Distributions and Log-Differentials on Wide Open Subspaces.

§1. Distributions.

Let V be a two dimensional $\mathbf{Q}_p$-vector space and let $\mathbf{P} := \mathbf{P}_V$ be the projective line associated to V. Now choose, once and for all, a coordinate system on V, i.e. identify V with $\mathbf{Q}_p^2$, whose elements we will write as row vectors (x, y). For any p-adic field K we use these coordinates to make the following identifications:

$$\mathbf{P}(K) = \mathbf{P}^1(K) = K \cup \{\infty\} \quad \left(\text{via } [x,y] \mapsto z := \frac{y}{x}\right);$$
$$\mathrm{Aut}_{\mathbf{Q}_p}(V) = GL_2(\mathbf{Q}_p);$$
$$\mathrm{Aut}_{\mathbf{Q}_p}(\mathbf{P}) = PGL_2(\mathbf{Q}_p).$$

The actions of the groups $GL_2(\mathbf{Q}_p)$ and $PGL_2(\mathbf{Q}_p)$ are given by matrix multiplication on the right. The coordinate system also also endows V and $\mathbf{P}$ with $\mathbf{Z}_p$-structures. In particular, we may define the groups $GL_2(\mathbf{Z}_p)$, $PGL_2(\mathbf{Z}_p)$ of automomorphisms that preserve the integral structure. We also have a reduction map $\mathbf{P}(\mathbf{Q}_p) \longrightarrow \mathbf{P}(\mathbf{F}_p) = \mathbf{F}_p \cup \{\infty\}$. The group $PGL_2(\mathbf{Z}_p)$ acts naturally on $\mathbf{P}(\mathbf{F}_p)$. The standard Iwahori subgroup is the subgroup $I \subseteq PGL_2(\mathbf{Z}_p)$ of elements that stabilize the point $\infty \in \mathbf{P}(\mathbf{F}_p)$.

For each point $s \in \mathbf{P}(\mathbf{Q}_p)$ we fix the following choice of a uniformizer at s:

$$w_s(z) := \begin{cases} z - s & \text{if } s \in \mathbf{Z}_p; \\ \frac{1}{z} - \frac{1}{s} & \text{if } s \notin \mathbf{Z}_p. \end{cases}$$

For $r \in |\mathbf{C}_p^\times|$ and $s \in \mathbf{P}^1(\mathbf{C}_p)$ we define the affinoid disk of radius r about s to be

$$B[s,r] := \left\{ z \in \mathbf{P}^1(\mathbf{C}_p) \;\middle|\; |w_s(z)| \leq r \right\}.$$

Note that in fact, $B[s,r]$ is a closed disk centered at s in the usual sense, but if $s \notin \mathbf{Z}_p$ then r is not what one usually calls the radius of this disk.

More generally, if $S \subseteq \mathbf{P}^1(\mathbf{Q}_p)$ is a non-empty compact subset, then we define

$$B[S,r] := \bigcup_{s \in S} B[s,r].$$

Because of our assumption that S is compact, the above union is actually a finite union of disjoint closed disks. It is possible that $B[S,r] = \mathbf{P}(\mathbf{C}_p)$, but if $B[S,r] \neq \mathbf{P}(\mathbf{C}_p)$, then $B[S,r]$ is the set of $\mathbf{C}_p$-points of a $\mathbf{Q}_p$-affinoid variety. In this case, we say r is an admissible radius for S. One checks easily that if $r < 1$ then r is admissible for every S.

Let S be a non-empty compact subset of $\mathbf{P}(\mathbf{Q}_p)$. Then for each $r \in |\mathbf{C}_p^\times|$ with r admissible for S we let $A[S,r]$ denote the $\mathbf{Q}_p$-Banach algebra of $\mathbf{Q}_p$-affinoid functions on

$B[S,r]$. For admissible r_1, r_2 with $r_1 > r_2$, the restriction map $A[S,r_1] \longrightarrow A[S,r_2]$ is completely continuous, injective, and has dense image. We define

$$\mathcal{A}(S) := \varinjlim_{r>0} A[S,r] \quad \text{and} \quad \mathcal{A}^\dagger(S,r_0) := \varinjlim_{r>r_0} A[S,r]$$

where r_0 is any admissible radius for S. (It should be pointed out that the set of all admissible radii for S does not have a largest element, hence the direct limit defining $\mathcal{A}^\dagger(S,r_0)$ is not vaccuous.) We endow each of these spaces with the inductive limit topology. Then $\mathcal{A}(S)$ is naturally identified with the space of locally analytic $\mathbf{Q}_p$-valued functions on S, while $\mathcal{A}^\dagger(S,r_0)$ is identified with the space of $\mathbf{Q}_p$-overconvergent functions on $B[S,r_0]$. Note that there are natural continuous inclusions

$$\mathcal{A}^\dagger(S,r_0) \hookrightarrow A[S,r_0] \hookrightarrow \mathcal{A}(S).$$

Moreover, the image of each of these maps is dense in its target space.

The distribution modules are defined as the dual spaces to the topological rings defined above. As before, we let $S \subseteq \mathbf{P}(\mathbf{C}_p)$ be a non-empty compact subset and let $r_0 \in |\mathbf{C}_p^\times|$ be admissible for S.

Definition. Define $\mathbf{D}[S,r_0]$ to be the space of continuous $\mathbf{Q}_p$-linear functionals on $A[S,r_0]$. We also define

$$\mathcal{D}(S) := \varprojlim_{r>0} \mathbf{D}[S,r] \quad \text{and} \quad \mathcal{D}^\dagger(S,r_0) := \varprojlim_{r>r_0} \mathbf{D}[S,r]$$

for an admissible radius r_0 for S. Each of these is endowed with the projective limit topology.

Equivalently, the spaces $\mathcal{D}(S)$, $\mathbf{D}[S,r_0]$, $\mathcal{D}^\dagger(S,r_0)$ are the spaces of continuous linear functionals on the topological vector spaces $\mathcal{A}(S)$, $A[S,r_0]$, $\mathcal{A}^\dagger(S,r_0)$ respectively, and the topology on each is the strong topology. By duality we have continuous linear injective maps

$$\mathcal{D}(S) \hookrightarrow \mathbf{D}[S,r_0] \hookrightarrow \mathcal{D}^\dagger(S,r_0).$$

If $G \subseteq PGL_2(\mathbf{Z}_p)$ is a subgroup that preserves the compact set $S \subseteq \mathbf{P}(\mathbf{Q}_p)$ then G also preserves $B[S,r]$ for all $r < 1$. On the other hand, if $r_0 > 0$ is any admissible radius for S such that G preserves $B[S,r_0]$ then G also acts naturally (and continuously) on of the spaces $\mathcal{A}(S)$, $A[S,r_0]$, and $\mathcal{A}^\dagger(S,r_0)$ and by duality on $\mathcal{D}(S)$, $\mathbf{D}[S,r_0]$, and $\mathcal{D}^\dagger(S,r_0)$. All of the maps defined above commute with this action. More precisely, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{Z}_p)$ be an element that preserves S. Then for $f \in \mathcal{A}(S)$ the function $\gamma f \in \mathcal{A}(S)$ is given by

$$(\gamma f)(z) := f(z\gamma) = f\left(\frac{b+dz}{a+cz}\right),$$

and for $\mu \in \mathcal{D}^\dagger(S)$, the distribution $\mu|\gamma \in \mathcal{D}^\dagger(S)$ is given by the integration formula

$$\int_S f \cdot d\mu|\gamma = \int_S \gamma f \cdot d\mu$$

for any $f \in \mathcal{A}^\dagger(S)$. These actions respect the filtrations $\mathcal{A}^\dagger(S) \subseteq A[S, r_0] \subseteq \mathcal{A}(S, r_0)$, and $\mathcal{D}(S) \subseteq \mathbf{D}[S, r_0] \subseteq \mathcal{D}^\dagger(S, r_0)$.

We will be interested in the special cases $S = \mathbf{P}(\mathbf{Q}_p)$, $\mathbf{Z}_p$, or $\mathbf{x}_\infty$ (the congruence class of $\infty \in \mathbf{P}(\mathbf{F}_p)$). We note that $\mathbf{P}(\mathbf{Q}_p)$ is the disjoint union of $\mathbf{Z}_p$ and $\mathbf{x}_\infty$, that $PGL_2(\mathbf{Z}_p)$ acts naturally (and continuously) on $\mathbf{P}(\mathbf{Q}_p)$, and that the Iwahori subgroup I acts naturally on $\mathbf{Z}_p$ and on $\mathbf{x}_\infty$. Moreover, I preserves $B[\mathbf{Z}_p, r_0]$ for any $r_0 < p$. To emphasize the special cases of interest to us, we make the following definitions.

$$\begin{array}{ll}
\mathcal{A} := \mathcal{A}(\mathbf{P}(\mathbf{Q}_p)) & \mathcal{D} := \mathcal{D}(\mathbf{P}(\mathbf{Q}_p)); \\
\mathcal{A}^\dagger := \mathcal{A}^\dagger(\mathbf{P}(\mathbf{Q}_p), 1/p) & \mathcal{D}^\dagger := \mathcal{D}^\dagger(\mathbf{P}(\mathbf{Q}_p), 1/p); \\
\mathcal{A}_0 := \mathcal{A}(\mathbf{Z}_p) & \mathcal{D}_0 := \mathcal{D}(\mathbf{Z}_p); \\
\mathcal{A}_0^\dagger := \mathcal{A}(\mathbf{Z}_p, 1) & \mathcal{D}_0^\dagger := \mathcal{D}^\dagger(\mathbf{Z}_p, 1).
\end{array}$$

We have canonical isomorphisms

$$\mathcal{A}^\dagger = \bigoplus_{\mathbf{x} \in \mathbf{P}(\mathbf{F}_p)} \mathcal{A}^\dagger(\mathbf{x}, 1/p) \quad \text{and} \quad \mathcal{D}^\dagger = \bigoplus_{\mathbf{x} \in \mathbf{P}(\mathbf{F}_p)} \mathcal{D}^\dagger(\mathbf{x}, 1/p).$$

Moreover, we have injective continuous maps

$$\mathcal{A}_0^\dagger \longrightarrow \bigoplus_{\substack{bx \in \mathbf{P}(\mathbf{F}_p) \\ \mathbf{x} \neq \mathbf{x}_\infty}} \mathcal{A}^\dagger(\mathbf{x}, 1/p) \hookrightarrow \mathcal{A}^\dagger$$

where the first is defined by restriction and the second is given by extension by zero, which is a strict inclusion. Extension by zero also gives us a continuous map $\mathcal{A}_0 \longrightarrow \mathcal{A}$, so by duality we obtain continuous maps

$$r : \mathcal{D}^\dagger \longrightarrow \mathcal{D}_0^\dagger \quad \text{and} \quad r : \mathcal{D} \longrightarrow \mathcal{D}_0$$

both of which will be called restriction to $\mathbf{Z}_p$. Summarizing, we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{D} & \hookrightarrow & \mathcal{D}^\dagger \\
r\downarrow & & r\downarrow \\
\mathcal{D}_0 & \hookrightarrow & \mathcal{D}_0^\dagger
\end{array}$$

in which the top horizontal arrow is a morphism of topological $PGL_2(\mathbf{Z}_p)$-modules, and all other arrows are morphisms of topological I-modules.

Finally, we remark that if $w_p \in GL_2(\mathbf{Q}_p)$ is any element of determinant p, whose entries are in $\mathbf{Z}_p$, and whose reduction modulo p is

$$w_p \equiv \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \pmod{p},$$

then $\mathbf{Z}_p \cdot w_p = \mathbf{x}_\infty$ and $\mathbf{x}_\infty \cdot w_p = \mathbf{Z}_p$. Such an element will be called an involution of I, since it normalizes I. For each $r \in |\mathbf{C}_p^\times|$ with $r < 1$, we have $B[\mathbf{Z}_p, r] \cdot w_p = B[\mathbf{x}_\infty, r/p]$. Thus pullback by w_p induces continuous linear functions

$$\mathcal{A}^\dagger \longrightarrow \mathcal{A}^\dagger(\mathbf{x}_\infty, 1/p) \xrightarrow{w_p} \mathcal{A}_0^\dagger$$

and, by duality, also a continuous map

$$\mathcal{D}_0^\dagger \xrightarrow{w_p} \mathcal{D}^\dagger(\mathbf{x}_\infty, 1/p) \longrightarrow \mathcal{D}.$$

§2. Differentials on wide open subspaces of $\mathbf{P}(\mathbf{C}_p)$.

We need a good method of representing overconvergent distributions. Whatever method we use should be well suited for calculating the action of $PGL_2(\mathbf{Z}_p)$ on it. Here's a method based on differential forms on wide open subspaces of $\mathbf{P}(\mathbf{C}_p)$ that I think might actually work fairly well. For an easy to read introduction to affinoids in $\mathbf{P}(\mathbf{C}_p)$, I recommend a quick reading of the first chapter or so of [1].

Let $S \subseteq \mathbf{P}(\mathbf{Q}_p)$ be a compact set and let $r_0 \in |\mathbf{C}_p^\times|$. For simplicity, I will assume r_0 is an integral power of p. We define

$$W(S, r_0) := \mathbf{P}^1(\mathbf{C}_p) \setminus B[S, r_0].$$

The space $W(S, r_0)$ is the standard example of a wide open subspace of $\mathbf{P}^1(\mathbf{C}_p)$.

Let $W = W(S, r_0)$. The ring of $\mathbf{Q}_p$-rigid analytic functions $A(W)$ on W is a topological $\mathbf{Q}_p$-algebra and the space $\Omega(W)$ of Kähler differentials on W is an $A(W)$-module. Recall that for each point $s \in \mathbf{P}(\mathbf{Q}_p)$ we have fixed the following choice of a uniformizer at s:

$$w_s(z) := \begin{cases} z - s & \text{if } s \in \mathbf{Z}_p; \\ \frac{1}{z} - \frac{1}{s} & \text{if } s \in \mathbf{x}_\infty. \end{cases}$$

Note that if $s \in S$ then $w_s^{-1} \in A(W)$ and consequently $w_s^{-2} dw_s \in \Omega(W)$.

Proposition. *Suppose $B[S, r]$ is the union $B[S, r] := \bigcup_{i \in I} B[s_i, r]$ of disojoint disks centered at s_i, $i \in I$. Then we have the following descriptions of $A(W)$, $\Omega(W)$:*

(1) *Every function $f \in A(W)$ has a unique representation in the form*

$$f = a + \sum_{i \in I} \sum_{n=1}^{\infty} a_n(i) w_{s_i}^{-n}.$$

(2) *Every $\omega \in \Omega(W)$ has a unique representation in the form*

$$\sum_{i \in I} \left(\sum_{n=0}^{\infty} a_n(i) w_{s_i}^{-n} \cdot \frac{dw_{s_i}}{w_{s_i}} \right)$$

where $\sum_{i\in I} a_0(i) = 0.$

(3) *Moreover, an expression of the form (1) (resp. (2)) represents an element of* $A(W)$ *(resp.* $\Omega(W)$*) if and only if for every real number* $t > r_0$ *the coefficients satisfy*

$$|a_n(i)| = o(t^n) \text{ as } n \to \infty.$$

Now fix a subgroup $G \subseteq PGL_2(\mathbf{Z}_p)$ that preserves W. Then G acts on $A(W)$ and on $\Omega(W)$. Explicitly, the action on $A(W)$ is described by the following formulas: if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $f \in A(W)$ then for all $z \in W$ we have

$$(\gamma f)(z) = f\left(\frac{b+dz}{a+cz}\right).$$

I prefer to write the action of G on $\Omega(W)$ as a *right* action. If $\omega \in \Omega(W)$ is a differential, and $\gamma \in G$, I will therefore write

$$\omega|\gamma := \gamma^{-1}(\omega).$$

To describe the action of G on expressions for f or ω of the form given in the last proposition, we only need to describe the action on the uniformizers and then plug these back into the expression for f. If $s \in S$ and $\gamma \in G$, then there is a unique power series $P_{\gamma,s}(T)$ such that

$$\gamma(w_{s\gamma}) = P_{\gamma,s}(w_s).$$

Indeed, $\gamma w_{s\gamma}$ is a rational function that is holomorphic on W that vanishes at s and therefore has an expansion of the desired type. It would be nice to have some good computer programs that compute this action efficiently to a fairly high order of approximation. I have never used Magma, but from what I have heard, it may be perfect for this type of calculation.

§3. Log-Differentials on Wide Open Subspaces.

In fact, for the application to distributions, we need a slightly larger space, $\Omega_{\log}(W)$, which contains $\Omega(W)$ as a subspace of codimension one. Let $\tilde{\mathbf{P}}$ denote affine space of V with the origin deleted. Thus, for an arbitrary field K, we have $\tilde{\mathbf{P}}(K) = V_K \setminus \{0\}$. We have a natural morphism $\pi : \tilde{\mathbf{P}} \longrightarrow \mathbf{P}$, whose fibers are copies of the multiplicative group. Let $\tilde{W}$ be the full preimage in $\tilde{\mathbf{P}}(\mathbf{C}_p)$ under π of W. Thus the fibers of the natural map

$$\pi : \tilde{W} \longrightarrow W$$

are copies of $\mathbf{C}_p^\times$. The space $\tilde{W}$ has a natural structure as $\mathbf{Q}_p$-rigid analytic space. The action of $\mathbf{Q}_p^\times$ on the fibers of π induces an action of $\mathbf{Q}_p^\times$ on $A(W)$ and on $\Omega(\tilde{W})$.

Define $\Omega_0(W) \subseteq \Omega(\tilde{W})$ to be the subspace on which $\mathbf{Q}_p^\times$ acts trivially. Notice that $\Omega_0(W)$ contains $\Omega(W)$ but this inclusion is far from an equality. Indeed, if L_1, L_2 are linearly independent $\mathbf{Q}_p$-linear forms on V, whose zeroes are in S, then every element $\omega \in \Omega_0(W)$ can be expressed uniquely in the form

$$\omega = f(z)\frac{dL_1}{L_1} + g(z)\frac{dL_2}{L_2}$$

where $f, g \in A(W)$. We have $\omega \in \Omega(W)$ if and only if $f + g = 0$. We define

$$\Omega_{\log}(W) := \left\{ \omega = f(z)\frac{dL_1}{L_1} + g(z)\frac{dL_2}{L_2} \in \Omega_0(W) \;\middle|\; f + g \text{ is a constant} \right\}.$$

Remark: It's not hard to verify directly that $\Omega_{\log}(W)$ is a G-invariant subspace of $\Omega_0(W)$, which contains $\Omega(W)$ as a subspace of codimension one. But to emphasize the connection with the logarithm, it's interesting to consider the multiplicative group $U(\tilde{W})$ of functions on $\tilde{W}$ defined by

$$U(\tilde{W}) := \left\{ u \in A(\tilde{W})^\times \;\middle|\; \forall (x,y) \in \tilde{\mathbf{P}}, \text{ the function } t \mapsto \frac{u(tx,ty)}{u(x,y)} \text{ is a character of } \mathbf{Q}_p^\times \right\}.$$

If $u \in U(\tilde{W})$ then $\log(u)$ is a well-defined locally analytic function on $\tilde{W}$, which satisfies the "homogeneity" relation

$$\log(u(tx,ty)) = k\log(t) + \log(u(x,y))$$

for some integer k, for all $(x,y) \in \tilde{W}$ and all $t \in \mathbf{C}_p^\times$. We also have an exact sequence

$$0 \longrightarrow \mathbf{Q}_p^\times \longrightarrow U(\tilde{W}) \xrightarrow{d\log} \Omega_{\log}(W)$$

where the image of d spans a dense $\mathbf{Q}_p$-subspace of $\Omega_{\log}(W)$.

For each $s \in S$ we define

$$\delta_s := \frac{dL}{L} \in \Omega_{\log}(W)$$

where L is any non-zero linear form that vanishes at s. For example

$$\delta_\infty = \frac{dX}{X} \quad \text{and} \quad \delta_0 = \frac{dY}{Y}.$$

The set of all δ_s where s ranges over S is invariant under the action of G. Indeed, $\delta_s\gamma = \delta_{s\gamma}$. Moreover, these elements are all the same modulo the space of holomorphic forms: if $s, t \in S$ are given in our coordinate system by $s = [a,b]$, $t = [c,d]$ then

$$\delta_t - \delta_s = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{dz}{(az-b)(cz-d)} \in \Omega(W).$$

We therefore have a natural exact sequence of G-modules

$$0 \longrightarrow \Omega(W) \longrightarrow \Omega_{\log}(W) \xrightarrow{\rho} \mathbf{Q}_p \longrightarrow 0$$

where the first map is the canonical inclusion and ρ is the "residue" map, which vanishes on $\Omega(W)$ and takes the value 1 on δ_s for every $s \in S$.

Here's a little proposition that exhibits the elements of $\Omega_{\log}(W)$ in a form convenient for computations. As before we write $W = W(S, r_0)$ and choose centers for the closed disks in the complement of W: say $B[S, r_0] := \bigcup_{i \in I} B[s_i, r_0]$.

Proposition. *Every $\omega \in \Omega_{\log}(W)$ has a unique representation in the form*

$$\omega = \sum_{i \in I} \left(a_0(i)\delta_{s_i} + \sum_{n=1}^{\infty} a_n(i) w_{s_i}^{-n} \frac{dw_{s_i}}{w_{s_i}} \right)$$

where for every real number $t > r_0$ the coefficients satisfy $|a_n(i)| = o(t^n)$ as $n \to \infty$. *In particular, the restriction maps $\Omega_{\log}(W(s_i, r_0)) \longrightarrow \Omega_{\log}(W)$ $(i \in I)$ induce a canonical isomorphism*

$$\bigoplus_{i \in I} \Omega_{\log}(W(s_i, r_0)) \xrightarrow{\cong} \Omega_{\log}(W).$$

If $r_1 < r_2$ in $|\mathbf{C}_p^\times|$ then $W(S, r_2) \subseteq W(S, r_1)$ and we have a continuous map

$$\Omega_{\log}(W(S, r_1)) \longrightarrow \Omega_{\log}(W(S, r_2)).$$

For any non-empty compact subset of $\mathbf{P}(\mathbf{Q}_p)$, let's define

$$\mathcal{H}_p(S) := \mathbf{P}(\mathbf{C}_p) \setminus S.$$

When $S = \mathbf{P}(\mathbf{Q}_p)$ we simply write $\mathcal{H}_p := \mathcal{H}_p(\mathbf{P}(\mathbf{Q}_p))$, which is the standard notation for the p-adic upper half plane. We define

$$\Omega_{\log}(\mathcal{H}_p(S)) := \varprojlim_{r > 0} \Omega_{\log}(W(S, r)),$$

so that in particular we have

$$\Omega_{\log}(\mathcal{H}_p) := \varprojlim_{r > 0} \Omega_{\log}(W(\mathbf{P}(\mathbf{Q}_p), r)).$$

§4. Distributions and Log-Differentials on wide open subspaces.

As in the last section, we fix a non-empty compact subset $S \subseteq \mathbf{P}(\mathbf{Q}_p)$. We define a $\mathbf{Q}_p$-linear map $\mu : \Omega_{\log}(W) \longrightarrow \mathcal{D}^\dagger(S, r)$ by the integration formula

$$\int_S f \cdot d\mu_\omega := \rho_{\partial W}(f\omega)$$

for any $f \in \mathcal{A}^\dagger(S,r)$. The "residue" $\rho_{\partial W}(f\omega)$ is defined in the obvious way as the sum of the residues over the oriented annuli at the "ends" of W. In fact, μ induces an isomorphism of topological vector spaces

$$\mu : \Omega_{\log}(W(S,r)) \xrightarrow{\cong} \mathcal{D}(S)$$

and passing to the limit as $r \to 0^+$, we also obtain an isomorphism

$$\mu : \Omega_{\log}(\mathcal{H}_p(S)) \xrightarrow{\cong} \mathcal{D}(S).$$

These isomorphisms are simple extensions of a theorem of Vishik (see the appendix of [2]) and Teitelbaum [5]. Indeed, they proved (in different languages) that there is an exact sequence

$$0 \longrightarrow \Omega(\mathcal{H}_p(S)) \xrightarrow{\mu} \mathcal{D}(S) \xrightarrow{\rho} \mathbf{Q}_p \longrightarrow 0$$

where ρ is defined by

$$\rho(\nu) := \int_S 1 \cdot d\nu.$$

We just have to extend the map μ from $\Omega(\mathcal{H}_p(S))$ to $\Omega_{\log}(\mathcal{H}_p(S))$. For a nice treatment of the correspondence between differentials and distributions, also see [4].

It is not hard to see that these isomorphisms commute with the action of G: namely,

$$\mu_{\omega|\gamma} = \mu_\omega|\gamma$$

for every $\omega \in \Omega_{\log}(W)$ and every $\gamma \in G$ where $G \subseteq PGL_2(\mathbf{Z}_p)$ is any subgroup that preserves S and $W(S,r)$.

Remark. Let $\omega \in \Omega_{\log}(W)$ be expressed as in the last proposition in the form:

$$\omega = \sum_{i \in I} \left(a_0(i)\delta_{s_i} + \sum_{n=1}^{\infty} a_n(i) w_{s_i}^{-n} \frac{dw_{s_i}}{w_{s_i}} \right).$$

For each $i \in I$, let $S_i = S \cap B[s_i, r]$. Then the coefficients in the above expansion have the following meaning in terms of the distribution μ_ω: for each $i \in I$ and each $n \geq 0$ we have

$$a_i(n) = \int_{S_i} w_{s_i}^n \cdot d\mu_\omega.$$

In particular, we have

$$\rho(\omega) = \int_S 1 \cdot d\mu_\omega.$$

Thus $\rho(\omega)$ may be interpreted as the "total measure" of μ_ω. It is perhaps also worth noting that for each $s \in S$, the differential δ_s corresponds under the map μ to the Dirac distribution supported at s, which is customarily denoted by the same symbol δ_{s_i}.

Henceforth, I will use the terms "distribution" and "log-differential" almost interchangeably.

§5. The difference equation $\mu|\Delta = \nu$ and the Kubota-Leopold p-adic L-function.

This is a side calculation that we need in order to lift classical modular symbols to overconvergent ones. It is also a good test case for doublechecking our work with differentials on wide open subspaces. I think it is also interesting in its own right because of its unexpected connection to the Kubota-Leopoldt p-adic L-function.

Consider the difference operator $\Delta := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - I$, which operates naturally on the spaces $\mathbf{D}^\dagger(\mathbf{x}_\infty, r)$ and $\mathbf{D}^\dagger(\mathbf{Z}_p, r)$ by $\mu \mapsto \mu|\Delta$.

Theorem. *For each $n \geq 0$ the following sequences are exact*

$$0 \longrightarrow \mathcal{D}^\dagger(\mathbf{Z}_p, p^{-n}) \xrightarrow{\Delta} \mathcal{D}^\dagger(\mathbf{Z}_p, p^{-n}) \xrightarrow{\rho} \mathbf{Q}_p \longrightarrow 0,$$
$$0 \longrightarrow \mathbf{Q}_p \xrightarrow{\cdot\omega_\infty} \mathcal{D}^\dagger(\mathbf{x}_\infty, p^{-1-n}) \xrightarrow{\Delta} \mathcal{D}^\dagger(\mathbf{x}_\infty, p^{-1-n}) \xrightarrow{\rho} \mathbf{Q}_p \longrightarrow 0.$$

Proof Sketch: First of all, it is clear from the definition of Δ that the image of Δ is contained in the kernel of ρ. Also, the kernel of Δ on $\mathcal{D}^\dagger(\mathbf{Z}_p)$ must be zero since there are no translation invariant analytic distributions on $\mathbf{Z}_p$. On $\mathbf{x}_\infty$ the only translation invariant distribution is δ_∞. So the heart of the proof is to show that every distribution (on either $\mathbf{Z}_p$ or $\mathbf{x}_\infty$) with total measure zero must be in the image of the difference operator. Rather than writing down a complete proof of this, let me instead describe an algorithm which, in either case, given $\nu \in \ker(\rho)$ produces a $\mu \in \mathcal{D}^\dagger$ such that $\nu = \mu|\Delta$.

Lemma. *Let W be the wide open subspace $W := \{ z \in \mathbf{P}(\mathbf{C}_p) \mid 1 < |z| < p \}$. For each $k \in \mathbf{Z}$ we define $\eta_k \in \Omega_{\log}(W)$ by*

$$\eta_k := \begin{cases} \sum_{n=k}^{\infty} \binom{n}{k} b_{n-k} \cdot z^{-n} \cdot \frac{dz}{z} & \text{if } k \neq 0 \\ \delta_0 + \sum_{n=1}^{\infty} b_n \cdot z^{-n} \cdot \frac{dz}{z} & \text{if } k = 0. \end{cases}$$

where the coefficients b_n are the Bernoulli numbers. Then for each integer k, η_k has an analytic continuation to an element $\eta_k \in \Omega_{\log}(\mathcal{H}_p)$. Moreover, the following assertions hold.

(1) η_k satisfies the difference equation

$$\eta_k|\Delta = \frac{k+1}{z^{k+1}} \cdot \frac{dz}{z};$$

(2) for each $n \geq 1$, η_k satisfies the distribution law

$$\eta_k = \frac{1}{p^{n(k+1)}} \cdot \sum_{a=0}^{p^n-1} \eta_k|\beta_n(a)$$

where $\beta_n(a) := \begin{pmatrix} 1 & a \\ 0 & p^n \end{pmatrix}$;

(3) if $k < 0$ *then* η_k *extends to a meromorphic differential on* $\mathbf{P}(\mathbf{C}_p)$ *with only one pole, namely a pole of order* $1 - k$ *at* ∞;

(4) If $k \geq 0$ *then* η_k *analytically continues to an element* $\eta_k \in \Omega_{\log}(\mathcal{H}_p(\mathbf{Z}_p))$;

(5) If $k > 0$, *then* $\eta_k \in \Omega(\mathcal{H}_p(\mathbf{Z}_p))$, *i.e.* η_k *is holomorphic.*

Proof of the lemma: To prove η_k extends to a log-differential on $\mathcal{H}_p$ it suffices to prove the much stronger assertions (3), (4), and (5).

If $k < 0$, then the sum defining η_k is a finite sum since $\binom{n}{k} = \binom{n}{n-k} = 0$ unless $n < 0$. In fact, the sum simplifies to $\eta_k = b_{-1-k}(z+1) \cdot dz$ where $b_m(z)$ is the mth Bernoulli polynomial. Since $b_{-1-k}(z)$ is a polynomial of degree $-1-k$ it has only one pole at ∞ of order $-1-k$. Also dz has a pole of order 2 at ∞ and is holomorphic elsewhere. So η_k has a pole of order $1-k$ at ∞, as claimed. Assertion (1) is an immediate consequence of the relation $b_m(z+1) - b_m(z) = mz^{m-1}$ for $m \geq 0$. Assertion (2) follows from the standard distribution relation satisfied by the Bernoulli polynomials.

Now suppose $k \geq 0$. We see at once that η_k extends to an element

$$\eta_k \in \Omega_{\log}(W(\mathbf{Z}_p, 1)).$$

Then a straightforward computation with power series shows that η_k is the unique element of $\Omega_{\log}(W(\mathbf{Z}_p, 1))$ satisfying the difference equation (1). Here are the details in the case $k = 0$. First of all we notice that $\delta_0 = \delta_\infty + dz/z$. Hence we may write $\eta_0 = \delta_\infty + \mu$ where

$$\mu = \sum_{n=0}^{\infty} b_n \cdot z^{-n} \cdot \frac{dz}{z}.$$

Since $\delta_\infty | \Delta = 0$, it therefore suffices to prove

$$\mu | \Delta = \frac{1}{z} \cdot \frac{dz}{z}.$$

Here's the calculation:

$$\begin{aligned} \mu \,\Big|\, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \sum_{n=0}^{\infty} b_n \cdot z^{-n} \cdot \frac{dz}{z} \,\Big|\, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \sum_{n=0}^{\infty} b_n \cdot (z-1)^{-n} \cdot \frac{dz}{z-1} \\ &= \sum_{n=0}^{\infty} b_n \cdot z^{-1-n}(1 - z^{-1})^{-1-n} \cdot dz. \end{aligned}$$

Now apply the binomial theorem to obtain

$$\begin{aligned}
\mu \Bigg| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \sum_{n=0}^{\infty} b_n \cdot z^{-1-n} \cdot \sum_{r=0}^{\infty} \binom{-1-n}{r} (-1)^r z^{-r} \cdot dz \\
&= \sum_{n=0}^{\infty} b_n \cdot z^{-1-n} \cdot \sum_{r=0}^{\infty} \binom{n+r}{r} z^{-r} \cdot dz \\
&= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+r}{r} b_n \cdot z^{-n-r} \cdot \frac{dz}{z} \\
&= \sum_{m=0}^{\infty} \sum_{r=0}^{m} \binom{m}{r} b_{m-r} \cdot z^{-m} \cdot \frac{dz}{z}.
\end{aligned}$$

A standard identity for Bernoulli numbers asserts that for $m > 0$,

$$\sum_{r=1}^{m} \binom{m}{r} b_{m-r} = \begin{cases} 1 & \text{if } m = 1; \\ \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with the last identity we obtain

$$\mu \Bigg| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \mu + \frac{1}{z} \cdot \frac{dz}{z}$$

and the claim is established.

To prove (2), we note that the right hand side of (2) lies in $\Omega_{\log}(W(\mathbf{Z}_p, 1))$. It therefore suffices to show that the right hand side satisfies the difference equation (1). Here's the calculation

$$\begin{aligned}
\left(\frac{1}{p^{n(k+1)}} \sum_{a=0}^{p^n-1} \eta_k | \beta_n(a) \right) \Bigg| \Delta &= \frac{1}{p^{n(k+1)}} \cdot \sum_{a=0}^{p^n-1} \eta_k | (\beta_n(a+1) - \beta_n(a)) \\
&= \frac{1}{p^{n(k+1)}} \cdot \eta_k | (\beta_n(p^n) - \beta_n(0)) \\
&= \frac{1}{p^{n(k+1)}} \cdot \eta_k | \Delta \beta_n(0) \\
&= \frac{k+1}{(p^n z)^{k+1}} \cdot \frac{dz}{z} \Bigg| \beta_n(0) \\
&= \frac{k+1}{z^{k+1}} \cdot \frac{dz}{z}.
\end{aligned}$$

To prove (4), we note that for each a, $\eta_k | \beta_n(a)$ is a log-differential on

$$W(\mathbf{Z}_p, 1) \cdot \beta_n(a) = W(a + p^n \mathbf{Z}_p, p^{-n})$$

and therefore the right hand side of (2) extends to the intersection of these wide open spaces, which is just $W(\mathbf{Z}_p, p^{-n})$. Hence, $\eta_k \in \Omega_{\log}(W(\mathbf{Z}_p, p^{-n}))$ for every $n > 0$. This proves η_k continues analytically to all of $\mathcal{H}_p(\mathbf{Z}_p)$. This completes the proof of the lemma.

Proof of the Theorem (continued). First we suppose $\nu \in \mathcal{D}^\dagger(\mathbf{Z}_p, p^{-n})$ with $\rho(\nu) = 0$. Write

$$\nu = \sum_{j=0}^{p^n-1} \nu_j$$

where each $\nu_j \in \mathcal{D}^\dagger(\mathbf{x}_j, p^{-n}) = \Omega_{\log}(W(\mathbf{x}_j, p^{-n}))$. Thus, for each j we may write

$$\nu_j = a_0(j)\delta_j + \sum_{m=1}^{\infty} a_m(j) w_j^{-m} \cdot \frac{dw_j}{w_j}.$$

Now for each $k \geq 0$, let $\eta_k \in \mathcal{D}$ be the kth Kubota-Leopoldt distribution described above. For $j = 0, 1, \ldots, p^n - 1$ let $s_0(j) = \sum_{r=0}^{j} a_0(r)$ and set

$$\mu_{\mathbf{x}_j} := -s_0(j)\delta_j + \sum_{m=1}^{\infty} \frac{a_m(j)}{m} \cdot \eta_{m-1} \left| \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \right. \in \mathcal{D}^\dagger(\mathbf{Z}_p, p^{-n}).$$

We then set

$$\mu := \sum_{j=0}^{p-1} \mu_{\mathbf{x}_j} \in \mathcal{D}^\dagger(\mathbf{Z}_p, p^{-n}).$$

This is the desired solution of the difference equation. To prove this, we have to show that the series defining μ converge on $W(\mathbf{Z}_p, p^{-n})$. This is a consequence of the von Staudt–Clausen Theorem (see [6]). I'll come back to this later. To show that μ solves the difference equation is a formal calculation.

Next we consider the case $\nu \in \mathcal{D}^\dagger(\mathbf{x}_\infty, p^{-1-n})$ with $\rho(\nu) = 0$. Then

$$\nu = a_0(\infty)\delta_\infty + \sum_{m=1}^{\infty} a_m(\infty) w_\infty^{-m} \cdot \frac{dw_\infty}{w_\infty}.$$

In this case we define

$$\mu := \sum_{n=1}^{\infty} \frac{a_m(\infty)}{m} \cdot b_m(z+1) \cdot dz \in \mathcal{D}^\dagger(\mathbf{x}_\infty, p^{-1-n})$$

where $b_m(z)$ is the nth Bernoulli polynomial. This is the desired solution of the difference equation, as one easily confirms.

To put this in perspective, the following result is of interest. The proof amounts to using the above formulas for η_k to explicitly calculate the special values of $L_p(\mu_k, s)$ at integers $s \leq 1$ and compare with the values of Kubota-Leopoldt (see for example [6]).

Proposition. *Let $k \geq 0$ and $\mu_k \in \mathcal{D}_0$ be the distribution associated to $\eta_k \in \Omega_{\log}(\mathcal{H}_p(\mathbf{Z}_p))$. Let $\mathcal{X}(\mathbf{C}_p)$ be the weight space:*

$$\mathcal{X}(\mathbf{C}_p) := \mathrm{Hom}_{cont}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$$

and let $L_p(\mu_k, s)$ be the analytic function defined for $s \in \mathcal{X}(\mathbf{C}_p)$ by

$$L_p(\mu_k, s) := \int_{\mathbf{Z}_p^\times} t^{1-s} \cdot d\mu_k(t).$$

Then

$$L_p(\mu_k, s) = \binom{1 - wt(s)}{k} (wt(s) + k - 1) \cdot \zeta_p(s + k),$$

where $\zeta_p(s)$ is the Kubota-Leopoldt p-adic zeta function and $wt(s) := \left.\dfrac{dt^s}{dt}\right|_{t=1}$.

§6. Hecke Operators at p

Let $\Gamma \subseteq SL_2(\mathbf{Z})$ be a congruence subgroup of tame level and let $\Gamma_0 := \Gamma \cap \Gamma_0(p)$. We define the Hecke operator

$$U_p : H_c^1(\Gamma, \mathcal{D}^\dagger) \longrightarrow H_c^1(\Gamma, \mathcal{D}^\dagger)$$

to be the following composition:

$$H_c^1(\Gamma, \mathcal{D}^\dagger) \xrightarrow{r_*} H_c^1(\Gamma_0, \mathcal{D}_0^\dagger) \xrightarrow{w_p} H_c^1(\Gamma_0, \mathcal{D}^\dagger) \xrightarrow{\mathrm{Tr}} H_c^1(\Gamma, \mathcal{D}^\dagger)$$

where these maps are explained as follows. The last map is the usual corestriction map (or trace). The map $r : \mathcal{D}^\dagger \longrightarrow \mathcal{D}_0^\dagger$ is defined by the integration formula

$$\int_{\mathbf{P}^1(\mathbf{Q}_p)} f \cdot dr(\mu) := \int_{\mathbf{Z}_p} f \cdot d\mu.$$

Let $w_p : \mathcal{D}_0^\dagger \longrightarrow \mathcal{D}^\dagger$ be the map defined at the end of §1. We then define

$$\begin{aligned} w_p : H_c^1(\Gamma_0, \mathcal{D}_0^\dagger) &\longrightarrow H_c^1(\Gamma_0, \mathcal{D}^\dagger) \\ \Phi &\mapsto \Phi | w_p, \end{aligned}$$

where $(\Phi|w_p)(D) = \Phi(w_p D)|w_p$ for all $D \in \Delta_0$. The operator $U_p : H_c^1(\Gamma_0, \mathcal{D}_0^\dagger) \longrightarrow H_c^1(\Gamma_0, \mathcal{D}_0^\dagger)$ is defined in the usual way in terms of double cosets. A straightforward calculation shows that the diagram

$$\begin{array}{ccc} H_c^1(\Gamma, \mathcal{D}^\dagger) & \xrightarrow{U_p} & H_c^1(\Gamma, \mathcal{D}^\dagger) \\ r_* \downarrow & & r_* \downarrow \\ H_c^1(\Gamma_0, \mathcal{D}_0^\dagger) & \xrightarrow{U_p} & H_c^1(\Gamma_0, \mathcal{D}_0^\dagger) \end{array}$$

is commutative.

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