

TERMINAL QUOTIENT SINGULARITIES IN DIMENSIONS THREE AND FOUR

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ABSTRACT. We classify isolated terminal cyclic quotient singularities in dimension three, and isolated Gorenstein terminal cyclic quotient singularities in dimension four. In addition, we give a new proof of a combinatorial lemma of G. K. White using Bernoulli functions.

Let X be a smooth algebraic variety over \mathbf{C} , and let ω_X be the canonical bundle of X . For each $n > 0$, if $\Gamma(X, \omega_X^{\otimes n}) \neq \emptyset$, there is a natural *pluricanonical map* $\phi_n: X \rightarrow \mathbf{P}\Gamma(X, \omega_X^{\otimes n})^*$. An algebraic variety is of *general type* if ϕ_n is a birational map for n sufficiently large. For a variety of general type, the pluricanonical images $\phi_n(X)$ are the most natural birational models of X to study.

Canonical singularities are the singularities which may occur in the pluricanonical models of varieties of general type. In dimension 1, the pluricanonical models are smooth so there are no canonical singularities; in dimension 2 the canonical singularities coincide with the classical *rational double points*. One characterization of the rational double points is as quotient singularities: if G is any finite subgroup of $\mathrm{Sl}(2, \mathbf{C})$, then the quotient \mathbf{C}^2/G has a rational double point, and every rational double point is analytically isomorphic to such a quotient singularity. Reid and Shepherd-Barron [10], and independently Tai [14], have given a condition for quotient singularities to be canonical in arbitrary dimensions (although not all canonical singularities are quotient singularities in dimensions greater than two).

Terminal singularities are a class of canonical singularities which play an important role in birational geometry (as evidenced by recent work of Mori [8], Reid [12], and Tsunoda [15]). In this note we study cyclic quotient singularities which are terminal. In dimension three we explicitly describe all isolated terminal cyclic quotient singularities, while in dimension four, we describe isolated terminal cyclic quotient singularities which are also Gorenstein. The description uses a combinatorial lemma due to G. K. White [17]; we have given a new proof of this lemma (Corollary 1.4 below) using Bernoulli functions.

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1. Bernoulli functions. The material in this section is all quite standard (except for the last two corollaries); we have adapted the presentation in the first chapter of Kubert and Lang [7] to suit our purposes.

If x is a rational number, we let $\langle x \rangle$ denote the rational number such that $x \equiv \langle x \rangle \pmod{\mathbf{Z}}$, and $0 \leq \langle x \rangle < 1$. Define

$$\mathbf{B}_1(x) = \begin{cases} \langle x \rangle - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

Let N be a natural number, let $C(N) = (1/N)\mathbf{Z}/\mathbf{Z}$, and let $G(N) = (\mathbf{Z}/N\mathbf{Z})$. $G(N)$ acts on $C(N)$ by multiplication, and there is an embedding $G(N) \rightarrow C(N)$ given by $a \rightarrow a/N$.

Let $\chi: G(N) \rightarrow \mathbf{C}^*$ be a character of conductor N and define

$$\mathbf{B}_{1,\chi} = \sum_{a \in G(N)} \chi(a) \mathbf{B}_1\left(\frac{a}{N}\right).$$

The following classical theorem is essentially due to Dirichlet; a nice proof can be found in [5, §2, Theorem 2].

THEOREM 1.1. *If χ is an odd character (that is, $\chi(-a) = -\chi(a)$ for all $a \in G(N)$), where N is the conductor of χ , then $\mathbf{B}_{1,\chi} \neq 0$.*

Now fix a natural number $N > 2$, and denote $C(N)$ and $G(N)$ by C and G , respectively. Let $V = \mathbf{C}(G)$ be the group algebra of G generated (as a \mathbf{C} -vector space) by elements σ_a for $a \in G$. We define a function $S: C \rightarrow V$ by

$$S(x) = \sum_{a \in G} \mathbf{B}_1(ax) \sigma_a.$$

Let W be the subspace of V generated by $\{S(x): x \in C\}$, and let $\Lambda = \text{Ann}(W) \subset V^*$. For each $a \in G$, let $\lambda_a = \sigma_a^* + \sigma_{-a}^* \in V^*$. Note that

$$\lambda_a(S(x)) = \mathbf{B}_1(ax) + \mathbf{B}_1(-ax) = 0$$

for all x since \mathbf{B}_1 is an odd function. Thus, $\lambda_a \in \Lambda$.

PROPOSITION 1.2. *Λ is generated (as a \mathbf{C} -vector space) by $\{\lambda_a: a \in G\}$. In particular, $\dim_{\mathbf{C}}(\Lambda) = \dim_{\mathbf{C}}(W) = \phi(N)/2$, where ϕ is Euler's ϕ -function.*

PROOF. Let χ be an (arbitrary) odd character on G , M the conductor of χ , and $H = G(M)$. There is a well-defined map $H \rightarrow C = (1/N)\mathbf{Z}/\mathbf{Z}$ given by $a \rightarrow a/M$. We define an element $w_\chi \in W$ by

$$w_\chi = \sum_{a \in H} \chi(a) S\left(\frac{a}{M}\right).$$

In terms of the given basis of V , we may write

$$w_\chi = \sum_{b \in G} \left(\sum_{a \in H} \chi(a) \mathbf{B}_1\left(\frac{ab}{M}\right) \right) \sigma_b.$$

Thus, the coefficient of σ_1 is

$$\sum_{a \in H} \chi(a) \mathbf{B}_1\left(\frac{a}{M}\right) = \mathbf{B}_{1,\chi} \neq 0,$$

so $w_\chi \neq 0$.

There is a natural action of G on $V = \mathbf{C}(G)$. Under this action, $c \in G$ sends w_χ to

$$\begin{aligned} \sum_{b \in G} \left(\sum_{a \in H} \chi(a) \mathbf{B}_1\left(\frac{ab}{M}\right) \right) \sigma_b &= \sum_{b' \in G} \sum_{a \in H} \chi(a) \mathbf{B}_1\left(\frac{ac^{-1}b'}{M}\right) \sigma_{b'} \\ &= \sum_{b' \in G} \sum_{a' \in H} \chi(a'c) \mathbf{B}_1\left(\frac{a'b'}{M}\right) \sigma_{b'} = \chi(c) w_\chi. \end{aligned}$$

Thus, w_χ lies in the χ -eigenspace of the G -action. Since each $w_\chi \neq 0$, we see that $\{w_\chi : \chi \text{ is an odd character}\}$ is a linearly independent subset of W , so $\dim_{\mathbf{C}}(W) \geq \phi(N)/2$. On the other hand, by considering the elements λ_a , we see that $\text{codim}_{\mathbf{C}}(W) = \dim_{\mathbf{C}}(\Lambda) \geq \phi(N)/2$. Since $\dim_{\mathbf{C}}(V) = \phi(N)$, the inequalities are equalities, and the proposition follows.

COROLLARY 1.3. *Let $a, b, c, d \in G$ and suppose, for all $x \in C$, $\mathbf{B}_1(ax) + \mathbf{B}_1(bx) + \mathbf{B}_1(cx) = \mathbf{B}_1(dx)$. Then, after reordering a, b, c , we have $a \equiv -b(N)$, and $c \equiv d(N)$.*

PROOF. Let $\lambda = \sigma_a^* + \sigma_b^* + \sigma_c^* - \sigma_d^* \in V^*$. By hypothesis, $\lambda \in \Lambda$. But Λ is generated by $\{\sigma_a^* + \sigma_{-a}^*\}$, so the corollary is clear.

COROLLARY 1.4 (WHITE [17, THEOREM 2], FRUMKIN [3] AND DANILOV [2]). *Let a, b, c be integers relatively prime to N . Suppose, for all $k \in \mathbf{Z} - N\mathbf{Z}$, $\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle > 1$. Then, after reordering a, b, c , we have $a + b \equiv 0(N)$.*

PROOF. First note that if $\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle \geq 2$, then

$$\left\langle -\frac{ak}{N} \right\rangle + \left\langle -\frac{bk}{N} \right\rangle + \left\langle -\frac{ck}{N} \right\rangle = 1 - \left\langle \frac{ak}{N} \right\rangle + 1 - \left\langle \frac{bk}{N} \right\rangle + 1 - \left\langle \frac{ck}{N} \right\rangle \leq 3 - 2 = 1,$$

which contradicts our hypothesis. Thus,

$$1 < \left\langle \frac{ak}{N} \right\rangle + \left\langle \frac{bk}{N} \right\rangle + \left\langle \frac{ck}{N} \right\rangle < 2.$$

Let $d = a + b + c$. Since

$$\left\langle \frac{ak}{N} \right\rangle + \left\langle \frac{bk}{N} \right\rangle + \left\langle \frac{ck}{N} \right\rangle \equiv \left\langle \frac{dk}{N} \right\rangle \pmod{\mathbf{Z}},$$

we must have

$$\left\langle \frac{ak}{N} \right\rangle + \left\langle \frac{bk}{N} \right\rangle + \left\langle \frac{ck}{N} \right\rangle = 1 + \left\langle \frac{dk}{N} \right\rangle > 1$$

for each $k \in \mathbf{Z} - N\mathbf{Z}$. In particular, d is relatively prime to N . In addition, by subtracting $\frac{3}{2}$ from both sides of the above equation, we get

$$\mathbf{B}_1(ax) + \mathbf{B}_1(bx) + \mathbf{B}_1(cx) = \mathbf{B}_1(dx)$$

for all $x = k/N \notin \mathbf{Z}$. On the other hand, this identity is trivially satisfied for $x = 0$. This corollary now follows from the preceding one.

2. Quotient singularities. Throughout this section Γ will be a finite subgroup of $\mathrm{Gl}(n, \mathbf{C})$, and $X_\Gamma = \mathbf{C}^n/\Gamma$. We will study the singularity of X_Γ at the origin $O \in \mathbf{C}^n$.

$g \in \Gamma$ is called a *quasi-reflection* if $\mathrm{rank}(g - I) = 1$. A classical theorem of Shephard and Todd [13] and Chevalley [1] says that X_Γ is smooth if and only if Γ is generated by quasi-reflections. If Γ is arbitrary, let Γ_q be the largest subgroup of Γ generated by quasi-reflections, and let $\tilde{\Gamma} = \Gamma/\Gamma_q$. Then it is clear that $X_\Gamma \cong X_{\tilde{\Gamma}}$; in studying quotient singularities, we may therefore restrict to groups Γ such that $\Gamma_q = \{I\}$. Such a group is called *small*.

The following lemma is quite standard (cf. [4 or 9] for a proof).

LEMMA 2.1. *Let $\Gamma \subset \mathrm{Gl}(n, \mathbf{C})$ be a small group, and let $S = \{x \in \mathbf{C}^n \mid g(x) = x \text{ for some } g \neq 1\}$. Then the singular locus of X_Γ is S/Γ .*

COROLLARY 2.2. *If $\Gamma \subset \mathrm{Gl}(n, \mathbf{C})$ is a small cyclic group of order N , then X_Γ has an isolated singularity if and only if all eigenvalues of a generator g of Γ are primitive N th roots of unity.*

The duality sheaf ω_{X_Γ} of a quotient singularity has been studied by Khinich [6], Watanabe [16], Reid and Shephard-Barron [10], Reid [11], and Tai [14]. We summarize their results as follows.

THEOREM 2.3. *Let $\Gamma \subset \mathrm{Gl}(n, \mathbf{C})$ be a small group. For each $g \in \Gamma$ of order $M \neq 1$ and each primitive M th root of unity ζ , write the eigenvalues of g as $\zeta^{a_1}, \dots, \zeta^{a_n}$, with $0 \leq a_i < M$, and define $e(g, \zeta) = a_1/M + \dots + a_n/M$.*

- (i) (Khinich and Watanabe) X_Γ is Gorenstein if and only if $\Gamma \subset \mathrm{Sl}(n, \mathbf{C})$.
- (ii) (Reid, Shephard-Barron, and Tai) X_Γ is canonical if and only if $e(g, \zeta) \geq 1$ for all primitive ζ and all $g \neq 1$.
- (iii) (Reid) X_Γ is terminal if and only if $e(g, \zeta) > 1$ for all primitive ζ and all $g \neq 1$.

We can now state our main result.

THEOREM 2.4. *Let $\Gamma \subset \mathrm{Gl}(n, \mathbf{C})$ be a small cyclic group of order N .*

- (i) *If $n = 3$, then X_Γ is an isolated terminal singularity if and only if, for each $g \in \Gamma$, $\det g$ is one of the eigenvalues of g .*
- (ii) *If $n = 4$, then X_Γ is an isolated Gorenstein terminal singularity if and only if $\Gamma \subset \mathrm{Sl}(2, \mathbf{C}) \times \mathrm{Sl}(2, \mathbf{C}) \subset \mathrm{Gl}(4, \mathbf{C})$ in such a way that the projections $\mathrm{pr}_i: \Gamma \rightarrow \mathrm{Sl}(2, \mathbf{C})$ are faithful representations of Γ .*

REMARK 2.5. Choose a primitive N th root of unity ζ . In suitable coordinates on \mathbf{C}^n , the conditions above are equivalent to

- (i) There is some a with $(a, N) = 1$ such that Γ is generated by

$$\begin{bmatrix} \zeta & & \\ & \zeta^a & \\ & & \zeta^{-a} \end{bmatrix}.$$

(ii) There is some a with $(a, N) = 1$ such that Γ is generated by

$$\begin{bmatrix} \zeta & & & \\ & \zeta^{-1} & & \\ & & \zeta^a & \\ & & & \zeta^{-a} \end{bmatrix}.$$

This provides the explicit description promised in the introduction.

PROOF. (i) Let g be a generator of Γ , ζ a primitive N th root of unity, and $\zeta^a, \zeta^b, \zeta^c$ the eigenvalues of g , with $0 \leq a, b, c < N$. By Corollary 1.2 X_Γ is isolated if and only if $(a, N) = (b, N) = (c, N) = 1$. On the other hand, by 2.3(iii), X_Γ is terminal if and only if $\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle > 1$ for each $k \in \mathbf{Z} - N\mathbf{Z}$. By Corollary 1.4 we have $a + b \equiv 0 (N)$ after rearranging, so the determinant is ζ^c , which is one of the eigenvalues.

(ii) Let g be a generator of Γ , ζ a primitive N th root of unity, and $\zeta^a, \zeta^b, \zeta^c, \zeta^d$ the eigenvalues of g with $0 \leq a, b, c, d < N$. By Corollary 1.2 and Theorem 2.3(i) and (iii), X_Γ is isolated terminal Gorenstein if and only if $a + b + c + d \equiv 0 (N)$, $(a, N) = (b, N) = (c, N) = (d, N) = 1$, and

$$\left\langle \frac{ak}{N} \right\rangle + \left\langle \frac{bk}{N} \right\rangle + \left\langle \frac{ck}{N} \right\rangle + \left\langle \frac{dk}{N} \right\rangle > 1$$

for all $k \in \mathbf{Z} - N\mathbf{Z}$. Since $a + b + c + d \equiv 0 (N)$, we see that

$$\left\langle \frac{ak}{N} \right\rangle + \left\langle \frac{bk}{N} \right\rangle + \left\langle \frac{ck}{N} \right\rangle + \left\langle \frac{dk}{N} \right\rangle$$

is an integer, so that this is true if and only if

$$\left\langle \frac{ak}{N} \right\rangle + \left\langle \frac{bk}{N} \right\rangle + \left\langle \frac{ck}{N} \right\rangle + \left\langle \frac{dk}{N} \right\rangle \geq 2$$

or

$$\left\langle \frac{ak}{N} \right\rangle + \left\langle \frac{bk}{N} \right\rangle + \left\langle \frac{ck}{N} \right\rangle \geq 2 - \left\langle \frac{dk}{N} \right\rangle = 1 + \left\langle -\frac{dk}{N} \right\rangle.$$

This is greater than 1, so we may again apply Corollary 1.4, to get $a + b \equiv 0 (N)$ after rearranging; hence, $c + d \equiv 0 (N)$ as well, and we see that $\Gamma \subset \text{Sl}(2, \mathbf{C}) \times \text{Sl}(2, \mathbf{C})$ with each induced representation in $\text{Sl}(2, \mathbf{C})$ being faithful.

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