DRAFT:
P-ADIC DEFORMATIONS OF
COHOMOLOGY CLASSES
OF SUBGROUPS OF $GL(N, \mathbb{Z})$: THE NON-ORDINARY CASE

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SECTION 1: INTRODUCTION

Our paper [AS-Barcelona] presented a control theorem on the ordinary part of a p-adic deformation of the cohomology of congruence subgroups of $GL(n, \mathbb{Z})$. In the current work, we present a different approach to this problem which enables us to work with a much larger part of the non-torsion part of the cohomology, including some of the non-ordinary part. What we lose is (1) the torsion in the cohomology; and (2) the parameter space for our deformation is now some undetermined polydisc of characters, rather than the whole space of characters. Our methods work just as well for any split Chevalley group, and probably for any arithmetic subgroup of any reductive $\mathbb{Q}$-group which is quasi-split at p. We have chosen in this paper to restrict ourselves to $GL(n)/\mathbb{Z}$ in order to make the exposition as clear as possible.

The basic setup is similar to [AS-Barcelona] except that we use a different base ring R of rigid analytic functions on the disc of characters rather than a completed group ring like Λ. We also find it easier to begin directly with the congruence subgroup that has p in the level. We follow Coleman's approach [C] based on Serre's p-adic theory [Se] of Banach modules over R with a completely continuous operator u.

Coleman's paper is written for $GL(2)$ and it was developed for modular symbols by the second author [Stevens]. Our principal innovations here compared with the $GL(2)$ case include the following: (1) Working with cohomology classes becomes trickier because now the cohomology is a subquotient of the cochains, whereas in the $GL(2)$ case the cohomology could be viewed as a subset of modular symbols. We have to play off the strong and weak topologies against each other to show that the coboundaries are closed in the space of cochains. (2) Because of this, we

have to lift the Hecke operator U_p to the cochain level. The cochains, but not the cohomology, will have an orthonormal basis in general. (3) We have to factor Fredholm determinants over R, which is no longer a one-dimensional ring.

We consider a congruence subgroup Γ of $GL(n, \mathbb{Z})$ of level divisible by p and a coefficient module $V(\mathbb{Q}_p)$ which is a lattice in $V(\mathbb{Q}_p)$, where V is an irreducible rational representation of $GL(n)$. We consider $H^*(\Gamma, V(\mathbb{Q}_p))$ as a module for a ring $\mathcal{H}$ of Hecke operators. To construct a p-adic deformation of this cohomology group, we form a large $\mathbb{Z}_p$-module $\mathbb{D}$ of measures on a certain coset space with maps from $\mathbb{D}$ to $V(\mathbb{Q}_p)$, for varying V's. These induce $\mathcal{H}$-maps on the cohomology and we determine the kernels.

We actually go in two steps: Depending on the highest weight κ of V we form a quotient space $\mathbb{D}_\kappa$ of $\mathbb{D}$ and determine the kernel of the induced $\mathcal{H}$-map on the cohomology. The module $\mathbb{D}_\kappa$ has $V(\mathbb{Q}_p)$ as a further quotient, and we prove that the induced map on cohomology is an isomorphism on the slope-less- than-or-equal-to $f(\kappa)$ part of the cohomology. The word "slope" here refers to the p-adic ordinal of the eigenvalue of a suitable U-operator. As in the $GL(2)$-case, U is a Hecke operator at p. $f(\kappa)$ is some simple piece-wise linear function of κ.

More precisely, we let Γ denote the intersection of a congruence subgroup of level prime to p with a certain subgroup of $GL(n, \mathbb{Z})$ of level $p^{(n-1)\nu}$. Then we construct

$$\phi_V^h \colon H^*(\Gamma, \mathbb{D})^h \to H^*(\Gamma, V(\mathbb{Q}_p))^h$$

where the superscript h denotes the slope-less- than-or-equal-to h part of the cohomology.

To specify the kernel, let K denote a polydisc in $T(\mathbb{Z}_p)$, where T is the torus of diagonal matrices in $GL(n)$. Set R to be the Banach algebra of rigid functions on K. Then $\mathbb{D}$ has an R-module structure commuting with the Γ-action which induces an R-action on $H^*(\Gamma, \mathbb{D})$ commuting with the $\mathcal{H}$-action. Let κ be the highest weight of V (with respect to an appropriately chosen Borel subgroup containing T) and let I_κ denote the kernel of the Banach-algebra homomorphism from R to $\mathbb{C}_p$ given by evaluation at κ. Then our main theorem states that if $h \leq f(\kappa)$, then the kernel of ϕ_V^h equals $I_\lambda H^*(\Gamma, \mathbb{D})^h$. In the actual theorem below we include a nebentype character.

We also can assert the surjectivity of ϕ_V^h under certain hypotheses.

As in [AS-Barcelona] the theorem should be thought of as giving p-adic deformations as follows: If α is a Hecke-eigenclass in $H^*(\Gamma, \mathbb{D})^h$, then its images in $H^*(\Gamma, V(\mathbb{Q}_p))^h$ as V varies will form a family of cohomology classes whose Hecke eigenvalues will be congruent modulo powers of p that depend on what congruences obtain among the highest weights κ modulo powers of p. The theorem maintains some control on when the images are nonzero. However, we cannot expect a freeness result for $H^*(\Gamma, \mathbb{D})^h$.

Since non-torsion Hecke eigenclasses in $H^*(\Gamma, V(\mathbb{Q}_p))^h$ give rise to irreducible automorphic representations on $GL(n)$, we are also getting p-adic deformations of certain automorphic forms.

Because of the way our construction works, we can only vary V in a small p-adic neighborhood. We do not know how to control the size of this neighborhood. As mentioned above, in the ordinary case [AS-Barcelona] we could vary V over all representations with highest weight congruent to $p \bmod p - 1$.

SECTION 2: THE BIG CELL

We introduce some notation. All the algebraic groups below should be thought of as group schemes over $\mathbb{Z}$:

Set $G = GL(n)$, $N =$ lower triangular unipotent matrices in G, $T =$ diagonal matrices in G, and $B = TN = NT$.

We denote by a raised "o" the transposed group. For example $B^o = TN^o$ is the opposite Borel subgroup to B.

Let $\mathbb{C}_p$ denote as usual the completion of an algebraic closure of $\mathbb{Q}_p$, and let $\mathbb{O}$ be its ring of integers. We denote by v the valuation on $\mathbb{C}_p$ such that $v(p) = 1$.

Let $I =$ Iwahori subgroup of $G(\mathbb{C}_p) = \{g \in G(\mathbb{O}) \mid g \mod p \in B^o(\mathbb{O}/p)\}$. We also define $I^* = \{g \in G(\mathbb{O}) \mid v(g_{ij}) > 0, i > j\}$ and $I_\nu = \{g \in G(\mathbb{O}) \mid v(g_{ij}) \geq \nu(i - j), i > j\}$.

Let $Y = N(\mathbb{C}_p)\backslash G(\mathbb{C}_p)$. It is a rigid analytic space. Let X' denote the image in Y of $TN^o(\mathbb{O})$, the "integral points of the big cell". It is a finite union of affinoid polydiscs.

Lemma 2.1. *(a) X' is the image of I^* in Y. (b) I^* acts on X' on the right rigidly. (c) $T(\mathbb{O})$ acts on X' on the left rigidly. (d) These two actions commute.*

Proof: The actions involve only rational functions without poles so (b) and (c) are clear. The other statements are easily checked.

We fix a congruence subgroup Γ' of $G(\mathbb{Z})$ of level prime to p.

We will now define some sets B_r on which $I_\nu \cap G(\mathbb{Z}_p)$ and hence $\Gamma_\nu := I_\nu \cap \Gamma'$ will act. Each B_r is of the form $T(\mathbb{Z}_p)D_r$ where D_r is an affinoid polydisc in $N^o(\mathbb{C}_p)$. Γ_ν will also preserve the $\mathbb{Z}_p$-points of B_r.

As in lemma 2.1, the action will be rigid and will commute with the left $T(\mathbb{Z}_p)$-action. We will only need one of these spaces in this paper, but their $GL(2)$ analogues were all needed in [Stevens] and may be useful later for defining p-adic L-functions.

Let r denote a tuple of rational numbers (α_{ij}) where $1 \leq i < j \leq n$. Set $D_r = \{[x_{ij} \in X' | x_{ii} = 1, v(x_{ij}) \geq \alpha_{ij} \forall i < j\}$. Set B_r to be the image in Y of $T(\mathbb{Z}_p)D_r$.

Note: we use $[a]$ to denote the image in Y of a matrix a in $G(\mathbb{C}_p)$.

We assume all $\alpha_{ij} < 0$ so that D_r contains $N^o(\mathbb{O})$. In fact, it is easy to verify the following lemma:

Lemma 2.2. *Given the nonnegative integer ν, assume that for all $i < j$, $0 > \alpha_{ij} > -1$ if $\nu = 0$ and $0 > \alpha_{ij} > (i - j)\nu$ if $\nu > 0$. Then I_ν leaves B_r stable, acts rigidly on it, commuting with the rigid action of $T(\mathbb{Z}_p)$ on the left.*

SECTION 3: SEMIGROUP ACTIONS ON VARIOUS BANACH SPACES

¿From now on for simplicity we fix a compatible pair ν and r as in Lemma 2.2 and drop them from the notation. To avoid confusion with the Borel subgroup, we will use X instead of B.

As in [AS-Barcelona], consider a semigroup $\tilde{S}$ in $G(\mathbb{Q}) \cap G(\mathbb{Z}_p)$ that contains Γ. We assume that S contains the diagonal matrices $d(\ell, a) = \mathrm{diag}(\ell, \ldots, \ell, 1, \ldots, 1)$ with a ℓ's for $0 \leq a \leq n$ and ℓ not dividing the level of Γ. (Remember that p divides the level of Γ.) We define S to be the semigroup generated by $\tilde{S}$ and the matrix $\pi = \mathrm{diag}(1, p, p^2, \ldots, p^{n-1})$.

We assume that $SS^{-1} \cap G(\mathbb{Z}) = \Gamma$, which is a harmless condition satisfied by all Hecke pairs in practical use.

We let $\mathcal{H} = \mathcal{H}(\Gamma, S)$ be the Hecke algebra of double cosets. We assume $\mathcal{H}$ is commutative. In [AS-Barcelona] we defined the $*$-action of S on Y. It is the restriction of the unique semigroup action of the semigroup generated by $G(\mathbb{O})$ and π on Y such that for every $b \in B^o(\mathbb{C}_p)$ and every $g \in G(\mathbb{O})$, $[b] * g = [bg]$ and $[b] * \pi = [\pi^{-1} b \pi]$. This will induce semigroup actions on spaces of functions and distibutions on X, and hence Hecke algebra actions on the cohomology of Γ with coefficients in such spaces.

Let $\mathcal{X}$ denote the space of all continuous group homomorphisms from $T(\mathbb{Z}_p)$ to $\mathbb{C}_p^\times$. For a finite extension L of $\mathbb{Q}_p$, let $\mathcal{X}(L)$ denote the L- rational points, i.e. those homomorphisms taking values in L.

Fix an L- rational point $\kappa_0 \in \mathcal{X}$ and let K denote an m-dimensional closed affinoid polydisc centered at κ_0 for some m from 0 to n. So if $m = 0$, K is just the point $\{\kappa_0\}$. In particular, we shall assume that we have rational coordinate functions $(b_1, \ldots, b_n)$ and $\mathbb{Q}$-rational numbers $(r_1, \ldots, r_n)$ such that κ_0 has the coordinates $(0, \ldots, 0)$ and $K = \{(b_1, \ldots, b_n) : |b_i| \le |r_i| \forall i\}$. Write $K(L)$ for $K \cap \mathcal{X}(L)$.

Let $A(K)$ denote the rigid analytic functions on K with values in $\mathbb{C}_p$. It is a Banach algebra, isomorphic to a Tate algebra. In the case where K is not a point, choose L-rational coordinates $(\beta_1, \ldots, \beta_n)$ on K so that κ_0 has coordinates $(0, \ldots, 0)$. Thus $A(K)$ is the set of functions λ on K such that λ is given by a single power series in the β's which converges on K. The norm on $A(K)$ is the sup norm of functions. Let I_0 be the ideal generated by $\beta_1, \ldots, \beta_n$, so $I_0 = \{\lambda \in A(K) | \lambda(\kappa_0) = 0\}$. If L is a finite extension of $\mathbb{Q}_p$ we write $A(K)(L)$ for the L-points of $A(K)$, i.e. those functions which take values in L on $K(L)$, or equivalently have coefficients in L when written as power series in the β's.

Recall that X is $T(\mathbb{Z}_p)$ times a closed affinoid polydisc D. Define $\mathbb{A}_K$ to be the set of functions $F : K \times X \to \mathbb{C}_p$ such that

(1) $F(\kappa, tx) = \kappa(t)F(\kappa, x)$ for all $\kappa \in K$, $t \in T(\mathbb{Z}_p)$, and $x \in X$, and

(2) writing $x = td$ with $t \in T(\mathbb{Z}_p)$ and $d \in D$, F should be locally analytic in the t variable and rigid analytic in the (k, n) variables. Since K and D are closed affinoid polydiscs, (2) means that F is given by a single convergent power series on $K \times D$, with coefficients being locally analytic functions on $T(\mathbb{Z}_p)$.

We put the sup norm on $\mathbb{A}_K$ and this makes it into a Banach module over $A(K)$, where the module action is given by $(\lambda F)(\kappa, x) = \lambda(\kappa)F(\kappa, x)$. It is also a left S-module with the action $(sF)(\kappa x) = F(\kappa, x * s)$. These actions commute. Let $\mathbb{A}_K(L)$ denote the L-rational points.

Define $\mathbb{D}_K$ to be the space of continuous $A(K)$-module homomorphisms $\mu : \mathbb{A}_K \to A(K)$. This is also a Banach module over $A(K)$ with the operator norm on μ, i.e. $|\mu| = sup\{|\mu(F)|/|F| : F \ne 0\}$. The module action is the usual one: $(\lambda\mu)(F) = \lambda(\mu(F))$. It also inherits a right S-action from $\mathbb{A}_K$ in the usual way, i.e. $(\mu s)(F) = (\mu(sF))$. Again, these two actions commute. Let $\mathbb{D}_K(L)$ denote the L-rational points.

We let $\mathbb{E}_K$ be the unit ball in $\mathbb{D}_K$. Let $\mathbb{E}_K(\mathbb{O}_L)$ denote the $\mathbb{O}_L$-rational points.

We want to prove that the sequences

$$0 \to I_0 \mathbb{D}_K \to \mathbb{D}_K \to \mathbb{D}(0) \to 0$$

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and the similar sequences with $\mathbb{E}_K$, $\mathbb{D}_K(L)$, and $\mathbb{E}_K(\mathbb{O}_L)$ are exact. If $n = 1$ this is fairly easy, but in general we have to introduce some intermediate spaces of measures in order to reduce to the $n = 1$ case. So we have to make some more definitions.

Without loss of generality, we assume K is n-dimensional until the end of the proof of Corollary 3.5. For $m = 0, \ldots, n$ let $K_m = \{(b_1, \ldots, b_m, 0, \ldots, 0)\} \subset K$. So $K_0 = \{\kappa_0\}$ and $K_n = K$. Similarly we set $T_m = \{diag(t_1, \ldots, t_m, 1, \ldots, 1)\} \subset T(\mathbb{Z}_p)$. If $m < l$ we have the obvious inclusions $K_m \subset K_l$, $T_m \subset T_l$, and projections $K_l \to K_m$ and $T_l \to T_m$ written as $\kappa \mapsto \hat{\kappa}$ and $t \mapsto \hat{t}$ where $(b_1, \ldots, b_l, 0, \ldots, 0)\hat{} = (b_1, \ldots, b_m, 0, \ldots, 0)$ and $diag(t_1, \ldots, t_l, 1, \ldots, 1)\hat{} = diag(t_1, \ldots, t_m, 1, \ldots, 1)$.

Define $\mathbb{A}_m$ to be $\mathbb{A}_{K_m}$ and $\mathbb{D}_m$ to be $\mathbb{D}_{K_m}$. For later use we point out that there are constants c_J such that $\{c_J \tilde{n}^J\}$ form an ONB for $\mathbb{A}_m$ over $A(K)$. Here $\tilde{n}^J$ denotes the unique extension of n^J as a function on D to $\mathbb{A}_m$ and n^J is the monomial taking the image $[n_{ij}]$ of the unipotent upper triangular matrix (n_{ij}) in D to $\prod n_{ij}^{J_{ij}}$. This uses the fact that the Gauss norm equals the spectral norm on closed affinoid polydiscs.

Because $|\kappa(t)|$ is absolutely bounded above and below for $t \in T(\mathbb{Z}_p)$ and $\kappa \in K$, we have a norm on A_m equivalent to the one above given by $|F| = sup\{|F(\kappa, n)| : \kappa \in K, n \in D\}$. In fact this is the norm we shall use.

Let $m \leq l$. Define $\Phi : \mathbb{A}_m \to \mathbb{A}_l$ and $\Psi : \mathbb{A}_l \to \mathbb{A}_m$ by the formulas $(\Phi F)(\kappa, tn) = (\kappa(t)/\hat{\kappa}(\hat{t}))F(\hat{\kappa}, \hat{t}n)$, for all $\kappa \in K_l$, $t \in T(\mathbb{Z}_p)$ and $n \in D$, and $(\Psi F)(\kappa, tn) = F(\kappa, tn)$, for all $\kappa \in K_m$, $t \in T(\mathbb{Z}_p)$ and $n \in D$.

Lemma 3.2. Φ and Ψ are maps of $\mathbb{C}_p$-vector spaces, defined over L, and they both are bounded operators with norm 1. The composition $\Psi \circ \Phi$ is the identity on $\mathbb{A}_m$. Ψ is an $A(K)$-module map but Φ is not.

Proof: One checks easily they map into the claimed targets, using the fact that if $F(\kappa, tn) = \kappa(t)F(\kappa, n)$ for all $\kappa \in K_m$, $t \in T(\mathbb{Z}_p)$ and $n \in D$ then F is in $\mathbb{A}_m$. The other assertions are also easy to check.

We have continuous homomorphisms (in fact with operator norm $= 1$):

$$i_{m,l} = i : A(K_m) \to A(K_l)$$

and

$$p_{m,l} = p : A(K_l) \to A(K_m)$$

which are the pull-backs of projection and inclusion respectively. That is, $(i\lambda)(\kappa) = \lambda(\hat{\kappa})$ for $\kappa \in K_l$ and $(p\lambda)(\kappa) = \lambda(\kappa)$ for $\kappa \in K_m$.

In particular, we have $p_{m,n} : A(K) \to A(K_m)$ via which we will automatically view any $A(K_m)$-module also as an $A(K)$-module. For instance $\mathbb{D}_m$ becomes an $A(K)$-module for any m.

Define $\pi_{m,l} = \pi : \mathbb{D}_l \to \mathbb{D}_m$ by

$$(\pi\mu)(F) = p_{m,l}(\mu(\Phi F)).$$

It's easy to see that π has operator norm less than or equal to 1, (so it is continuous), that its image lies in $\mathbb{D}_m$ as claimed, and that it is an $A(K)$-module map.

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Theorem 3.3. *The sequence (where the first nontrivial map is the inclusion ι and the second is π)*

$$0 \to (\beta_{m+1}, \ldots, \beta_l)\mathbb{D}_l \to \mathbb{D}_l \to \mathbb{D}_m \to 0$$

is an exact sequence of continuous $A(K)$-module maps defined over L with operator norm less than or equal to 1.

Proof of Theorem 3.3: We must check that (1) π is surjective and that (2) the kernel of π is the image of ι.

(1) Fix $m < l$. We are assuming that K is an n-dimensional polydisc. So there is a set K' which is an $(l - m)$-dimensional polydisc such that for any point with coordinates $(b_1, \ldots, b_m, 0, \ldots, 0) \in K$ and any $(b_{m+1}, \ldots, b_l) \in K'$ then $(b_1, \ldots, b_m, b_{m+1}, \ldots, b_l)$ is in K.

For any $(b_{m+1}, \ldots, b_l) \in K'$, define $\Psi_{(b_{m+1}, \ldots, b_l)} : \mathbb{A}_l \to A_m$ by

$$(\Psi_{(b_{m+1}, \ldots, b_l)}(F))((b_1, \ldots, b_m, 0, \ldots, 0), x) = F((b_1, \ldots, b_m, b_{m+1}, \ldots, b_l), x).$$

This is $\mathbb{C}_p$-linear and has operator norm 1.

We then define a map $\nu \to \tilde{\nu}$ from $\mathbb{D}_m$ to $\mathbb{D}_l$ by

$$\tilde{\nu}(F)(b_1, \ldots, b_l, 0, \ldots, 0) = \nu(\Psi_{(b_{m+1}, \ldots, b_l)}(F))(b_1, \ldots, b_m, 0, \ldots, 0).$$

Of course $\tilde{\nu}$ depends on the choice of $(b_{m+1}, \ldots, b_l) \in K'$, which we suppress from the notation. One easily checks that this map has operator norm less than or equal to 1 and is an $A(K_l)$-module map.

Now choose $(b_{m+1}, \ldots, b_l) = (0, \ldots, 0)$. We claim that $\pi(\tilde{\nu}) = \nu$. For any $G \in \mathbb{A}_m$ and any $(b_1, \ldots, b_m, 0, \ldots, 0)$, we have $(\pi(\tilde{\nu}))(G)(b_1, \ldots, b_m, 0, \ldots, 0) = p_{m,l}(\tilde{\nu})(\Phi G))(b_1, \ldots, b_m, 0, \ldots, 0) = \nu(\Psi_{(0,\ldots,0)}\Phi G)(b_1, \ldots, b_m, 0, \ldots, 0)$. So it is enough to show that $\Psi_{(0,\ldots,0)}\Phi G = G$. But we have by definition $\Psi_{(0,\ldots,0)}\Phi G((b_1, \ldots, b_m, 0, \ldots, 0), tn) = (\Phi G)((b_1, \ldots, b_m, 0, \ldots, 0), tn) = (\kappa(t)/\hat{\kappa}(\hat{t}))G(\hat{\kappa}, \hat{t}n)$ where $\kappa = \hat{\kappa} = (b_1, \ldots, b_m, 0, \ldots, 0)$. Therefore the last expression above equals $(\kappa(t/\hat{t}))G(\kappa, \hat{t}n)$.

On the other hand, since $G \in \mathbb{A}_m$,

$$G(\kappa, \hat{t}n) = (\kappa(\hat{t}))G(\kappa, n)$$

so that $(\kappa(t/\hat{t}))G(\kappa, \hat{t}n) = \kappa(t)G(\kappa, n) = G(\kappa, tn) = G((b_1, \ldots, b_m, 0, \ldots, 0), tn)$.

(2) First we check that $(\beta_{m+1}, \ldots, \beta_l)\mathbb{D}_l$ is contained in the kernel of π. If $\mu \in \mathbb{D}_l$ and $m + 1 \leq j \leq l$, then $\pi(\beta_j\mu) = b_j\pi(\mu)$ since π is an $A(K)$-module map. But β_j acts on $\mathbb{D}_m$ via the restriction map from $A(K)$ to $A(K_m)$. In particular β_j acts via 0, so $b_j\pi(\mu) = 0$.

To prove the converse, we put m, l back in the notation, writing $\pi = \pi_{m,l}$. It's easy to see that if $m \leq s \leq l$ then $\pi_{m,s} \circ \pi_{s,l} = \pi_{m,l}$.

We will now prove by induction on $l - m \geq 1$ that if $\mu \in \mathbb{D}_l$ and $\pi_{m,l}(\mu) = 0$ then $\mu \in (\beta_{m+1}, \ldots, \beta_l)\mathbb{D}_l$. First we do the induction step. Assume that $l - m \geq 2$ and the statement is true for $\pi_{m+1,l}$ and $\pi_{m,m+1}$. Then

$$\pi_{m,l}(\mu) = 0 \Rightarrow \pi_{m,m+1}(\pi_{m+1,l}(\mu)) = 0 \Rightarrow \pi_{m+1,l}(\mu) = \beta_{m+1}\nu$$

for some $\nu \in \mathbb{D}_{m+1}$. Lift ν to $\tilde{\nu} \in \mathbb{D}_l$ so that $\pi_{m+1,l}(\tilde{\nu}) = \nu$. Then $\pi_{m+1,l}(\mu - \beta_{m+1}\tilde{\nu}) = 0$ since $\pi_{m+1,l}$ is $A(K)$-linear. By the induction hypothesis again, $\mu - \beta_{m+1}\tilde{\nu} \in (\beta_{m+2}, \ldots, \beta_l)\mathbb{D}_l$. It follows that $\mu \in (\beta_{m+1}, \ldots, \beta_l)\mathbb{D}_l$ as desired.

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It remains to do the case where $l = m+1$ so $\pi : \mathbb{D}_{m+1} \to \mathbb{D}_m$. Suppose $\mu \in \mathbb{D}_{m+1}$ and $\pi(\mu) = 0$. We want to show that $\mu \in \beta_{m+1}\mathbb{D}_{m+1}$.

Let F be any element of $\mathbb{A}_m$. Then $(\pi\mu)(F) = 0$ means that $p_{m,m+1}\mu\Phi F = 0$. In other words, for any $(b_1, \ldots, b_m, 0, \ldots, 0) \in K_m$, $\mu(\Phi F)(b_1, \ldots, b_m, 0, \ldots, 0) = 0$. That is $\mu(\Phi F)$ is an element of $A(K_{m+1})$ which vanishes on the locus of $b_{m+1} = 0$, and therefore $\mu(\Phi F) \in \beta_{m+1}\mathbb{A}(K_{m+1})$ (e.g. by the Weierstrass preparation theorem.)

Now for any $G \in \mathbb{A}_{m+1}$, write out the power series representation for G:

$$G(\kappa, n) = \sum_{i=0}^{\infty} \gamma_i(\hat{\kappa}, n) b_{m+1}^i$$

where $\kappa = (b_1, \ldots, b_{m+1}, 0, \ldots, 0)$ and $\hat{\kappa} = (b_1, \ldots, b_m, 0, \ldots, 0)$. For each i there is a unique extension ϕ_i of γ_i to an element of $\mathbb{A}_m$, defined by

$$\phi_i(\hat{\kappa}, tn) = \hat{\kappa}(t)\gamma_i(\hat{\kappa}, n).$$

We see that $(\Phi\phi_i)(k, n) = \phi_i(\hat{\kappa}, n) = \gamma_i(\hat{\kappa}, n)$. Thus G and $\sum \beta_{m+1}^i \Phi(\phi_i)$ agree on (κ, n) and are both in $\mathbb{A}_{m+1}$ and hence are equal. Therefore, using that μ is a continuous, $A(K_{m+1})$-linear map,

$$\mu(G) = \sum \beta_{m+1}^i \mu(\Phi(\phi_i)) = \sum_{i \geq 1} \beta_{m+1}^i \mu(\Phi(\phi_i)) + \mu(\Phi(\phi_0)).$$

We had earlier shown that $\mu(\Phi F) \in \beta_{m+1}\mathbb{A}(K_{m+1})$ for any F and in particular $\mu(\Phi(\phi_0)) \in \beta_{m+1}\mathbb{A}(K_{m+1})$. We conclude that for any G, $\mu(G) \in \beta_{m+1}\mathbb{A}(K_{m+1})$.

So define $\nu : \mathbb{A}_{m+1} \to A(K_{m+1})$ by $\mu(G) = \beta_{m+1}\nu(G)$. We will be finished if we show that $\nu \in \mathbb{D}_{m+1}$, i.e. that ν is bounded and is $A(K_{m+1})$-linear. The second point is easy, since μ and multiplication by $1/\beta_{m+1}$ are $A(K_{m+1})$-linear. For the first point, since μ is bounded, it will suffice to show that there exists an $M > 0$ such that $|\nu(G)| \leq M|\mu(G)|$ for all $G \in \mathbb{A}_{m+1}$.

If we write $\nu(G) = \lambda$, then $\mu(G) = \beta_{m+1}\lambda$. Thus it is enough to show that there exists M such that $|\lambda| \leq M|\beta_{m+1}\lambda|$ for all $\lambda \in A(K_{m+1})$. In fact, let's show that there exists an M such that $|\lambda| = M|\beta_{m+1}\lambda|$ for all $\lambda \in A(K_{m+1})$.

However, this is a consequence of the fact that the Gauss norm and the spectral norm are equal on a polydisc. We have assumed that K and hence K' has the form $\{b : |b_i| \leq |r_i|\}$ for some rational numbers r_i. Using multi-index notation, write $\nu(G) = \sum a_I(b/r)^I$ and $\mu(G) = r_{m+1}(b_{m+1}/r_{m+1}) \sum a_I(b/r)^I$. Thus $|\mu(G)/r_{m+1}| = |\nu(G)|$ since the maximum of the coefficients a_I are the same in the two sums. Since r_{m+1} is a norm-multiplicative element in the Banach algebra, we're finished.

Let a superscript 0 denote the unit ball in a normed space. Since all the auxilliary operators used in the proof had operator norm less than or equal to one, the same method of proof applies to show:

Theorem 3.4. *The sequence*

$$0 \to (\beta_{m+1}, \ldots, \beta_l)\mathbb{E}_l \to \mathbb{E}_l \to \mathbb{E}_m \to 0$$

is an exact sequence of continuous $A(K)^0$-module maps defined over $\mathbb{O}_L$.

Corollary 3.5. *The sequence $(\beta_n, \ldots, \beta_1)$ is a $\mathbb{D}_K$-regular (resp. $\mathbb{E}_K$-regular) sequence in $A(K)$ (resp. $A(K)^0$).*

Proof: We give the proof for $\mathbb{D}_K$. The proof for $\mathbb{E}_K$ is similar. Let $\mu \in \mathbb{D}_K$ represent an element $\bar{\mu} \in \mathbb{D}_K/(\beta_n, \ldots, \beta_{j+1})\mathbb{D}_K$ such that $\beta_j \bar{\mu} = 0$. We must show $\bar{\mu} = 0$. We have $\beta_j \mu = \sum_{i=j+1}^{i=n} \beta_i \nu_i$ for some $\nu_i \in \mathbb{D}_K$. Hence for any $F \in \mathbb{A}_K$ and any $(b_1, \ldots, b_n) \in K$

$$(\beta_j(\mu(F)))(b_1, \ldots, b_n) = b_j \mu(F)(b_1, \ldots, b_n) = \sum b_i \nu_i(F)(b_1, \ldots, b_n)$$

that is, $\beta_j(\mu(F)) = \sum b_i(\nu_i(F))$. Writing $\mu(F)$ as $\sum b_i \lambda_i + \alpha$ where α is a power series in $(b_1, \ldots, b_j)$, we get

$$0 = \beta_j(\mu(F)) - \beta_j(\mu(F)) = \sum \beta_i((\nu_i(F)) - \beta_j \lambda_i) - \beta_j \alpha$$

and comparing coefficients of monomials gives that $\beta_j \alpha$ and hence α equal 0. Thus for every F, $\mu(F)$ lies in the ideal generated by $(\beta_n, \ldots, \beta_{j+1})$.

This means that for every F, $\mu(F)$ restricted to K_j is identically 0. In particular, for any $G \in \mathbb{A}_j$, $(\pi\mu)(G) = p_{j,n}(\mu(\Phi G)) = 0$ since $p_{j,n}$ is just restriction to K_j. So $\pi\mu = 0$. We conclude from Theorem 3.3 that μ lies in $(\beta_n, \ldots, \beta_{j+1})\mathbb{D}_K$.

If K consists of the single point κ_0, write $\mathbb{A}(0)$ and $\mathbb{D}(0)$ for $\mathbb{A}_K$ and $\mathbb{D}_K$ respectively. Later when we may let κ_0 be a variable κ, we will write $\mathbb{A}(\kappa)$ and $\mathbb{D}(\kappa)$.

We recall the following cohomological lemma which is Lemma 1.2 in [AS-Barcelona].

Lemma 3.6. *Let R be a commutative ring, G a group, M a right RG-module, I an ideal of R. Suppose I is generated by an M-regular sequence $(x_1, \ldots, x_r)$. Then the image of the map*

$$i_*: H^*(G, IM) \to H^*(G, M),$$

induced by the inclusion $i: IM \to M$, equals $IH^*(G, M)$.

SECTION 4: COHOMOLOGY AND U-OPERATORS

We fixed the Hecke pair, i.e. congruence subgroup $\Gamma = \Gamma_\nu$ and semigroup S in Section 3. We have the Hecke algebra $\mathcal{H} = \mathcal{H}(\Gamma, S)$. For any right S-module M, we will denote the $\mathcal{H}$-module $H^*(\Gamma, M)$ by $H(M)$.

We denote the Hecke operator corresponding to the double coset $\Gamma \pi \Gamma$ by u. We can lift u to the cochain level in a non-canonical way as follows:

Wwrite $\Gamma \pi \Gamma$ as the disjoint union of single cosets ΓA_i. We may and shall assume that each A_i is upper triangular, integral, with determinant equal to $\det(\pi)$ and with diagonal part $= \mathrm{diag}\,(1, p^m, p^{2m}, \ldots, p^{(n-1)m})$ Then $\Gamma \pi \Gamma$ is the disjoint union of single cosets $\Gamma_\nu A_i$.

We fix a resolution F_* of $\mathbb{Z}_p$ by free, finitely generated $\mathbb{Z}_p \Gamma$ - modules. We use F to compute the cohomology of Γ and $\pi^{-1}\Gamma\pi \cap \Gamma$ in terms of cochains. For the group, $\pi^{-1}\Gamma\pi$ we use $F^\bullet$ where the underlying groups of $F^\bullet$ are the same as in F, and the group action is defined by $f^\bullet \pi^{-1}\gamma\pi = f\gamma$. We also fix a homotopy equivalence τ between the two $\pi^{-1}\Gamma\pi \cap \Gamma$-resolutions F and $F^\bullet$.

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By definition, the Hecke operator u on $H(M)$ equals $tr \circ res \circ \pi$, where $\pi \colon H^*(\Gamma, M) \to H^*(\pi^{-1}\Gamma\pi, M)$ is induced by conjugation $\pi^{-1}\Gamma\pi \to \Gamma$ ($\gamma \mapsto \pi\gamma\pi^{-1}$) and the right action on M ($x \mapsto x\pi$), $res \colon H^*(\pi^{-1}\Gamma\pi, M) \to H^*(\pi^{-1}\Gamma\pi \cap \Gamma, M)$ is the restriction map and $tr \colon H^*(\pi^{-1}\Gamma\pi \cap \Gamma, M) \to H^*(\Gamma\, M)$ is the transfer map.

As in Formulae 4.3 of [AS-Barcelona] we can lift u to an operator U on the cochains as follows. Write $\gamma_i = \pi^{-1}A_i$.

Formula 4.1. *For a cocycle z in $Hom_\Gamma(F_k, M)$, uz is represented by the cochain*

$$f \mapsto U(f) = \Sigma_i z(\tau(f\gamma_t^{-1}))A_i.$$

We write $C_k(M)$ for the cochains $Hom_\Gamma(F_k, M)$ and $C(M)$ for $\oplus_k C_k(M)$, $Z(M)$ for the cocycles and $B(M)$ for the boundaries. Thus $H(M) = C(M)/Z(M)$. Assume that M has a topology. Since the coboundary maps on the cochains are continuous, $Z(M)$ is closed. However, $B(M)$ need not be closed in general, a source of much complication.

Lemma 4.2. *Fix K to be a closed affinoid polydisc or a single point in $\mathcal{X}$. The operator U acts completely continuously on the $A(K)^0$-module $C(\mathbb{E}_K(\mathbb{O}_L))$.*

Proof: The module $\mathbb{A}_K(\mathbb{O}_L)$ has the ONB over $A(K)^0$ given in Section 3, namely, $\{c_J \tilde{n}^J\}$. Because the $n_{ij}, i < j$, are positive root vectors for the torus, π acts on n^J by a positive power of p that tends to infinity as J grows. In other words, multiplication by π is completely continuous on $\mathbb{A}_K(\mathbb{O}_L)$. Therefore it is completely continuous on the dual space $\mathbb{E}_K(\mathbb{O}_L)$, and hence on the cochains $C_k(\mathbb{E}_K(\mathbb{O}_L)) \approx \mathbb{E}_K(\mathbb{O}_L)^{m_k}$ where m_k is the rank of F_k over $\mathbb{Z}_p\Gamma$. We give $C_k(\mathbb{E}_K(\mathbb{O}_L))$ the structure of Banach $A(K)^0$-module induced by the isomorphism with $\mathbb{E}_K(\mathbb{O}_L)^{m_k}$.

Then U is a finite sum of terms, each of which is the composition of a continuous map, π and an invertible map. Each term is completely continuous and hence so is U. QED.

Lemma 4.3. *Fix K as above. The coboundaries $B(\mathbb{E}_K(\mathbb{O}_L))$ are closed in $C(\mathbb{E}_K(\mathbb{O}_L))$.*

Proof: So far we have been dealing with the (strong) topology on $C(\mathbb{E}_K(\mathbb{O}_L))$ induced by the operator norm topology on $\mathbb{E}_K(\mathbb{O}_L)$. There is also the weak topology, defined by saying that a sequence of elements $e_i \in \mathbb{E}_K(\mathbb{O}_L)$ has a limit e if and only if for every $F \in \mathbb{A}_K(\mathbb{O}_L)$, $\lim e_i(F) = e(F)$. A standard diagonal argument shows that $\mathbb{E}_K(\mathbb{O}_L)$ is compact in the weak topology.

Now let $b_i \in B(\mathbb{E}_K(\mathbb{O}_L))$ be a sequence which converges in the strong topology to a point $c \in C(\mathbb{E}_K(\mathbb{O}_L))$. Then it also converges to c in the weak topology. Write $b_i = da_i$ where d is the coboundary operator. Passing to a subsequence, we may assume that a_i converges in the weak topology, say to a_∞. Since d is continuous (in both topologies) we have that b_i converges to da_∞ in the weak topology. But the weak topology is Hausdorff, so $c = da_\infty$, i.e. c is a coboundary. QED.

Note: It follows that $H(\mathbb{E}_K(\mathbb{O}_L))$ inherits the structure of a topological $A(K)^0$-module but it does not obviously have any canonical Banach module structure. The obstruction to finding the latter is related to the possible existence of torsion.

Throughout this section, R is either a Banach algebra over L with a multiplicative norm, or the unit ball in such an algebra. In particular, R is a complete valued ring containing L or $\mathbb{O}_L$ where the norm restricts to the usual one on L or $\mathbb{O}_L$ respectively. We also assume R is noetherian. For example, R could be a Tate algebra over L or over O_L. By Theorem 1 page 207 of BGR, the former is noetherian, and by the second part of the appendix to this section below, the latter is noetherian.

If $a \in R$ and $|a| = p^{-h}$, we say "a has slope h".

We have the completely continuous $A(K)^0$-module endomorphism U on $C(\mathbb{E}_K(\mathbb{O}_L))$ which induces the Hecke operator u on $H(\mathbb{E}_K(\mathbb{O}_L))$. Fix a rational number h. We want to derive a "slope $\leq h$"- decomposition of $H(\mathbb{E}_K(\mathbb{O}_L))$ with respect to u that doesn't depend on the choice of the lifting U. But the methods we are using only work after tensoring with L. First we do this at the cochain level.

Note that $\mathbb{D}_K(L) = \mathbb{E}_K(\mathbb{O}_L) \otimes L$ since $\mathbb{E}_K(\mathbb{O}_L)$ is the unit ball in $\mathbb{D}_K(L)$. Also by the Universal Coefficient Theorem, $H(\mathbb{E}_K(\mathbb{O}_L)) \otimes L = H(\mathbb{D}_K(L))$ as $\mathcal{H}$-modules.

If $f(T)$ is a polynomial of degree d, let $f^*(T) = T^d f(T^{-1})$. In our notation, we will use Y to denote a polynomial of the type Q^*, where in Coleman's notation a Fredholm determinant factors into $Q(T)S(T)$. (In such a factorization we shall always assume $Q(0) = 1$.)

Definition. *If $Y(T)$ is a monic polynomial with coefficients in R, we say Y has slope $\leq h$ if and only if $Y(0)$ is a unit in R and every root of Y^* in an algebraic closure of the fraction field of R has norm $\leq p^h$, where the norm is the unique one induced from that on R.*

Remark: Since the roots of Y are the inverses of the roots of Y^* this is equivalent to saying that every root of Y has slope $\leq h$.

Recall [BGR] that if $f(T) = T^m + a_1 T^{m-1} + \cdots + a_m$ then $sp(f)$ is the maximum of the $|a_i|^{1/i}$ for $1 \leq i \leq m$. We extend this definition to any polynomial g whose leading coefficient is a unit b by defining $sp(g) = sp(b^{-1}g)$. Then the roots of g all have norm $\leq sp(g)$, and at least one root has norm equal to $sp(g)$.

Then Y has slope $\leq h$ if and only if $sp(Y^*) \leq p^h$. Suppose that $R = A(K)$ with the Gauss norm, If $\xi \in K$, and f is a power series with coefficients in R, we denote by f_ξ the specialization of f at ξ, i.e. the power series obtained by evaluating the coefficients of Y at ξ. Then $sp(Y_\xi^*) \leq sp(Y^*)$ for all $\xi \in K$. It follows that if Y has slope $\leq h$, then Y_ξ has slope $\leq h$, i.e. any root of any specialization of Y to a point in K has slope $\leq h$.

Definition: For any R-module M with an R-linear endomorphism u, a slope $\leq h$ decomposition is an $R[u]$-submodule M_1 satisfying the following properties (1)-(3):

(1) M_1 is finitely generated over R.

(2) There is a monic polynomial $Y(T) \in R[T]$ of slope $\leq h$ such that $Y(u)$ annihilates M_1.

(3) For any monic polynomial $P(T) \in R[T]$ of slope $\leq h$, $P(u)$ acts left pseudo-invertibly on M/M_1.

Definition: If V is an R-module and V_1 a sub-module and $f : V \to V$ an R-module map that preserves V_1, we say f acts left pseudo-invertibly on V/V_1 if there exists an R-module map $g : V \to V$ that preserves V_1 and a positive integer m such that $g \circ f$ induces multiplication by p^m on V/V_1.

Notations: (1) For a polynomial such as Y in (2) we shall write Y_M.

(2) We shall use the notation $M^h = M_1 \otimes L$.

Lemma 5.1. *If M is an $R[u]$-module with a slope $\leq h$ decomposition, then M^h is unique up to isomorphism.*

Proof. Suppose M_1 and M_1' are both slope $\leq h$ decompositions of M, equipped with polynomials $Y(T) = Y_M(T)$ and $Y'(T) = Y_{M'}(T)$. Consider $x' \in M_1'$. Let an overline denote the image of an element of M in M/M_1.

Then $0 = Y'(u)x'$ implies that $0 = Y'(u)\bar{x}'$. Let g be a left pseudo-inverse to $Y'(u)$ on M/M_1. Then $0 = gY'(u)\bar{x}' = p^m \bar{x}'$ shows that $p^m x'$ lies in M_1 for some m. Since M_1' is finitely generated, we obtain that for some m, $p^m M_1' \subset M_1$ and by symmetry $p^m M_1 \subset M_1'$. After tensoring with L we get that $M'^h = M_1^h$.

Definition: If M is an $R[u]$-module, even if it doesn't necessarily have a slope $\leq h$ decomposition, we define $M^{(h)}$ to be the sub-$R \otimes L$-module of $M \otimes L$ generated by all generalized eigenvectors of u whose generalized eigenvalues (in an algebraic closure of $R \otimes L$) are of slope $\leq h$.

Lemma: If M is an $R[u]$-module with a slope $\leq h$ decomposition, then $M^{(h)} = M^h$.

Proof: Since $Y_M(u)M^h = 0$ we have $M^{(h)} \supset M^h$. Let v be in $M^{(h)}$ and let $\bar{v}$ be its image in $M^{(h)}/M^h$. Then $\bar{v}$ is annihilated by $P(u)$ where P is some polynomial of slope $\leq h$. Let β be the left inverse to $P(u)$ on $M^{(h)}/M^h$. (Since we have tensored with L, the left pseudo-inverse is an inverse.) Then $\bar{v} = \beta P(u)\bar{v} = 0$ shows that $v \in M^h$.

Theorem 5.2. *Given K and h as above, then after possibly shrinking K (without decreasing its dimension) there exists a slope $\leq h$ decomposition of the $A(K)(L)^0[U]$-module of cochains, $C = C(\mathbb{E}_K(\mathbb{O}_L))$.*

In fact, there exist continuous operators q_1 and q_2 on C with the following properties:

    (1) *they commute with U;*
    (2) *there exists $m \geq 0$ such that $q_1^2 = q_2^2 = p^{2m}$, $q_1 q_2 = q_2 q_1 = 0$, and $q_1 + q_2 = p^m$;*
    (3) *Image$(q_1) \subset C$ is a slope $\leq h$ decomposition of C.*

Moreover, the same is true when C is replaced by its subgroup of cocycles Z, or its subgroup of boundaries B or by the cohomology $H = Z/B$.

Note: We call a pair (q_1, q_2) as in item (2) above a "pair of pseduo-projectors".

Proof: First we invoke the theorem in the Appendix to this section that shows (possibly after shrinking K) that the Fredholm determinant $f(T)$ of U on the ONable $A = A(K)(\mathbb{O}_L)$-module $C = C(\mathbb{E}_K(\mathbb{O}_L))$ has a factorization

$$f(T) = Q(T)S(T)$$

such that S is entire, Q is a polynomial relatively prime to S whose leading coefficient is a unit, $Q(0) = 1$, the slope of Q^* is $\leq h$, and the slope of the inverse of every root of every specialization of S to a point in K is $> h$.

We apply Theorem 4.2 of [Coleman] to derive the existence of a direct sum decomposition $C \otimes \mathbb{Q}_p = N(Q) \oplus F(Q)$ into closed submodules, where the projections are given by entire power series in $A[[T]]$ evaluated at $T = U$, and such that $Q^*(U)N(Q) = 0$ and $Q^*(U)$ is invertible on $N(Q)$.

Morever, if $P(T)$ is any monic polynomial of slope $\leq h$ it is easy to see that P^* is relatively prime to S: Saying that it is relatively prime to S means that the resultant $Res(P^*, S)$ is a unit. Taking the resultant, which is just a certain determinant made from the coefficients of the power series involved, commutes with specialization. So it suffices to show that for any ξ in the disk, $Res(P_\xi^*, S_\xi) \neq 0$. This will be true if for each ξ, P_ξ^* and S_ξ together generate the unit ideal. This is true because they have no common root on the disk. Indeed the inverse roots of P_ξ^* all have slope $\leq h$ and those of S_ξ all have slope $> h$.

It follows then from Lemma 4.0 in [Coleman] that $P(U)$ acts invertibly on $F(Q)$ with an inverse again given by evaluating an entire power series at U.

Since the powers of p in the denominators of the coefficients of an entire power series is bounded, we can multiply the projectors through by an appropriate power of p and obtain item (2). Since the projectors are power series in U, they commute with U, hence item (1). Finally, item (3) follows easily from the preceding two paragraphs.

Because Z and B are closed U-stabele submodules in C and the pseudo-projectors q_i are given by convergent power series in U, they induce a pair of pseduo-projectors on Z, B and H. Clearly the properties (1), (2), and (3) carry over. The ring $A(K)(O_L)$ is noetherian, so the finite generation of the slope $\leq h$ part of Z follows from that of C.

We get the following corollary:

Corollary 5.3. *There is a splitting $H \otimes L = N \oplus F$ as $A(K(L))[u]$-modules where N is the image of $p^{-m} q_1$ and F is the image of $p^{-m} q_2$. This is the unique splitting such that $N \subset H \otimes L$ is a slope $\leq h$ decomposition.*

Proof: It is easy to check the uniqueness. The existence follows immediately from the Theorem.

Remarks: (1) $H \otimes L$ may not have a canonical Banach A-module structure since the obvious "norm" may only be a semi- norm.

(2) If K is just a single point, we can derive all of the above using Serre's theory, without Coleman's additions.

(3) If K' is subdisk of K, possibly of smaller dimension, the structure of the pseudo-projectors with properties (1) - (3) passes over to $\mathbb{E}_{K'}$ compatibly with the restriction maps on cochains etc. induced by $\mathbb{E}_K \to \mathbb{E}_{K'}$.

We now obtain the first step of the Control Theorem:

Theorem 5.4. *Given $\kappa_0 \in K$ and h, there exists (after possibly shrinking K) an exact sequence of $\mathcal{H}$ and $A(K)$-modules (with commuting actions), for any i,*

$$0 \to I_0 H^i(\mathbb{D}_K(L))^h \to H^i(\mathbb{D}_K(L))^h \to H^i(\mathbb{D}_0(L))^h \to H^{i+1}(I_0 \mathbb{D}_K(L))(h)$$

where the first nontrivial map is inclusion and the second is induced by $\pi_{0,n}$.

Proof: We have the exact sequence (with all maps defined over L)

$$0 \to I_0 \mathbb{D}_K \to \mathbb{D}_K \to \mathbb{D}_0 \to 0$$

given from Theorem 3.3 with $m = 0$ and $l = n$. Therefore

$$0 \to I_0 \mathbb{D}_K(L) \to \mathbb{D}_K(L) \to \mathbb{D}_0(L) \to 0$$

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is also exact. Take the long exact sequence of cohomology (which is $\mathcal{H}$-equivariant by [AS-Crelle]) and apply Lemma 3.6. It is easy to see using the direct sum decomposition into slope leh and $\geq h$ parts that the slope $\leq h$ part of $I_0 H(\mathbb{D}_K(L))$ is $I_0 H(\mathbb{D}_K(L))^h$. The rest of the proof is similar. That the image of the last arrow lies in the (h)-part is obvious from the definitions.

APPENDICES:

(1) FACTORIZATION OF THE FREDHOLM DETERMINANT

Let $A(K) = k < x_1, \ldots, x_n >$ be the Tate algebra of functions on the closed unit disk K given by convergent power series with coefficients in k, where k is a finite extension of $\mathbb{Q}_p$. We give it the usual norm, which is multiplicative. In the course of this appendix we will speak of "shrinking the disk". This means we will replace K by a smaller closed disk K' (or more generally a polydisk) whose radii are elements in $|k|$. The Tate algebra $A(K')$ of K' is isomorphic to that of K and we will identify $A(K)$ as a subring of $A(K')$ by restriction.

Let $A[T]^{ent}$ denote the ring of entire power series, so an element of it is a power series $f(T) = \sum a_i T^i$ with $M^i |a_i| \to 0$ for any $M > 0$. Give it the norm $|f| = sup|a_i|$. If f is any such power series, and ξ is a point in K, we denote by f_ξ the entire power series in $k[[T]]^{ent}$ obtained by evaluating all the coefficients of f at ξ.

If $q(T)$ is a polynomial of degree d, we denote $q^* = T^d q(T^{-1})$. Thus if $q(0) = 1$ then q^* is monic.

Theorem. *Let h be a rational number. Let $f \in A[T]^{ent}$ such that $|f| = 1$ and $f(0) = 1$. Suppose there exists a polynomial $q(T) \in k[T]$ and an entire power series $s(T) \in k[T]^{ent}$ such that $q(0) = s(0) = 1$, all inverse roots of q have slope less than or equal to h, all inverse roots of s have slope greater than or equal to h, and $f_0 = qs$. Then after possibly shrinking the disk, there exists a polynomial $Q(T) \in A[T]$ and an entire power series $S(T) \in A[T]^{ent}$ such that (1) $Q(0) = S(0) = 1$, (2) for all ξ in the disk, all inverse roots of Q_ξ have slope less than or equal to h and all inverse roots of S_ξ have slope greater than or equal to h, and (3) $f = QS$. Moreover, Q has for leading coefficient a unit and Q is relatively prime to S.*

Proof: Suppose d is the largest index with $|a_d| = 1$. After shrinking the disk, we may assume that a_d is a unit in A. Then f is T-distinguished of degree d. Then by the Weierstrass Preparation Theorem (BGR p. 201) we have that $f = we$ where w is a distinguished polynomial of degree d, $|w| = 1$ and e is a unit in $A[[T]]$. It is easy to check that e is in fact entire. If d is larger than 0 we have accomplished something, and we can continue inductively with e.

Of course we may have $d = 0$. So now suppose more generally that the Newton polygon for f_0 has first slope $\lambda \geq 0$. This means that there is some $d > 0$ such that d is the largest index for which $ord(a_d(0)) = \lambda d$ and $ord(a_i(0)) \geq \lambda i$ for all $i < d$ and $ord(a_j(0)) = \lambda j$ for all $j > d$.

Then we can shrink the disk so that $ord(a_i(\xi))$ is constant for all $i \leq d$ and for all ξ in the disk. Moreover, we can shrink the disk further if necessary so that $ord(a_j(\xi)) > \lambda j$ for all $j > d$ and for all ξ in the disk. This is because the fact that f is entire implies that $ord(a_j(\xi)) - \lambda j$ tends to infinity uniformly in ξ as $j \to \infty$. So just pick J sufficiently large that $ord(a_j(\xi)) > \lambda j$ for all $j \geq J$ and all ξ in the disk, and then shrink the disk so that that $ord(a_i(\xi))$ is constant for all $i < J$ and for all ξ in the disk.

Having shrunk the disk, we've arranged things so that for all ξ in the disk, the Newton polygon for f_ξ has first segment of slope λ and horizontal length d. Now choose a number algebraic b over k whose slope is λ, and change variables by setting $g(T) = f(bT)$. Then the Newton polygon for g_ξ has first segment of slope 0 and horizontal length d for every ξ in the disk. Now we are in the situation of the first paragraph of this proof. We apply the Weierstrass Prepartion Theorem as in that paragraph and then change variables back by $T \to b^{-1}T$.

Continuing in this way, since the Newton polygon of the original f has only a finite number of slopes less than or equal to h, we obtain a factorization $f = QS$ as stated in the theorem, except that the coefficients may be in a finite extension of k.

Suppose we had two factorizations $f = Q_i S_i$, $i = 1, 2$, with the properties of the statement of the theorem. Then for any specialization at a point ξ of the disk, and for each i, the factorization $f_\xi = (Q_i)_\xi (S_i)_\xi$ is the usual slope $\leq h$ factorization of the entire power series f_ξ, which is unique. Hence $Q_1 = Q_2$ and $S_1 = S_2$. Now let σ be any element of the absolute Galois group of k. Then $f = \sigma(Q)\sigma(S)$ is another such factorization. Hence $Q = \sigma(Q)$ and $S = \sigma(S)$. Therefore our desired factorization is defined over k.

Finally, in order to use this factorization in Theorem 4.2 of Coleman's red preprint, we need Q to have leading coefficient a unit, and we need Q to be relatively prime to S. By our construction, having shrunk the disk, the slope $\leq h$ part of the Newton polygon of f_ξ is independent of ξ in the disk. In particular, the degree of Q_ξ is constant, so the leading coefficient of Q is nonvanishing on the disk. Hence it is a unit. Saying that Q is relatively prime to S means that the resultant $Res(Q, S)$ is a unit. Taking the resultant, which is just a certain determinant made from the coefficients of the power series involved, commutes with specialization. So it suffices to show that for any ξ in the disk, $Res(Q_\xi, S_\xi) \neq 0$. This will be true if for each ξ, Q_ξ and S_ξ together generate the unit ideal. This is true because they have no common root on the disk. See Corollary 5 p. 263 of BGR.

(2) THE UNIT BALL IN THE TATE ALGEBRA IS NOETHERIAN

The following lemma and reference was kindly provided by Keith Conrad.

Lemma. *Let (R, m) be a discrete valuation ring with respect to the m-adic topology. Then the Tate ring $R < X_1, \ldots, X_n >$ is noetherian.*

Proof. The ideal m is principal, say $m = pR$. The Tate ring is the completion of the polynomial ring $R[X_1, \ldots, X_n]$ relative to the ideal (p). The lemma follows from paragraph (23.K), p.169 of H. Matsumura's book Commutative Algebra, second edition, Benjamin, Reading, MA, 1980.

SECTION 6: HIGHEST WEIGHTS AND REPRESENTATIONS, NEBENTYPE CHARACTERS

We recall the from [AS-Barcelona] the definition of the $*$ action on certain finite dimensional modules.

We let λ denote the highest weight of an irreducible right rational $G = GL(n)$-module V with respect to (B, T). We fix a highest weight vector v_λ in $V(\mathbb{Q}_p)$, so that $vn = v$ for $n \in N$ and $vt = \lambda(t)v$ for $t \in T$.

Althought S is in $G(\mathbb{Q}_p)$, we only let $\tilde{S}$ act on V in the usual way. We extend this $\tilde{S}$-action on V to the semigroup S generated by $\tilde{S}$ and π as follows: Let the usual

action of $GL(n, \mathbb{Z}_p)$ on V be denoted by juxtaposition, and define the $*$-action of π by

$$w * \pi = \lambda^{-1}(\pi)(w\pi).$$

It is easily checked that putting these together defines a group action by the group generated by $GL(n, \mathbb{Z}_p)$ and π, and in particular a semigroup action of S. It induces an action of the Lie algebra $\mathfrak{g}$ of $GL(n, \mathbb{Z}_p)$ which is compatible with the action of π. That is, if $Z \in \mathfrak{g}$, $w * Z\pi = w * \pi(Ad(\pi)(Z))$, where Ad refers to the right adjoint action.

Warning: We are forced to use the $*$-action on X because the ordinary action doesn't preserve X. Then we are forced to introduce the $*$-action on V so that Lemma 7.2 will be true. As a consequence, our Hecke operator u corresponding to the double coset of π is $\lambda^{-1}(\pi)$ times the usual Hecke operator T_π. This normalization has to be taken into account when translating our results to more standard language.

We let ε denote a nebentype character (which may be trivial), i.e., a character $\varepsilon \colon T(\mathbb{Z}/p^\nu) \to \mathbb{O}_L^\times$. We can also then denote by ε the extended characters $\varepsilon \colon \tilde{S} \to \mathbb{O}_L^\times$ and $\varepsilon \colon X \to \mathbb{O}_L^\times$ where $\varepsilon([x])$ depends only on the values of the diagonal entries of the upper triangular representative x modulo p^ν. We extend ε from $\tilde{S}$ to S by setting $\varepsilon(\pi) = 1$.

Given a representation V, we can twist the action of S on V by ε, which we denote by $V(\varepsilon) = V \otimes (\mathbb{O}_L)_\varepsilon$.

Finally, we obtain actions of the Hecke algebra $\mathcal{H}$ on $H(\mathbb{D}_\lambda)$ and $H(V_\lambda(\varepsilon))$.

<center>SECTION 7: COMPARISON THEOREM
BETWEEN THE COHOMOLOGY OF $\mathbb{D}_\lambda$ AND V_λ.</center>

Fix a highest weight λ. In this section, K is a single point $\{\lambda\}$. Write $\mathbb{D}_K = \mathbb{D}_\lambda$ and $\mathbb{A}_K = \mathbb{A}_\lambda$. We identify $A(K)$ with $\mathbb{C}_p$. We leave out the κ variable when writing elements in A_λ. Thus A_λ consists of functions $F(td)$ which are locally analytic in t and rigid analytic in d, and such that $F(tx) = \lambda(t)F(x)$.

We define

$$\phi_\lambda \colon \mathbb{D}_\lambda \to V_\lambda$$

by $\phi_\lambda(\mu) = \int_X v_\lambda x \, d\mu(x)$. We define $\phi_{\lambda,\varepsilon} \colon \mathbb{D}_\lambda(\varepsilon) \to V_\lambda(\varepsilon)$ by twisting ϕ_λ by ε.

What the integral means: Let $\{v_i\}$ be a basis for $V(\mathbb{Q}_p)$ and for any upper triangular matrix g that represents an element in X, write $v_\lambda g = \sum v_i f_i(g)$ where the $f_i(g)$ are regular functions (i.e. rational functions without poles). Then for each i, $f_i([g])$ is in $\mathbb{A}_\lambda$ where $x = [g]$ runs over the points of X. To see this, note that f_i is locally analytic in t and rigid analytic in d. Since v_λ is a highest weight vector, $\lambda(t)v_\lambda g = v_\lambda tg = \sum v_i f_i(tg)$ shows that $f_i(tg) = \lambda(t)f_i(g)$. The value of the integral is then by definition the vector $\sum \mu(f_i)v_i$.

Lemma 7.1. *The map $\phi_{\lambda,\varepsilon}$ is equivariant with respect to S.*

Proof. It suffices to check this when $\varepsilon = 1$.

First, if $s \in S$ has determinant prime to p, we have $\phi_\lambda(\mu s) = \int_X v_\lambda x \, d(\mu s)(x) = \int_X v_\lambda(xs) \, d\mu(x) = [\int_X v_\lambda x \, d\mu(x)]s = \phi_\lambda(\mu)s$.

We now check the equivariance with respect to π: $\phi_\lambda(\mu\pi) = \int_X v_\lambda x \, d(\mu\pi)(x) = \int_X v_\lambda(x * \pi)d\mu(x) = \int_X v_\lambda(\pi^{-1}x\pi) \, d\mu(x) = \lambda^{-1}(\pi)[\int_X v_\lambda x \, d\mu(x)]\pi = \phi_\lambda(\mu) * \pi$.

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Lemma 7.2. *The map $\phi_{\lambda,\varepsilon}$ induces an $\mathcal{H}$-equivariant map on cohomology:*

$$\phi_{\lambda,\varepsilon} \colon H(\mathbb{D}_\lambda(\varepsilon)) \to H(V_\lambda(\varepsilon)) \,.$$

Proof. This follows from the equivariance of cohomology with respect to the Hecke operators, given a morphism on coefficients that is equivariant with respect to the semigroup.

We want to show that if h is not too large then the map in Lemma 7.2 induces an isomorphism on the slope $\leq h$ parts of the cohomology. First, define the element $\mu_\lambda \in \mathbb{D}_\lambda$ as the delta function at the origin in X. In other words, if $F \in \mathbb{A}_\lambda$, $\mu_\lambda(F) = F[1]$.

Since the elements of $\mathbb{D}_\lambda$ are continuous $\mathbb{C}_p$-linear functionals in $\mathbb{A}_\lambda$, and since A_λ consists of locally analytic functions, we can differentiate the group actions and view both $\mathbb{D}_\lambda$ and A_λ as Lie-algebra modules for the Lie algebra of the compact open subgroup $I_\nu \cap G(\mathbb{Z}_p)$ of $G(\mathbb{Q}_p)$, i.e. for $\mathfrak{g} =$ the Lie algebra of $G(\mathbb{Q}_p)$. If $Z \in \mathfrak{g}$, we have $\mu Z(F) = \mu(ZF)$. The action of Z is computed in the usual way using the one-parameter subgroup e^{sZ}. Then they also become modules for the universal enveloping algebra $\mathcal{U}$ of $\mathfrak{g}$. The $*$-action of π on these modules is compatible with the $\mathfrak{g}$ action in the usual way (cf. the similar comment in section 6.)

Lemma 7.3. *μ_λ is a maximal vector of weight λ with respect to (B,T).*

Proof: If $g \in N(\mathbb{Z}_p) \cap I_\nu$, $(\mu_\lambda g)(F) = (gF)[1] = F[g] = F[1] = \mu_\lambda(F)$, so μ_λ is killed by the Lie algebra of $N(\mathbb{Z}_p)$. Similarly if $t \in T(\mathbb{Z}_p)$, $(\mu_\lambda t)(F) = (tF)[1] = F[t] = \lambda(t)F[1] = \lambda(t)\mu_\lambda(F)$.

Lemma 7.4. *The map in Lemma 7.2 is surjective.*

Proof: One easily checks that it maps μ_λ to v_λ. Since V is irreducible, it must be surjective.

Recall that we have the standard coordinate functions n_{ij} $(i < j)$ on D, extended in the unique way to functions $\tilde{n}_{ij}$ in $\mathbb{A}_\lambda$. For any (a,b) let Z_{ab} denote the generator of the Lie algebra corresponding to the (a,b)'th entry. (If we identify the Lie algebra with $gl(n)$ then Z_{ab} is the matrix which is all 0's except a 1 in the (a,b)'th place.) A simple calculation shows that $Z_{ab}\tilde{n}_{ij} = \tilde{n}_{ij}$ if $b = j, i < a$, $= 1$ if $b = j, i = a$, and $= 0$ otherwise.

For any multi-index J we define μ_J to be the functional on power series in the n_{ij} that picks out the coefficient of n^J. We view μ_J as a functional on $\mathbb{A}_\lambda$ by setting $\mu_J(F) = \mu_J(F[d])$ where $F[d]$ is the power series giving F on D, written in terms of the n_{ij}.

Lemma 7.5. *The L-subspace C_λ of $\mathbb{D}_\lambda$ spanned by the translates of μ_λ under $\mathcal{U}(L)$ is equal to the L-span of the μ_J.*

Proof: Since μ_λ is a maximal vector, it suffices to see what happens when we repeatedly apply the various derivations Z_{ab}'s for $a < b$, using the formula above. It's clear that $\mu_J Z_{ab}$ is a linear combination of $\mu_{J'}$'s of lower weight. So it only remains to show that each μ_J appears in $\mu_\lambda \mathcal{U}(L)$. For any power series $\sum a_I \tilde{n}^I$ we have $(\mu_J Z_{ab}) \sum a_I \tilde{n}^I = \mu_J(\sum a_I(Z_{ab}\tilde{n}^I))$. It's easy to construct an argument now by induction on the weight of μ_J with respect to a total ordering on the roots induced by the choice (B,T).

Note that each monomial in the $\tilde{n}^J$ is a weight vector for the torus. Given h, partition these monomials into the set $\mathcal{M}_1$ of those whose weight, evaluated on π, is also a weight appearing in V_λ or has slope $\leq h$, and $\mathcal{M}_2$ consisting of the others. Let $\mathbb{A}_i$ be the closure in $\mathbb{A}_\lambda$ of the L-span of the monomials in $\mathcal{M}_i$, $i = 1, 2$. Then $\mathbb{A}_1$ is finite dimensional and $\mathbb{A}_\lambda = \mathbb{A}_1 \oplus \mathbb{A}_2$.

Also define $\mathbb{D}_i = \{\mu \in \mathbb{D}_\lambda : \mu(F) = 0 \forall F \in \mathbb{A}_{i+1}\}$ (viewing the subscripts mod 2.) Then for $i = 1, 2$, $\mathbb{D}_i$ is the continuous L-dual of $\mathbb{A}_i$, they are closed in $\mathbb{D}_\lambda(L)$, and $\mathbb{D}_\lambda(L) = \mathbb{D}_1 \oplus \mathbb{D}_2$. Also, $\mathbb{D}_1$ is finite dimensional.

Lemma 7.6. *Let K_λ be the kernel of the map $\phi_{\lambda,\varepsilon}$ in Lemma 7.2. Then $\mathbb{D}_2 \subset K_\lambda$.*

Proof: Let w denote the operator given by the action of π on $\mathbb{A}_2$. It is completely continuous, and if σ is any weight appearing in V_λ we have that $1 - \sigma(\pi)^{-1}w$ acts invertibly on $\mathbb{A}_2$. Therefore its transpose $1 - \sigma(\pi)^{-1}w'$ acts invertibly on D_2 and so does their product W over all such σ. Therefore W acts invertibly on the image of $\mathbb{D}_2$ under $\phi_{\lambda,\varepsilon}$. But W annihilates the target of the map. So the image must vanish.

Let α_1, $1 \leq i \leq n-1$, run over the simple roots of $SL(n)$ with respect to (B, T). For any two roots α, β we use the standard notation $< \beta, \alpha > = 2(\beta, \alpha)/(\alpha, \alpha)$. It is linear in β.

Lemma 7.7. *Define $f(\lambda)$ to be the minumum of $< \lambda, \alpha_i > +1$ for all i. If $h < f(\lambda)$ then π acting on K_λ is given by $p^h \Pi$ for some operator Π that preserves the unit ball of K_λ.*

Proof: By Lemma 7.6, K_λ is the sum of $\mathbb{D}_2$ and the finite dimensional space $K_\lambda \cap \mathbb{D}_1$. So it suffices to prove that $p^{-h}\pi$ preserves the unit ball of each of these spaces. By definition this is true for $\mathbb{A}_2$ and hence for its dual $\mathbb{D}_2$.

For the second we can prove more: $\mathbb{D}_1$ is a subspace of C_λ defined in Lemma 7.5, and we'll prove that $p^{-h}\pi$ preserves the unit ball of $K_\lambda \cap C_\lambda$. We use the fact that C_λ is a "standard cyclic module" of weight λ in the terminology of [Humphreys]. Any "standard cyclic module" is a quotient of the Verma module (denoted $Z(\lambda)$ in [Humphreys]) and has a unique irreducible quotient module, which in our case must be $V_\lambda \simeq C_\lambda/(K_\lambda \cap C_\lambda)$. It follows from Theorem 21.4 in [Humphreys] that $K_\lambda \cap C_\lambda$ is generated by vectors of the form $\mu_\lambda y_i^{m_i+1} Z$, $1 \leq i \leq n-1$. Here y_i is the simple root vector corresopnding to $-\alpha_i$, or in our notation above, $y_i = Z_{i,i+1}$, $m_i = < \lambda, \alpha_i >$, and Z runs over the monomials in Z_{ab} with $a < b$. In particular, for every Z, $\pi^{-1}Z\pi = p^e Z$ for some positive power e. Since $-\alpha_i(\pi) = p$, we get that π acts on $\mu_\lambda y_i^{m_i+1}$ as multiplication by p^{m_i+1}. (By our conventions $\mu_\lambda \pi = \mu_\lambda$.) Therefore, under the condition on h, the operator norm of $p^{-h}\pi$ on $K_\lambda \cap C_\lambda$ is ≤ 1. QED.

Remark: Following the warning in section 6 we might want to translate the condition on h into "classical" language. Let the highest weight of V_λ by given by the n-tuple $(b_1, \ldots, b_n)$, where this notation means the character that takes the diagonal matrix $diag(t_1, \ldots, t_n)$ to $\prod t_i^{b_i}$. Then λ dominant means that $b_1 \geq b_2 \geq \cdots \geq b_n$. The simple roots are $(0, \ldots, 0, 1, -1, 0, \ldots, 0)$ with the 1 in the ith place, $i = 1, \ldots, n-1$. The inner product on weight space is the usual dot product, and $< \lambda, \alpha_i > = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) = (\lambda, \alpha_i) = b_i - b_{i+1}$. Then the condition in Lemma 7.7 is that $h < b_i - b_{i+1} + 1$ for all $i = 1, \ldots, n-1$.

If we want to state this in terms of the slope of the usual Hecke operator T_π acting on $H(V_\lambda)$, we must remember that $u = \lambda^{-1}(\pi)T_\pi$. Therefore to find the required bound on the slope of the eigenvalue of the usual T_π we should "subtract" the logarithm base p of $\lambda^{-1}(\pi) = p^{-(b_2+2b_3+\cdots+(n-1)b_n)}$ We get that our main theorem applies when the slope of the eigenvalue of T_π acting on $H(V_\lambda)$ is less than

$$h < b_i - b_{i+1} + b_2 + 2b_3 + \cdots + (n-1)b_n + 1$$

for all $i = 1, \ldots, n-1$.

For example, if $n = 2$, and $\lambda = (g, 0)$, we get the classical slope on weight $g + 2$ cuspforms must be less than $g + 1$, which agrees with results of Coleman and Stevens.

Lemma 7.8. *Let U be a lift of the Hecke operator u to cochains as given in Formula 4.1. If $h < f(\lambda)$, then U acting on the cochains $C(K_\lambda(L))$ equals $p^h U'$ for some completely continuous operator U' that preserves the unit ball of $C(K_\lambda(L))$.*

Proof: The assertion about complete continuity follows in the same way as in the proof of Lemma 4.2. The rest follows immediately from Formula 4.1 and Lemma 7.7.

Corollary 7.9. *If $h < f(\lambda)$, then with respect to U, the cochains $C = C(K_\lambda(L))$ have a slope $\leq h$ decomposition $C_1 \oplus C_2$ with $C_1 = 0$ and $C_2 = C$. The same is true where C is replaced by its unit ball C^0.*

Proof: The only thing that needs to be proved is that if $P(T)$ is any polynomial with coefficients in L of slope $\leq h$ then $P(U)$ acts invertibly on C and the inverse is given by a convergent power series in U. We may assume P is linear, so $P(U) = (U - \rho)$ with $\rho \in L$ of slope $\leq h$.

Let h' be a little bigger than h but still satisfying the condition of Lemma 7.8. Then applying Lemma 7.8 to h', we see that $p^{-h}U$ has operator norm strictly less than 1 on C. Therefore it preserves the unit ball C^0. It also follows that the Fredholm determinant $\det(1 - \rho^{-1}UT)$ of $\rho^{-1}U$ on C is congruent to 1 modulo the maximal ideal in $\mathbb{O}_L$. Therefore it can't have $T = 1$ as a root. By Proposition 11 and its proof in [Serre], $1 - \rho^{-1}U$ is invertible on C and its inverse is given by a convergent power series in U. So the same is true for $(U - \rho)$.

Lemma 7.10. *If $h < f(\lambda)$, then for any monic polynomial P over $\mathbb{C}_p$ with slope $\leq h$, there exists an inverse β to $P(u)$ acting on the cohomology $H = H(K_\lambda(L))$. Moreover, β is induced from a convergent power series in u acting on $H(K_\lambda(L)^0)$.*

Proof: This is clear from Corollary 7.9, since $H(K_\lambda(L)^0)$ is a subquotient Z/B of $C^0 = C(K_\lambda(L)^0)$ where Z and B are closed.

The next theorem is the second step in the Control Theorem.

Theorem 7.11. *If $h < f(\lambda)$, then with respect to u, then the map $\phi_{\lambda,\varepsilon}$ in Lemma 7.2 induces an isomorphism $H(\mathbb{D}_\lambda)^h \to H(V_\lambda)^h$.*

Proof: From the short exact sequence

$$0 \to K_\lambda^0 \to \mathbb{D}_\lambda^0 \to V_\lambda^0 \to 0$$

obtain the long exact sequence of cohomology:

$$H^i(K_\lambda^0) \to H^i(\mathbb{D}_\lambda^0) \to H^i(V_\lambda^0) \to H^{i+1}(K_\lambda^0)$$

Taking slope $\leq h$ parts, we obtain $H^i(\mathbb{D}_\lambda)^h \to H^i(V_\lambda)^h$. We claim this map is an isomorphism. For suppose $x \in H^i(\mathbb{D}_\lambda)^h$. Then there is some monic polynomial Y of slope $\leq h$ such that $Y(u)x = 0$. Now x comes from some element y in $H^i(K_\lambda)^h$. Multiplying everything by a high power of p we may assume that y is in $H^i(K_\lambda^0)^h$. Let β be the inverse to $Y(u)$ on $H^i(K_\lambda^0)^h$, given by a convergent power series f in u. Then $y = \beta Y(u)y$ goes to $f(u)Y(u)x = 0$ in $H^i(\mathbb{D}_\lambda^0)^h$. In other words, $x = 0$. So the map is injective.

To see that it is surjective, consider (without loss of generality) $x \in H^i(V_\lambda^0)^h$ and let y denote its image in $H^{i+1}(K_\lambda^0)^h$. Again, choose Y as above, so that $Y(u)x = 0$. Then $Y(u)y = 0$ but $Y(u)$ acts invertibly on $H^{i+1}(K_\lambda^0)^h$. So $y = 0$. Thus x comes from $H^i(\mathbb{D}_\lambda^0)$. It is easy to see it must come from the slope $\leq h$ part.

We can combine this with Theorem 5.5, setting $\lambda = \kappa_0$ to obtain our main Control Theorem:

Theorem 7.12. *Given $\kappa_0 \in K$ and $h << \kappa_0, \alpha_i > +1$ for all i, there exists (after possibly shrinking K) an exact sequence of $\mathcal{H}$ and $A(K)$-modules (with commuting actions)*

$$0 \to I_0 H(\mathbb{D}_K(L))^h \to H(\mathbb{D}_K(L))^h \to H(V_{\kappa_0}(L))^h$$

where the first nontrivial map is inclusion and the second is induced by $\pi_{0,n}$ composed with $\phi_{\lambda,\varepsilon}$.

Remark: Since $H(\mathbb{D}_K(L))^h$ is a finitely generated module over the noetherian Banach ring $A(K)(L)$, it follows a posteriori from Proposition 3, p.164 of [BGR] that it has a complete $A(K)(L)$-module norm, and that all such are equivalent.

Section 8: Lifting cuspidal classes

We ought to show that there are some interesting nontrivial classes in the cohomology of $\mathbb{D}_K$. We shall do this in this section. It seems to be difficult to show that there are families of dimension greater than 0 when the ambient group is $GL(n)$ for $n \geq 3$. We will leave that for a subsequent paper.

Assume $n \geq 3$, and set $m = n^2/4 + n/2 - 1$ if n is even and $m = (n-1)^2/4 + n - 1$ if n is odd. Then m is the highest dimension in which cuspidal cohomology can exist for a congruence subgroup of $GL(n, \mathbb{Z})$. Let λ be a highest weight and $\alpha \in H^m(V_\lambda)$ an $\mathcal{H}$-eigenvector (as always we suppress the group Γ_ν from the notation for cohomology.)

We assume that α is "strongly quasi-cuspidal" mod p. By the term in quotation marks we mean that if the reduction mod p of the package of $\mathcal{H}$-eigenvalues occurring on α also occurs on $\beta \in H^i(V_{\lambda'})$ for any i and any highest weight λ', then β must be a cuspidal cohomology class, i.e. coming from a cuspidal automorphic form.

For example, heuristically speaking, think of the p-adic Galois representation that is conjectured to be attached to α. If it is irreducible mod p then it couldn't be attached to a cohomology class that comes from the Borel-Serre boundary or a discrete, non-cuspidal automorphic representation, because they are Eisensteinian, and would yield reducible Galois representations. So such an α would be "strongly quasi-cuspidal". When $n = 3$ it is not hard to show, using the last section of [AS Crelle's] that α is "strongly quasi-cuspidal" mod p if any of its Hecke polynomials is irreducible mod p. This doesn't depend on any conjectures.

If $V_{\lambda'}$ restricted to $SL(n)$ is not selfdual, it is a theorem of Borel and Wallach that $H^i_{cusp}(V_{\lambda'}) = 0$ for all i. In such a case β would be forced to be 0.

For any Hecke module, let the subscript ϕ denote the sum of all the generalized $\mathcal{H}$-eigenspaces whose eigenvalues are strongly quasi-cuspidal mod p.

Theorem 8.1. *Given λ, and $m(n)$ as defined above, let $\alpha \in H^m(n)(V_\lambda)$ be "strongly quasi-cuspidal" and an $\mathcal{H}$-eigenvector of slope $h < f(\lambda)$. Then, possibly after shrinking K, α is in the image of the last map in Theorem 7.12.*

Proof: Set $i = m(n)$. We use the notation of section 3, where we set $\kappa_0 = \lambda$. In particular, we choose coordinates β_j on K centered at λ as in section 3, and for $m < l$ we have the exact sequence of Theorem 3.3:

$$0 \to (\beta_{m+1}, \ldots, \beta_l)\mathbb{D}_l \to \mathbb{D}_l \to \mathbb{D}_m \to 0.$$

We obtain the long exact cohomology sequence

$$0 \to H^i((\beta_{m+1}, \ldots, \beta_l)\mathbb{D}_l(L)) \to H^i(\mathbb{D}_l(L)) \to H^i(\mathbb{D}_m(L)) \to H^{i+1}((\beta_{m+1}, \ldots, \beta_l)\mathbb{D}_l(L)).$$

We will prove by induction on $l - m \geq 1$ that any $\alpha \in H^i(\mathbb{D}_m(L))^h_\phi$ lifts to $H^i(\mathbb{D}_l(L))$.

Lemma 8.2. $H^{i+1}(\mathbb{D}_l(L))^h_\phi = 0.$

Proof: Taking the ϕ-eigenspaces in Theorem 7.12 gives us the exact sequences, for any $\kappa \in K_l(L)$

$$0 \to I_\kappa H^{i+1}(\mathbb{D}_l(L))^h_p hi \to H^{i+1}(\mathbb{D}_l(L))^h_p hi \to H^{i+1}(V_\kappa(L))(h)_p hi.$$

Since $i + 1$ is outside the cuspidal range, $H^{i+1}(V_\kappa(L))^h_p hi = 0$ for all κ. Let L_a denote an algebraic closure of L. Then if we set M to be the finitely generated $A(K_l)(L_a)$-module $H^{i+1}(\mathbb{D}_l(L_a))^h_p hi$, we have that $M \subset \cap I_\kappa H^{i+1}(\mathbb{D}_l(L_a))^h_p hi$ where the intersection is taken over all $\kappa \in K_l(L_a)$. The conclusion now follows by Nakayama's lemma: M is a module with empty support (cf. Corollary 7 page 376 of BGR.)

Lemma 8.3. $H^{i+1}(\beta_l \mathbb{D}_l(L))(h)_\phi = 0.$

Proof: By Corollary 3.5, multiplication by β_l defines an isomorphism of $A(K_l)(L)[\Gamma]$-modules $\mathbb{D}_l(L) \simeq \beta_l \mathbb{D}_l(L)$. So this lemma follows immediately from the preceding one.

Now suppose $l - m = 1$. Then αg goes to 0 in $H^{i+1}(\beta_l \mathbb{D}_l(L))(h)_\phi = 0$ and hence lifts to $H^i(\mathbb{D}_l(L))^h_\phi$.

Finally we do the induction step. If $l - m \geq 2$, use the induction on the long exact sequences stemming from the exact sequences

$$0 \to \beta_{m+1} \mathbb{D}_{m+1} \to \mathbb{D}_{m+1} \to \mathbb{D}_m \to 0$$

and

$$0 \to (\beta_{m+1}, \ldots, \beta_l)\mathbb{D}_l \to \mathbb{D}_l \to \mathbb{D}_{m+1} \to 0$$

to lift α first to $H^i(\mathbb{D}_{m+1}(L))^h_\phi$ and then to $H^i(\mathbb{D}_l(L))^h_\phi$.

References

[AS]. A. Ash and G. Stevens, *Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues*, J. f. d. Reine u. Angew. Math. **365** (1986), 192–220.

[B]. K. Brown, *Cohomology of Groups*, Springer, 1982.

[C]. R. Coleman, *P-adic Banach spaces and families of modular forms*, preprint.

[GS]. R. Greenberg and G. Stevens, *p-adic L-functions and p-adic periods of modular forms*, Inv. Math. **111** (1993), 407–447.

[Hi]. H. Hida, *Control theorems of p-nearly ordinary cohomology groups for $SL(n)$*, to appear in Bull. Soc. Math. France.

[L]. S. Lang, *Algebra*, second ed., Addison-Wesley, 1984.

[S1]. G. Stevens, *Rigid analytic modular symbols*, preprint.

[S2]. G. Stevens, *Cohomology of some p-adic Banach spaces*, preliminary version.

[TU]. J. Tilouine and E. Urban, *Families p-adique a trois variables de formes de Siegel et de representations galoisiennes*, C.R. Acad. Sci. Paris **321** (1995), 5–10.