhood  $Z$  of  $x$  in  $\mathcal{X}_\varphi$ , an irreducible affinoid neighborhood  $Z'$  of  $x'$  in  $\mathcal{X}_{\varphi'}$ , and an isomorphism  $f : Z' \xrightarrow{\sim} Z$  such that  $f(x') = x$  and  $f^*\varphi_Z = \varphi'_{Z'}$ .

Let  $\tilde{\varphi}_0 = \widetilde{\text{Tw}}(\varphi_0) : \mathcal{H} \rightarrow \overline{\mathbb{Q}_p}[[\mathbb{Z}_p^\times]]$  be the universal twist of  $\varphi_0$ , and let  $x_0 \in \mathcal{X}_{\tilde{\varphi}_0}(\overline{\mathbb{Q}_p})$  be the point associated to the trivial character of  $\mathbb{Z}_p^\times$ . It follows from Proposition 3.4 that if  $\varphi_0$  is arithmetic, then the restriction of  $\tilde{\varphi}_0$  to the connected component of  $x_0$  is an arithmetic deformation of  $\varphi_0$  at  $x_0$ .

DEFINITION 3.9. An arbitrary deformation  $(\varphi, x)$  of  $\varphi_0$  will be called a *twist deformation* of  $\varphi_0$  if there is a morphism  $f : \tilde{\varphi}_0 \rightarrow \varphi$  such that  $f^\#(x) = x_0$ .

Note that if  $(\varphi, x)$  is a twist deformation of  $\varphi_0$ , then every specialization  $\varphi_z$ ,  $z \in \mathcal{X}_\varphi(\overline{\mathbb{Q}_p})$ , is a twist of  $\varphi_0$  by some character  $\mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}_p}^\times$ .

Clearly, every twist deformation of an arithmetic  $\overline{\mathbb{Q}_p}$ -eigenpacket  $\varphi_0$  is arithmetic. If the converse is true, we say that  $\varphi_0$  is arithmetically rigid.

DEFINITION 3.10. An arithmetic  $\overline{\mathbb{Q}}_p$ -eigenpacket is said to be *arithmetically rigid* if every arithmetic deformation of  $\varphi_0$  is a twist deformation.

In Section 9 of this paper, we will give examples of arithmetic  $K$ -eigenpackets that are arithmetically rigid.

#### 4. The universal $p$ -ordinary arithmetic eigenpacket

In this section we construct the universal  $p$ -ordinary arithmetic eigenpacket  $\varphi_{\text{ord}}$  and use it to give a criterion for rigidity of  $p$ -ordinary arithmetic  $\overline{\mathbb{Q}}_p$ -eigenpackets. Here  $K$ , as always, denotes a local field finite over  $\mathbb{Q}_p$ .

DEFINITION 4.1. A  $\overline{\mathbb{Q}}_p$ -valued eigenpacket  $\varphi$  is said to be  $p$ -ordinary if  $\varphi(U) \in \mathcal{O}_{\overline{\mathbb{Q}}_p}^\times$ . More generally, a  $p$ -analytic eigenpacket  $\varphi$  is said to be  $p$ -ordinary if  $\varphi_x$  is  $p$ -ordinary for every  $x \in \mathcal{X}_\varphi(\overline{\mathbb{Q}}_p)$ .

We have the following important result.

THEOREM 4.2. *There is a  $p$ -ordinary arithmetic eigenpacket  $\varphi_{\text{ord}}$  satisfying the following universal property: for every  $p$ -ordinary arithmetic eigenpacket  $\varphi$  there is a unique morphism  $\varphi_{\text{ord}} \rightarrow \varphi$ . Moreover, a point  $x \in \mathcal{X}_{\varphi_{\text{ord}}}$  is arithmetic if and only if  $x$  has dominant integral weight. In other words  $\mathcal{A}(\varphi_{\text{ord}}) = \mathcal{X}_{\varphi_{\text{ord}}}^+$ .*

Our construction of  $\varphi_{\text{ord}}$  will use the ordinary part of the cohomology of the universal highest weight module  $\mathbb{D}$  constructed in Section 2. Consider the  $\mathcal{H}$ -module  $\mathbb{H}$  defined by

$$\mathbb{H} := \bigoplus_i H^i(\Gamma, \mathbb{D})^0.$$

We define  $R'_{\text{ord}}$  to be the image of  $\mathcal{H}$  in  $\text{End}_\Lambda(\mathbb{H})$  and consider the associated  $\Lambda$ -morphism

$$\varphi'_{\text{ord}} : \mathcal{H} \rightarrow R'_{\text{ord}}.$$

We know from Theorem 2.1 that  $\mathbb{H}$  is a finitely generated  $\Lambda$ -module, and from this it follows that  $R'_{\text{ord}}$  is a finite  $\Lambda$ -algebra. Thus  $\varphi'_{\text{ord}}$  is a  $p$ -analytic eigenpacket. Let  $\mathcal{X}_{\varphi'_{\text{ord}}}$  be the associated rigid analytic space, and let  $\mathcal{A}(\varphi'_{\text{ord}})$  be the set of arithmetic points on  $\mathcal{X}_{\varphi'_{\text{ord}}}$  as defined in the last section.

LEMMA 4.3. *Let  $\varphi$  be a  $p$ -ordinary  $\overline{\mathbb{Q}}_p$ -valued eigenpacket. Then  $\varphi$  is arithmetic if and only if  $\varphi = (\varphi'_{\text{ord}})_x$  for some  $x \in \mathcal{X}_{\varphi'_{\text{ord}}}^+$ . In particular,  $\mathcal{A}(\varphi'_{\text{ord}}) = \mathcal{X}_{\varphi'_{\text{ord}}}^+$ .*

The proof of Lemma 4.3 will be given in the next section. In this section, we assume Lemma 4.3 and prove Theorem 4.2 as a consequence.

*Proof of Theorem 4.2.* Let  $I_{\text{arith}} \subseteq R'_{\text{ord}}$  be the ideal defined by

$$I_{\text{arith}} := \bigcap_{x \in \mathcal{A}(\varphi'_{\text{ord}})} \ker(x).$$

We then let  $R_{\text{ord}} := R'_{\text{ord}}/I_{\text{arith}}$  and define

$$\varphi_{\text{ord}} : \mathcal{H} \longrightarrow R_{\text{ord}}$$

to be the composition of  $\varphi'_{\text{ord}}$  with  $R'_{\text{ord}} \rightarrow R_{\text{ord}}$ . By definition, we see that  $\varphi_{\text{ord}}$  is arithmetic. Let  $J := \ker(\varphi_{\text{ord}}) \subseteq \mathcal{H}$ .

Now let  $\varphi : \mathcal{H} \rightarrow A_\varphi$  be an arbitrary  $p$ -ordinary arithmetic eigenpacket. We will show  $J \subseteq \ker(\varphi_x)$  for every  $x \in \mathcal{A}(\varphi)$ . Indeed, if  $x \in \mathcal{A}(\varphi)$  then, since  $\varphi_x$  is arithmetic, Lemma 4.3 tells us that there is a point  $y \in \mathcal{A}(\varphi'_{\text{ord}}) = \mathcal{X}_{\varphi'_{\text{ord}}}^+$  such that  $\varphi_x = (\varphi'_{\text{ord}})_y$ . However, we clearly have  $J \subseteq \ker((\varphi'_{\text{ord}})_y)$ , so  $J \subseteq \ker(\varphi_x)$ , as claimed.

It now follows easily that  $J \subseteq \ker(\varphi)$ . Indeed, from the last paragraph we have

$$\varphi(J) \subseteq \bigcap_{x \in \mathcal{A}(\varphi)} \ker(x),$$

and the intersection on the right is  $\{0\}$  since  $\varphi$  is arithmetic. Thus we have shown that  $\ker(\varphi_{\text{ord}}) \subseteq \ker(\varphi)$ . Since  $\varphi_{\text{ord}} : \mathcal{H} \rightarrow R_{\text{ord}}$  is surjective by definition, we conclude that  $\varphi : \mathcal{H} \rightarrow A_\varphi$  factors through  $\varphi_{\text{ord}}$ . This proves the existence of a morphism  $\varphi_{\text{ord}} \rightarrow \varphi$ . Uniqueness follows from the surjectivity of  $\varphi_{\text{ord}}$ . This completes the proof of Theorem 4.2.  $\square$

It is natural to ask whether or not the morphism  $\varphi'_{\text{ord}} \rightarrow \varphi_{\text{ord}}$  is an isomorphism, or equivalently, whether  $\varphi'_{\text{ord}}$  is arithmetic. Hida’s theory answers this in the affirmative when  $n = 2$ , but we know almost nothing in this direction when  $n \geq 3$ . The eigenpacket  $\varphi'_{\text{ord}}$  deserves to be of interest independent of whether or not it is arithmetic. For example,  $\varphi'_{\text{ord}}$  parametrizes  $p$ -ordinary torsion in the cohomology of the  $\mathbb{Z}_p$ -lattices  $L_\lambda$ ,  $\lambda \in \mathcal{W}^+$ . If  $\varphi'_{\text{ord}} \rightarrow \varphi_{\text{ord}}$  is an isomorphism, then it can be shown that every eigenpacket occurring in the  $p$ -ordinary torsion must be congruent to some  $p$ -ordinary arithmetic  $\overline{\mathbb{Q}}_p$ -eigenpacket.

DEFINITION 4.4. We define the  $p$ -ordinary eigenscheme to be the formal scheme  $X_{\text{ord}} := \text{Spf}(R_{\text{ord}})$ . We also let  $\mathcal{X}_{\text{ord}} := \mathcal{X}_{\varphi_{\text{ord}}}$  and call this the  $p$ -ordinary eigenvariety.

If  $\mathcal{O}_K$  is the ring of integers in a finite extension  $K/\mathbb{Q}_p$  then any continuous homomorphism  $R_{\text{ord}} \rightarrow K$  takes values in  $\mathcal{O}_K$ . Hence the canonical map

$$X_{\text{ord}}(\mathcal{O}_K) \xrightarrow{\sim} \mathcal{X}_{\text{ord}}(K)$$

is a bijection. Thus the scheme  $X_{\text{ord}}$  remembers the  $K$ -points of  $\mathcal{X}_{\text{ord}}$ . However, the scheme  $X_{\text{ord}}$  contains additional information related to congruences, and this will be important to us below in the definition of quasicuspidality. For  $x \in X_{\text{ord}}(K)$  we let  $P_x$  be the kernel in  $R_{\text{ord}}$  of  $x$ . Then  $P_x$  is a prime ideal in  $R_{\text{ord}}$ .

DEFINITION 4.5. If  $x, x' \in \mathcal{X}_{\text{ord}}(\overline{\mathbb{Q}}_p)$  are two  $\overline{\mathbb{Q}}_p$ -points, then we say that  $x$  and  $x'$  are congruent if  $P_x$  and  $P_{x'}$  are contained in the same maximal ideal of  $R_{\text{ord}}$ .

We note that two points  $x, x' \in \mathcal{X}_{\text{ord}}(\overline{\mathbb{Q}}_p)$  are congruent if and only if there is an automorphism  $\tau$  of  $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$  such that for every  $\alpha \in \mathcal{H}$  we have the congruence  $(\varphi_{\text{ord}})_x(\alpha) \equiv \tau((\varphi_{\text{ord}})_{x'}(\alpha))$  modulo the maximal ideal of the integers of  $\overline{\mathbb{Q}}_p$ . This is the usual notion of congruence between two systems of Hecke eigenvalues. Indeed, since  $R_{\text{ord}}$  is complete and semilocal, the set of maximal ideals of  $R_{\text{ord}}$  is in one-to-one correspondence with the set of connected components of  $X_{\text{ord}}$  and these are in one-to-one correspondence with the set of congruence classes of points on  $\mathcal{X}_{\text{ord}}$ .

We close this section by showing that  $\varphi_{\text{ord}}$  is isomorphic to  $\text{Tw}(\varphi_{\text{ord}})$  and deriving some simple consequences.

**THEOREM 4.6.** *The unique morphism  $\varphi_{\text{ord}} \rightarrow \text{Tw}(\varphi_{\text{ord}})$  is an isomorphism. Moreover, for every specialization  $\varphi_0$  of  $\varphi_{\text{ord}}$  at a point of  $\mathcal{X}_{\text{ord}}(\overline{\mathbb{Q}}_p)$ , there is a morphism*

$$\varphi_{\text{ord}} \longrightarrow \widetilde{\text{Tw}}(\varphi_0),$$

where  $\widetilde{\text{Tw}}(\varphi_0)$  is the universal twist of  $\varphi_0$  defined in Section 3.

*Proof.* Recall that if  $A$  is a  $\Lambda$ -module, then  $A(\mathbf{1})$  denotes its twist by  $\det$  (Definition 3.3). Let  $\mathbb{D} \cong \mathbb{D}(X)$  be the  $\Lambda[S]$ -module described in Section 2. We define the  $\Lambda$ -morphism  $\text{Tw} : \mathbb{D} \rightarrow \mathbb{D}(\mathbf{1})$  by sending a measure  $\mu \in \mathbb{D}$  to the measure  $\text{Tw}(\mu) \in \mathbb{D}(\mathbf{1})$  defined by the integration formula

$$\int_X f(x) d\text{Tw}(\mu)(x) = \int_X \det(x) f(x) d\mu(x)$$

for  $f \in C(X)$ . Clearly  $\text{Tw}$  is an isomorphism. On the other hand, for  $\sigma \in S$  we have  $\text{Tw}(\mu|\sigma) = \det(\sigma)\text{Tw}(\mu)|\sigma$ . In particular,  $\text{Tw}$  is a  $\Gamma$ -morphism and therefore induces a  $\Lambda$ -isomorphism

$$\mathbb{H} \xrightarrow{\sim} \mathbb{H}(\mathbf{1}).$$

Conjugation induces a  $\Lambda$ -isomorphism  $\text{Tw} : \text{End}_\Lambda(\mathbb{H}) \xrightarrow{\sim} \text{End}_\Lambda(\mathbb{H}(\mathbf{1}))$ . A computation shows that the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\text{Tw}} & \mathcal{H}(\mathbf{1}) \\ \downarrow & & \downarrow \\ \text{End}_\Lambda(\mathbb{H}) & \xrightarrow{\text{Tw}} & \text{End}_\Lambda(\mathbb{H}(\mathbf{1})) \end{array}$$

is commutative where the vertical arrows are both given by  $\varphi'_{\text{ord}}$ . Hence  $\text{Tw}$  induces an isomorphism  $\text{Tw} : \varphi'_{\text{ord}} \rightarrow \varphi'_{\text{ord}}$ . By Proposition 3.4, this morphism preserves the ideal  $I_{\text{arith}}$  and therefore induces an isomorphism  $\text{Tw} : \varphi_{\text{ord}} \rightarrow \text{Tw}(\varphi_{\text{ord}})$ .

Now let  $\varphi_0$  be the specialization of  $\varphi_{\text{ord}}$  at a point  $x_0 \in \mathcal{X}_{\text{ord}}(\overline{\mathbb{Q}}_p)$ . For each integer  $n$ , let  $x_n := \text{Tw}^n(x_0) \in \mathcal{X}_{\text{ord}}(\overline{\mathbb{Q}}_p)$  and note that we then have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\varphi_{\text{ord}}} & R_{\text{ord}} \\ \downarrow & & \downarrow \\ \overline{\mathbb{Q}}_p[[\mathbb{Z}_p^\times]] & \xrightarrow{\psi_n^\dagger} & \overline{\mathbb{Q}}_p \end{array}$$

Here the left vertical arrow is  $\widetilde{\text{Tw}}(\varphi_0)$ , the right vertical arrow is  $x_n$ , and  $\psi_n^\dagger$  is the map with structure character the  $n$ th power character  $\mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ . However, since  $\bigcap_n \ker(\psi_n^\dagger) = \{0\}$ , we see that  $\widetilde{\text{Tw}}(\varphi_0)$  maps every  $\alpha \in \ker(\varphi_{\text{ord}})$  to 0. Finally, since  $\varphi_{\text{ord}}$  is surjective, we conclude that  $\widetilde{\text{Tw}}(\varphi_0)$  factors through  $\varphi_{\text{ord}}$ . This gives us a morphism  $\varphi_{\text{ord}} \rightarrow \widetilde{\text{Tw}}(\varphi_0)$ . This completes the proof of Theorem 4.6. □

If  $\varphi = (\varphi_{\text{ord}})_x$  is the eigenpacket associated to an arbitrary point  $x \in \mathcal{X}_{\text{ord}}(\overline{\mathbb{Q}}_p)$ , then the last theorem gives us a morphism  $R_{\text{ord}} \rightarrow \overline{\mathbb{Q}}_p[[\mathbb{Z}_p^\times]]$ . If we compose this morphism with the morphism associated to the canonical projection  $\mathbb{Z}_p^\times \rightarrow 1 + p\mathbb{Z}_p$  (or to  $1 + 4\mathbb{Z}_p$  if  $p = 2$ ) we

obtain a  $\Lambda$ -morphism

$$\tilde{x} : R_{\text{ord}} \longrightarrow \overline{\mathbb{Q}}_p[[1 + p\mathbb{Z}_p]]$$

with kernel a prime ideal  $P_{\tilde{x}} \subseteq R_{\text{ord}}$ .

DEFINITION 4.7. For arbitrary  $x \in \mathcal{X}_{\text{ord}}(\overline{\mathbb{Q}}_p)$  we let  $Z(\tilde{x}) \subseteq \mathcal{X}_{\text{ord}}$  be the irreducible Zariski closed subset of  $\mathcal{X}_{\text{ord}}$  cut out by  $P_{\tilde{x}}$ .

We have the following corollary of Theorem 4.6.

COROLLARY 4.8. If  $Z$  is an irreducible component of  $\mathcal{X}_{\text{ord}}$  and  $x \in \mathcal{X}_{\text{ord}}(\overline{\mathbb{Q}}_p)$  then  $x \in Z(\overline{\mathbb{Q}}_p) \Rightarrow Z(\tilde{x}) \subseteq Z$ .

*Proof.* Since  $R_{\text{ord}}$  is finite over  $\Lambda$ , we have  $\mathcal{X}_{\text{ord}}$  is a noetherian topological space (with the Zariski topology), and hence has only finitely many irreducible components. Consider the powers  $\text{Tw}^n$  of the twist morphism  $\text{Tw} : \mathcal{X}_{\text{ord}} \rightarrow \mathcal{X}_{\text{ord}}$  of Theorem 4.6 for  $n \equiv 0$  modulo  $(p - 1)$  (or modulo 2 if  $p = 2$ ). Clearly,  $\text{Tw}^n(Z)$  is an irreducible component of  $\mathcal{X}_{\text{ord}}$  for every such  $n$ ; thus there is an integer  $N \equiv 0$  modulo  $(p - 1)$  (again modulo 2 if  $p = 2$ ) such that  $\text{Tw}^n(Z) = \text{Tw}^m(Z)$  whenever  $n \equiv m$  modulo  $N$ . In particular, we have  $\text{Tw}^n(Z) = Z$  whenever  $N|n$ . From this it follows that the set  $\{\text{Tw}^n(x) \mid N|n\} \subseteq Z(\overline{\mathbb{Q}}_p)$ . However, this set is a Zariski dense subset of  $Z(\tilde{x})$ . Therefore  $Z(\tilde{x}) \subseteq Z$  and Corollary 4.8 is proved.  $\square$

THEOREM 4.9. Let  $\varphi_0$  be a  $p$ -ordinary arithmetic  $\overline{\mathbb{Q}}_p$ -eigenpacket, and let  $x_0 \in \mathcal{X}_{\text{ord}}^+$  be the corresponding point. Then  $\varphi_0$  is rigid if and only if  $Z(\tilde{x}_0)$  is an irreducible component of  $\mathcal{X}_{\text{ord}}$ .

*Proof.* If  $\varphi_0$  is rigid and  $Z$  is an irreducible component of  $\mathcal{X}_{\text{ord}}$  containing  $x_0$ , then by the definition of rigidity,  $Z$  must be a twist-deformation of  $x_0$ . Hence every arithmetic point of  $Z$  is a twist of  $x_0$ , and therefore every arithmetic point of  $Z$  is in  $Z(\tilde{x}_0)$ . Since the arithmetic points of  $Z$  are Zariski dense in  $Z$ , we conclude that  $Z \subseteq Z(\tilde{x}_0)$ . Therefore by Corollary 4.8, we have  $Z = Z(\tilde{x}_0)$  as claimed.

Conversely, suppose that  $Z(\tilde{x}_0)$  is an irreducible component of  $\mathcal{X}_{\text{ord}}$ , and let  $(\varphi, z_0)$  be any arithmetic deformation of  $\varphi_0$ . Since  $\varphi_0$  is ordinary, we can choose an irreducible affinoid neighborhood  $Z$  of  $z_0$  in  $\mathcal{X}_\varphi$  such that  $\varphi_Z$  is  $p$ -ordinary. Hence there is a morphism  $f : \varphi_{\text{ord}} \rightarrow \varphi_Z$  such that  $f^\#(z_0) = x_0$ . Since  $Z$  is irreducible, we have also  $f^\#(Z)$  is irreducible in  $\mathcal{X}_{\text{ord}}$ . Let  $W$  be an irreducible component containing  $f^\#(Z)$ . By Corollary 4.8,  $Z(\tilde{x}_0) \subset W$ . Since  $Z(\tilde{x}_0)$  is an irreducible component,  $Z(\tilde{x}_0) = W$ . Thus  $f^\#(Z) \subseteq Z(\tilde{x}_0)$ . Hence  $\varphi_Z$  is a twist of  $\varphi_0$  and it follows that  $(\varphi, z_0)$  is a twist deformation of  $\varphi_0$ . This proves  $\varphi_0$  is rigid and Theorem 4.9 is proved.  $\square$

### 5. Proof of Lemma 4.3

In this section, we prove a general cohomological result (Theorem 5.1) and deduce Lemma 4.3 as a consequence.

Let  $M$  be a profinite  $\Lambda[S]$ -module, and let  $H^*(M)^0$  be the ordinary part of the cohomology  $H^*(\Gamma, M)$  with respect to the operator  $T(\pi)$ . Then we may (and do) regard  $H^*(M)^0$  as an  $\mathcal{H}$ -module. Define  $R(M)$  to be the image of  $\mathcal{H}$  in  $\text{End}_\Lambda(H^*(M)^0)$  and define  $A(M)$  to be  $R(M)^{\text{red}}$ , that is,  $R(M)$  modulo its nilradical.

**THEOREM 5.1.** *Let  $M$  be a profinite  $\Lambda[S_0]$ -module, and suppose that  $H^*(M)^0$  is a finitely generated  $\Lambda$ -module. Let  $I$  be an ideal in  $\Lambda$  that is generated by a finite  $M$ -regular sequence. Then  $H^*(M/IM)^0$  is a finitely generated  $\Lambda/I$ -module. Moreover there is a natural isomorphism*

$$R(M)/\text{Rad}_{R(M)}(I) \cong A(M/I),$$

where  $\text{Rad}_{R(M)}(I)$  is the radical of  $IR(M)$  in  $R(M)$ .

Before proving Theorem 5.1, we first indicate how it implies Lemma 4.3.

*Proof of Lemma 4.3.* If we take  $M = \mathbb{D}$  then we have  $R(\mathbb{D}) = R'_{\text{ord}}$ . Moreover, we know (see Lemma 2.2) that for each dominant integral weight  $\lambda$ , the ideal  $I_\lambda \subseteq \Lambda$  is generated by a  $\mathbb{D}$ -regular sequence and that  $\mathbb{D}/I_\lambda\mathbb{D} \cong \mathbb{D}_\lambda$ . Hence, by Theorem 5.1, we have  $A(\mathbb{D}_\lambda) \cong R'_{\text{ord}}/\text{Rad}_{R'_{\text{ord}}}(I_\lambda)$ . Moreover, from Lemma 2.3 we have  $A(L_\lambda) \cong A(\mathbb{D}_\lambda)$ . Thus we have

$$A(L_\lambda) \cong R'_{\text{ord}}/\text{Rad}_{R'_{\text{ord}}}(I_\lambda),$$

from which Lemma 4.3 is an immediate consequence. □

We now turn to the proof of Theorem 5.1.

*Proof of Theorem 5.1.* Our proof is by induction on the length of an  $M$ -regular sequence of generators for  $I$ . We begin with the case where  $I$  is principal, generated by an  $M$ -regular element  $\alpha \in \Lambda$ , that is an element  $\alpha$  such that  $M$  has no  $\alpha$ -torsion. In this case, we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} M \longrightarrow M/\alpha M \longrightarrow 0.$$

Now moving on to the long exact cohomology sequence, we obtain an exact sequence

$$0 \longrightarrow H/\alpha H \longrightarrow H^*(M/\alpha M)^0 \longrightarrow H[\alpha] \longrightarrow 0,$$

where  $H := H^*(M)^0$  and  $H[\alpha]$  is the  $\alpha$ -torsion in  $H$ . We first construct a homomorphism  $R(M) \rightarrow A(M/\alpha M)$ . For this, we let  $x \in \mathcal{H}$  be such that  $x$  annihilates  $H$ . Then the last exact sequence implies that  $x^2$  annihilates  $H^*(M/\alpha M)^0$ . Hence  $x$  maps to an element of the nilradical of  $R(M/\alpha M)$  and it follows that  $x$  maps to 0 in  $A(M/\alpha M)$ . Thus the canonical map  $\mathcal{H} \rightarrow A(M/\alpha M)$  factors through the canonical map  $\mathcal{H} \rightarrow R(M)$ . Hence we have a canonical surjective map

$$\tilde{\varphi} : R(M) \longrightarrow A(M/\alpha M).$$

We need to show that  $\ker(\tilde{\varphi}) = \text{Rad}_{R(M)}(\alpha)$ . The inclusion  $\supseteq$  is obvious.

Therefore let  $x \in \ker(\tilde{\varphi})$ . From the above exact sequence, we conclude that  $xH \subseteq \alpha H$ . Since, by hypothesis,  $H$  is finitely generated over  $\Lambda$ , there is a positive integer  $m$  such that  $H[\alpha^{m+1}] = H[\alpha^m]$ . In particular, we see that  $\alpha^m H$  has no  $\alpha$ -torsion, and from this it follows that  $\alpha : \alpha^m H \rightarrow \alpha^{m+1} H$  is an isomorphism of  $\Lambda$ -modules. Let  $\beta : \alpha^{m+1} H \rightarrow \alpha^m H$  be the inverse map. Let  $y \in \text{End}_\Lambda(H)$  be the composition

$$y : H \xrightarrow{x^{m+1}} \alpha^{m+1} H \xrightarrow{\beta} \alpha^m H \subseteq H.$$

Then  $\alpha y = x^{m+1}$ .

Now define

$$R' := \{ y \in \text{End}_\Lambda(H) \mid \exists k \in \mathbb{Z}^+, z \in R(M) \text{ such that } \alpha^k y = \alpha^m z \}.$$

The endomorphism  $y$  constructed in the last paragraph is an element of  $R'$ . Clearly,  $R'$  is a finite  $\Lambda$ -algebra containing  $R(M)$ . Moreover, for every element of  $\rho \in R'$  we have  $\alpha^k \rho \in R(M)$

for some  $k \geq 0$ . Since  $R'$  is finitely generated over  $\Lambda$ , there is an exponent  $N$  such that  $\alpha^N R' \subseteq R(M)$ .

Now consider  $x^{(m+1)(N+1)} = (\alpha y)^{N+1} = \alpha(\alpha^N y^{N+1})$ . Since  $y^{N+1} \in R'$  we have  $z := \alpha^N y^{N+1} \in R(M)$ . Hence  $x^{(m+1)(N+1)} \in \alpha R(M)$ . This proves  $x \in \text{Rad}_{R(M)}(\alpha)$ . Hence  $\ker(\varphi) \subseteq \text{Rad}_{R(M)}(\alpha)$ . This completes the proof in the special case where  $I$  is generated by a single  $M$ -regular element of  $\Lambda$ .

Now we suppose that  $r \geq 1$  and that the theorem is true whenever  $I$  is generated by an  $M$ -regular sequence of length  $r$ . Let  $\alpha_1, \dots, \alpha_r, \alpha$  be an  $M$ -regular sequence of length  $r + 1$ . Let  $I$  be the ideal generated by  $\alpha_1, \dots, \alpha_r$  in  $\Lambda$ , and let  $J$  be the ideal generated by  $I$  and  $\alpha$ . We define  $\psi$  to be the composition of the surjective homomorphisms

$$\psi : R(M) \xrightarrow{\varphi} R(M/IM) \longrightarrow A(M/JM).$$

Let  $x \in \ker(\psi)$ . Then by the principal case proved in the last paragraph, we have  $y := \varphi(x) \in \text{Rad}_{R(M/IM)}(\alpha)$ . Thus there is an  $m \in \mathbb{Z}^+$  such that  $y^m \in \alpha R(M/IM)$ . This means that  $\varphi(x^m) \in \alpha R(M/IM)$ . Thus there is an element  $z \in R(M)$  such that  $\varphi(x^m) = \alpha\varphi(z)$ . It then follows from the induction hypothesis that  $x^m - \alpha z \in \text{Rad}_{R(M)}(I)$ . There is therefore a positive integer  $N$  such that

$$(x^m - \alpha z)^N \in IR(M).$$

From this we see that  $x^{mN} \in JR(M)$ . Hence  $x \in \text{Rad}_{R(M)}(J)$ . This proves  $\ker(\psi) \subseteq \text{Rad}_{R(M)}(J)$ . However, the opposite inclusion is immediate. Hence  $\ker(\psi) = \text{Rad}_{R(M)}(J)$ . This completes the proof of Theorem 5.1. □

### 6. Cuspidal and quasicuspidal eigenpackets

In this section we recall briefly the connection between cohomology and automorphic representations and review some results from the theory of mod  $p$  cohomology of congruence subgroups  $\Gamma$  of  $\text{SL}(n, \mathbb{Z})$  in order to have a sufficient condition for arithmetic eigenpackets to be cuspidal automorphic.

We denote the adèles of  $\mathbb{Q}$  by  $\mathbb{A}$ . Let  $\Gamma = \Gamma_0(N)$  for some  $N$ , and let  $M$  be a compact open subgroup of  $\text{GL}(n, \mathbb{A}_f)$  such that its intersection with  $\text{GL}(n, \mathbb{Q})$  is  $\Gamma$ . Let  $\tilde{\mathfrak{g}}$  denote the Lie algebra of  $\text{SL}(n, \mathbb{R})$ ,  $K = \text{O}(n)$ ,  $Z$  the center of  $\text{SL}(n, \mathbb{R})$ . Let  $E$  be any finite-dimensional irreducible algebraic complex representation of  $\text{GL}(n, \mathbb{R})$ .

The cuspidal cohomology  $H_{\text{cusp}}^*(\Gamma, E)$  is defined in [6]. By [9, Lemma 3.15, p. 121], there is a canonical surjective map

$$\Psi : \bigoplus_{\Pi} H^*(\tilde{\mathfrak{g}}, K; \Pi_{\infty} \otimes E) \otimes \Pi_f^M \longrightarrow H_{\text{cusp}}^*(\Gamma, E),$$

where  $\Pi$  runs over a certain finite set of cuspidal automorphic representations of  $\text{GL}(n, \mathbb{A})$ . The source of  $\Psi$  is adelic and so computes exactly the cuspidal cohomology of a certain adelic double coset space which has one of its topological components isomorphic to the locally symmetric space for  $\Gamma$ . Therefore  $\Psi$  is surjective, but not necessarily an isomorphism. (Note: although  $\text{O}(n)$  is not contained in  $\text{SL}(n, \mathbb{R})$ , it acts on it by conjugation, and  $\Psi$  is displayed correctly. See [9, pp. 113–4].)

If  $\alpha \in H_{\text{cusp}}^*(\Gamma, E)$  is an  $\mathcal{H}$ -eigenclass, then by the strong multiplicity one theorem for  $\text{GL}(n)$ , there is a unique  $\Pi_{\alpha}$  such that  $\alpha \in \Psi(H^*(\tilde{\mathfrak{g}}, K; \Pi_{\alpha, \infty} \otimes E) \otimes \Pi_{\alpha, f}^M)$ .

We fix an isomorphism of  $\mathbb{C}_p$  and  $\mathbb{C}$ . Recall that  $\Gamma_0$  denotes the subset of  $\Gamma$  consisting of those matrices that are upper triangular modulo  $p$ . If  $\alpha \in H^*(\Gamma_0, L_{\lambda})$ , then we say that  $\alpha$  is cuspidal if its image in  $H^*(\Gamma_0, L_{\lambda} \otimes \mathbb{C}) \simeq H^*(\Gamma_0, L_{\lambda}) \otimes_{\mathbb{Z}_p} \mathbb{C}$  is cuspidal.



DEFINITION 6.1. An  $\mathcal{O}_K$ -valued eigenpacket is said to be cuspidal if it occurs as the system of  $\mathcal{H}$ -eigenvalues on a cuspidal cohomology class. A point on  $\mathcal{X}_{\text{ord}}(K)$  is cuspidal if  $\varphi_x$  is cuspidal.

We will need the following fact. Set  $n = 3$  and let  $\partial Y$  denote the boundary of the Borel–Serre compactification of  $\Gamma \backslash \text{GL}(n, \mathbb{R})/KZ$ . Then

$$H^3_{\text{cusp}}(\Gamma, E) \oplus H^3(\partial Y, E) \simeq H^3(\Gamma, E).$$

See, for instance, Proposition 2.1 and its proof in [2] which, though stated there for trivial coefficients, remains valid for  $E$ -coefficients.

If  $\varphi$  is an  $\mathcal{O}_K$ -valued eigenpacket such that  $\varphi(T(\ell, k)) = a(\ell, k)$  then the Hecke polynomial of  $\varphi$  at  $\ell$  is defined to be

$$P_{\varphi, \ell}(X) = \sum_{k=0}^n (-1)^k a(\ell, k) \ell^{k(k-1)/2} X^k \in \mathcal{O}_K[X].$$

If  $x \in \mathcal{X}_{\text{ord}}(K)$  then we will write  $P_{x, \ell}$  for  $P_{\varphi_x, \ell}$ .

DEFINITION 6.2. An  $\mathcal{O}_K$ -valued eigenpacket  $\varphi$  is said to be  $\ell$ -quasicuspidal if  $P_{\varphi, \ell}$  is irreducible modulo the maximal ideal of  $\mathcal{O}_K$ . We say that  $\varphi$  is quasicuspidal if  $\varphi$  is  $\ell$ -quasicuspidal for some prime  $\ell \neq p$ .

We say that a point  $x \in \mathcal{X}_{\text{ord}}(K)$  is  $\ell$ -quasicuspidal or quasicuspidal if  $\varphi_x$  is  $\ell$ -quasicuspidal or quasicuspidal, respectively.

The following proposition follows immediately from the definitions.

PROPOSITION 6.3. *If  $x \in \mathcal{X}_{\text{ord}}(K)$  is quasicuspidal then every point in the connected component of  $x$  in  $\mathcal{X}_{\text{ord}}(K)$  is also quasicuspidal.*

CONJECTURE 6.4. Every quasicuspidal arithmetic eigenpacket is cuspidal.

THEOREM 6.5. *If  $n = 2$  or  $3$  then conjecture 6.3 is true.*

*Proof.* This follows from [3, Proposition 3.2.1], which says that a Hecke eigenclass in  $H^3(\partial Y, E)$  has all its Hecke polynomials reducible plus the fact that a non-cuspidal Hecke eigenclass must restrict non-trivially to the Borel–Serre boundary, if  $n \leq 3$ . □

### 7. Removing $p$ from the level

In this section only, in order to conform with the convention of the references, all representation spaces will be left modules.

Let  $I$  denote the Iwahori subgroup of  $\text{GL}(n, \mathbb{Z}_p)$  consisting of all matrices which are upper triangular modulo  $p$ . When  $n = 3$ , our computers are not powerful enough to compute the cohomology of  $\Gamma_0 := \Gamma \cap I$ . We can only compute for the larger group  $\Gamma$ . Therefore we need a theorem that allows us to remove  $p$  from the level of a cohomology class.

Let  $G = \text{GL}(n)$  and let  $(B, T)$  be the pair of upper triangular and diagonal matrices, respectively. We use the corresponding ordering on the roots, so that the positive roots are

those occurring in  $B$ . Let  $\delta$  be the character on  $T$  which is the  $p$ -adic absolute value of the product of the positive root characters:  $\delta(\text{diag}(t_1, \dots, t_n)) = |t_1|^{n-1}|t_2|^{n-3} \dots |t_n|^{1-n}$ .

Recall that we have identified the Hecke algebras  $\mathcal{H}$  and  $\mathcal{H}_0$  for  $\Gamma$  and  $\Gamma_0$ , and so  $\mathcal{H}$  acts on the cohomology of both  $\Gamma$  and  $\Gamma_0$ . We then have the following theorem.

**THEOREM 7.1.** *Let  $\Gamma$  be a congruence subgroup of  $GL(n, \mathbb{Z})$  of level prime to  $p$ , and let  $\Gamma_0 = \Gamma \cap I$  as above. Let  $\lambda$  be the highest weight  $(a_1, a_2, \dots, a_n)$ , so that  $a_1 \geq a_2 \geq \dots \geq a_n$ . Let  $\phi$  be a packet of Hecke eigenvalues (for  $\mathcal{H}$ ) that occurs in  $H_{\text{cusp}}^*(\Gamma_0, L_\lambda(\mathbb{C}))^0$ . Suppose that  $a_1 > a_2 > \dots > a_n$ . Then there is an  $\mathcal{H}$ -eigenclass  $\alpha \in H_{\text{cusp}}^*(\Gamma, L_\lambda(\mathbb{C}))$  such that for any  $l \neq p$  and any  $i = 1, \dots, n$ , we have  $T(l, i)\alpha = \phi(T(l, i))\alpha$ .*

*Proof.* Let  $|\cdot|$  denote the  $p$ -adic absolute value normalized so that  $|p| = p^{-1}$ . Let  $\alpha \in H_{\text{cusp}}^*(\Gamma_0, L_\lambda)^0$  be an ordinary cuspidal cohomology class which is a Hecke eigenclass with eigenvalues given by  $\phi$ . Let  $\Pi$  be the irreducible automorphic representation of  $GL(n, \mathbb{A})$  associated to  $\alpha$  and write  $\Pi_f = \otimes \Pi_v$ .

Then  $\Pi_p$  has a non-zero  $I$ -invariant vector and therefore is isomorphic to a subrepresentation of the induced representation  $\mathbb{I}(\chi)$  from  $B(\mathbb{Q}_p)$  to  $G(\mathbb{Q}_p)$  of some unramified character  $\chi$  on  $T(\mathbb{Q}_p)$  (see [8, Theorem 3.8]). This is normalized induction, so that  $\mathbb{I}(\chi)$  consists of the locally constant functions  $f : G(\mathbb{Q}_p) \rightarrow \mathbb{C}$  such that  $f(bg) = \delta(b)^{1/2}\chi(b)f(g)$  for  $b \in B$  and  $g \in G$  (where  $\chi$  is extended from  $T$  to  $B$  by making it trivial on the unipotent radical of  $B$ .)

By a straightforward but rather complicated explicit computation in the induced representation, we can determine the unramified character  $\chi$ , given the eigenvalues of the Hecke operators at  $p$ . Let  $U(p, m)$  denote the Hecke operator  $U(1, \dots, 1, p, \dots, p)$  with  $m$  copies of  $p$ ,  $m = 0, \dots, n$ . We may choose  $\alpha$  so that it is also an eigenclass for all the  $U(p, m)$  operators. Let  $\beta_m$  be the eigenvalue of  $U(p, m)$  on the cohomology class  $\alpha$  with respect to the  $\star$ -action. By assumption, the algebraic integers  $\beta_i$  are  $p$ -adic units. Note that  $\beta_0 = \beta_n = 1$ , since  $U(p, 0)$  and  $U(p, n)$  act via the  $\star$ -action as the identity.

To compare the cohomology with the automorphic theory, we have to remove the  $\star$ -action. With respect to the usual action, the eigenvalue of  $U(p, m)$  on  $\alpha$  is  $p^{a_n + a_{n-1} + \dots + a_{n-m+1}} \beta_m$ .

Let  $d_m$  be the diagonal matrix with 1 along the diagonal except for a  $p$  at the  $m$ th place. The explicit computation alluded to above gives the following values, for  $m = 1, \dots, n$ :

$$\chi \delta^{1/2}(d_m) = p^{a_{n-m+1} - n + 2m - 1} \beta_m \beta_{m-1}^{-1}.$$

It follows that if  $r_m = a_{n-m+1} - (n - 2m + 1)/2$ , then

$$\chi(d_m) = p^{r_m} \beta_m \beta_{m-1}^{-1}.$$

Define characters  $\chi_i$  of  $\mathbb{Q}_p$  by  $\chi(\text{diag}(t_1, \dots, t_n)) = \chi_1(t_1) \dots \chi_n(t_n)$ . Since  $\chi$  is unramified, each  $\chi_i(t) = |t|^{c_i}$  for some complex number  $c_i$ .

It is a result of Bernstein and Zelevinsky ([5, Theorem 4.2]) that  $\mathbb{I}(\chi)$  is irreducible unless  $\chi_i \chi_j^{-1} = |\cdot|$  for some  $i \neq j$ . See of [14, Section 1] for a discussion of question of reducibility of representations induced from parabolic subgroups.

We compute

$$\chi_i \chi_j^{-1}(p) = p^{r_i - r_j} \beta_i \beta_{i-1}^{-1} \beta_j^{-1} \beta_{j-1}.$$

Note that since  $a_1 > a_2 > \dots > a_n$ , the  $r_m$  are a strictly increasing sequence of integers with gaps of at least 2 between successive terms. Taking into account the fact that the  $\beta_m$  are  $p$ -adic units, we see that  $\chi_i \chi_j^{-1}(p)$  never equals  $p^{-1} = |p|$ . Therefore we conclude that  $\mathbb{I}(\chi)$  is irreducible and so  $\Pi_p = \mathbb{I}(\chi)$ .

Let  $\Psi$  be the map from  $(\mathfrak{g}, K)$  cohomology discussed in Section 6. As in that section,  $M$  denotes a compact subgroup of  $G(\mathbb{A}_f)$  intersecting  $G(\mathbb{Q})$  in  $\Gamma$ . We let  $M_0 =$

$\{y \in M \mid y_p \in I\}$ . Then  $\alpha = \Psi(c \otimes x)$  for some generator  $c$  of  $H^*(\tilde{\mathfrak{g}}, K; \Pi_{\alpha, \infty} \otimes E)$  and for some  $x = \otimes x_v \in \Pi_f^{M_0}$ .

It follows from [8, p. 152] that  $\Pi_p$  contains a vector  $x'_p$  fixed under  $\text{GL}(n, \mathbb{Z}_p)$ . (Thus, in fact,  $\Pi$  is unramified at  $p$ .) Set  $x' = \otimes x'_v$ , where  $x'_v = x_v$  if  $v \neq p$ . Then  $\alpha' = \Psi(c \otimes x')$  is a Hecke eigenclass in  $H^*_{\text{cusp}}(\Gamma, L_\lambda(\mathbb{C}))$  with the same packet of  $\mathcal{H}'$ -eigenvalues as  $\alpha$ .  $\square$

### 8. Essential selfduality

We begin this section with some considerations of selfduality. If  $B$  is a ring and  $\pi$  a representation of  $\text{GL}(n, B)$ , then the contragredient of  $\pi$  is the representation  $\pi'$  defined by  $\pi'(g) = \pi({}^t g^{-1})$ . We say that  $\pi$  is ‘essentially selfdual’ if there is a character  $\chi$  of  $B^\times$  such that  $\pi'$  is isomorphic to  $\pi \otimes (\chi \circ \det)$ .

In parallel to this, if  $H$  is a Hecke algebra over  $B$  with involution  $\iota$  and  $\phi : H \rightarrow A$  is a  $B$ -homomorphism, then we will say that  $\phi$  is ‘essentially selfdual’ if there is a character  $\chi$  of  $A^\times$  such that for every double coset  $T$  in  $H$ , we have  $\phi(\iota(T)) = \chi(\det(T))\phi(T)$ .

DEFINITION 8.1. (a) If  $(\Gamma, S)$  is a Hecke pair for  $\text{GL}(n)$  and  $E$  is a finite-dimensional  $S$ -module over a ring of characteristic 0, then a Hecke eigenclass in  $H^*(\Gamma, E)$  is called an essentially selfdual cohomology class if its associated automorphic representation (as in Section 6) is an essentially selfdual representation.

(b) If  $\phi$  is an  $\mathcal{H}$ -eigenpacket (as in Section 3), then it is called an essentially selfdual eigenpacket if  $\phi|\mathcal{H}'$  is essentially selfdual, where  $\mathcal{H}'$  is the Hecke algebra of the semigroup  $S'$  defined in Section 1 and  $\iota$  is the involution of it given by transpose-inverse composed with conjugation by the matrix  $\text{diag}(N, 1, \dots, 1)$ .

It follows from the definitions that an essentially selfdual cohomology eigenclass affords an essentially selfdual eigenpacket.

The (contravariant) dominant integral weights with respect to  $(B^{\text{opp}}, T)$  are given by  $n$ -tuples of integers  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Such an  $n$ -tuple corresponds to the character  $t := \text{diag}(t_1, t_2, \dots, t_n) \mapsto t^\lambda := t_1^{\lambda_1} t_2^{\lambda_2} \dots t_n^{\lambda_n}$  on the torus  $T$ . The highest weight of the contragredient of  $V_\lambda$  is the dominant integral weight  $\lambda^\vee := (-\lambda_n, \dots, -\lambda_1)$ . Therefore  $\lambda$  is essentially selfdual if and only if  $\lambda_i + \lambda_{n+1-i} = \lambda_j + \lambda_{n+1-j}$  whenever  $1 \leq i, j \leq n$ .

DEFINITION 8.2. We say that a point  $k = (k_1, k_2, \dots, k_n) \in \mathcal{W}(K)$  of the weight space is *essentially selfdual* if  $k_i + k_{n+1-i} = k_j + k_{n+1-j}$  whenever  $1 \leq i, j \leq n$ . We let  $\mathcal{W}^{\text{esd}}$  be the set of all essentially selfdual weights in  $\mathcal{W}$ . We also define

$$\mathcal{X}_{\text{ord}}^{\text{esd}} := \{x \in \mathcal{X}_{\text{ord}} \mid \text{wt}(x) \in \mathcal{W}^{\text{esd}}\}.$$

We note that  $\mathcal{W}^{\text{esd}}$  is a closed subspace of  $\mathcal{W}$  of dimension  $[n/2] + 1$  and that  $\mathcal{X}_{\text{ord}}^{\text{esd}}$  is therefore a closed subspace of  $\mathcal{X}_{\text{ord}}$  of dimension at most  $[n/2] + 1$ .

By [7, Theorem 6.12 II], the cuspidal cohomology of an arithmetic group with coefficients in a finite-dimensional irreducible rational module  $E$  over a field  $L$  of characteristic 0 vanishes unless  $E$  is essentially selfdual as an  $L$ -module. We shall call such  $E$  an essentially selfdual coefficient module, and its highest weight is an essentially selfdual weight.

It is easy to see that an essentially selfdual eigenpacket has an essentially selfdual weight. The converse is not true, as shown by the examples computed in [2].

**THEOREM 8.3.** *Every cuspidal point of  $\mathcal{X}_{\text{ord}}$  is contained in  $\mathcal{X}_{\text{ord}}^{\text{esd}}$ .*

*Proof.* The statement is immediate from the Borel–Wallach theorem and the fact that  $\mathcal{X}_{\text{ord}}^{\text{esd}}$  is Zariski closed. □

We now suppose that  $n = 3$ . We will use the notation  $\lambda(g) = (2g, g, 0)$  for  $g \geq 0$ . Any essentially selfdual highest weight is of the form  $\lambda(g) \otimes \det^j$  for some  $g \geq 0$  and  $j \in \mathbb{Z}$ .

**THEOREM 8.4.** *Let  $K$  be a finite extension of  $\mathbb{Q}_p$ . Let  $g_0 \geq 0$  and set  $\lambda_0 = \lambda(g_0)$ . Let  $x_0 \in \mathcal{X}_{\text{ord}}(K)$  be a quasicuspidal arithmetic eigenpacket of weight  $\lambda_0$  and suppose that  $x_0$  is not arithmetically rigid. Then for every integer  $g \equiv g_0$  modulo  $(p - 1)$  (or modulo 2 if  $p = 2$ ), there is a cuspidal automorphic point  $x \in \mathcal{X}_{\text{ord}}(\overline{\mathbb{Q}}_p)$  of weight  $\lambda(g)$  that is congruent to  $x_0$ .*

*Proof.* Since  $x_0$  is not arithmetically rigid, let  $Z$  be an irreducible component of  $\mathcal{X}_{\text{ord}}$  containing  $x_0$ . Then Corollary 4.8 and Theorem 4.9 tell us that  $Z(\tilde{x}_0)$  is properly contained in  $Z$ . In particular, we know that  $\dim Z \geq 2$ . On the other hand, since  $x_0$  is quasicuspidal, every arithmetic point on  $Z$  is quasicuspidal and is therefore cuspidal (Theorem 6.4). Therefore Theorem 8.3 tells us  $Z \subseteq \mathcal{X}_{\text{ord}}^{\text{esd}}$ . However, since  $n = 3$ , we know that  $\mathcal{X}_{\text{ord}}^{\text{esd}}$  has dimension at most 2. It follows that  $Z$  is an irreducible component of  $\mathcal{X}_{\text{ord}}^{\text{esd}}$  and that  $\dim Z = 2$ . Since  $R_{\text{ord}}$  is finite over  $\Lambda$ , the image of  $Z$  under the weight map is a Zariski-closed irreducible subset of  $\mathcal{W}^{\text{esd}}$  also having dimension 2. However,  $\mathcal{W}^{\text{esd}}$  is two-dimensional and therefore the image of  $Z$  is precisely the connected component of  $\mathcal{W}$  containing  $\lambda_0$ .

However, if  $g$  is a non-negative integer  $g \equiv g_0$  modulo  $(p - 1)$  (or modulo 2 if  $p = 2$ ), then  $\lambda(g)$  is in the same connected component of  $\mathcal{W}$  as  $\lambda_0$ . Thus, there is a point  $x \in Z(\overline{\mathbb{Q}}_p)$  lying over  $\lambda(g)$ . By Lemma 4.3,  $x$  is arithmetic. Clearly  $x$  and  $x_0$  are congruent (see Definition 4.5 and the accompanying remarks). Hence  $x$  is quasi-cuspidal and is therefore cuspidal automorphic of weight  $\lambda(g)$ . This completes the proof of the theorem. □

The following corollary is immediate from the theorem and the definitions. Recall from the introduction that we denote by  $V_g$  the representation  $V_{\lambda(g)}$  of highest weight  $\lambda(g)$ .

**COROLLARY 8.5.** *Let  $\varphi_0$  be a quasicuspidal arithmetic eigenpacket of weight  $\lambda(g_0)$  occurring in  $H^3(\Gamma, V_{g_0}(K))$  and suppose that  $\varphi_0$  is not arithmetically rigid. Then for every integer  $g \equiv g_0$  modulo  $(p - 1)$  (or modulo 2 if  $p = 2$ ), there is a finite extension  $L/K$  and a cuspidal automorphic eigenpacket  $\varphi$  occurring in  $H^3(\Gamma, V_g(L))$  such that  $\varphi$  is congruent to  $\varphi_0$ .*

### 9. Examples of $p$ -adically rigid classes via computation

In this section, we assume that  $n = 3$ . According to Corollary 8.5, to check that a given quasicuspidal arithmetic eigenpacket is arithmetically rigid, it suffices to show that  $H^3_{\text{cusp}}(\Gamma, V_h(\mathbb{C})) = 0$  for suitable  $h$ .

For any cohomology group  $H^*(\cdot)$  with coefficients in a module which is a vector space over a field  $k$ , let  $h^*(\cdot)$  denote its dimension over  $k$ . Let  $s_w(N)$  denote the dimension of the space of classical holomorphic cuspforms of weight  $w$ , level  $N$  and trivial nebentype for  $\text{GL}(2)/\mathbb{Q}$ .

**LEMMA 9.1.** *Let  $q$  be a prime. Then if  $h$  is even,*

$$h^3(\Gamma_0(q), V_h(\mathbb{C})) = h^3_{\text{cusp}}(\Gamma_0(q), V_h(\mathbb{C})) + 2s_{h+2}(q) + 3$$

and if  $h$  is odd,

$$h^3(\Gamma_0(q), V_h(\mathbb{C})) = h_{\text{cusp}}^3(\Gamma_0(q), V_h(\mathbb{C})).$$

*Proof.* The proof follows from the fact that  $h_{\text{cusp}}^3(\Gamma, E) + h^3(\partial Y, E) = h^3(\Gamma, E)$  (see Section 4). To compute the cohomology of the Borel–Serre boundary, take  $\Gamma_0(q)$ -invariants in [15, Theorem 7.6(4), p. 116] (where  $m = q$ ). Compare Proposition 2.3 and Lemma 3.9 of [2].  $\square$

Rather than computing over  $\mathbb{C}$ , we compute over  $\mathbb{Z}/r$  for some prime  $r$ . In fact we took  $r = 149, 151$  or  $163$ . The following lemma follows immediately from the Universal coefficient theorem.

LEMMA 9.2. *Let  $\Gamma = \Gamma_0(q)$  for some odd prime  $q$ , and let  $r$  be a prime. Then  $h^3(\Gamma, V_h(\mathbb{Z}/r)) = h^3(\Gamma, V_h(\mathbb{C})) + e$  for some non-negative integer  $e$ .*

LEMMA 9.3. *Let  $\Gamma = \Gamma_0(q)$  for some odd prime  $q$ ,  $h \geq 0$  be an even integer, and let  $r$  be a prime. If  $h^3(\Gamma, V_h(\mathbb{Z}/r)) = 2s_{h+2}(q) + 3$  then  $h_{\text{cusp}}^3(\Gamma, V_h(\mathbb{C})) = 0$  and  $e = 0$ .*

*Proof.* The proof follows from the previous two lemmas.  $\square$

From [2] we derive the following two examples of  $p$ -adically rigid cohomology classes. (On the other hand, we have no glimmer of any non-rigid, non-essentially selfdual cohomology classes, consistent with the conjecture we made in the introduction.)

THEOREM 9.4. (1) *Let  $\varphi$  be either of the two eigenpackets in  $H_{\text{cusp}}^3(\Gamma_0(89), \mathbb{C})$ . Then  $\varphi$  is 3-adically arithmetically rigid.*

(2) *Let  $\psi$  be either of the eigenpackets in  $H_{\text{cusp}}^3(\Gamma_0(61), \mathbb{C})$ . Then  $\psi$  is 5-adically arithmetically rigid up to twist.*

*Proof.* (1) A simple calculation from the Hecke data given in [2] shows that for  $p = 3$ , we have  $\varphi$  is 2-quasicuspidal and ordinary. Therefore by Corollary 8.5, if  $\varphi$  were not 3-adically arithmetically rigid, then there would be a non-trivial cuspidal cohomology in  $H_{\text{cusp}}^3(\Gamma_0(89), V_2(\mathbb{C}))$ . However, the computations reported upon in the introduction to this paper say that  $H_{\text{cusp}}^3(\Gamma_0(89), V_2(\mathbb{C})) = 0$ .

(2) Similarly, for  $p = 5$ ,  $\psi$  is 2-quasicuspidal and ordinary. Since our computations say that  $H_{\text{cusp}}^3(\Gamma_0(61), V_4(\mathbb{C})) = 0$ , the same reasoning shows that  $\psi$  is 5-adically arithmetically rigid up to twist.  $\square$

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