Deforming the Gauss-Manin Connection

Glenn Stevens

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§0. The Gauss-Manin Connection.

Let $p > 2$ be a prime and $N \geq 4$. The p-adic modular curve $X_1(Np)$ will be denoted X. We let $X = W_0 \cup W_\infty$ denote the usual decomposition of X as the union of two wide open sets. Hence W_i ($i = 0$ or ∞) is a wide open neighborhood of the ordinary component Z_i containing the cusp i ($= 0$ or ∞) and the intersection $W = W_\infty \cap W_0$ is the union the supersingular annuli. We have $Z_\infty := W_\infty \setminus W$ and $Z_0 := W_0 \setminus W$. Let $X(v)$ be the largest connected wide open neighborhood of Z_∞ on which the canonical subgroup is defined. Let $\pi : E \longrightarrow X$ denote the universal generalized elliptic curve over X and let $\mathcal{H} := H^1_{DR}(E/X)$ denote the De Rham cohomology sheaf on X. Let $\nabla : \mathcal{H} \longrightarrow \mathcal{H}$ denote the Gauss-Manin connection on $\mathcal{H}$. One knows from Katz that $\mathcal{H}$ has a natural decomposition as

$$\mathcal{H} = \underline{\omega}^{-1} \oplus \underline{\omega}$$

where $\underline{\omega} := \pi_* \Omega^1_{E/X}\big(\log(\text{cusps})\big)$. Now choose two sections X, Y of $\mathcal{H}(X(v))$ as follows. Choose $X \in \underline{\omega}(X(v)) \subseteq \mathcal{H}(X(v))$ corresponding to the weight one Eisenstein series E_1, and let Y be the generator of $\underline{\omega}^{-1}(X(v))$ for which the cup product of X and Y is 1. It follows then that the q-expansions of X and Y at the ∞ cusp are

$$X = E_1(q)\omega_{can},$$
$$Y = E_1(q)^{-1}\eta_{can}.$$

There is a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of regular 1-forms on $X(v)$ such that

$$\nabla \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Since $\langle X, Y \rangle = 1$ we have $\langle \nabla X, Y \rangle + \langle X, \nabla Y \rangle = 0$. This is equivalent to the assertion $\alpha + \delta = 0$. Hence the above matrix has trace zero. In particular, we may write ∇ in terms of the Lie algebra $sl(2)$ as

$$\nabla = (\alpha h + \beta x + \gamma y)$$

where $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

§1. Representations of $sl(2)$ and the Gauss-Manin connection.

Our basic observation is that the above description of ∇ gives us a natural way of making vector bundles with connection on X out of representations of $sl(2)$, as follows.

Consider a category $\mathcal{R}$ of representations of $sl(2)$. I won't give a precise definition of what this category ought ot be right now, but I'll need to give one at some point, since

I want to allow representations to be infinite dimensional, and to have coefficients which are functions on the space $\mathcal{X}$ defined below. The simplest thing to do would be to require the objects to be Banach modules, but I think this may be too restrictive.

Also consider a category $\mathcal{C}$ of quasi-coherent $\mathcal{O}$-modules with connection on $Z_\infty^\dagger$. Again, I should be more precise, but I'm not ready to commit to a particular definition of what an object in this category should be. For now, let me just say that I want to include $\mathcal{O}^\dagger$-modules of infinite rank. For example, we could fix a radius v of overconvergence and then consider orthonormalizable $\mathcal{O}(X(v))$-Banach modules. This is a good model to start with, but again, I think it is too restrictive.

In the last section we saw how the choice of a basis for $\mathcal{H}$ over $Z_\infty^\dagger$ gives rise to an element

$$\nabla \in sl(2) \otimes \mathcal{O}^\dagger$$

Now we make the simple observation that ∇ gives rise to a functor from $\mathcal{R}$ to $\mathcal{C}$ in the obvious way. Namely, if V is an object of $\mathcal{R}$ then

$$\begin{aligned} \nabla : V\hat{\otimes}\mathcal{O}^\dagger &\longrightarrow V\hat{\otimes}\Omega^\dagger \\ v \otimes 1 &\longmapsto h(v) \otimes \alpha + x(v) \otimes \beta + y(v) \otimes \gamma \end{aligned}$$

defines an object of $\mathcal{C}$.

I have to add another word of caution. In practice, I will often want to take for V the dual of a Banach space. In that case, I will want the above $\hat{\otimes}$ product to be a completed tensor with respect to the *weak* topology on V, not the strong (Banach space) topology. The moral of the story is that it will be pretty clear what we want to do in examples, but I'm having some trouble saying it at the right level of generality.

§2. Verma modules.

Let $\mathcal{X} := \mathrm{Hom}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$. We view $\mathbf{Z}$ as embedded in $\mathcal{X}$ by $n \mapsto (t \mapsto t^n)$. The building blocks of our Verma modules will be the formal symbols

$$X^{[\kappa_1]}Y^{\kappa_2}, \quad (\kappa_1, \kappa_2) \in \mathcal{X}^2.$$

The notation is intended to suggest that we are taking something like the κ_1-divided power of X and the usual κ_2-power of Y. More precisely, if k_1 (resp. k_2) is the weight of κ_1 (resp. κ_2), then we want $X^{[\kappa_1]}$ (resp. Y^{κ_2}) to behave like the k_1-divided power of X (resp. the k_2-power of Y) with respect to differential operators. Here k_1, k_2 are the weights of κ_1, κ_2 respectively. I prefer to keep the κ_i's in the notation because it will be more natural later on when we exponentiate to obtain an action of the group

$$G_n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z}_p) \;\middle|\; \begin{matrix} c \equiv 0 \bmod p \\ b \equiv 0 \bmod p^n \end{matrix} \right\}$$

for some sufficiently large n.

For $\kappa \in \mathcal{X}$ define $\mathbf{V}_\kappa$, $\mathbf{V}_\kappa^-$, and $\mathbf{D}_\kappa$ by

$$\mathbf{V}_\kappa := \left\{ \sum_{n=0}^{\infty} \lambda_n X^{[\kappa-n]} Y^n \;\middle|\; \lambda_n \in \mathbf{C}_p, \quad \lim_{n\to\infty} |n!\lambda_n| = 0 \right\},$$

$$\mathbf{V}_\kappa^- := \left\{ \sum_{n=1}^{\infty} \lambda_n X^{[\kappa+n]} Y^{[-n]} \;\middle|\; \lambda_n \in \mathbf{C}_p, \quad |(n-1)!\lambda_n| = O(1) \right\},$$

$$\mathbf{D}_\kappa := \left\{ \sum_{n=0}^{\infty} \lambda_n X^n Y^{\kappa-n} \;\middle|\; \lambda_n \in \mathbf{C}_p, \quad |n!\lambda_n| = O(1) \right\},$$

where we let $sl(2)$ act by

$$x := Y \frac{\partial}{\partial X}$$
$$y := X \frac{\partial}{\partial Y}$$
$$h := X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y}.$$

More generally, if $S \subseteq \mathcal{X}$ is a rigid subspace of $\mathcal{X}$ we may define $\mathbf{V}_S$, $\mathbf{V}_S^-$, and $\mathbf{D}_S$ similarly, by allowing the coefficients λ_n to be rigid functions on S satisfying the stated conditions pointwise on S. (Here again, I'm not sure what the "right" condition should be).

If $\kappa \in \mathcal{X}$ is an arithmetic character of weight $k \geq 0$ then we may also define

$$V_\kappa := \mathbf{C}_p[X, Y]_k,$$

the finite dimensional space of homogeneous polynomials of degree k, with the action of $sl(2)$ induced by the above formulas for x, y, h.

Proposition. *If $\kappa \in \mathcal{X}$ is an arithmetic character of weight $k \geq 0$, then there is a natural exact sequence*

$$0 \longrightarrow V_\kappa \longrightarrow \mathbf{D}_\kappa \longrightarrow \mathbf{V}_\kappa^- \longrightarrow 0,$$

where the map $V_\kappa \longrightarrow \mathbf{D}_\kappa$ is defined in the obvious way, and the map $\mathbf{D}_\kappa \longrightarrow \mathbf{V}_\kappa^-$ is defined termwise by

$$X^n Y^{\kappa-n} \longmapsto \begin{cases} \frac{(-1)^{n-k-1} n!}{(n-k-1)!} \cdot X^{[\kappa+n-k]} Y^{[k-n]} & \text{if } n > k; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is a straightforward calculation (assuming I've stated this correctly).

Remark. It is important to notice that the map $\mathbf{D}_\kappa \longrightarrow \mathbf{V}_\kappa^-$ is defined only at arithmetic points κ and is not interpolizable on any affinoid subdomain $S \subseteq \mathcal{X}$ by a map $\mathbf{D}_S \longrightarrow \mathbf{V}_S^-$.

§3. Quasi-coherent $\mathcal{O}^\dagger$-modules and the Gauss-Manin connection.

Now we apply the functor of §1 to the $sl(2)$-modules described in §2 to obtain $\mathcal{O}^\dagger$-modules $\mathcal{V}_\kappa$, $\mathcal{V}_\kappa^-$, $\mathcal{D}_\kappa$ on $Z_\infty^\dagger$ corresponding to the $sl(2)$-modules $\mathbf{V}_\kappa$, $\mathcal{V}_\kappa^-$, $\mathcal{D}_\kappa$ respectively. Here again, I haven't written down a precise definition of these things. Presumeably, κ should encode both the weight and the nebentype. To keep things simple for now I will take κ in the identity component of $\mathcal{X}$ and close enough to the trivial character so that the nebentype is trivial. Eventually, we will have to change point of view and replace X^κ (which corresponds to the family E_1^κ, whatever that means) by the family E_κ, $\kappa \in \mathcal{X}$.

If κ is arithmetic, we also let $\mathcal{H}_\kappa$ be the coherent $\mathcal{O}^\dagger$-module associated to V_κ. This is the sheaf you have been considering in your papers.

We are interested in the long exact cohomology sequence associated to the short exact sequence in the proposition of the last section.

Theorem. *Let κ be an arithmetic character in $\mathcal{X}$ and let $\mathcal{M}_\kappa^\dagger$ denote the space of overconvergent modular forms of weight κ. Then we have natural isomorphisms*

$$\nu : \mathcal{M}_{-\kappa}^\dagger \xrightarrow{\sim} H^0(Z_\infty^\dagger, \mathcal{D}_\kappa)$$
$$\mu : \mathcal{M}_{2+\kappa}^\dagger \xrightarrow{\sim} H^0(Z_\infty^\dagger, \mathcal{V}_\kappa^-)$$

Moreover, the cohomology sequence attached to the short exact sequence of the last proposition corresponds to an exact sequence

$$0 \longrightarrow H^0(Z_\infty^\dagger, \mathcal{H}_\kappa) \longrightarrow \mathcal{M}_{-\kappa}^\dagger \xrightarrow{\theta^{k+1}} \mathcal{M}_{2+\kappa}^\dagger \xrightarrow{\omega} H^1(Z_\infty^\dagger, \mathcal{H}_\kappa) \longrightarrow \cdots$$

where θ is the usual θ-operator and ω is the composition of Kodaira-Spencer and the standard map $(\mathcal{H}_\kappa \otimes \Omega^1)(Z_\infty^\dagger) \longrightarrow H^1(Z_\infty^\dagger, \mathcal{H}_\kappa)$

Proof Sketch: The existence of the isomorphism ν is a consequence of transversality of ∇, exactly as in your papers. (The module $\mathcal{V}_\kappa$ is filtered (not as $sl(2)$-module, just as $\mathcal{O}$-module) and ∇ defines a linear isoomorphism from Fil^r/Fil^{r+1} to Fil^{r-1}/Fil^r for every $r > -k$.) A similar argument proves the existence of μ. The exact sequence is an easy consequence of the definitions of ν and μ.

§4. Frobenius Structure.

Let φ be the canonical lifting of Frobenius to $\mathcal{H}$ and let $F : \varphi^*\mathcal{H} \longrightarrow \mathcal{H}$ be the corresponding horizontal isomorphism. Then $(\mathcal{H}, \nabla, \varphi, F)$ is an overconvergent F-crystal on Z_∞. Then one can also define a Frobenius structure on $(\mathcal{D}_\kappa, \nabla)$ for any κ (not necessarily arithmetic. We just need to define a horizontal isomorphism

$$F : \varphi^*\mathcal{D}_\kappa \longrightarrow \mathcal{D}_\kappa.$$

For this, we let e be the overconvergent function defined by its q-expansion

$$e(q) = E_1(q^p)/E_1(q).$$

Then, according to Katz (page Ka-109, SLN 350), we have

$$F(\varphi^* X) = peX$$
$$F(\varphi^* Y) = GeX + e^{-1} Y$$

where G is the overconvergent weight two modular form whose q-expansion is given by $G(q) = (p\varphi(P) - P)/12$ (so here we may need to assume $p > 3$). So, for arbitrary κ we define

$$\begin{aligned} F(\varphi^*(X^r Y^{\kappa - r})) &= p^r e^r X^r (GeX + e^{-1} Y)^{\kappa - r} \\ &= p^r e^r X^r \sum_{m=0}^{\infty} e^{m - 2\kappa} \left(\sum_{n=0}^{m} p^n \lambda_n \binom{k-n}{m-n} G^{m-n} \right) X^m Y^{\kappa - m} \end{aligned}$$

Moreover, if S is a rigid subspace of $\mathcal{X}$ contained in some sufficiently small neighborhood of the origin, then this frobenius is analytic in $\kappa \in S$. Thus we also obtain a Frobenius on the family $\mathcal{D}_S$ of Verma modules:

$$F : \varphi^* \mathcal{D}_S \longrightarrow \mathcal{D}_S.$$

If κ is arithmetic, we can also define a Frobenius structure on $\mathcal{V}_\kappa$ and on $\mathcal{V}_\kappa^-$. However, it should be emphasized that it is not possible to define a Frobenius structure on $\mathcal{V}_\kappa$ nor on $\mathcal{V}_\kappa^-$ unless $k \in \mathbf{Z}$. In particular we do not obtain a family of Frobenii on the $\mathcal{V}_\kappa$. However, there does exist a "Verschiebung"-structure on these families.

Proposition. *For any arithmetic point $\kappa \in \mathcal{X}$ the exact sequence of the last proposition commutes with Frobenius.*

§1. Some BIG Verma modules.

Let's start by defining the biggest Verma modules one can imagine. Let $\mathcal{X} := \mathrm{Hom}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$. We view $\mathbf{Z}$ as embedded in $\mathcal{X}$ by $n \mapsto (t \mapsto t^n)$. The building blocks of our Verma modules will be the formal symbols

$$X^{[\kappa_1]}Y^{\kappa_2}, \quad (\kappa_1, \kappa_2) \in \mathcal{X}^2.$$

The notation is intended to suggest that we are taking something like the κ_1-divided power of X and the usual κ_2-power of Y. More precisely, if k_1 (resp. k_2) is the weight of κ_1 (resp. κ_2), then we want $X^{[\kappa_1]}$ (resp. Y^{κ_2}) to behave like the k_1-divided power of X (resp. the k_2-power of Y) with respect to differential operators. Here k_1, k_2 are the weights of κ_1, κ_2 respectively. I prefer to keep the κ_i's in the notation because it will be more natural later on when we exponentiate to obtain an action of the group

$$G_n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z}_p) \;\middle|\; \begin{matrix} c \equiv 0 \bmod p \\ b \equiv 0 \bmod p^n \end{matrix} \right\}$$

for some sufficiently large n.

For $(\kappa_1, \kappa_2) \in \mathcal{X} \times \mathcal{X}$ define

$$\mathbf{V}_{\kappa_1,\kappa_2} := \left\{ \sum_{n \in \mathbf{Z}} \lambda_n X^{[\kappa_1 - n]} Y^{\kappa_2 + n} \;\middle|\; \lambda_n \in \mathbf{C}_p,\ |\lambda_{-n}/n!| = O(1),\ |\lambda_n| = o(1) \text{ as } n \to \infty \right\}.$$

We endow $\mathbf{V}_{\kappa_1,\kappa_2}$ with an action of the Lie algebra $sl(2)$ (or even $gl(2)$) defined term by term by the formulas

$$\begin{aligned} x(X^{[\kappa_1]}Y^{\kappa_2}) &= (k_1+1)k_2 X^{[\kappa_1+1]}Y^{\kappa_2-1}; \\ y(X^{[\kappa_1]}Y^{\kappa_2}) &= X^{[\kappa_1-1]}Y^{\kappa_2+1}; \\ h(X^{[\kappa_1]}Y^{\kappa_2}) &= (k_1 - k_2) X^{[\kappa_1]}Y^{\kappa_2} \end{aligned}$$

for any $(\kappa_1, \kappa_2) \in \mathcal{X} \times \mathcal{X}$. (For $z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in gl(2)$ we define $z(X^{[\kappa_1]}Y^{\kappa_2}) = (k_1 + k_2)X^{[\kappa_1]}Y^{\kappa_2}$.)

More generally, if $S \subseteq \mathcal{X} \times \mathcal{X}$ is an affinoid subspace (note: S need not be a sub*domain*. It could be a point, or a disk $\times$ a point, or a disk $\times$ a disk), then we define

$$\mathbf{V}_S := \left\{ \sum_{n \in \mathbf{Z}} \lambda_n(\kappa_1, \kappa_2) X^{[\kappa_1 - n]} Y^{\kappa_2 + n} \;\middle|\; \lambda_n \in A(S), \quad \|\lambda_n\|_S = O(1), \quad \lim_{n \to \infty} \|\lambda_n\|_S = 0 \right\}.$$

If $S \subseteq \mathcal{X} \times \mathcal{X}$ is an arbitrary rigid subspace, we define $\mathbf{V}_S$ in the obvious way (as an inverse limit over affinoid subdomains of S). In any case, the above formulas define an action of $sl(2)$ (or $gl(2)$) on $\mathbf{V}_S$.

Examples: Here are three important examples. Fix $n \geq 0$ and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n$.

1. Let $S = \mathcal{X} \times \{0\}$. For $\kappa \in \mathcal{X}$ let

$$(aX + cY)^{[\kappa]} := \sum_{n=0}^{\infty} \frac{a^{\kappa-n}c^n}{n!} X^{[\kappa-n]}Y^n \in \mathbf{V}_{(\kappa,0)}.$$

Letting κ range over $\mathcal{X}$ we may view $(aX + cY)^{[\kappa]}$ as an element of $\mathbf{V}_{\mathcal{X}\times\{0\}}$.

2. Let $S = \{0\} \times \mathcal{X}[-n]$. (Remark: $\mathcal{X}[-n] := \{\kappa \in \mathcal{X} \,|\, \mathrm{ord}_p(k) \geq -n\}$.) For $\kappa \in \mathcal{X}[-n]$ let

$$(bX + dY)^{\kappa} := \sum_{m=0}^{\infty} k(k-1)\cdots(k-m+1)b^m d^{\kappa-m} X^{[m]}Y^{\kappa-m} \in \mathbf{V}_{(0,\kappa)}.$$

Letting κ range over $\mathcal{X}$ we may view $(bX + dY)^{\kappa}$ as an element of $\mathbf{V}_{\{0\}\times\mathcal{X}[-n]}$.

3. Multiplying the above two expressions together formally, we get

$$(aX + cY)^{[\kappa_1]}(bX + dY)^{\kappa_2} = \sum_{N=-\infty}^{\infty} \mu_N(\kappa_1, \kappa_2) X^{[\kappa_1-N]}Y^{\kappa_2+N} \in \mathbf{V}_{\kappa_1,\kappa_2}$$

where

$$\mu_N(\kappa_1, \kappa_2) := \sum_{m=0}^{\infty} \binom{k_2}{m} \binom{k_1 - N}{m} \frac{m!}{(N+m)!} a^{\kappa_1-N-m} d^{\kappa_2-m} c^{N+m} b^m,$$

which clearly converges to a rigid function on $\mathcal{X} \times \mathcal{X}$. Hence, letting (κ_1, κ_2) range over $\mathcal{X} \times \mathcal{X}$, we regard $(aX + cY)^{[\kappa_1]}(bX + dY)^{\kappa_2}$ as an element of $\mathbf{V}_{\mathcal{X}\times\mathcal{X}}$.

For arbitrary $\nu = \sum_{n\in\mathbf{Z}} \nu_n(\kappa_1, \kappa_2) X^{[\kappa_1-n]}Y^{\kappa_2+n} \in \mathbf{V}_{\mathcal{X}\times\mathcal{X}}$ and $\gamma \in G_0$ as above we define

$$\gamma(\nu) := \sum_{n\in\mathbf{Z}} \nu_n(\kappa_1, \kappa_2)(aX + cY)^{[\kappa_1-n]}(bX + dY)^{\kappa_2+n} \in \mathbf{V}_{\mathcal{X}\times\mathcal{X}}.$$

This defines an action of G_0 on $\mathbf{V}_{\mathcal{X}\times\mathcal{X}}$. Moreover, if $S \subseteq \mathcal{X} \times \mathcal{X}$ is a rigid subspace, then these formulas also define an action of G_0 on the space $\mathbf{V}_S$.

Finally, we observe that the above action of G_n can be differentiated to obtain an action of the Lie algebra $sl(2)$ (or even $gl(2)$) on $\mathbf{V}_S$ for any rigid space $S \subset \mathcal{X} \times \mathcal{X}$. In particular we have

$$\begin{aligned} x(X^{[\kappa_1]}Y^{\kappa_2}) &= (k_1 + 1)k_2 X^{[\kappa_1+1]}Y^{\kappa_2-1}; \\ y(X^{[\kappa_1]}Y^{\kappa_2}) &= X^{[\kappa_1-1]}Y^{\kappa_2+1}; \\ h(X^{[\kappa_1]}Y^{\kappa_2}) &= (k_1 - k_2)X^{[\kappa_1]}Y^{\kappa_2}. \end{aligned}$$

§1. Analytic families

Let S be an arbitrary subset of $\mathbf{Z}_p \times \mathbf{Z}_p$ satisfying the relation

$$(a, b) \in S \Longleftrightarrow (a + 1, b - 1) \in S.$$

Let

$$\mathbf{V}_S^{nv} := \left\{ f : W_\infty \times S \longrightarrow \mathbf{C}_p \;\middle|\; f(z, a, b) \text{ is rigid analytic in the first variable} \right\}.$$

We define an action of the Lie algebra $sl(2)$ on $\mathbf{V}_S^{nv}$ by defining

$$\begin{aligned}(xf)(z, a, b) &:= a \cdot f(z, a - 1, b + 1)\\ (yf)(z, a, b) &:= b \cdot f(z, a + 1, b - 1)\\ (hf)(z, a, b) &:= (a - b) \cdot f(z, a, b)\end{aligned}$$

A straightforward calculation shows that this does indeed define an action of $sl(2)$. Now define

$$\begin{aligned}\nabla : \mathbf{V}_S^{nv} &\longrightarrow \mathbf{V}_S^{nv} \otimes \Omega^1(W_\infty)\\ f &\longmapsto (\alpha hf + \beta xf + \gamma yf) \otimes dz\end{aligned}$$

Now suppose S is also an analytic subspace of $\mathbf{Z}_p^2$. Then we define the subspace $\mathbf{V}_S \subseteq \mathbf{V}_S^{nv}$ by

$$\mathbf{V}_S := \left\{ f : W_\infty \times S \longrightarrow \mathbf{C}_p \;\middle|\; f(z, a, b) \text{ is rigid analytic on } W_\infty \times S \right\}.$$

It is clear from the above formulas that $\mathbf{V}_S$ is preserved by the action of $sl(2)$ and also by the action of ∇. Hence we obtain a connection

$$\nabla : \mathbf{V}_S \longrightarrow \mathbf{V}_S \otimes \Omega^1(W_\infty).$$

Proposition. *Suppose k is a nonnegative integer and that $(k, 0) \in S$. Then the map*

$$\begin{aligned}\sigma_k : \mathbf{V}_S &\longrightarrow Symm^k(\mathcal{H})\\ f &\longmapsto \sum_{r=0}^{k} f(z, k - r, r) \cdot \frac{X^{k-r}}{(k - r)!} \cdot \frac{Y^r}{r!}\end{aligned}$$

commutes with ∇, i.e. is horizontal.

Proof: The proof is a straightforward calculation. For the sake of completeness, we include the calculation here. First of all, we have

$$\begin{aligned}
\sigma_k(\nabla f) = &\sum_{r=0}^{k} (\nabla f)(z, k-r, r) X^{[k-r]} Y^{[r]} \\
=&\alpha(z) \sum_{r=0}^{k} (k-2r) f(z, k-r, r) X^{[k-r]} Y^{[r]} \otimes dz + \\
&\beta(z) \sum_{r=0}^{k} (k-r) f(z, k-r-1, r+1) X^{[k-r]} Y^{[r]} \otimes dz + \\
&\gamma(z) \sum_{r=0}^{k} r f(z, k-r+1, r-1) X^{[k-r]} Y^{[r]} \otimes dz.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\nabla(\sigma_k(f)) = &\sum_{r=0}^{k} f(z, k-r, r) \cdot \nabla(X^{[k-r]} Y^{[r]}) \\
=&f(z,k,0) \nabla(X^{[k]}) + \left(\sum_{r=1}^{k-1} f(z, k-r, r) \nabla(X^{[k-r]} Y^{[r]}) \right) + f(z, 0, k) \nabla(Y^{[k]}) \\
=&f(z,k,0) \cdot (k\alpha(z) X^{[k]} + \gamma(z) X^{[k-1]} Y^{[1]}) \otimes dz + \\
&\alpha(z) \sum_{r=1}^{k-1} (k-2r) f(z, k-r, r) \cdot X^{[k-r]} Y^{[r]} \otimes dz + \\
&\beta(z) \sum_{r=1}^{k-1} (k-r+1) f(z, k-r, r) \cdot X^{[k-r+1]} Y^{[r-1]} \otimes dz + \\
&\gamma(z) \sum_{r=1}^{k-1} (r+1) f(z, k-r, r) \cdot X^{[k-r-1]} Y^{[r+1]} \otimes dz + \\
&f(z, 0, k) \big(-k\alpha(z) Y^{[k]} + \beta(z) X^{[1]} Y^{[k-1]} \big) \otimes dz \\
=&\alpha(z) \sum_{r=0}^{k} (k-2r) f(z, k-r, r) \cdot X^{[k-r]} Y^{[r]} \otimes dz + \\
&\beta(z) \sum_{r=0}^{k} (k-r) f(z, k-r-1, r+1) \cdot X^{[k-r]} Y^{[r]} \otimes dz + \\
&\gamma(z) \sum_{r=0}^{k} r f(z, k-r+1, r-1) \cdot X^{[k-r]} Y^{[r]} \otimes dz.
\end{aligned}$$

Comparing this with the above expression for $\sigma_k(\nabla f)$ we see that $\sigma_k(\nabla f) = \nabla(\sigma_k(f))$ and the proposition is proved.

Remark. In light of this proposition, we view $\nabla : \mathbf{V}_S \longrightarrow \mathbf{V}_S \otimes \Omega^1(W_\infty)$ as an analytic family of differential operators with the property that ∇ specializes to the classical Gauss-Manin connection at integral weights $k \geq 0$.